Journal of Economic Behavior & Organization
Vol. 64 (2007) 406–425

JOURNAL OF
Economic Behavior
& Organization

www.elsevier.com/locate/econbase

Networks and clubs[☆]

Frank H. Page Jr.[a,c,*], Myrna Wooders[b,d,1]

[a] *Department of Economics, Indiana University, Bloomington, IN 47405, United States*
[b] *Department of Economics, Vanderbilt University, Nashville, TN 37235, United States*
[c] *Centre d'Economie de la Sorbonne, Universite Paris 1, Pantheon-Sorbonne, France*
[d] *Department of Economics, University of Warwick, Coventry CV4 7AL, UK*

Received 17 September 2005; accepted 4 January 2007
Available online 17 June 2007

Abstract

We formulate a club model where players' have identical single-peaked preferences over club sizes as a network formation game. For situations with "many" clubs, we provide necessary and sufficient for non-emptiness of the farsighted core and the direct (or myopic) core. With "too few" clubs, if players are farsighted then the farsighted core is empty. In this same case, if players are myopic then the direct core is always nonempty and, for any club network in the direct core, clubs are of nearly equal size (i.e., clubs differ in size by at most one member).

JEL Classification: A14; D20; J00

Keywords: Clubs; Network formation games; Path dominance core; Nash club equilibria

1. Introduction

The study of group formation and group activities in economies has had a long history in economics, going back at least to Tiebout (1956) and Buchanan (1965). Groups may form for the purposes of provision of public goods, either 'local' or pure, for the purpose of mutual insurance,

☆ This paper was completed while Page and Wooders were visiting CERMSEM at the University of Paris 1 in June-July 2005 and June 2006.

* Corresponding author at: Department of Economics, Indiana University, Bloomington, IN 47405, United States. Tel.: +1 812 855 4764; fax: +1 812 855 3736.
E-mail addresses: fpage@indiana.edu (F.H. Page Jr.), m.wooders@vanderbilt.edu (M. Wooders).
URLs: http://mypage.iu.edu/~fpage/index.html (F.H. Page Jr.), http://www.myrnawooders.com (M. Wooders).
1 Tel.: +1 615 343 0461; fax: +1 615 343 8494.

0167-2681/$ – see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.jebo.2007.01.003

to keep prices high (cartels), to enjoy each other's company, or for a multitude of other reasons. We shall call such groups 'clubs.' Here we offer a new approach to the study of clubs. In particular, modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if players behave farsightedly in choosing their club memberships, as well as club networks that are stable if players are myopic. Thus we bring together two strands of the literature: club theory[2] and the theory of social and economic networks initiated by Kirman (1983).

Unlike the random graph theoretic approach taken by Kirman (1983), here we follow an approach similar to that taken by Jackson and Wolinsky (1996) in their study of networks and focus exclusively on strategic considerations in club network formation. The basic setup of our model is closely related to the model of Konishi et al. (1997, 1998), among others.[3] They examine, however, free mobility equilibrium of a local public goods economy (an assignment of players to clubs, locations, or jurisdictions that partitions the population and has the property that no individual can gain either by moving to any other existing club or creating his own club). The partition derived from the players' strategy choices is thus stable against unilateral deviations by individuals; that is, the partition is *Nash stable*.

In contrast to much of the prior literature on clubs (discussed further below), we allow *strategic coalitional moves* and permit players to be farsighted.[4] Using the farsighted core introduced in Page and Wooders (2004) as our stability notion,[5] we show that if players' payoffs are single-peaked on the domain of club sizes, if players agree on the club size at which payoffs peak (i.e., players agree on the optimal club size) and if there are sufficiently many players and clubs to allow for the partition of players into clubs of optimal size, then a necessary *and* sufficient condition for the farsighted core to be nonempty is that the set of players can be partitioned into clubs of optimal size. In contrast to the case with farsighted players, we show that if there are sufficiently many players and clubs and players are myopic, then a necessary and sufficient condition for the direct core (or the myopic core) to be nonempty is that players who end up in smaller-than-optimal size clubs (i.e., 'left-over' players) have no incentive to switch their memberships to already existing clubs of optimal size. We note that in this case, the outcome of myopic behavior corresponds to outcomes of myopic behavior as in Arnold and Wooders (2005) and the set of outcomes in the direct core corresponds to the set of Nash club equilibrium outcomes.[6] We also show that if players are farsighted and there are too few clubs, so that the average number of players per club is larger than the optimal club size, then the farsighted core is empty. If players are myopic and there are too few clubs, then the direct core is nonempty, and for any club network in the direct core the club structure is such that clubs are of nearly equal size (i.e., clubs differ in size by at most one member).

---

[2] For surveys of club theory from several perspective see, for example, Demange (2005), Kovalenkov and Wooders (2005), Conley and Smith (2005), Le Breton and Weber (2005), and Jaramillo et al. (2005).

[3] In fact, there are a multitude of papers on clubs, both modeled as games in characteristic form (or as hedonic games, by which we mean that preferences are defined directly over coalitions) and modelled with more details about the economic structure. A comprehensive survey is beyond the scope of this paper; instead, we discuss only those relationships to the literature that we believe may be most informative to the reader.

[4] In addition to using a network model, our approach differs from the cooperative/price-taking approach in much of the literature on clubs in that coalitions behave strategically and there is 'free entry'.

[5] Stated loosely, a club network is contained in the farsighted core if no group of agents has an incentive to alter their club memberships, taking into account club membership changes that might take place in the future.

[6] The Nash club equilibrium concept of Arnold and Wooders requires that only individuals within the same club can coordinate their actions.

We demonstrate via an example why, if there are too few clubs relative to the number of players, the farsighted core is empty. In particular, we show via our example that this emptiness problem is caused by the fact that farsighted players, unlike myopic players, may switch their club memberships to already overcrowded or optimally-sized clubs, temporarily making themselves worse off, if they believe that switching might induce an out-migration that makes them ultimately better off. This ceases to be the case if players are myopic because myopic players will switch memberships if and only if switching makes them immediately strictly better off. As a result, we are able to show in general, and illustrate via our example, that with myopic players even when there are too few clubs, a club structure in which all clubs are of nearly equal size is immune to coalitional defections. We note that Arnold and Wooders reach similar conclusions in a dynamic model of club formation with myopic players.

We also demonstrate via an example the importance of the rules of network formation in determining stable club outcomes. In our model of club network formation we assume free mobility, meaning any player *or group of players* can move freely and unilaterally from one club to another. We illustrate via an example the implications for stability with farsighted players versus stability with myopic players of assuming that only one player at a time can move freely and unilaterally from one club to another. To the best of our knowledge, the differences in equilibrium outcomes of club economies, depending on the rules of network formation and on whether individuals are farsighted or myopic, has not previously been noted in the literature.

Our framework builds on a canonical model of club formation: individuals are homogeneous and have single-peaked preferences over club size. The model captures the idea that individuals are positively affected by the number of members who share their clubs, but eventually congestion effects set in so that there is a most preferred club size, which we shall call 'optimal.'[7] This canonical model, as a special case, and extensions of the model in a number of directions have a long history in the literature, going back to club economies with essentially homogeneous players modeled as games in characteristic function form, see for example, Pauly (1970) and Shubik and Wooders (1982, 1983) and continuing to the more recent literature, for example, Banerjee et al. (2001), Bogomolnaia and Jackson (2002), and Diamantoudi and Xue (2003). Pauly, for a situation with homogeneous players and transferable utility, and Shubik and Wooders (1982, 1983), for non-transferable utility games, describe conditions ensuring nonemptiness of cores and approximate cores.[8] Banerjee et al. focus on the core in simple coalition formation games while Bogomolnaia and Jackson study various solution concepts such as Nash stability and individual stability in hedonic games. Diamantoudi and Xue study the farsighted stable set and the largest consistent set (Chwe, 1994) in these games. While these papers go beyond the canonical model used herein, they perhaps underscore the importance of understanding this model.

Our paper adds a network structure to the canonical model. In the special case where the set of players can be divided into clubs of optimal size, then outcomes of all solution concepts coincide; nonemptiness of the core of the cooperative game is well known[9] and the robustness of the stability of partitions of the players into clubs of optimal size is reinforced by our results. In the other case, however, when there are 'too few' clubs, the results for cooperative games mentioned above do not

[7] Note that this club size may not be Pareto-optimal; it is instead the preferred club size of an individual. The optimal club size may be three, for example, but Pareto optimality may require that, in a five-person economy, all individuals are in one club.

[8] These two papers are based on results for general non-transferable utility games in Wooders (1983). Kovalenkov and Wooders (2003) provide the most recent results in this area.

[9] See, for example, Pauly (1970) and Wooders (1978) (for the one-private good case).

apply. In particular, how is the characteristic function (stating the worth of each possible coalition) to be defined? The standard definition from economic theory defines the worth of a coalition as the most that it can guarantee itself no matter what the complementary coalition does. In our network framework (as in Konishi et al., 1998), a coalition cannot prevent others from joining; that is, there is free entry.[10] Under the standard definition the worth of a coalition is equal to the minimum payoff the coalition can guarantee to its members. As illustrated by an example, the 'free entry' condition of our club model is not compatible with the standard definition. (This is not at all meant as a criticism of the cooperative game approach; it addresses important, but different, sorts of situations.) The same comments apply to hedonic games (with ordinal preferences over coalitions of membership). The differences between our network approach and the cooperative game approach also appear in the set of equilibrium outcomes. We will elaborate on this with some examples in the paper and in the penultimate section of the paper.

On a more abstract level, in each of the papers on cooperative games noted above, there are essentially two primitives, a set of alternatives for each coalition and a dominance relationship. In our approach there are four primitives: the feasible set of networks, the preferences of players, the rules of network formation, and a dominance relation. In this paper, given the player population, the feasible set of networks is determined by the number of club locations, the preferences of players are single-peaked over club size, the rules of network formation are free entry, and the dominance relations discussed are indirect and direct dominance. We focus primarily on indirect dominance (i.e., farsighted dominance). The importance of how one might arrive at a core point in a cooperative game has long been recognized in the literature on networks based on cooperative games (cf., Slikker and van den Nouweland, 2001 or van den Nouweland, 2005 for a survey). Our framework allows us to consider this question in a club context.

We shall proceed as follows. In Section 2, we introduce the notion of a club network and state the assumptions of our model. In Section 3, we define the farsighted dominance relation over the feasible set of club networks, and we define the farsighted path dominance relation. In Section 4, we define the abstract club network formation game with respect to the farsighted path dominance relation and we define the farsighted core of the club network formation game. Finally, in Section 4, we state our main result giving necessary and sufficient conditions for nonemptiness of the farsighted core for the case in which there are sufficiently many clubs.

2. Club networks

We begin by introducing the notion of a club network. Using bipartite networks we are able to represent any club structure in a compact and precise way.

Let N be a finite set of players consisting of two or more players with typical element denoted by i and let C be a finite set of club types (or alternatively, a set of club labels or club locations) with typical element denoted by c.

Definition 1 *(Club networks)*. A club network g is a nonempty subset of $N \times C$ such that $(i, c) \in g$ if and only if player i is a member of club c.

Given club network g,

$$g(c) := \{i \in N : (i, c) \in g\}$$

[10] See also Demange (1994) and, for a survey, Demange (2005).

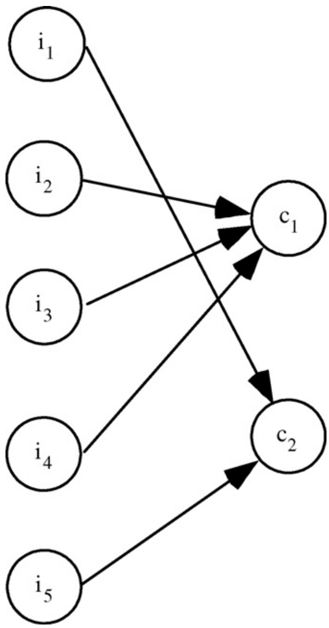

Fig. 1. Club network g_0.

(i.e., the section of g at c) is the set of members of club c in network $g \subseteq N \times C$, while the set

$$g(i) := \{c \in C : (i, c) \in g\}$$

(i.e., the section of g at i) is the set of clubs to which player i belongs in network $g \subseteq N \times C$.

In club network g, the set of all clubs to which *some* member of club c' belongs is given by

$$g^2(c') = g(g(c')) = \cup_{i \in g(c')} g(i).$$

Moreover, in network g the set of players who share membership in *some* club with player i' is given by

$$g^2(i') = g(g(i')) = \cup_{c \in g(i')} g(c).$$

Note that if each player can belong to only one club, then $g^2(i')$ is simply the set of players who belong to the same club as player i'.

Example 1. To illustrate, suppose there are five players $N = \{i_1, i_2, i_3, i_4, i_5\}$ and two clubs $C = \{c_1, c_2\}$. Further, suppose that c_1 denotes a chess club while c_2 denotes a fencing club. Club network g_0 depicted in Fig. 1 represents one possible club structure given N and C.

In club network g_0 the chess club has three members

$$g_0(c_1) = \{i_2, i_3, i_4\},$$

while the fencing club has two members

$$g_0(c_2) = \{i_1, i_5\}.$$

Note that in club network g_0 each player is a member of one and only one club. Thus, for example

$$g_0(i_5) = \{c_2\},$$

that is, player i_5 is a member of the fencing club, but is not a member of the chess club. Below we will formalize the single club membership property of this example in an assumption that we will maintain throughout the paper. Finally, note for example that under single club membership,

$$g^2(c_1) = c_1 \quad \text{and} \quad g^2(i_4) = \{i_2, i_3, i_4\}.$$

The collection of all club networks given N and C is given by the collection of all *nonempty* subsets of $N \times C$, denoted by $P(N \times C)$. We shall denote by $|g(c)|$ the number of members of club c (i.e., the club size) in network g and by $|g(i)|$ the number of clubs to which i belongs in network g. In Example 1, the chess club has three members, that is $|g_0(c_1)| = 3$, and player i_5 belongs to one club - the fencing club - and thus $|g_0(i_5)| = 1$.

We shall maintain the following assumptions throughout:

A-1 (single club membership). The feasible set of club networks, $\mathbb{K} \subset P(N \times C)$, is given by

$$\mathbb{K} \subset \{g \in P(N \times C) : |g(i)| = 1 \quad \text{for all} \quad i \in N\}.$$

Thus, in each feasible club network $g \in \mathbb{K}$ each player is a member of one and only one club. Again note that club network g_0 in Example 1 satisfies the single club membership assumption [A-1]. Also note that under assumption [A-1] the collection $\{g(c) : c \in C\}$ forms a partition of the set of players.

A-2 (identical payoff functions depending on club size). Players have identical payoff functions, $u(\cdot)$, and payoffs are a function of club size only. In Example 1, player i_5 is a member of the fencing club, that is, $g_0(i_5) = \{c_2\}$, and this club has a membership set given by

$$g_0(g_0(i_5)) := g_0^2(i_5) = \{i_1, i_5\}.$$

Thus, in network g_0 player i_5 has a payoff given by

$$u(|g_0(g_0(i_5))|) = u(|g_0^2(i_5)|) = u(|\{i_1, i_5\}|) = u(2).$$

In general, given any club network g, $|g^2(i)|$ denotes the total number of club members in the club to which player i belongs.

A-3 (single-peaked payoffs). There exists a club size s^* with $1 \leq s^* < |N|$ such that payoffs are increasing in club size up to club size s^* and decreasing thereafter.

A-4 (free mobility). Each player or group of players can move freely and unilaterally from one club to another. This means that a player can drop his membership in any given club and join any other club without bargaining and without seeking the permission of any player or group of players. In this sense our model of club formation as a game over club networks is noncooperative. The assumption of free mobility is quite common in models of noncooperative network formation (see, for example, Bala and Goyal, 2000), as well as in the club literature (see, for example, Demange, 2005 and the references contained therein).

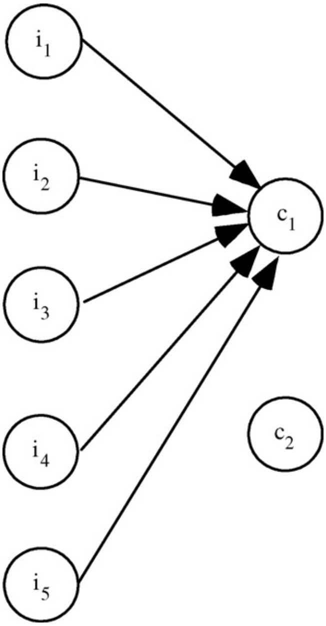

Fig. 2. Club network g_1.

Example 2. It is important to note that our assumptions do not rule out the possibility that some clubs have no members (i.e., are empty). Thus, in some feasible club networks $g \in \mathbb{K}$, it may be the case that $g(c) = \emptyset$ for some club type $c \in C$. If club c has no members, then $|g(c)| = |\emptyset| = 0$. Fig. 2 depicts just such a situation.

In moving from club network g_0 in Example 1 to club network g_1 above, players i_1 and i_5 have freely and unilaterally dropped their memberships in the fencing club and joined the chess club. Thus, in club network g_1 the fencing club c_2 has no members.[11]

Before leaving this section, we note that, while we have introduced utility functions in the description of our model, this is not necessary for our results to hold. We could instead have preferences given by preference relations directly over club memberships; that is, we could equally well have had hedonic preferences (see Bogomolnaia and Jackson, 2002 and, for a formulation with networks, Page and Wooders, 2005).

3. Dominance relations over club networks

Under the assumption of free mobility players can alter any existing club network by simply switching their memberships. Such membership changes however can trigger further membership

[11] While we assume that in moving from club network g_0 to club network g_1 agents i_1 and i_5 act freely and unilaterally in switching their memberships, our model does not address the question of how agents i_1 and i_5 come to switch their memberships simultaneously, whether by communication and collusion or by serendipity. In order to address this question formally additional structure would have to be added to the current model. Page et al. make a start on addressing this question via the introduction of the supernetwork (i.e., a network of networks) in which the arcs represent coalitional moves and coalitional preferences (see also Page and Wooders, 2004).

changes by other players that in the end leave some or all of the players who initially switched not better off and possibly worse off. Here we will consider two types of dominance relations over club networks: direct dominance and indirect dominance. Under direct dominance players are only concerned with the immediate consequences of their membership decisions. Under indirect dominance, players are farsighted and are concerned with the long run consequences of their membership decisions. We begin by formalizing the notions of direct and indirect dominance.

3.1. Direct and indirect dominance

Throughout the following let S denote a *nonempty* subset of N.

Definition 2 *(Direct and indirect dominance).* Let g_0 and g_1 be two club networks in $\mathbb{K}$ ($g_0 \neq g_1$).

(1) (Feasible change) We say that the coalition S can feasibly change club network g_0 to club network g_1, denoted

$$g_0 \underset{S}{\rightarrow} g_1,$$

if the move from network g_0 to network g_1 only involves a change in club memberships by players in S, leaving unchanged the memberships of players outside group S; that is,

$$g_0(i) = g_1(i) \quad \text{for all players} \quad i \in N \setminus S \text{ (i.e, } i \text{ not contained in } S\text{)}.$$

(2) (Improvement) We say that club network g_1 is an improvement over club network g_0 for players $i \in S$, denoted $g_1 \succ_S g_0$

$$\text{if } u(|g_1^2(i)|) > u(|g_0^2(i)|) \quad \text{for players } i \in S.$$

(3) (Direct dominance) We say that club network g_1 directly dominates club network g_0, denoted

$$g_1 \rhd g_0 \quad \text{if for some set of players } S,\ g_0 \rightarrow_S g_1 \quad \text{and} \quad g_1 \succ_S g_0.$$

We shall sometimes write $g_1 \rhd_S g_0$ to indicate the set of players S who can change network g_0 to network g_1 and for whom g_1 dominates g_0.

(4) (Indirect dominance) We say that club network $g_* \in \mathbb{K}$ indirectly dominates club network $g \in \mathbb{K}$, denoted

$$g_* \rhd\rhd g$$

if there exists a finite sequence of club networks, $g_0, \ldots, g_n$, with $g := g_0$ and $g_* := g_n$, and a corresponding sequence of sets of players, $S_1, \ldots, S_n$, such that for $k = 1, 2, \ldots, n$,

$$g_{k-1} \underset{S_k}{\rightarrow} g_k \quad \text{and} \quad g_* \succ_{S_k} g_{k-1}.$$

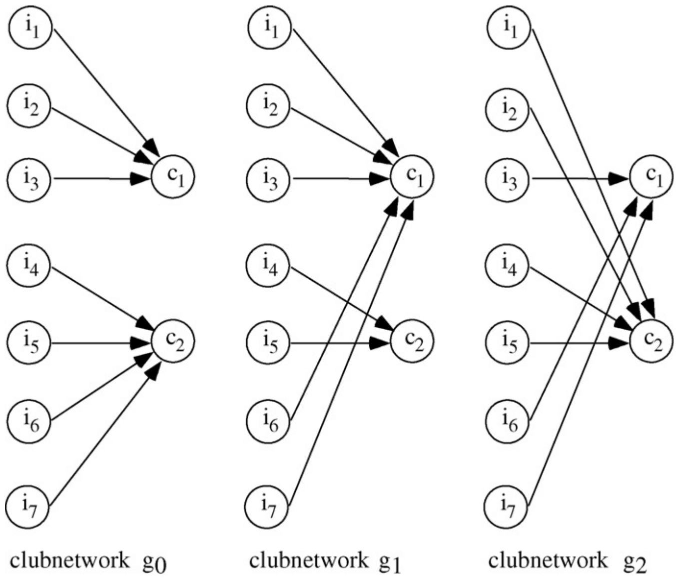

Fig. 3. Three possible club structures.

Thus, club network g_* indirectly dominates club network g if (i) there is a finite sequence of feasible changes to network g ending with network g_*, and if (ii) payoffs

$$(u(|g_*^2(i)|))_{i \in N}$$

in *ending* club network g_* are such that for each k and for the players in each coalition S_k, payoffs in the ending club network g_* are greater than the payoffs players in S_k would have received in club network g_{k-1} (i.e., in the club network that players in S_k changed); that is, for each k

$$u(|g_*^2(i)|) := u(|g_n^2(i)|) > u(|g_{k-1}^2(i)|) \quad \text{for} \quad i \in S_k.$$

The definition of indirect dominance above is a network rendition of Chwe's definition of farsighted dominance.

The following example illustrates indirect dominance. We will return to this example later in our discussions of the direct and farsighted cores.

Example 3. Suppose that there are seven players and two clubs and that the optimal club size is 3. Fig. 3 depicts three feasible club networks, g_0, g_1, and g_2. Club network g_2 indirectly dominates club network g_0.

To see this, consider the following sequence of moves. First, players i_6 and i_7 switch their memberships from club c_2 to club c_1. This feasible move by players i_6 and i_7 changes club network g_0 to club network g_1 and is denoted by

$$g_0 \underset{\{i_6,i_7\}}{\rightarrow} g_1.$$

Second, players i_1 and i_2 switch their memberships from club c_1 to club c_2. This feasible move by players i_1 and i_2 changes club network g_1 to club network g_2 and is denoted by

$$g_1 \underset{\{i_1,i_2\}}{\rightarrow} g_2.$$

Given an optimal club size of 3 and given the assumption of single-peaked payoffs, the initial moves by players i_6 and i_7 makes them worse off. In particular, players i_6 and i_7 start out in club c_2 in network g_0 with four members $\{i_4, i_5, i_6, i_7\}$ and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4),$$

and move to club c_1 creating a new club network g_1 in which club c_1 has five members $\{i_1, i_2, i_3, i_6, i_7\}$. As a result, players i_6 and i_7 are made worse off with payoffs given by

$$u(|g_1^2(i_6)|) = u(|g_1^2(i_7)|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5).$$

However, due to the second round of moves by players i_1 and i_2, players i_6 and i_7 end up in a smaller club c_1 in club network g_2, and thus end up better off. In particular, in the second round of moves, players i_1 and i_2 leave club c_1 and move to club c_2 changing club network g_1 to club network g_2. This move makes players i_1 and i_2 better off, but also makes players i_6 and i_7 better off. In particular, players i_1 and i_2 move from club c_1 in network g_1 with five members $\{i_1, i_2, i_3, i_6, i_7\}$ and payoffs given by

$$u(|g_1^2(i_1)|) = u(|g_1^2(i_2)|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5),$$

to club c_2 in network g_2 with four members $\{i_1, i_2, i_4, i_5\}$ and payoffs given by

$$u(|g_2^2(i_1)|) = u(|g_2^2(i_2)|) = u(|\{i_1, i_2, i_4, i_5\}|) = u(4).$$

These second round moves by players i_1 and i_2 leave players i_6 and i_7 in a smaller club c_1 and thus make players i_6 and i_7 better off. Thus, players i_6 and i_7 who started out in club c_2 in network g_0 with four members $\{i_4, i_5, i_6, i_7\}$ and payoffs given by

$$u(|g_0^2(i_6)|) = u(|g_0^2(i_7)|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4)$$

end up in club c_1 in network g_2 with 3 members, $\{i_3, i_6, i_7\}$ and payoffs given by

$$u(|g_2^2(i_6)|) = u(|g_2^2(i_7)|) = u(|\{i_3, i_6, i_7\}|) = u(3).$$

3.2. *Path dominance*

Let $>$ denote an irreflexive dominance relation on the set of club networks $\mathbb{K}$.[12] In what follows, we shall assume that $>$ is either direct or indirect (as defined above), but for now, we shall simply assume that $>$ is any irreflexive relation on $\mathbb{K}$.

We say that a sequence of club networks $\{g_k\}_k$ in $\mathbb{K}$ is a *path domination sequence* (or a $<$-path) if for any two consecutive networks g_{k-1} and g_k,

$$g_{k-1} < g_k.$$

Using the terminology of graph theory, we can think of the relation $g_{k-1} < g_k$ between networks g_k and g_{k-1} as defining a $<$-arc *from* network g_{k-1} *to* network g_k. The *length* of $<$-path $\{g_k\}_k$ is defined to be the number of $<$-arcs in the path. We say that network $g_1 \in \mathbb{K}$ is *$<$-reachable* from network $g_0 \in \mathbb{K}$ if there exists a $<$-path of *finite* length *in* $\mathbb{K}$ from g_0 to g_1.

[12] If dominance relation $>$ on $\mathbb{K}$ is irreflexive, this means that it is not possible to have $g > g$ for any $g \in \mathbb{K}$.

Using the notion of $<$-reachability we can define a new relation on the feasible set of club networks $\mathbb{K}$. In particular, for any two networks g_0 and g_1 in $\mathbb{K}$ define

$$g_1 \trianglerighteq_{\mathbb{K}} g_0 \text{ if and only if } \begin{cases} g_1 \text{ is } < -\text{reachable from } g_0, \text{ or} \\ g_1 = g_0. \end{cases} \tag{1}$$

For any dominance relation $<$ the induced relation $\trianglerighteq_{\mathbb{K}}$ is a weak ordering on $\mathbb{K}$. In particular, $\trianglerighteq_{\mathbb{K}}$ is reflexive ($g \trianglerighteq_{\mathbb{K}} g$) and $\trianglerighteq_{\mathbb{K}}$ is transitive ($g_2 \trianglerighteq_{\mathbb{K}} g_1$ and $g_1 \trianglerighteq_{\mathbb{K}} g_0$ implies that $g_2 \trianglerighteq_{\mathbb{K}} g_0$). We shall refer to the relation $\trianglerighteq_{\mathbb{K}}$ as the path dominance relation.[13]

If the dominance relation $>$ is given by a direct dominance relation $\triangleright$, we shall refer to the induced path dominance relation $\trianglerighteq_{\mathbb{K}}$ as a *direct path dominance* relation; and if $>$ is given by an indirect dominance relation $\triangleleft\triangleleft$, we shall refer to $\trianglerighteq_{\mathbb{K}}$ as a *farsighted path dominance* relation

Note that if network g_1 directly dominates network g_0 for players $i \in S$, then g_1 also dominates g_0 with respect to the direct path dominance relation $\trianglerighteq_{\mathbb{K}}$. Thus

$$\text{if } g_1 \triangleright_S g_0 \text{ for some coalition } S, \text{ then } g_1 \trianglerighteq_{\mathbb{K}} g_0.$$

This applies even if the coalition S consists of a single player, that is, even if $S = \{i\}$ for some player $i \in N$. Thus

$$\text{if } g_1 \triangleright_{\{i\}} g_0 \text{ for some player } i \in N, \text{then } g_1 \trianglerighteq_{\mathbb{K}} g_0.$$

Also note that if network g_1 directly dominates network g_0, then g_1 also indirectly dominates g_0. However, if g_1 indirectly dominates g_0, then g_1 may or may not directly dominate g_0.

4. Club formation games and the path dominance core

An abstract club formation game (in the sense of von Neumann Morgenstern) is given by the pair $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$, where $\mathbb{K}$ is the feasible set of club networks and $\trianglerighteq_{\mathbb{K}}$ is the path dominance relation on $\mathbb{K}$.

One of the most fundamental stability notions in game theory is the core. Here, we define the notion of core for club formation games with respect to path dominance.[14] We call this notion of the core the *path dominance core*.

Definition 3 *(The path dominance core).* Let $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ be a club formation game. A subset $\mathbb{C}$ of club networks in $\mathbb{K}$ is said to be the path dominance core of $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ if for each club network $g \in \mathbb{C}$ there does not exist a club network $g' \in \mathbb{K}$, $g' \neq g$, such that $g' \trianglerighteq_{\mathbb{K}} g$ and if there does not exist a strict superset of $\mathbb{C}$ with this property. If $\trianglerighteq_{\mathbb{K}}$ is induced by a direct dominance relation, we shall refer to the path dominance core as the direct core and, if $\trianglerighteq_{\mathbb{K}}$ is induced by an indirect dominance relation, we shall refer to the path dominance core as the farsighted core.

Note that any club network g contained in the path dominance core $\mathbb{C}$ (direct or farsighted) is a Nash club network, and in fact is a strong Nash club network.[15] Letting $\mathbb{NE}$ denote the set of

[13] The relation $\trianglerighteq_{\mathbb{K}}$ is sometimes referred to as the transitive closure in $\mathbb{K}$ of the dominance relation $>$ on $\mathbb{K}$.

[14] We use the classic approach to the core of Gillies (1959).

[15] A club network $g \in \mathbb{K}$, is a Nash club network if there does not exist another club network $g' \in \mathbb{K}$ such that $g' \triangleleft_{\{i\}} g$ for some agent $i \in N$.

A club network $g \in \mathbb{K}$, is a strong Nash club network if there does not exist another club network $g' \in \mathbb{K}$ such that $g' \triangleleft_S g$ for some coalition S.

Nash club networks in $\mathbb{K}$ and letting $\mathbb{SNE}$ denote the set of strong Nash club networks in $\mathbb{K}$, we can conclude from our definition of the path dominance core that

$$\mathbb{C} \subseteq \mathbb{SNE} \subseteq \mathbb{NE}.$$

Our first results give necessary and sufficient conditions for the farsighted core to be nonempty.

Theorem 1 (Necessary and sufficient conditions for nonemptiness of the farsighted core). *Consider a club network formation game* $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ *with* $|N|$ *players,* $|C|$ *clubs, and optimal club size* s^*, $1 \leq s^* < |N|$. *Suppose that assumptions* (A-1)–(A-4) *hold and let*

$$|N| = rs^* + l \text{ for positive integer } r \text{ and non-negative integer } l \text{ such that } 0 \leq l < s^*.$$

If the path dominance relation $\trianglerighteq_{\mathbb{K}}$ *is induced by an indirect dominance relation, that is, if* $\trianglerighteq_{\mathbb{K}}$ *is given by*

$$g_1 \trianglerighteq_{\mathbb{K}} g_0 \text{ if and only if } \begin{cases} g_1 \text{ is } \lhd\lhd \text{ −reachable from } g_0, \text{ or} \\ g_1 = g_0, \end{cases}$$

then the following statement holds:

(1) *If* $|C| \geq |N|/s^*$, *that is, if the number of club locations is large enough to allow for the formation of the maximum number of clubs of optimal size* (*i.e., r clubs of size* s^*), *then the farsighted core is nonempty if and only if all the players can be divided into clubs of optimal size* (*i.e., if and only if* $l = 0$). *Moreover, club network* g_* *is contained in the farsighted core if and only if* g_* *has r clubs of size* s^*.
(2) *If* $|C| < |N|/s^*$ *and* s^* and $|C|$ *are greater than or equal to* 2, *then the farsighted core is empty.*

Proof. We divide the proof into two parts, depending on whether there are enough clubs for all players to be in clubs of optimal size.

(1) Suppose that $|C| \geq |N|/s^*$. First, we show that nonemptiness implies that players can be divided into clubs of optimal size. Suppose that players cannot be divided into clubs of optimal size; that is, suppose $0 < l < s^*$. Consider a club network g_1 with r clubs of size s^* and one club of size l. Note that we need only consider networks with r clubs of size s^* and one club of size l because these networks are the only possible candidates for inclusion in the path dominance core (if a club is too large a group could benefit by moving to another club, and if there are two clubs smaller in size than s^*, then a group could benefit by moving from one of these clubs to the other). To begin suppose that clubs c_1 and c_2 are such that

$$|g_1(c_1)| = s^* \quad \text{and} \quad |g_1(c_2)| = l,$$

and suppose players $g_1(c_2)$ (i.e., all l members of club c_2) leave club c_2 and join club c_1. In the new network g_2,

$$|g_2(c_1)| = s^* + l \quad \text{and} \quad |g_2(c_2)| = 0.$$

All other clubs remain unchanged. Next, suppose that the l players who just moved from club c_2 to club c_1 join with $s^* - l$ of the players originally in club c_1 in network g_1 and move to club c_2. In the resulting new network g_3,

$$|g_3(c_1)| = l \quad \text{and} \quad |g_3(c_2)| = s^*.$$

Finally, note that network g_3 indirectly dominates network g_1. In particular, players $g_1(c_2)$ who initiated this sequence of changes are better off in network g_3 than in network g_1, and the l players $g_1(c_2)$ and $s^* - l$ of the players $g_1(c_1)$ who together changed network g_2 to network g_3 are better off. Thus, $g_3 \rhd\rhd g_1$, and thus our only reasonable candidate for membership in the path dominance core cannot be in the path dominance core. We must conclude that nonemptiness of the path dominance core implies that players can be divided into clubs of optimal size without left-overs.

Second, let g be a club network such that, for all clubs $c \in C$, either $|g(c)| = s^*$ or $|g(c)| = 0$. Thus, network g partitions the players into clubs of optimal size. Because $u(s^*) > u(|g(c)|)$ for all clubs c with $|g(c)| \neq s^*$, no group of players can initiate a change in network g, which will lead to another network, making the players in the group better off. Thus, any such club network is contained in the path dominance core.

(2) Suppose that $|C| < |N|/s^*$ and s^* and $|C|$ are greater than or equal to 2. Consider a club network g_0 in which clubs c_1 and c_2 are such that

$$|g_0(c_1)| = s^* + k_1 \quad \text{and} \quad |g_0(c_2)| = s^* + k_2$$

for nonnegative integers k_1 and k_2.

Case 1. Suppose $k_1 \geq k_2 + 1$. First, suppose a coalition $S_1 \subset g_1(c_1)$ of size k_1 from club c_1 moves to club c_2. In the new club network g_1,

$$|g_1(c_1)| = s^* \quad \text{and} \quad |g_1(c_2)| = s^* + k_1 + k_2.$$

Next, suppose a coalition $S_2 \subset g_1(c_2)$ of size k_1 from club c_2, satisfying $S_1 \cap S_2 = \emptyset$, moves to club c_1. In the resulting network g_2,

$$|g_2(c_1)| = s^* + k_1 \quad \text{and} \quad |g_2(c_2)| = s^* + k_2.$$

Finally, note that network g_2 indirectly dominates network g_0. In particular, the players in coalition S_1 from player set $g_0(c_1)$ who initiated this sequence of changes is better off in network g_2 than in network g_0, because they start out in club c_1 of size $|g_0(c_1)| = s^* + k_1$ and end up in club c_2 of smaller size $|g_2(c_2)| = s^* + k_2$. The players in coalition S_2 from player set $g_1(c_2)$ who changed network g_1 to network g_2 are better off because in network g_1 they are in club c_2 of size $s^* + k_1 + k_2$ while in network g_2 they are in club c_1 of smaller size $s^* + k_1$. Thus, $g_2 \rhd\rhd g_0$.

Case 2. Suppose $k_1 = k_2 > 0$. Again consider clubs c_1 and c_2 in network g_0. We have

$$|g_0(c_1)| = s^* + k_1 \quad \text{and} \quad |g_0(c_2)| = s^* + k_2.$$

Suppose that a two-player coalition $S_1 \subseteq g_0(c_1)$ from club c_1 moves to club c_2. In the new club network g_1

$$|g_1(c_1)| = s^* + k_1 - 2 \quad \text{and} \quad |g_1(c_2)| = s^* + k_2 + 2.$$

Next suppose that a three-player coalition $S_2 \subset g_1(c_2)$ from club c_2, satisfying $S_1 \cap S_2 = \emptyset$, moves to club c_1. In the resulting network g_2,

$$|g_2(c_1)| = s^* + k_1 - 2 + 3 \quad \text{and} \quad |g_2(c_2)| = s^* + k_2 + 2 - 3.$$

Finally, note that network g_2 indirectly dominates network g_0. In particular, the players in coalition S_1 from player set $g_0(c_1)$ who initiated this sequence of changes is better off in network g_2 than in network g_0- because they start out in club c_1 of size $|g_0(c_1)| = s^* + k_1$ and end up in club c_2 of smaller size $|g_2(c_2)| = s^* + k_2 - 1$. The players in coalition S_2 from player set $g_1(c_2)$ who changed network g_1 to network g_2 are better off because in network g_1 they are in club c_2 of size $|g_1(c_2)| = s^* + k_2 + 2$ while in network g_2 they are in club c_1 of smaller size $|g_2(c_1)| = s^* + k_1 + 1$. Thus, $g_2 \rhd\rhd\, g_0$. □

It should be noted that if there is only one club (i.e., if $|C| = 1$), then the farsighted core is nonempty. By Theorem 1, if there are at least two clubs and the optimal club size is at least 2, but there are too few clubs to allow players to be divided into clubs of optimal size, then farsightedness leads to instability and the farsighted core is empty in general. However, as our next result will demonstrate, if players are myopic rather than farsighted, then stability is possible even if players cannot be divided into clubs of optimal size. Thus, in club formation games with free entry, myopia can lead to stability.

Theorem 2 (Necessary and sufficient conditions for nonemptiness of the direct core). *Consider a club network formation game* $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ *with* $|N|$ *players*, $|C|$ *clubs, and optimal club size* s^*, $1 \leq s^* < |N|$. *Suppose that assumptions* (A-1)–(A-4) *hold and let*

$$|N| = rs^* + l \text{ for positive integer } r \text{ and } \textit{non-negative integer } l \textit{ such that } 0 \leq l < s^*.$$

If the path dominance relation $\trianglerighteq_{\mathbb{K}}$ *is induced by a direct dominance relation, that is, if* $\trianglerighteq_{\mathbb{K}}$ *is given by*

$$g_1 \trianglerighteq_{\mathbb{K}} g_0 \text{ if and only if } \begin{cases} g_1 \textit{ is } \lhd -\text{reachable from } g_0, \text{or} \\ g_1 = g_0, \end{cases}$$

then the following statements are true:

(1) *If* $|C| \geq |N|/s^*$, *that is, if the number of club locations is large enough to allow for the formation of the maximum number of clubs of optimal size* (*i.e., r clubs of size* s^*), *then the direct core is nonempty if and only if either* $l = 0$ *or* $u(l) \geq u(s^* + 1)$. Moreover, club network g_* *is contained in the direct core if and only if* g_* *has r clubs of size* s^* *and one club of size l.*
(2) *If* $|C| < |N|/s^*$, *that is, if the number of club locations is not large enough to allow for the formation of the maximum number of clubs of optimal size, then the direct core is nonempty.*

Moreover, club network g_* is contained in the direct core if and only if the club structure induced by g_* is such that clubs differ in size by at most one member, that is, if and only if

$$|(|g_*(c)| - |g_*(c')|)| \leq 1$$

for all clubs c and c' in C.

Proof. (1) If $|C| \geq |N|/s^*$ and $l = 0$, then (1) follows from conclusion 1 of Theorem 1. Suppose now that

$$|C| \geq \frac{|N|}{s^*} \quad \text{and} \quad u(l) \geq u(s^* + 1) \quad \text{where} \quad 0 < l < s^*.$$

Consider a club network g_1 with r clubs of size s^* and one club of size l. Let S^* be the group of players such that each player i in S^* is a member of as s^* club (i.e., a club of size s^*) and let L be the group of players in the club of size l. Because

$$u(|g_1^2(i)|) \geq u(|g^2(i)|) \quad \text{for all} \quad g \in \mathbb{K} \quad \text{and all} \quad i \in S^*,$$

no coalition requiring the participation of players from S^* will be able to change club network g_1 to another club network making the participates from S^* better off. Moreover, because

$$u(l) \geq u(s^* + 1) \text{ and payoffs are single peaked,}$$

no coalition of players from L alone will be able to change club network g_1 to another club network making the players from L better off. Thus, for any club network g_1 with r clubs of size s^* and one club of size l, there does not exist a club network $g_2 \in \mathbb{K}$, $g_2 \neq g_1$, such that $g_2 \trianglerighteq_{\mathbb{K}} g_1$. Therefore, if $|C| \geq |N|/s^*$ and $u(l) \geq u(s^* + 1)$, then any club network g_1 with r clubs of size s^* and one club of size l is in the path dominance core.

Suppose now that $|C| \geq |N|/s^*$ but that $u(l) < u(s^* + 1)$. Let $g \in \mathbb{K}$, and given g define the following club subcollections:

$$C_g^+ := \{c \in C : |g(c)| > s^*\}, \quad C_g^* := \{c \in C : |g(c)| = s^*\} \quad \text{and} \quad C_g^- := \{c \in C : |g(c)| < s^*\}.$$

Given that $|C| \geq |N|/s^*$, $C_g^- \neq \emptyset$ for all $g \in \mathbb{K}$.

Let $g \in \mathbb{K}$ and suppose that $C_g^+ \neq \emptyset$. Consider clubs $c_1 \in C_g^+$ and $c_2 \in C_g^-$ and let S_1 be a coalition of players from club c_1 of size $s^* - |g(c_2)|$. Observe that if players in coalition $S_1 \subset g(c_1)$ switch their memberships to club c_2, then the new larger club c_2 will be of optimal size s^* and all members of coalition S_1 will be made better off by making the switch. Let $g' \in \mathbb{K}$ be the club network which results from this switch. Then we have

$$g' \triangleright_{S_1} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g.$$

Let $g \in \mathbb{K}$ and suppose that $C_g^+ = \emptyset$. If $|C_g^*| = r$, then there is a player i in some club $c_1 \in C_g^-$ who can switch his membership to some club $c_2 \in C_g^*$ and be made better off because $u(l) < u(s^* + 1)$. Letting $g' \in \mathbb{K}$ be the club network resulting from this switch we have

$$g' \triangleright_{\{i\}} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g.$$

If $|C_g^*| < r$ (maintaining the assumption that $C_g^+ = \emptyset$) then sufficiently many players from clubs in C_g^- can switch their memberships to some club $c' \in C_g^-$ resulting in a new, larger club c' of optimal size s^*. Moreover, all players making this membership switch will be better off.

Letting S' denote the coalition of players making the switch and letting $g' \in \mathbb{K}$ be the resulting club network we have

$$g' \triangleright_{S'} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g. \quad \square$$

Example 3 above is particularly interesting as it demonstrates that farsighted behavior may generate quite different outcomes than myopic behavior and strong Nash equilibria (or Nash club equilibria). In Example 3, the number of clubs is not sufficiently large to permit all players to be in clubs of optimal size (i.e., $|C| < |N|/s^*$ for $|C| = 2$, $|N| = 7$, and $s^* = 3$). By part (2) of Theorem 2 above, club networks g_0 and g_2 in Fig. 3 are contained in the direct core. In particular, in both networks g_0 and g_1 club sizes differ by 1 and no group of players (nor any single player) can improve upon his own payoff, but nevertheless the farsighted core is empty. This is because farsighted players will switch their club memberships to an already overcrowded or optimal club, temporarily making themselves worse off, if they believe that switching will induce an out migration that ultimately makes them better off. In Example 3, players i_6 and i_7 move from club c_2 in network g_0 to club c_1 thereby changing network g_0 to network g_1. This move by i_6 and i_7 then induces players i_1 and i_2 to move from club c_1 in network g_1 to club c_2 thereby changing network g_1 to network g_2 and making players i_6 and i_7 better off by leaving them in a smaller club.

In our model of club network formation we assume free mobility, meaning any player *or group of players* can move freely and unilaterally from one club to another. In the our last example we show how stable club outcomes may change with a change in the rules.

Example 4 *(Noncooperative free mobility).* As indicated by Theorems 1 and 2 above, depending on the rules of network formation, direct path dominance and farsighted path dominance can yield very different results. For example, suppose we assume that the rules of network formation are *noncooperative and free entry is allowed*. Thus, only one player at a time can freely and unilaterally change his club membership. Also, suppose that there are 12 players and 6 club locations:

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, \quad C = \{c_1, c_2, c_3, c_4, c_5, c_6\}.$$

Finally, suppose the optimal club size is 3 and that all players' single-peaked preferences over club sizes are given by

$$u(3) > u(2) > u(1), \quad \dots u(5) < u(4) < u(3), \quad u(4) > u(2) > u(1).$$

Graphically, payoffs look like Fig. 4.

The club configuration

$$\boxed{\underset{c_1}{\{1,2,3,4\}} \quad \underset{c_2}{\{5,6,7,8\}} \quad \underset{c_3}{\{9,10,11,12\}} \quad \underset{c_4}{\emptyset} \;\; \underset{c_5}{\emptyset} \;\; \underset{c_6}{\emptyset}}$$

with corresponding club network denoted by g_0 is in the direct core but is not in the farsighted core under *noncooperative*, free entry rules. To see this consider the following sequence of noncooperative moves. First, player 1 moves from club c_1 to club c_4. After this move by player 1 the club configuration is

$$\boxed{\underset{c_1}{\{2,3,4\}} \quad \underset{c_2}{\{5,6,7,8\}} \quad \underset{c_3}{\{9,10,11,12\}} \quad \underset{c_4}{\{1\}} \;\; \underset{c_5}{\emptyset} \;\; \underset{c_6}{\emptyset}}.$$

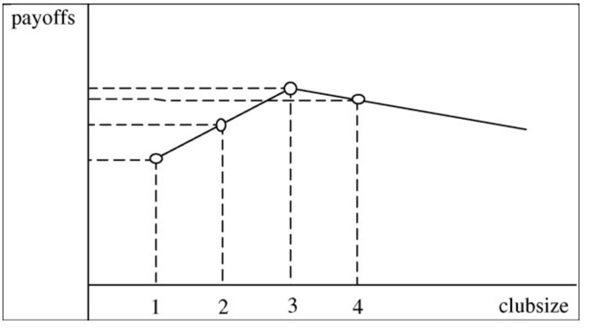

Fig. 4. Payoffs to club size.

Let g_1 be corresponding club network. Second, player 5 moves from club c_2 to club c_4. After this move by player 5 the club configuration is

$$\boxed{\underset{c_1}{\{2,3,4\}} \quad \underset{c_2}{\{6,7,8\}} \quad \underset{c_3}{\{9,10,11,12\}} \quad \underset{c_4}{\{1,5\}} \quad \underset{c_5}{\emptyset} \quad \underset{c_6}{\emptyset}}.$$

Let g_2 be corresponding club network. Third, player 9 moves from club c_3 to club c_4. After this move by player 9 the club configuration is

$$\boxed{\underset{c_1}{\{2,3,4\}} \quad \underset{c_2}{\{6,7,8\}} \quad \underset{c_3}{\{10,11,12\}} \quad \underset{c_4}{\{1,5,9\}} \quad \underset{c_5}{\emptyset} \quad \underset{c_6}{\emptyset}}.$$

Let g_3 be corresponding club network. Club network g_3 is not only in the farsighted core, it is also in the direct core. Note that club network g_3 indirectly dominates club network g_0, and thus, club network g_0, while being contained in the direct core, is not contained in the farsighted core. To see that g_3 indirectly dominates g_0, note that

$$g_0 \to_{\{1\}} g_1 \to_{\{5\}} g_2 \to_{\{9\}} g_3, \qquad g_0 \prec_{\{1\}} g_3, \quad g_1 \prec_{\{5\}} g_3 \quad \text{and} \quad g_2 \prec_{\{9\}} g_3.$$

Finally, note that if the rules of network formation had allowed any group of players to move freely and unilaterally from one club to another, then the club configuration

$$\boxed{\underset{c_1}{\{1,2,3,4\}} \quad \underset{c_2}{\{5,6,7,8\}} \quad \underset{c_3}{\{9,10,11,12\}} \quad \underset{c_4}{\emptyset} \quad \underset{c_5}{\emptyset} \quad \underset{c_6}{\emptyset}}$$

would no longer be in the direct core. For example, players 4, 8, and 12 could move to club c_4 producing club configuration

$$\boxed{\underset{c_1}{\{1,2,3\}} \quad \underset{c_2}{\{5,6,7\}} \quad \underset{c_3}{\{9,10,11\}} \quad \underset{c_4}{\{4,8,12\}} \quad \underset{c_5}{\emptyset} \quad \underset{c_6}{\emptyset}}.$$

Letting g_4 be corresponding club network, under free mobility, g_4 directly dominates g_0 because

$$g_0 \to_{\{4,8,12\}} g_4 \quad \text{and} \quad g_0 \prec_{\{4,8,12\}} g_4.$$

Thus, under free mobility g_0 is not contained in the direct core.

5. Further relationships to the literature

As noted in the introduction, in view of the very central and important nature of our canonical model, there are many papers in the literature studying this model or of other models that include the canonical model as a special case. Here we note just a few of these papers.

Within the context of cooperative games, where free entry is not permitted, it would be standard to use u as the worth function and then allow the game to be essentially superadditive; that is, to allow any group of players to divide into smaller groups or clubs without any externalities between the clubs.[16] To explore the differences with our approach, consider the following example:

Example 5 *(Further Relationships to the Literature).* Suppose that

$$u(1) = 0, \quad u(2) = 1, \quad u(3) = 3, \qquad u(n) = 0, \quad n \geq 4.$$

Let N be the set of players. When $|N| > 3$ and players can freely form coalitions (and exclude other players) the core is nonempty if and only if $|N|$ is divisible by three. But in our model we assume that the number of clubs (or club locations) is exogenously given and we allow free entry into clubs. Here is where our approach, like that of Konishi et al. (1997, 1998) among others, diverges from cooperative game theory.

Continuing the example, suppose that there is more than one club. With free entry, if $|N| \geq 4$, the most that any coalition can *guarantee* itself is zero. Thus, the approach of cooperative game theory would give us a cooperative game (N, v) where $v(S) = 0$ for all coalitions S! The unique point in the core would be the zero vector. With free entry into clubs, this is clearly not a fruitful approach.

Other notions of the core are applicable, for example, the gamma-core proposed by Chander and Tulkens (1995, 1997). The gamma core assumes that the complementary coalition will do the best possible for itself. In some contexts, such as environmental economics, this is an eminently reasonable and fruitful approach. But it also does not give the same outcomes in all situations as our approach.

Models incorporating the canonical model as a special case and using noncooperative approaches or mixes of non-cooperative and cooperative approaches to analyze stable divisions of players into clubs or coalitions include Konishi et al. (1997, 1998), Bogomolnaia and Jackson (2002), and Arnold and Wooders (2005), among others. Konishi et al. treat models of economies with local public goods; the most salient similarities between their model and ours is that they have a fixed number of jurisdictions and free entry is allowed. See also Demange (1994, 2005). All these papers consider some form of *Nash stability* of partitions of players into clubs. A partition of players into clubs is *Nash stable* if no player would prefer to leave his current club and join another. A full treatment of Nash stability in club networks is beyond the scope of our paper. Moreover, it is treated in a more general setting allowing hedonic games as a special case in Page and Wooders (2005), where in fact we introduce a concept of farsighted Nash stability and provide some characterization results.[17]

A paper successfully combining aspects of cooperative game theory and free entry is Bogomolnaia and Jackson (2002). These authors consider a hedonic (cooperative) game but introduce an approach and equilibrium concepts that allow them to treat free-entry equilibrium. The authors define a partition Π of players into clubs as contractually individually stable if there does not exist a player i and a club $S_k \in \Pi$, $i \notin S_k$, with the property that i prefers the club membership $\{i\} \cup S_k$ to the membership of the club to which he belongs under the partition Π and, for all members of S_k, the club $\{i\} \cup S_k$ is at least as good as the club S_k. To illustrate some of the differences between our work and theirs, first consider a case of our model where the number of clubs

[16] See, for example, Pauly (1970) and Wooders (1978).

[17] For the basic model considered in this paper, Nash stability is also discussed in Arnold and Wooders (2005).

is at least as great as the number of players; it then follows from Proposition 2 of Bogomolnaia and Jackson that both an individually stable coalition partition and a Nash stable partition exist. However, when we have a smaller number of clubs, their result cannot be immediately applied. We note, however, that in the context of our model, a contractually individual stable partition always exists. In particular, consider an assignment of individuals to clubs where there are as many clubs of optimal size as possible, and all excess players are in one club. Suppose that there are, for example, three clubs, the optimal club size s^* is 3, and there are 11 players. Consider a partition where there are two clubs of size 3 and one of size 5. This is a contractually individually stable partition, even though the Bogomolnaia–Jackson conditions are not satisfied.

6. Conclusions

An aspect of our work which we find particularly interesting is relationships between the outcomes of the dynamic process in Arnold and Wooders (2005) and the outcomes of farsighted strategic behavior. Research in progress addresses these questions and also further develops network models of clubs, in a number of directions.

Acknowledgement

The authors thank CERMSEM and Paris 1 for their hospitality.

References

Arnold, T., Wooders, M., 2005. Dynamic club formation with coordination. Working Paper No. 05-w22, Department of Economics, Vanderbilt University.

Bala, V., Goyal, S., 2000. A noncooperative model of network formation. Econometrica 68, 1181–1229.

Banerjee, S., Konishi, H., Sonmez, T., 2001. Core in a simple coalition formation game. Social Choice and Welfare 18, 135–153.

Bogomolnaia, A., Jackson, M.O., 2002. The stability of hedonic coalition structures. Games and Economic Behavior 38, 201–230.

Buchanan, J.M., 1965. An economic theory of clubs. Economica 33, 1–14.

Chander, P., Tulkens, H., 1995. A core-theoretic solution for the design of cooperative agreements on transfrontier pollution. International Tax and Public Finance 2, 279–293.

Chander, P., Tulkens, H., 1997. The core of an economy with multilateral environmental externalities. International Journal of Game Theory 26, 379–401.

Chwe, M., 1994. Farsighted coalitional stability. Journal of Economic Theory 63, 299–325.

Conley, J., Smith, S., 2005. Coalitions and clubs: Tiebout equilibrium in large economies. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 246–265.

Demange, G., 2005. Group formation: the Interaction of increasing returns and preference diversity. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 171–208.

Demange, G., 1994. Intermediate preferences and stable coalition structures. Journal of Mathematical Economics 23, 45–48.

Diamantoudi, E., Xue, L., 2003. Farsighted stability in hedonic games. Social Choice and Welfare 21, 39–61.

Gillies, D.B., 1959. Solutions to general non-zero-sum games. In: Tucker, A.W., Luce, R.D. (Eds.), In: Contributions to the Theory of Games, Vol. 4. Princeton University Press, Princeton, pp. 47–85.

Jackson, M.O., Wolinsky, A., 1996. A strategic model of social and economic networks. Journal of Economic Theory 71, 44–74.

Jaramillo, F., Kempf, H., Moizeau, F., 2005. Inequality and growth clubs. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 381–429.

Kirman, A., 1983. Communication in markets: a suggested approach. Economic Letters 12, 101–108.
Konishi, H., Le Breton, M., Weber, S., 1998. Equilibrium in a finite local public goods economy. Journal of Economic Theory 79, 224–244.
Konishi, H., Le Breton, M., Weber, S., 1997. Free mobility equilibrium in a local public goods economy with congestion. Research in Economics 51, 19–30.
Kovalenkov, A., Wooders, M.H., 2003. Approximate cores of games and economies with clubs. Journal of Economic Theory 110, 81–120.
Kovalenkov, A., Wooders, M.H., 2005. Games and economies with near exhaustion of gains to scale. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 209–245.
Le Breton, M., Weber, S., 2005. Session-proof cost allocations and stable group structures in models of horizontal differentiation. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 266–285.
Page, Jr., F. H., Wooders, M. H., 2004. Strategic basins of attraction, the farsighted core, and network formation games. FEEM, Working Paper 36.05.
Page, Jr., F. H., Wooders, M. H., 2005. Strategic basins of attraction, the path dominance core, and network formation games. Working Paper No. 06-w14, Department of Economics, Vanderbilt University.
Pauly, M., 1970. Cores and clubs. Public Choice 9, 53–65.
Shubik, M., Wooders, M. H., 1982. Near markets and market games. Cowles Foundation Discussion Paper No. 657.
Shubik, M., Wooders, M.H., 1983. Approximate cores of replica games and economies. Part II. Set-up costs and firm formation in coalition production economies. Mathematical Social Sciences 6, 285–306.
Slikker, M., van den Nouweland, A., 2001. Social and Economic Networks in Cooperative Game Theory. Kluwer Academic Publishers, Boston.
Tiebout, C., 1956. A pure theory of local expenditures. Journal of Political Economy 64, 416–424.
van den Nouweland, A., 2005. Models of network formation in cooperative games. In: Demange, G., Wooders, M. (Eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press, Cambridge, pp. 58–88.
Wooders, M.H., 1978. Equilibria, the core, and jurisdiction structures in economies with a local public good. Journal of Economic Theory 18, 328–348.
Wooders, M.H., 1983. The epsilon core of a large replica game. Journal of Mathematical Economics 11, 277–300.