Near-Markets and Market Games

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1. Introduction

The market game, both in its sidepayment and nosidepayment versions, has provided a valuable tool for the utilization of game theoretic analysis for the study of exchange economies. It is known that exchange economies map into totally balanced games[1] for both sidepayments and nosidepayments. It is known that totally balanced sidepayment games map into exchange economies and it is conjectured, but not yet proved or counterexampled, that every totally balanced nosidepayment game is representable by an exchange economy.[2]

A natural question to ask is are there other economic phenomena which give rise to market games or "near-market games"?

In this paper we consider the relationship of market games and near-market games to economies with complexities beyond that of the exchange economy. We argue that a broad class of replication economies generate near-market games, including private goods economies with nonconvexities, coalition production economies, and economies with local public goods. We note that economies with pure public goods do not, without special restrictions, give rise to near-market games.

More specifically, we say that a sequence of replica games is a sequence of near-market games if the games are superadditive and the sequence satisfies a "near-minimum efficient scale for coalitions" property—all increasing returns to coalition size are eventually exhausted. The near-minimum efficient scale property ensures that the sequence is asymptotically totally balanced—given any epsilon greater than zero and any subgame of any game in the sequence, when the set of players in that subgame is replicated sufficiently often, the epsilon-core of the replicated subgame is non-empty.

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1) We note that every subgame of a totally balanced game has a non-empty core.

2) Clearly the mappings are not one to one in both directions, as the market game contains far less information than does the exchange economy.

A model of a sequence of replica economies with coalition production and local public goods, where agents are allowed to be members of possibly more than one jurisdiction, is developed and the derived sequence of games is shown to be a sequence of near-market games. Few restrictions are placed on the model; the major ones are that the asymptotic growth of utility functions is no more than linear and the production correspondences are such that positive outputs do not become virtually free in per-capita terms as the economies become large.

The model in this paper differs from Shubik and Wooders (1983a, 1983b) in that here both local public goods and coalition production are incorporated; in the previous papers, except for a few examples, local public goods were not considered. Moreover, coalition structures are not restricted to partitions; instead, a class of allowable coalition structures is taken as given and in an allowable coalition structure it is possible that an agent can belong to one or more jurisdictions and one or more productive coalitions simultaneously. The primary purpose of the development of the model herein is to manifest the strength of the near-minimum efficient scale property. This natural and nonrestrictive property ensures the asymptotic total balancedness. The assumption of the sidepayments property enables us to keep the proofs relatively simple.

Except under some very special conditions (satiation or "asymptotic" satiation, *i.e.*, marginal utilities go to zero as the amount of the public good increases) replica economies with pure public goods do not generate near-market games. Since games derived from economies with pure public goods may well be totally balanced, this suggests that, in line with the incentives literature it is when we consider sequences of economies that game-theoretic properties of private-goods or "market-like" economies and of pure public goods economies differ.

As indicated above, in this paper we analyze replication sequences of economies–ones with a fixed number of types of agents and increasing numbers of agents of each type. This keeps the analysis relatively straightforward and facilitates development and exposition of the model. To extend this analysis to economies with a continuum of agents or with simply a "large" number of agents appears to pose different problems than extensions of this nature for private goods exchange economies. The technical results in this paper are based on results concerning non-emptiness of approximate cores of large replica games (See Wooders (1983) and Shubik and Wooders (1983a)); these results are for games without sidepayments but apply to the sidepayments case. The extension of much of the analysis to economies with a large number of agents (but not necessarily replica situations) could, we believe, be carried out using the more recent game-theoretic framework in Wooders and Zame (1984) and, to the continuum, by using the Kaneko and Wooders (1984) framework (Kaneko and Wooders is also for "without sidepayments").

In Section 2 a mathematical structure and analysis for near-market games is presented. Section 3 contains our model of a sequence of replica economies whose derived games are near-market games. Section 4 concludes the paper. All proofs are contained in the appendix.

2. Near Market Games

2.1 Games

We first review some game-theoretic concepts.

A game (with sidepayments) is an ordered pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set, called the set of *players*, and v is a real-valued function mapping subsets of N into R_+ with $v(\varnothing) = 0$. Two players i and j are *substitutes* if given any subset S of N where $i \notin S$ and $j \notin S$, we have $v(S \cup \{i\}) = v(S \cup \{j\})$. A *subgame* (S, v) of (N, v) is an ordered pair consisting of a non-empty subset S of N and the function v restricted to subsets of S. The game (N, v) is *superadditive* if for all disjoint subsets S and S' of N, we have $v(S) + v(S') \leq v(S \cup S')$. A *payoff* for the game is a vector $B = (B^1, \ldots, B^n) \in R^n_+$, the non-negative orthant of the n-fold Cartesian product of the reals. A payoff B is *feasible* if $\sum_{i \in N} B^i \leq v(N)$. Given $\varepsilon \geq 0$, a payoff is in the (weak) *ε-core*[3] if it is feasible and if, for all non-empty subsets S of N, $\sum_{i \in S} B^i \geq v(S) - \varepsilon |S|$ where $|S|$ denotes the cardinal number of the set S. When $\varepsilon = 0$, the ε-core is called simply the *core*.

Given a game (N, v), let $(N, \tilde{v})$ denote the *totally balanced cover* of (N, v); the function $\tilde{v}$ is the smallest real-valued function such that, for all non-empty subsets S of N, the subgame $(S, \tilde{v})$ has a non-empty core and $v(S) \leq \tilde{v}(S)$.

2.2 Sequences of Games

A sequence of games $(N_r, v_r)_{r=1}^{\infty}$ is *superadditive* if each game (N_r, v_r) is superadditive. It is *per-capita bounded* if there is a constant K, independent of r, such that $v_r(N_r)/|N_r| \leq K$ for all r.

Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of games where, for some positive integer T, for each r the set of players N_r contains rT players, denoted by $N_r = \{(t, q) : t = 1, \ldots, T, q = 1, \ldots, r\}$. For each r and each t, let $[t]_r = \{(t, q) : q = 1, \ldots, r\}$. The sequence is a *sequence of replica games* if

(1) $N_r \subset N_{r+1}$ for all r;

(2) for each r and all subsets S of N_r, $v_r(S) \leq v_{r'}(S)$ whenever $r' \geq r$;

(3) for each r and each t all players in $[t]_r$ are substitutes for each other.

Throughout the following, given a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ we define $[t]_r$ as above and call the members of $[t]_r$ *players of type t*. We also assume there are T types of players and denote the set of players N_r as above.

Given a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ and a subset S of N_r for some r, define the vector $\rho(S) = (s_1, \ldots, s_T)$ by its coordinates $s_t = |S \cap [t]_r|$; the vector $\rho(S)$ is called the *profile* of S and is simply a list of the numbers of players of each type contained in S. Let I denote the T-fold Cartesian product of the nonnegative integers. Observe that for any r and any subset S of N_r, we have $\rho(S) \in I$. Also, since players of the same type are substitutes, if S and S' are two subsets with the same profiles then for any r such that $S \subseteq N_r$ and $S' \subseteq N_r$, we have $v_r(S) = v_r(S')$. Consequently the function v_r can be completely defined by a mapping from a subset of I to the reals. In the following, given r and a profile s of a subset of N_r, we define $v_r(s)$ as $v_r(S)$ for any $S \subseteq N_r$ with $\rho(S) = s$.[4] Given $s \in I$ we write $|s| = \sum_{t=1}^{T} s_t$ since when $\rho(S) = s$, we have $|S| = \sum_{t=1}^{T} s_t$.

We say a sequence of replica games satisfies the property of "*near minimum efficient scale*" (for

3) This concept was introduced by Shapley and Shubik (1966).

4) This abuse of notation should create no confusion. We note that we typically denote subsets by upper case letters and profiles by lower case ones.

coalitions), NMES, if it is per-capita bounded. If the sequence is also superadditive, we say the sequence is a sequence of *near-market games.*

We remark that when a sequence of replica games is per-capita bounded and superadditive, both $\tilde{v}(N_r)/|N_r|$ and $v(N_r)/|N_r|$ converge and to the same limit. In this case, for r sufficiently large, the per-capita gains to forming a coalition larger than N_r are small. This motivates our term "near-minimum efficient scale."

A sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ is *asymptotically totally balanced* if, given any r, any subset S of N_r, and any $\varepsilon > 0$, there is an n^* such that for all $n \geq n^*$ we have

$$\frac{\tilde{v}_{nr}(S_n)}{|S_n|} - \frac{v_{nr}(S_n)}{|S_n|} < \varepsilon,$$

where v_{nr} denotes the function $v_{r'}$ with $r' = nr$ and S_n is any subset of $N_{r'}$ with $\rho(S_n) = n\rho(S)$. It can easily be verified (and follows from well-known results, cf. Shapley (1967)), that given any game (N, v), we have $\tilde{v}(N)/|N| - v(N)/|N| < \varepsilon$ if and only if the ε-core of the game is non-empty. Consequently, given any subset S of N_r for some r and any sequence of subsets (S_n) satisfying the properties required above, for all n sufficiently large the subgames (S_n, v_{nr}) have non-empty ε-cores.

The following theorem provides sufficient conditions for asymptotic total balancedness of sequences of replica games.

Theorem 1. Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of near-market games. Then the sequence is asymptotically totally balanced.

Proof. All theorems are proven in the Appendix.

3. Near-Market Economies

3.1 Introduction to Near-Market Economies

In this section we develop a model of a sequence of replica economies with private goods, local public goods, and coalition production. Minimal restrictions are imposed on the model yet we are able to show that the sequence of derived games is superadditive and per-capita bounded and thus is asymptotically totally balanced. Therefore, the class of replication economies we consider are near-market economies–the derived games are near-market games for large replications.

3.2 The Model

The model may look more formidable than it actually is. Therefore, before the formal statement of the model, we will describe the main components.

An economy $\mathscr{E}_r$ is defined for each replication number r. The set of agents N_r consists of r agents of each of T types, where all agents of the same type have the same endowments and preferences. We take $\mathbb{R}^l$ as the private commodity space and $\mathbb{R}^m_+$ as the public commodity space. Utility functions then map $\mathbb{R}^{m+l}_+$ into R^1_+. Only private commodities are initially endowed so endowments are in R^l_+.

A jurisdiction structure is a specification of a collection of jurisdictions where all the agents in each jurisdiction jointly consume (local) public goods and where every agent belongs to at least one jurisdiction. In this model, we do not restrict allowable jurisdiction structures[5] to partitions.

Therefore, for any set of agents $S \subset N_r$ we take as given a set of allowable jurisdiction structures $\mathscr{J}_r(S)$; thus an element $J \in \mathscr{J}_r(S)$ is an allowable jurisdiction structure of S.

We also take as given a public goods production correspondence Z_r which associates a public goods production set with every possible jurisdiction (every non-empty subset of N_r). Thus allowable jurisdiction structures are given by the correspondence $\mathscr{J}_r(S)$ and production possibilities by $\mathscr{Z}_r$.

Allowable firm structures and production possibilities for private goods are defined in a manner analogous to the definitions of allowable juristiction structures and production possibilities for public goods.

A *sequence of replica economies* $(\mathscr{E}_r)_{r=1}^{\infty}$ is defined as a sequence of septuples

$$\mathscr{E}_r = (N_r, \mathbb{R}^l, \mathbb{R}^m_+, U_r, W_r, (\mathscr{J}_r, Z_r), (\mathscr{F}_r, Y_r))$$

where $N_r = \{(t, q): t = 1, \ldots, T, q = 1, \ldots, r\}$ is the set of agents;

$\mathbb{R}^l$ is the private commodity space;

$\mathbb{R}^m_+$ is the public commodity space;

$U_r = \{u^{tq}: (t, q) \in N_r\}$ is an indexed collection of utility functions mapping $\mathbb{R}^{m+l}_+$ into R^1_+ with the property that for some linear function L and some real number c we have $u^{tq}(x,y) \leq L(x,y) + c$ for all x in $\mathbb{R}^m_+$ and y in R^l_+ and for all (t,q) in N_r;[6)]

$W_r = \{w^{tq} \in R^l: (t, q) \in N_r\}$ is an indexed collection of initial endowment vectors, each in R^l_+ (no public goods are initially endowed);

$(\mathscr{J}_r, Z_r)$ is a pair of correspondences, where $\mathscr{J}_r$, called the *allowable jurisdiction structure correspondence*, maps non-empty subsets S of N_r into collections of non-empty subsets of S and Z_r, called the *public goods production correspondence*, maps subsets S of N_r into subsets of $\mathbb{R}^m_+ \times \mathbb{R}^l$;

and $(\mathscr{F}_r, Y_r)$ is another pair of correspondences where $\mathscr{F}_r$, called the *allowable firm structures correspondence*, maps non-empty subsets S of N_r into collections of subsets of S and Y_r, the *private goods production correspondence*, maps non-empty subsets S of N_r into subsets of $\mathbb{R}^l$.

Before proceeding, some further clarifying remarks may be in order. Throughout the paper, as above, a vector of public goods will be denoted by x, x', *etc.* and a vector of private goods by y, y', *etc.* The allowable jurisdiction structure correspondence simply describes, given a subset of agents S, the jurisdiction structures of S which are permitted; this correspondence is taken as given, and must satisfy certain conditions which will be stated later. The public goods production correspondence associates a production possibility set for public goods with each subset of agents; this correspondence is also taken as given and will also be required to satisfy certain conditions. The allowable firm structure correspondence is analogous to the allowable jurisdiction structure correspondence and specifies, for any given set of agents, those collections of subsets which can engage in production of private goods. The private goods production correspondence associates a production possibility set (for private goods) with each subset of agents.

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5) The reader may find it useful to think of a jurisdiction as a club, (and *not* as a "locality" in a spatial model).

6) This property is needed even to ensure that a sequence of *private* goods exchange economies generates a sequence of near-market games (see Shapley and Shubik (1966)) and is nonrestrictive.

A number of further specifications are made on the components of a sequence of replica economies:

(1) $N_r \subset N_{r+1}$ for each r (this relation is for technical convenience).

(2) For each t, each q and q' in $\{q'': q'' = 1, \dots, r\}$, and all r, $u^{tq} = u^{tq'}$ and $w^{tq} = w^{tq'}$; *i.e.*, all agents of the same type have the same utility functions and the same initial endowments. Also, $u^{tq}(w^{tq}) > 0$ for all $(t, q) \in N_r$ and for all r (this assumption is for technical convenience).

(3) Given r, $S \subseteq N_r$, and $J(S) \in \mathscr{J}_r(S)$, $J(S)$ is *an allowable jurisdiction structure of* S and a member of $J(S)$ is called a *jurisdiction*. Allowable jurisdiction structures $J(S)$ are required to satisfy the properties that

(3a) $S \subseteq \bigcup_{S' \in J(S)} S'$ (allowable jurisdiction structures of S cover S);

(3b) if $J(S) \in J_r(S)$ and $J(S') \in J_r(S')$, where S and S' are non-empty, disjoint subsets of agents, then $\{S'' \subseteq N_r : S'' \in J(S) \cup J(S')\} \in \mathscr{J}_r(S \cup S')$;

(3c) given $S \subseteq N_r$ and $r' \geq r$, if $J(S)$ is in $\mathscr{J}_r(S)$, then $\mathscr{J}(S)$ is in $\mathscr{J}_{r'}(S)$;

(3d) if S and S' are non-empty subsets of N_r with the same profiles, then there is a one-to-one mapping, say ψ, of $\mathscr{J}_r(S)$ onto $\mathscr{J}_r(S')$ such that if $\psi(J(S)) = J(S')$, then the collection of profiles of members of $J(S)$ (not all necessarily distinct), equals those of $J(S')$.

As stated earlier, $\mathscr{J}_r$ describes the set of allowable jurisdiction structures of S for each subset S of N_r. Given S, a member of $\mathscr{J}_r(S)$, say $J(S)$, must satisfy certain properties. The first property (3a) is that each agent must be in some jurisdiction. Property (3b) states that if $J(S)$ is an allowable jurisdiction structure of S and $J(S')$ is an allowable jurisdiction of S', where S and S' are disjoint, then the jurisdiction structure of $S \cup S'$ consisting of the union of members of $J(S)$ and $J(S')$ is allowable. Note that partitions satisfy (3a) and (3b) above but (3a) and (3b) also admit jurisdiction structures which are not partitions. Property (3c) is simply that the set of allowable jurisdiction structures of a set of agents does not decrease as the size of the set of agents increase. Finally, (3d) simply states that the set of allowable jurisdiction structures of a set of agents depends only on the number of agents of each type in that set.

To ensure that the meaning of (3) is clear, following is an example. Let $N_2 = \{(1,1), (2,1), (3,1), (1,2), (2,2), (3,2)\}$ so N_2 has 6 agents, 2 agents of each of three types. Let $S = \{(1,1), (2,1), (3,1)\}$. Then let $\mathscr{J}_2(S)$ be the partitions of S unioned with $J(S) = \{\{(1,1), (2,1)\}, \{(2,1), (3,1)\}\}$; for some unspecified reason, in addition to the partitions of S, the allowable jurisdiction structures of S include one where agent $(2,1)$ can be in a jurisdiction with $(1,1)$ and in a jurisdiction with $(3,1)$ simultaneously. Let $S' = \{(1,2), (2,2), (3,2)\}$. Since S and S' have the same profiles, we must have $\mathscr{J}_2(S')$ equal to the partitions of S unioned with $J(S') = \{\{(1,2), (2,2)\}, \{(2,2), (3,2)\}\}$. Also, since S and S' are disjoint $\mathscr{J}_2(S \cup S')$ must include, for example $J(S) \cup \{\{(1,2)\}, \{(2,2), (3,2)\}\}$ from (3b).

(4) The public goods production correspondence is required to satisfy the properties that

(4a) given $S \subseteq N_r$ and $r' > r$, $Z_r(S) \subseteq Z_{r'}(S)$ (the public goods production possibility set available to a coalition does not decrease when the economy becomes larger);

(4b) if S and S' are non-empty subsets of N_r for any r, with the same profiles, then

$Z_r(S) = Z_r(S')$, (the production possibility set for a subset of N_r depends only on the profile of that coalition);

(4c) $0 \in Z_r(S)$ for all non-empty subsets S of N_r and for all r.

(5) Given a subset S of N_r, *allowable firm structures* $F(S) \in \mathscr{F}_r(S)$ are required to satisfy the same properties as allowable jurisdiction structures, *i.e.* (3a), (3b), (3c), and (3d) of (3) above. Also, the private goods production correspondence is assumed to satisfy the same properties as the public goods production correspondence, (4a), (4b), and (4c) above.

(6) For private goods, we define the *aggregate production correspondence.* Given r and $S \subseteq N_r$, define

$$\bar{Y}_r(S) = \bigcup_{F(S) \in \mathscr{F}_r(S)} \sum_{S' \in F(S)} Y_r(S');$$

then $\bar{Y}_r(\cdot)$ is the rth *aggregate production correspondence.* Note that $\bar{Y}_r(\cdot)$ *is superadditive*; given any two disjoint, non-empty subsets S and S' of N_r, we have $\bar{Y}_r(S) + \bar{Y}_r(S') \subseteq \bar{Y}_r(S \cup S')$.

The above consists of a description of the components of a sequence of replica economies and members of the sequence. Note that at this point very little structure has been imposed on the model. In the following, we introduce additional definitions which enable us to relate production decisions to consumption decisions and to define feasible states of an economy.

An *N_r-allocation* is a vector $(x, y) = (x^{11}, \ldots, x^{Tr}, y^{11}, \ldots, y^{Tr}) \in \mathbb{R}_+^{Tr(m+l)}$ where (x^{tq}, y^{tq}) is a commodity bundle for the (t, q)th agent. Given any r and any non-empty subset S of N_r, an *S-allocation* is an N_r-allocation where $x^{tq} = 0$ and $y^{tq} = 0$ if $(t, q) \notin S$.

Given r and a non-empty subset S of N_r, an *S-private goods production plan* is a vector $y \in \bar{Y}_r(S)$.

In this paper it is assumed that public goods produced in one jurisdiction are not transferable to another. This necessitates a different approach to the definition of public goods production plans than that used for private goods (which can be exchanged between agents, productive coalitions, and/or jurisdiction). Specifically, for public goods we keep track of the jurisdiction structure associated with a public goods production plan. Given a non-empty subset S of N_r, an *S-public goods production plan* is an ordered pair, $\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$ where $J(S) \in \mathscr{J}_r(S)$.

Given r and a non-empty subset S of N_r, an *S-state of the economy, $e(S)$*, is an ordered triple, $e(S) = (\bar{y}, \varphi(S), (x, y))$ where $\bar{y}$ is a private goods production plan for S, $\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$ is an S-public goods production plan, and (x, y) is an S-allocation where $x^{tq} = \sum_{\substack{tq \in S' \\ S' \in J(S)}} x_{S'}$ for all $(t, q) \in S$ (the (t, q)th agent consumes the total outputs of public goods produced in all jurisdictions of which he is a member). The state is *S-feasible* if $\sum_{(t,q) \in S} (y^{tq} - w^{tq}) + \sum_{S' \in J(S)} z_{S'} = \bar{y}$. An N_r-state of the economy is called simply *a state of the* rth *economy*, or, when no confusion is likely to arise, simply a *state of the economy.*

Given $S \subseteq N_r$, let $A_r(S) = \{(x, y) :$ there is an S-feasible state of the economy with the associated S- allocation $(x, y)\}$. The set $A_r(S)$ is called the set of *S-attainable allocations.* We note that if $r' \geq r$, then

$$\text{Proj}_S\, A_r(S) \subseteq \text{Proj}_S\, A_{r'}(S)$$

where $\text{Proj}_S\, A_{r''}(S)$ denotes the projection of the set $A_{r''}(S)$ onto the subset of $\mathbb{R}^{|S|(m+l)}$ associated with the members of S for any replication number r''.

3.3 The Derived Games

We now define the sequence of games derived from the sequence of economies.

Given r and $S \subseteq N_r$, define $v_r(S) = \sup_{(x,y) \in A_r(S)} \sum_{(t,q) \in S} u^{tq}(x^{tq}, y^{tq})$ when $S \neq \varnothing$. Define $v_r(S) = 0$ when $S = \varnothing$.

Observe that the pair (N_r, v_r) is a game with sidepayments. The generation of a game with sidepayments presumes, as usual, the existence of a freely transferable medium of exchange; there is no need to introduce this explicitly. It is straightforward to verify that the sequence of derived games is a sequence of replica games.

3.4 Near-Market Economies

Without further restrictions on the economies, in particular on production, there is no assurance that the derived sequence of games is a sequence of near-market games. The restrictions required are, informally, that positive production does not become virtually free as the economies become large. Formally, we assume

A1. There is a closed convex cone $Y^* \subset \mathbb{R}^l$, with $-\mathbb{R}^l_+ \subseteq Y^*$ and $Y^* \cap \mathbb{R}^l_+ = \{0\}$, such that $Y_r(S) \subseteq Y^*$ for all subsets S of N_r and for all r.

A2. There is a closed convex cone $\mathscr{Z}^* \subseteq \mathbb{R}^m_+ \times \mathbb{R}^l$, with $\{0\} \times -\mathbb{R}^l \subseteq \mathscr{Z}^*$ and $\mathscr{Z}^* \cap \mathbb{R}^m_+ \times \mathbb{R}^l_+ = \{0\}$, such that for any r, any non-empty subset $S \subseteq N_r$, and any allowable jurisdiction structure $J(S) \in \mathscr{J}_r(S)$, we have $\sum_{S' \in J(S)} (x_{S'}, z_{S'}) \in \{(x,z) \in \mathbb{R}^{m+l}: (|S|x, z) \in Z^*\}$ for all $(x_{S'}, z_{S'}) \in Z(S')$ and for all $S' \in J(S)$.

The first assumption is clear. The second is that there is some set $\mathscr{Z}^*$ satisfying the properties of a standard private goods production set and public goods are never cheaper per capita than private goods would be if they were produced with the "production" set $\mathscr{Z}^*$.

An example of production correspondences which satisfy A2 in the one-private-good, one-public-good case is given by the production functions $x + z/|S| = 0$ for each subset S of N_r and for all r with $\mathscr{J}_r(S)$ equal to the set of partitions of S. Here the per capita costs of the public good, in terms of the inputs, is constant and independent of the size and composition of the jurisdiction structure.

To see what A2 rules out, suppose all coalitions have the production set Z determined by the production function $x + z = 0$, there is only one private good and one public good, and again $\mathscr{J}_r(S)$ is the set of all partitions of S. To show that A2 is not satisfied, let $x_r = 2/r$ and $z_r = -(2/r)$ for each positive integer r. Observe that $(x_r, z_r) \in Z(S)$ for all $S \subseteq N_r$ and for all r. Choose a sequence of subsets $S_r \subseteq N_r$ for each r such that $|S_r| = r$. We then have $\lim_{r\to\infty} |S_r| x_r = 2$ and $\lim_{r\to\infty} z_r = 0$. This contradicts A2 since $\mathscr{Z}^*$ is closed and $\mathscr{Z}^* \cap \mathbb{R}^2_+ = \{0\}$.

The consequences of A1 are described in Shubik and Wooders (1983b).

Together, the assumptions on Z_r and Y_r and on the utility functions imply that the sequence of derived games is per-capita bounded. Since the sequence of derived games is also superadditive, it is a sequence of near-market games and asymptotically totally balanced.

Theorem 2 Let $(\mathscr{E}_r)_{r=1}^{\infty}$ be a sequence of replica economies satisfying A1 and A2. Then the derived sequence of replica games is a sequence of near-market games and is asymptotically totally balanced.

In the Appendix, we show that the derived sequence of games is per-capita bounded and superadditive. Theorem 2 then follows as a Corollary to Theorem 1.

4. Conclusions

In this paper we have introduced concepts of sequences of near-market games and market economies. It was demonstrated that when "free" consumption (in per-capita terms) is ruled out, then sequences of economies with local public goods and coalition production generate sequences of near-market games—a property of sequences of private-goods-exchange economies. Also, this has been done with few restrictions; non-monotonicity of utility functions, non-convexities, and indivisibilities are allowed. The analysis suggests that many of the specifics of the particular model investigated could be varied and the same results still obtained. Thus it appears that diverse models of economic structures have the near-market game property. The fundamental condition for this property to obtain is that of a near-minimum efficient scale for coalitions.

Appendix

In this appendix, Theorem 1 is proven. Whenever possible, for the sake of brevity, we use results currently available for sequences of replica games. Also, throughout this appendix, the sequences of games are assumed to be sequences of per-capita bounded, superadditive replica games.

When a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ is superadditive and per-capita bounded, it can be shown that for any r', and any profile $s \leq \rho(N_{r'})$, the sequence of subgames $(S_n, v_{nr'})_{n=1}^{\infty}$ is superadditive and per-capita bounded, where $v_{nr'}$ is the characteristic function of the nr'th game and S_n is any subset of $N_{nr'}$ with profile equal to ns. Thus, to show that the sequence $(N_r, v_r)_{r=1}^{\infty}$ is asymptotically totally balanced, we need only show that the sequence is asymptotically balanced, *i.e.*, $\lim_{r\to\infty} \frac{v_r(N_r)}{|N_r|} = \lim_{r\to\infty} \frac{\tilde{v}_r(N_r)}{|N_r|}$. Then every subsequence of subgames $(S_n, v_{nr'})_{n=1}^{\infty}$ as above is asymptotically balanced.

Observe that from Wooders ((1983), Lemma 8), the limits, as r goes to infinity, of both $v_r(N_r)/r$ and $\tilde{v}_r(N_r)/r$ exist and are equal. This, and our observations above, prove Theorem 1. Also, this theorem can easily be obtained as a consequence of Wooders ((1983), Theorem 1).

Proof of Theorem 2

To prove the theorem, we need only prove that the sequence of derived games is superadditive and per-capita bounded. It is straightforward to verify that the sequence is superadditive; thus we omit the proof.

To prove per-capita boundedness of the sequence of games, we construct another sequence of economies, say the *-economies, so that the sequence of games derived from the *-economies, denoted by $(N_r, v_r^*)_{r=1}^{\infty}$, is per-capita bounded and has the property that for all r and for all non-empty subsets S of N_r, we have $v_r(S) \leq v_r^*(S) + c|S|$.

For the *-economies, let Y^* be the private goods production possibility set available to all coalitions $S \subseteq N_r$ for all r. Observe that for any firm structures, say $F_r(S)$ and $F'_r(S)$ we have

$$\sum_{S' \in F_r(S)} Y^* = \sum_{S' \in F'_r(S)} Y^*$$

so the firm structure will be irrelevant. Of course, $Y_r(S) \subseteq Y^*$ and for any firm structure $F(S)$, $\sum_{S' \in F(S)} Y^* \subseteq Y^*$.

Given r and a subset S of N_r, define $Z^*(S) = \{(x, z) \in \mathbb{R}^{m+l} : (|S|x, z) \in \mathscr{Z}^*\}$. Note that $Z^*(S)$ is a closed convex cone with vertex $\{0\}$ and $\{0\} \times -\mathbb{R}^l_+ \subseteq Z^*(S)$; this follows from the assumptions on $\mathscr{Z}^*$. Let $\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$ be an S-public goods production plan. From assumption A.2, there is an $(x, z) \in Z^*(S)$ such that

(a) for each $(t, q) \in S$, we have $\sum_{\{S' \in J(S)\ (t,q) \in S'\}} x_{S'} \leq x$

and

(b) $\sum_{S' \in J(S)} z_{S'} \leq z$.

Informally, there is an (x, z) in $Z^*(S)$ "at least as good as" any S-public goods production plan in the sense that with the production possibility set $Z^*(S)$, the agents can consume as much of the local public goods while using no more of the inputs. (Note, however, because agents do not necessarily have monotonic increasing preferences for the local public goods, they may not prefer to have more of them.)

For the purposes of this theorem, we can restrict our attention to states of the *-economies with associated jurisdiction structures $\{\{(t, q) : (t, q) \in N_r\}$ since with this jurisdiction structure all agents can be made "at least as well-off" as with any other jurisdiction structure. To see this, given any r and any non-empty subset S of N_r, let $(x, z) \in Z^*(S)$ so $(|S|x, z) \in \mathscr{Z}^*$ and $(x, z/|S|) \in Z^*(\{(t, q)\})$ for each $(t, q) \in S$; thus with the jurisdiction structure $\{\{(t, q)\} : (t, q) \in N_r\}$ each agent in S can consume x and total inputs are unchanged.

In the *-economies, the agents will all have the same utility functions. For each r and for all $(t, q) \in N_r$, define $u^{*tq}(x, y) = L(x, y)$ where L is a linear function such that $u^{tq}(x, y) \leq L(x, y) + c$ for some constant c for all $(x, y) \in \mathbb{R}^m_+ + \mathbb{R}^l_+$ and for all $(t, q) \in N_1$.

For each r and each non-empty subset S of N_r, let $A^*_r(S)$ denote the set of S-attainable allocations for the *-economies.

For each r and all non-empty subsets S of N_r, define

$$v^*_r(S) = \sup_{(x,y) \in A^*_r(S)} \sum_{(t,q) \in S} L^*(x^{tq}, y^{tq}).$$

The finiteness of this "sup" is ensured by the linearity of the function L^* and boundedness of $A^*_r(S)$. Also,

$$v^*_r(S) = \sup_{(x,y) \in A^*_r(S)} L\Big(\sum_{(t,q) \in S} x^{tq}, \sum_{(t,q) \in S} y^{tq}\Big).$$

Let $\mathscr{K}$ be a real number such that $K > v^*_1(N_1)$. We will show that $\mathscr{K}$ is a per-capita bound for the sequence of games $(N_r, v^*_r)^\infty_{r=1}$. Suppose not. Then there is an r' and a feasible state of the r'th *-economy, say $e^*(N_{r'}) = (\bar{y}, (J(N_{r'}), \{(x_{S'}, z_{S'}) \in Z^*(S') : S \in J(N_{r'})\})), (x, y))$, such that $L(\sum_{(t,q) \in N_{r'}} x^{tq}, \sum_{(t,q) \in N_{r'}} y^{tq}) > \mathscr{K} r'$. We can assume without any loss that $F(N_{r'}) = \{N_{r'}\}$

and $J(N_{r'}) = \{\{(t, q)\}: (t, q) \in N_{r'}\}$. Also, we can assume that $x^{tq} = x^{t'q'}$ and $y^{tq} = y^{t'q'}$ for all (t, q) and (t', q') in $N_{r'}$. We claim that there is an $(x', y') \in A_1^*(N_1)$ with $x'^{tq} = x^{tq}$ and $y'^{tq} = y^{tq}$ for all $(t, q) \in N_1$, which will yield a contradiction. Since $e^*(N_{r'})$ is feasible (for the r'th *-economy), we have $\bar{y} \in Y^*$. Observe that $\bar{y}/r' \in Y^*$. Also, for some z^{tq} for each $(t, q) \in N_{r'}$, we have $(x^{tq}, z^{tq}) \in Z^*(\{(t, q)\})$, and $\sum_{(t,q) \in N_{r'}} (y^{tq} - w^{tq}) + \sum_{(t,q) \in N_{r'}} z^{tq} = \bar{y}$ so $\sum_{(t,q) \in N_1} (y^{tq} - w^{tq}) + \sum_{(t,q) \in N_1} z^{tq} = \bar{y}/r'$.[7] This proves our assertion that $(x', y') \in A_1^*(N_1)$. But then $L(\sum_{(t,q) \in N_1} x'^{tq}, \sum_{(t,q) \in N_1} y'^{tq}) = \frac{1}{r'} L(\sum_{(t,q) \in N_{r'}} x^{tq}, \sum_{(t,q) \in N_{r'}} y^{tq}) > \mathscr{K}$ which is a contradiction. Therefore the sequence $(N_r, v_r^*)_{r=1}^{\infty}$ is per-capita bounded.

Since $u^{tq}(x, y) \leq L(x, y) + c$ for all $(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^l$, and from the construction of the production possibilities set for the *-economies, we have $v_r(N_r) \leq v_r^*(N_r) + c|N_r|$ for all r. Therefore the sequence $(N_r, v_r)_{r=1}^{\infty}$ is per-capita bounded.

Q.E.D.

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7) The vector z^{tq} can be taken equal to $z_{S'}$ when $\{(t, q)\} = S'$, where $S' \in J(N_{r'})$.