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STABILITY OF JURISDICTION STRUCTURES IN ECONOMIES WITH LOCAL PUBLIC GOODS*

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Individuals of different types can form groups, i.e. jurisdictions, for the purposes of collective consumption and production of local public goods by the members of the jurisdictions. Also, the utility of an individual may be affected by the composition and size of the jurisdiction of which he is a member. Jurisdiction formation is endogenous. Trade of private goods can occur within jurisdictions and within collections of jurisdictions. A stable partition of individuals is shown to exist for all sufficiently large economies. This stability depends, partially, upon the extent of 'satisficing' behavior or alternatively, jurisdiction formation costs, both of which can be made arbitrarily small. The major noteworthy assumption is that positive outputs cannot become virtually free in per-capita terms as the economy is replicated; this ensures that the public goods are 'local' rather than 'pure'; otherwise assumptions on production sets are minimal and, in particular, convexity is not required. To obtain stability with coalition formation costs, additional assumptions are made ensuring that there is a 'minimum efficient scale' for coalitions.

Key words: Jurisdiction; public goods; utility; economies; local; convexity; minimum efficient scale.

1. Introduction

It is not difficult to imagine that the jointness in consumption of a local public good entails interaction among the consumers. It is typically assumed that this interaction can be captured by having preferences depend on the number of consumers.[1] However, it is clear that, in addition to the size of the group, consumers may also be affected by the composition of the group of agents jointly consuming the good.[2]

[1] See Bewley (1981) for some review of the literature.

[2] See Shaked (1982) and Schweizer (1983) for examples of situations where only composition of the group matters.

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For example, swimmers are affected not only by the number of people in the pool but also by the types of swimming activities chosen. Production of local public goods may also involve interaction by the producers/consumers. If the production possibility set for the good involves increasing and then decreasing returns to group size for some relevant region, then the importance of group size is clear. Different abilities and complementarities in skills may make the composition of the agents involved in production an important factor. In this paper we consider a situation where preferences and/or production possibilities depend on both the size and composition of the group for (and by) whom the goods are provided.

A model of a replica economy with endogenous jurisdiction formation is developed and it is shown that stable states of the economy exist for all sufficiently large replications. A stable state of the economy, which includes a partition of the set of agents into jurisdictions as one of its components, has the property that no coalition of agents could, using only its own resources, significantly improve upon that state for its own membership. The existence of stable states is partially due to satisficing behavior or, alternatively, coalition formation costs. Both the extent of satisficing behavior and coalition formation costs are parametrized by a positive number, ε, and ε can be allowed to become arbitrarily small for sufficiently large economies. As ε goes to zero, in the case of satisficing behavior the 'extent' of satisficing (i.e., the difference from exact optimization) becomes small, and, in the case of coalition formation costs, these costs go to zero.

Very informally, the concepts of stability involve some pseudo-dynamics. A state of the economy is given and coalitions of agents then determine whether or not it is worthwhile to attempt to rearrange themselves into different jurisdiction structures (partitions of the agents in the coalitions) and reallocate their endowments. This 'pseudo-dynamic' approach becomes apparent in the case of coalition formation costs when one perceives that in the given state of the economy, agents can achieve levels of satisfaction which would not be achievable if they had to 'move' to that state from another state – they would have to use up resources in jurisdiction formation.

Obviously, both concepts of stability are related to the core and, if $\varepsilon = 0$, a stable state of the economy is in the core.

In addition to whether or not stable states of the economy exist, a natural question to ask is to what extent these states can be supported by competitive prices. Addressing this question is beyond the scope of this paper. However, in another paper, Wooders (1986), a notion of a competitive equilibrium is developed, and the results herein are used to show existence of the equilibrium. Essentially, it is shown that if a state of the economy and all replications of that state are stable, then the state is an approximate competitive equilibrium. From results herein it follows that such states exist.

Compared to the situation where only the *size* of a group affects preferences and/or production, the introduction of complementarities between types of agents greatly complicates the problem of existence of stable states. With only one private good or if prices for private goods are given, when only group size is relevant all states of the economy in the core have homogeneous jurisdictions. In these cases, one can determine a 'type-optimal' jurisdiction size, which maximizes the utility of a representative agent of the given type, for each type of agent separately. Then the nonemptiness of approximate cores of sufficiently large economies is relatively straightforward (see Wooders, 1980). However, when complementarities are present, the optimal composition of a jurisdiction depends on the relative abundance of agents of each type and the problem does not separate into a number of simpler problems, one for each type of agents. Therefore the arguments must be more essentially of a 'general equilibrium' nature.

In this paper, we utilize results in Wooders (1983) concerning approximate cores of large replica games without side payments to obtain our stability results. In that paper, conditions are demonstrated under which large replica games have nonempty approximate cores and a number of useful lemmas are proven. No 'balancedness' assumptions are required. The conditions are that the sequence of games is superadditive and per-capita bounded, and that the payoff sets are convex.[3] With the exception of the convexity requirement, these conditions are satisfied by sequences of games derived from a broad class of sequences of replica economies, for example, ones with private goods, coalition production, and local public goods. It was conjectured that the results could be applied to replication models of economies whose derived games do not necessarily satisfy the convexity requirement if there is an infinitely divisible good that is a substitute for every other good. This paper illustrates such an application. It appears that the overall strategy of the proofs can be applied to obtain analogous theorems to those therein for diverse economic models.

Relative to a *given* jurisdiction structure, our assumptions on consumption sets and preferences are standard and intended to keep the arguments relatively straightforward. To ensure per-capita boundedness and to model *local* public goods rather than pure, we assume that positive outputs of public goods do not become virtually free in per-capita terms as the group producing the goods becomes large, and we assume that individuals do not derive more and more utility from belonging to larger and larger groups. Otherwise our assumptions on both private and public goods production sets are minimal; in particular, non-convexities are allowed. Also, it is possible that public goods can be produced only in indivisible units - for example, only integral numbers of units. To obtain stability with arbitrarily small coalition formation costs, we in addition require that all 'increasing returns to coalition size' are achieved by some finite economy, i.e. there is a minimum efficient scale for coalitions.

Before concluding this introduction, a few remarks relating to the extent literature on local public goods economies may be helpful. The model and results of this paper significantly extend the nonemptiness of approximate core results in Wooders

[3] These terms are formally defined later.

(1980), (where preferences and production possibilities depend only on numbers of agents in groups). In conjunction with Wooders (1986), the model, concepts and results extend those in Wooders (1980) and manifest, in a general setting, 'competitive' aspects of economies with local public goods. The overall approach differs from (most of) the examples considered by Bewley (1981) in that our results are for large economies (ones with many agents) and endogenous formation of relatively small jurisdictions, whereas Bewley considers only 'fixed size' economies.[4]

In an insightful paper, Schweizer (1983) shows that allocations which satisfy a certain notion of 'within-club efficiency' can be characterized by prices. A complete description of Schweizer's model and the relationship of his work to ours is beyond the scope of this paper. However, we remark that our results, again in conjunction with Wooders (1986), extend those of Schweizer to a situation with a total population constraint and with the endogenous determination of both public goods provision and jurisdiction structures.

The paper contains 5 sections. In the following section, the model is developed. The third section contains the stability results; these are proven in the fourth section. In Section 5, we discuss some extensions of the results.

2. The model

The following notation and terminology will be used; R^n: the n-dimensional Cartesian product of the real numbers; R^n_+: the non-negative orthant of R^n; R^n_{++}: the positive orthant of R^n; given a set S, $|S|$ denotes the cardinal number of S. The unit vector in R^n is denoted by $\underline{1}=(1,1,\dots,1)\in R^n$.

We follow the convention that given x and y in R^n, $x \geq y$ means $x_i \geq y_i$ for all i; $x>y$ means $x\geq y$ and, for at least one i, $x_i>y_i$; and $x \gg y$ means $x_i>y_i$ for all i.

Given $x\in R^n$, $\|x\| = \max_i |x_i|$ where $|x_i|$ is the absolute value of the ith coordinate of x.

2.1. Agents

The set of agents of the rth replica economy is denoted by $N_r=\{(1,1),\dots,(t,q),\dots,(T,r)\}$ where (t,q) is called the *qth agent of type t*. Given N_r and $t\in\{1,\dots,T\}$, let $[t]_r=\{(t,q): q\in\{1,\dots,r\}\}$; the set $[t]_r$ is the set of *agents of type t of the rth replica economy*.

Given $S\subset N_r$, let s be the vector whose tth coordinate is defined by $s_t=|S\cap[t]_r|$; s is called the *profile of* S and is simply a list of the numbers of agents of each type in S. When S has profile s, we write $\varrho(S)=s$. Let I denote the T-fold Cartesian product of the non-negative integers excluding the zero vector; then for every r and

[4] Although Bewley has, in some examples, a continuum of consumers, he has a fixed finite number of relatively large jurisdictions. See Wooders (1986) for further discussion.

every non-empty $S \subset N_r$, $s \in I$ where $s = \varrho(S)$. We denote the set of elements of I whose tth coordinate is non-zero by $I(t)$; a member of $I(t)$ is the profile of a subset containing an agent of type t. The set of profiles of subsets of N_r is denoted by $I_r = \{s \in I : s \leq r\underline{1}\}$ and the set of profiles of subsets of N_r containing an agent of type t is denoted by $I_r(t) = \{s \in I : s \in I(t) \cap I_r\}$.

2.2. Goods

The economy has K private goods and M public goods. A vector of the public goods is denoted by $x = (x_1, \ldots, x_m, \ldots, x_M) \in R^M$ and a vector of private goods, by $y = (y_1, \ldots, y_k, \ldots, y_K) \in R^K$.

2.3. Endowments and preferences

It is assumed that each agent has a positive endowment of each private good and that there are no endowments of the public goods. Write w_k^{tq} *for the endowment of the* (t,q)*th agent of the* k*th private good* and let $w^{tq} = (w_1^{tq}, \ldots, w_k^{tq}, \ldots, w_K^{tq})$. All agents of type t are assumed to have the same endowment.

The utility function of the (t,q)th agent is denoted by $u^{tq}(.,.,.)$ and maps $I(t) \times R_+^M \times R_{++}^K$ into R_+.[5] The utility functions of all agents of type t are identical: let $u^t(.,.,.)$ denote the utility function of a representative agent of type t.

We require the following assumptions on $u^t(.,.,.)$:

(a) For any $s \in I(t)$, $u^t(s,.,.)$ is a continuous, quasi-concave function,

(b) Given any $s \in I(t)$, $(x,y) \in R_+^M \times R_{++}^K$ where $y' > y$, we have $u^t(s,x,y') > u^t(s,x,y)$ (monotonicity),

(c) Given any $s \in I(t)$, $s' \in I(t)$, $(x,y) \in R_+^M \times R_{++}^K$ and $x' \in R_+^M$, there is a $y' \in R_{++}^K$ such that $u^t(s,x',y') \geq u^t(s',x,y)$,

(d) For each t, we have $u^t(s,0,w^t) > 0$ when s is the profile with $s_t' = 1$ if $t' = t$ and $s_{t'} = 0$ otherwise, and

(e) There is an r^* such that for any r, any $(t,q) \in N_r$, and any $(x,y) \in R_+^M \times R_{++}^K$, for some $s \in I_{r^*}(t)$ we have $u^{tq}(s,x,y) \geq u^{tq}(s',x,y)$ for all $s' \in I(t)$.

Assumption (c) above ensures that agents in 'less desirable' jurisdictions can be compensated by increased allocations of the private goods; an analogous assumption is used in Wooders (1980, p. 1470). Assumption (d) is simply so that the results of Wooders (1983) can be more easily applied and is non-restrictive. Assumption (e) limits increasing returns to group size; individuals do not derive more and more utility from being part of a larger and larger group.

Given (t,q) and any two subsets containing (t,q), say S and S', we write $u^{tq}(S,x,y) > u^{tq}(S',x',y')$ if $u^{tq}(s,x,y) > u^{tq}(s',x',y')$ and $u^{tq}(S,x,y) = u^{tq}(S',x',y')$ if $u^{tq}(s,x,y) = u^{tq}(s',x',y')$ where $\varrho(S) = s$ and $\varrho(S') = s'$.

[5] The assumption that the domain of $u^{tq}(s,x,.)$ is R_{++}^K is more restrictive than required. Essentially, what is needed is the presence of *one* infinitely divisible good which is necessary for consumption.

2.4. Jurisdiction structures

A *jurisdiction structure of* $S \subset N_r$ is a partition of S, denoted by $\mathcal{S} = \{S_1, \dots, S_l, \dots, S_L\}$. A jurisdiction structure of N_r is called simply a *jurisdiction structure* and denoted by $\mathcal{N}_r = \{J_1, \dots, J_g, \dots, J_G\}$.

Given $(t,q) \in S$ and $\mathcal{S} = \{S_1, \dots, S_l, \dots, S_L\}$, a jurisdiction structure of S, define $u^{tq}(\mathcal{S}, x, y) = u^{tq}(S', x, y)$ where $S' \in \mathcal{S}$ and $(t,q) \in S'$; this definition does nothing more than simplify notation.

2.5. Allocations

Given a non-empty subset S of N_r and $\mathcal{S}$, a jurisdiction structure of S, an *allocation for S relative to* $\mathcal{S}$, or simply *an allocation for S*, is a pair (x^S, y^S) where $x^S \in R_+^{MS}$ and $y^S R_{++}^{KS}$ *such that for each* $S' \in \mathcal{S}$ and for all (t,q) and (t',q') in S', we have $x^{tq} = x^{t'q'}$ (all agents in each jurisdiction are allocated the same amount of the public goods).

Given an allocation for S relative to $\mathcal{S}$, say (x^S, y^S), the associated total consumption of the agent (t,q) is $(\mathcal{S}, x^{tq}, y^{tq})$.

2.6. Production

The production possibility set for public goods available to a jurisdiction depends on the profile of that jurisdiction. We take as given a correspondence, Y_0, from the set of profiles I to closed,[6] non-empty subsets of $R_+^M \times -R_+^K$. An element of $Y_0(s)$ is denoted by (x,z) where x represents outputs of the public goods and z represents inputs of private goods. We assume that $Y_0(s) \cap R_+^{M+K} = \{0\}$. Given a non-empty subset S of N_r for some r with $\varrho(S) = s$, we define $Y_0(S) = Y_0(s)$.

The production possibilities for the public goods relative to a jurisdiction structure $\mathcal{S} = \{S_1, \dots, S_l, \dots, S_L\}$ of S will be denoted by $Y_0(\mathcal{S})$. We assume that $Y_0(\mathcal{S}) = \prod_{l=1}^{L} Y_0(S_l)$; there are no externalities in production between jurisdictions. An element of $Y_0(\mathcal{S})$ is denoted by $\beta(\mathcal{S})$. Note that, given $\beta(\mathcal{S})$, for some $(x_l, z_l) \in Y_0(S_l)$ for each l, we have $\beta(\mathcal{S}) = \prod_{l=1}^{L} (x_l, z_l)$.

We now impose restrictions on Y_0 to ensure that in the derived games, all 'increasing returns to coalition size' are eventually exhausted.[7]

(a) There is a closed, convex cone Y_0^* with $Y_0^* \cap R_+^{M+K} = \{0\}$, and if $(x,z) \in Y_0(S)$, then $(|S|x, z) \in Y_0^*$ for any non-empty subset S of N_r and for any r.

We observe that no convexity assumptions are made on the production sets themselves either for private or public goods production.

[6] The closedness property simplifies the proofs but is not essential.

[7] These assumptions are discussed and illustrated in Shubik and Wooders (1986) for the transferable utility case.

The production possibility set for private goods is not dependent upon the jurisdiction structure. We denote this production possibility set by Y_1 and an element of Y_1 is denoted by $z \in R^K$. We assume that $Y_1 \cap R_+^K = \{0\}$ and that Y_1 is closed. To rule out 'free production' in the limit as coalitions become large, we assume:

(b) there is a closed, convex cone Y^* such that

(i) $Y_1 \subset Y^*$,

(ii) $Y^* \cap R_+^K = \{0\}$.

The (entire) production possibility set for S relative to the jurisdiction structure $\mathcal{S}$ is denoted by $Y(\mathcal{S})$ where $Y(\mathcal{S}) = Y_0(\mathcal{S}) \times Y_1$. An element of $Y(\mathcal{S})$ is called a *production for S relative to $\mathcal{S}$* or simply a *production for $\mathcal{S}$* and is denoted by $(\beta(\mathcal{S}), z)$ where $\beta(\mathcal{S}) \in Y_0(\mathcal{S})$ and $z \in Y_1$.

2.7. States of the economy

Given a non-empty subset $S \subset N_r$, and $\mathcal{S} = \{S_1, \ldots, S_l, \ldots, S_L\}$, a jurisdiction structure of S, *a state of the economy for* S relative to $\mathcal{S}$ is an ordered pair $\alpha(\mathcal{S}) = ((x^S, y^S), (\beta(\mathcal{S}), z))$ where (x^S, y^S) is an allocation for S relative to $\mathcal{S}$ and $(\beta(\mathcal{S}), z)$ is a production for S relative to $\mathcal{S}$ such that, given $\beta(\mathcal{S}) = \prod_{l=1}^{L} (x_l, z_l)$, for each S_l we have $x_l = x^{tq}$ for all $(t,q) \in S_l$ (the consumption of the public goods by the members of each jurisdiction equals output of the public goods by that jurisdiction). The state of the economy for S relative to $\mathcal{S}$ is *feasible* for S if

$$\sum_{tq \in S} (y^{tq} - w^{tq}) \leq z + \sum_{l=1}^{L} z_l .$$

3. Stability

We introduce two concepts of stability and demonstrate conditions under which stable states of the economy exist for all sufficiently large replications of the economy.

3.1. $s(\varepsilon)$-stability

We will define a feasible state of the economy as $s(\varepsilon)$-stable if no coalition of agents could 'significantly' improve upon that state for the membership of the coalition where whether or not an improvement is 'significant' depends on ε. In other words, a feasible state is $s(\varepsilon)$-stable if it is stable with 'satisficing' and the extent of satisficing is determined by ε.

Formally, given $\varepsilon \geq 0$, a replication number r, and a feasible state of the rth economy, say $\alpha(N_r)$, the state is *$s(\varepsilon)$-stable* if for all non-empty subsets S of N_r there does not exist a feasible state for S, say $\alpha'(S)$, with $u^{tq}(S, x'^{tq}, y'^{tq}) > u^{tq}(N_r, x^{tq}, y^{tq}) + \varepsilon$ for all $(t,q) \in S$ where (x^{N_r}, y^{N_r}) is the allocation associated with $\psi(N_r)$ and (x'^S, y'^S) is that associated with $\alpha'(S)$.

Theorem 1. *Given $\varepsilon>0$ there is an r' such that for all $r\geq r'$, an $s(\varepsilon)$-stable state of the economy exists.*

We note that Theorem 1 does not require that all 'increasing returns to coalition size' are exhausted by some *finite* economy (i.e. there is no 'minimum efficient scale' property, defined in the next subsection).

3.2. $c(\varepsilon)$-stability

This notion of stability is based on the view that coalition formation is costly. As stated in the introduction this concept implicitly involves 'pseudo-dynamics' since no resources are used up in establishing a given stable state while to move to another state involves coalition formation costs. One could imagine the given state as having been created 'last period' and the resources required to form the state used up then.

We take as given a mapping c from $I\times R^1_+$ to R^K. Given $s\in I$ and $\varepsilon\in R^1_+$, $c(s;\varepsilon)$ represents the vector of inputs of private goods required by a subset S with profile s to form a jurisdiction consisting of the members of S. We follow the convention that inputs are non-positive so $c(s;\varepsilon)\in -R^K_+$. We assume that there is a $\bar{z}\in R^K$ where $\bar{z}<0$ such that given any profile s, we have $c(s;\varepsilon)=\varepsilon|s|\bar{z}$ where $|s|=\sum_{t=1}^{T}s_t$, so jurisdiction formation costs are linear[8] in $|s|$.

Given $S\subset N_r$, we define $c(S;\varepsilon)$ by $c(S;\varepsilon)=c(s;\varepsilon)$ when $\varrho(S)=s$.

Given a non-empty subset S of N_r, and a state of the economy for S relative to $\mathcal{S}=\{S_1,\dots,S_L\}$, say $\psi(S)=((x^S,y^S),(\beta(S),z))$ where $\beta(S)=\prod_{l=1}^{L}(x_l,z_l)$, the state is *$c(\varepsilon)$-feasible for S* if

$$\sum_{tq\in S}(y^{tq}-w^{tq})\leq z+\sum_{l=1}^{L}z_l+\sum_{l=1}^{L}c(S_l;\varepsilon).$$

Given $\varepsilon\geq 0$ and a replication number r, a feasible state of the economy $\psi(N_r)$ with associated allocation (x^{N_r},y^{N_r}) is *$c(\varepsilon)$-stable* if there does not exist a non-empty subset S of N_r and a $c(\varepsilon)$-feasible state of the economy for S relative to $\mathcal{S}$ with associated allocation (x'^S,y'^S) such that

$$u^{tq}(\mathcal{S},x'^{tq},y'^{tq})>u^{tq}(N_r,x^{tq},y^{tq})\text{ for all }(t,q)\in S.$$

Informally, a feasible state is $c(\varepsilon)$-stable if no coalition S can improve upon the state after using up the resources required for coalition and jurisdiction formation.

For the following theorems, we assume that there is a *minimum efficient scale for jurisdictions* (MES). Specifically, we assume that there is an r^* such that given $r\geq r^*$, for any $\varepsilon\geq 0$ if $\psi(N_r)$ is a $c(\varepsilon)$-feasible state of the economy with associated allocation (x^{N_r},y^{N_r}) then there is a $c(\varepsilon)$-feasible state of the economy for N_r, say $\psi(N_r')$ with associated allocation, (x'^{N_r},y'^{N_r}) such that

[8] This assumption is more restrictive than required, but facilitates the proofs.

(a) $u^{tq}(N_r', x'^{tq}, y'^{tq}) \geq u^{tq}(N_r, x^{tq}, y^{tq})$,

(b) for all $S \in N_r'$ we have $\varrho(S) \leq \varrho(N_{r^*})$.

We call r^* an *MES bound*.

Informally, the assumption of a minimum efficient scale for jurisdictions ensures that all increasing returns to jurisdiction size can be realized by jurisdiction structures where the profile of each jurisdiction is bounded above by some given profile. This assumption is made in this form for convenience and is more restrictive than actually required. For the next theorem, what is essential are assumptions ensuring that, for $\varepsilon > 0$, the set of equal-treatment payoffs derived from the $c(\varepsilon)$-feasible states are bounded away from those derived from the feasible states. Theorem 3 requires that the equal-treatment payoffs of the derived balanced cover games are eventually non-increasing (a property of games derived from replication models of private-goods-exchange economies).

Theorem 2. *Assume there is a minimum efficient scale for jurisdictions. Given $\varepsilon > 0$ there is a replication number r' such that for all $r \geq r'$, a $c(\varepsilon)$-stable state of the economy exists.*

Theorem 3. *Assume there is a minimum efficient scale for jurisdictions. Given any $\varepsilon \geq 0$ there is an r^0 such that the r_jth economy has a $c(\varepsilon)$-stable state where $r_j = lr^0$ for all positive integers l.*

4. Proofs of the theorems

4.1. An introduction to the proofs

Since the model and additional notation which will be required are complicated, the overall strategy of the proofs may be difficult to perceive. Consequently, before beginning the proofs, we provide an overview.

First, we construct the sequence of games derived from the $c(\varepsilon)$-feasible states of the economy. A game in this sequence is an ordered pair (N_r, V_r^ε) where V_r^ε is a correspondence mapping non-empty subsets of N_r into R^{rT} and $V_r^\varepsilon(S)$ represents the payoffs, in terms of utilities, achievable by the coalition S. When $\varepsilon = 0$, we denote V_r^ε simply by V_r. Also, we can represent equal-treatment payoffs as subsets of R^T, say $E(r; \varepsilon)$ and $E(r)$ for the games (N_r, V_r^ε) and (N_r, V_r) respectively. We denote the associated balanced cover games[9] by $(N_r, \tilde{V}_r^\varepsilon)$ and $(N_r, \tilde{V}_r)$ and their equal-treatment payoffs by $\tilde{E}(r; \varepsilon)$ and $\tilde{E}(r)$. We first show that the derived games are per-capita bounded; i.e. there is a compact subset K of R^T such that $E(r; \varepsilon) \cap R_+^T$ and $\tilde{E}(r; \varepsilon) \cap R_+^T$ are contained in K for all r and ε.

[9] These are formally defined later. For now, we note that the balanced cover games have non-empty cores (Scarf, 1967).

Some of the results in Wooders (1983) now apply. In particular,

(1) $E(r;\varepsilon)\subset\tilde{E}(r;\varepsilon)$;

(2) $\tilde{E}(r;\varepsilon)\subset\tilde{E}(r+1;\varepsilon)$ for all r;

(3) the sequence $(\tilde{E}(r;\varepsilon))$ has a closed limit;

(4) given any r' there is an n^0 such that for all positive integers l we have $\tilde{E}(r';\varepsilon)\subset E(ln^0r';\varepsilon)$.

It is a consequence of (3) that given any $\varepsilon>0$ there is an r' such that, for some $\alpha\in E(r')$, α represents an equal-treatment payoff in the ε-core of $(N_r, \tilde{V}_r)$ for all $r\geq r'$. (This result is obtained in Wooders, 1983.)

We now describe the strategy of the proof of Theorem 1. From the preceeding, there is an r' and an α^* in $\tilde{E}(r')$ such that α^* represents an equal treatment payoff in the $\varepsilon/2$-core of $(N_r, \tilde{V}_r)$, i.e., a feasible payoff which cannot be improved upon by more than $\varepsilon/2$ for all members of any coalition. Note that, from (4), for some n^0, $\tilde{E}(r')\subset E(n^0r')$. Given any $r\geq n^0r'$, write $r=nr'+j$, where n is the largest integer such that $r\geq nr'$. Let $B_j=(N_r-N_{nr'})$; informally, we can think of agents in B_j as 'left-overs', who may not be able to realize payoffs as large as α^*. Since those agents in $N_{nr'}$ (or any subset of N_r with the same profile) can do at least as well as any payoff in $\tilde{E}(r')$, we can take away a small amount of private goods, say Δy, from each agent in $N_{nr'}$ without significantly affecting them. The total amount of private goods taken away from agents in $N_{nr'}$ can be given to agents in B_j. Since $nr'T\Delta y$ becomes arbitrarily large, since the number of agents in B_j is bounded ($|B_j|\leq r'T$), and, from overriding desirability of the private goods (assumption (2.3(c)), eventually the agents in B_j become as well-off as those in $N_{nr'}$ and no coalition can significantly improve upon the given payoff. From the construction of the derived games, this shows existence of an $s(\varepsilon)$-stable state. (It also shows that the closed limit of $(E(r;\varepsilon))$ equals that of $\tilde{E}(r;\varepsilon)$ for any $\varepsilon\geq 0$.)

The basic idea behind the proof of Theorem 2 is in showing that, given $\varepsilon>0$, for all r sufficiently large $\tilde{E}(r;\varepsilon)\subset E(r)$. Then there is a state of the rth economy which is associated with an equal-treatment payoff, say α^*, of (N_r, V_r), and this state is $c(\varepsilon)$-stable since α^* is not in the interior of $V_r^{\varepsilon}(S)$ for any coalition S. The relationship $\tilde{E}(r;\varepsilon)\subset E(r)$ follows from monotonicity and the fact that for $\varepsilon>0$, $c(\varepsilon)$-feasibility is more restrictive (i.e., fewer resources can be used in consumption) than feasibility and, for r large, taking the balanced cover does not significantly change the set of equal-treatment payoffs.

Through the remainder of the paper, given r and $S\subset N_r$, it is to be understood that $S\neq\varnothing$. Also we continue to write $\underline{1}=(1,1,\ldots,1)$ for the vector of 1's and the dimension of the vector is to be inferred from the context - this should create no confusion.

4.2. Some game-theoretic definitions and results

In this subsection we review some game-theoretic results which will be used in the proofs in the next subsection. For convenience, some results are stated in slightly different forms than they originally appeared.

A *game without side payments*, or simply a *game*, is an ordered pair (A, V) where A, called the set of *players*, is a finite set and V is a correspondence from the set of non-empty subsets of A into subsets of R^A such that

(i) for every non-empty $S \subset A$, $V(S)$ is a non-empty, proper, closed subset of R^A containing some member, say, α, where $\alpha \gg 0$;

(ii) if $\alpha \in V(S)$ and $\alpha' \in R^A$ with $\alpha^i = \alpha'^i$ for all $i \in S$, then $\alpha' \in V(S)$;

(iii) $V(S)$ is bounded relative to R_+^S i.e., for each S, there is a vector $k(S) \in R^A$, where, for all $\alpha \in V(S)$, $\alpha^i \le k^i(S)$ for all $i \in S$.

The above definition differs from the usual definitions of a game in that we've required each payoff set $V(S)$ to contain a strictly positive member. This requirement is simply for technical convenience.

Let (A, V) be a game. A vector $\alpha \in R^A$, where the coordinates of α are superscripted by the members of A, is called a *payoff* for the game. A payoff α is *feasible* if $\alpha \in V(A)$. Given a payoff α and players i and j, let $\sigma[\alpha; i, j]$ denote the payoff formed from α by permuting the values of the coordinates associated with i and j. Players i and j are *substitutes* if: for all $S \subset A$ where $i \notin S$ and $j \notin S$, given any $\alpha \in V(S \cup \{i\})$, we have $\sigma[\alpha; i, j] \in V(S \cup \{j\})$; and, for all $S \subset A$ where $i \in S$ and $j \in S$, given any $\alpha \in V(S)$, we have $\sigma[a; i, j,] \in V(S)$. The game is *superadditive* if whenever S and S' are disjoint, non-empty subsets of A, we have $V(S) \cap V(S') \subset V(S \cup S')$. It is *comprehensive* if for any non-empty subset S of A, if $\alpha \in V(S)$ and $\alpha' \le \alpha$ then $\alpha' \in V(S)$.

Given a game (A, V) and $\varepsilon \ge 0$, a payoff α is in the *ε-core* of (A, V) if (a) α is feasible and if, (b) for all non-empty subsets S of A, there does not exist an $\alpha' \in V(S)$ such that $\alpha' \gg \alpha + \varepsilon \underline{1}$. When $\varepsilon = 0$, the ε-core is simply the *core*.

We review the concepts of balancedness and the balanced cover of a game. Let (A, V) be a game. Consider a family β of subsets of A and let $\beta_i = \{S \in \beta: i \in S\}$. A family β of subsets of A is *balanced* if there exists positive 'balanced weights' ω_s for S in β with $\sum_{S \in \beta_i} \omega_S = 1$ for all $i \in A$. Let $B(A)$ denote the collection of all balanced families of subsets of A. Define $\tilde{V}(A) = \bigcup_{\beta \in B(A)} \bigcap_{S \in \beta} V(S)$. Define $\tilde{V}(S) = V(S)$ for all $S \subset A$ with $S \ne A$. Then $\tilde{V}$ maps subsets of A into R^A and is called the *balanced cover of V*. The game $(A, \tilde{V})$ is called the *balanced cover of* (A, V). If the game (A, V) has the property that $\tilde{V}(A) = V(A)$, the game (A, V) is *balanced*, and from Scarf's theorem (1967), the core of the game is non-empty.

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of games where, for each r, $A_r \subset A_{r+1}$ and $A_r = \{(t, q): t \in \{1, \dots, T\}, q \in \{1, \dots, r\}\}$. Write $\alpha = (\alpha_1, \dots, \alpha_q, \dots, \alpha_r)$ for a payoff for the rth game where $\alpha_q = (\alpha^{1q}, \dots, \alpha^{tq}, \dots, \alpha^{Tq})$ and α^{tq} is the component of the payoff associated with the (t, q)th player. Given r and t, define $[t]_r$ by $[t]_r = \{(t, q) \in A_r: q \in \{1, \dots, r\}\}$; the set $[t]_r$ consists of *the players of type t of the rth game*. The sequence $(A_r, V_r)_{r=1}^{\infty}$ is *a sequence of replica games* if:

(a) for each r and each $t = 1, \dots, T$, all players of type t of the rth game are substitutes for each other;

(b) for any r' and r'' where $r'<r''$ and any $S \subset A_{r'}$, we have $V_{r'}^P(S) \subset V_{r''}^P(S)$ where $V_r^P(S)$ denotes the projection of $V_r(S)$ onto R^S (i.e., the set of utility vectors achievable by the coalition S does not decrease as r increases).

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. A payoff for the (A_r, V_r) is said to have the *equal treatment property* if, for each t, we have $\alpha^{tq'} = \alpha^{tq''}$ for all q' and q''; players of the same type are allocated the same payoff. The sequence of games is *superadditive* if (A_r, V_r) is a superadditive game for all r. The sequence is *per-capita bounded* if there is a constant K such that for all r and for all equal-treatment payoffs α in $V_r(A_r)$ we have $\alpha^{tq} \leq K$.

A sequence of games $(A_r, V_r)_{r=1}^{\infty}$ is said to satisfy the assumption of *minimum efficient scale* (for coalitions), MES, if there is an r^* such that for all $r \geq r^*$ given $\alpha \in \tilde{V}_r(A_r)$ there is a balanced collection β of subsets of A_r with the properties that (1) $\varrho(S) \leq \varrho(A_{r^*})$ for all $S \in \beta$ and (2) $\alpha \in \bigcap_{S \in \beta} V_r(S)$. We call r^* an *MES BOUND.*

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. We say that the sequence has a non-empty *strong approximate core* if given any $\varepsilon > 0$ there is an r^* sufficiently large so that for all $r \geq r^*$, the ε-core of (A_r, V_r) is non-empty. The sequence has a non-empty *weak approximate core* if given any $\varepsilon > 0$ and any $\lambda > 0$ there is an r^* such that for all $r \geq r^*$, for some $\tilde{\alpha} \in R^{A_r}$ and some $\alpha \in V_r(A_r)$, we have

(a) $$|\{(t,q) \in A_r : \alpha^{tq} \neq \tilde{\alpha}^{tq}\}| < \lambda |A_r|$$

and

(b) $\tilde{\alpha}$ cannot be ε-improved upon by any coalition S; i.e. there does not exist an $S \subset A_r$ and an $\alpha' \in V_r(S)$ such that $\alpha' \gg \alpha + \varepsilon \underline{1}$. The next two theorems are variations of theorems in Wooders (1983).

Theorem 4. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive replica games with MES bound r^*. For any $r > r^*$, the core of the game $(A_r, \tilde{V}_r)$ is non-empty and contains a payoff with the equal treatment property.*

The non-emptiness of the core in Theorem 4 is Scarf's Theorem (1967).

In Wooders (1983), Theorem 3 is stated with the additional assumption of quasi-transferable utility, QTU, – the payoff sets are assumed to not have segments of their boundaries in the positive orthant parallel to the coordinate planes. However, it is shown that a game can be approximated by one with QTU property. The original theorem can be applied to obtain an equal-treatment payoff in the core of the approximating game. A limit of some subsequence of the payoffs in the cores of the approximating games is an equal-treatment payoff in the core of the game.

Before stating the next theorem we introduce the following notation and definitions. Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of the replica games.

$$\text{Define: } E(r) = \left\{\alpha \in R^T : \prod_{i=1}^{r} \alpha \in V_r(A_r)\right\} \quad \text{and}$$

$$\tilde{E}(r) = \left\{\alpha \in R^T : \prod_{i=1}^{r} \alpha \in \tilde{V}_r(A_r)\right\}.$$

In Wooders (1983) it is shown that if $(A_r, V_r)_{r=1}^{\infty}$ is a superadditive, per-capita bounded sequence of replica games, then the closed limit[10] of $(\tilde{E}(r))$ exists; denote this limit by $L(\tilde{E})$. Let $\|V, W\|$ denote the Hausdorff distance (with respect to the sup norm) between two subsets V and W of R^n.

Theorem 5. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of per-capita bounded, superadditive replica games with the property that $\|E(r), L(\tilde{E})\| \to 0$ as $r \to \infty$. Then given $\varepsilon > 0$ there is an r^* such that for all $r \geq r^*$, the ε-core of (A_r, V_r) is non-empty, i.e., the sequence has a non-empty strong approximate core.*

We remark that in Wooders (1983) this theorem is stated requiring convexity of the payoff sets $V_r(A_r)$ for each r rather the convergence of $E(r)$ to $L(\tilde{E})$. The only role of convexity however is in showing that $\|E(r), L(\tilde{E})\| \to 0$ as $r \to \infty$. In this paper, we will use properties of the economies underlying the derived games to show that for these games $\|E(r), L(\tilde{E})\| \to 0$ as $r \to \infty$.

The next theorem is proven in Shubik and Wooders (1983a).

Theorem 6. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive, per-capita bounded replica games. Then the weak approximate core is non-empty.*

4.3. The derived games

Throughout the remainder of the paper we assume that for some $\varepsilon^0 > 0$, we have $\varepsilon < \varepsilon^0$ and ε^0 is sufficiently small so that $-\varepsilon^0 \bar{z} < w^t$ for each type t where $\bar{z}$ is the vector satisfying $c(s, \varepsilon) = \varepsilon |s| \bar{z}$. (Note that this implies that for any r and any subset S of N_r, we have $-c(S, \varepsilon) < \sum_{tq \in S} w^{tq}$; any coalition has enough resources to form a jurisdiction.) We also require that for each type t, we have $u^t(s, 0, w^{tq} - \varepsilon^0 \bar{z}) > 0$ where s is the profile with $s_t = 1$ and $s_{t'} = 0$ for all $t' \neq t$; this is possible from assumptions 2.3(a) and 2.3(d). These assumptions entail no loss of generality since if a state of the economy is ε-stable (either $s(\varepsilon)$-stable or $c(\varepsilon)$-stable) then it is ε'-stable for all $\varepsilon' \geq \varepsilon$.

Given r and $\varepsilon \geq 0$ we associate a correspondence $\hat{V}_r^{\varepsilon}$ with the rth economy where $\hat{V}_r^{\varepsilon}$ maps subsets S of N_r into R^{N_r}. For each subset S of N_r, define $\hat{V}_r^{\varepsilon}(S)$ as the set of vectors $\alpha \in R^{N_r}$ with the property that for some jurisdiction structure $\mathcal{S}$ of S and some $c(\varepsilon)$-feasible state of the economy for S relative to $\mathcal{S}$, say $\psi(S)$ with associated allocation (x^S, y^S), we have $\alpha^{tq} \leq u^{tq}(S, x^{tq}, y^{tq})$ for each $(t,q) \in S$.

The coordinates of $\alpha \in \hat{V}_r^{\varepsilon}(S)$ are superscripted by the members of N_r and ordered $\alpha = (\alpha^{11}, \dots, \alpha^{t1}, \dots, \alpha^{T1}, \dots, \alpha^{1q}, \dots, \alpha^{tq}, \dots, \alpha^{Tq}, \dots, \alpha^{1r}, \dots, \alpha^{tr}, \dots, \alpha^{Tr})$. The coordinate α^{tq} of α is called the *payoff of the* (t,q)*th player* and α is called a *payoff.*

We remark that $\hat{V}_r^{\varepsilon}(S)$ is non-empty for any r and any subset S of N_r since there is an $\alpha \gg 0$ in $\hat{V}_r^{\varepsilon}(S)$ with $\alpha^{tq} = u^{tq}(S', 0, w^{tq} - \varepsilon \bar{z})$ where $S' = \{(t,q)\}$ for each (t,q) in

[10] See Hildenbrand (1974), pp. 15–17 for a definition of the closed limit and some properties.

S. Also, $\hat{V}_r^\varepsilon(S)$ contains all its limit points except those for sequences of payoffs whose associated states of the economy have y_k^{tq} converging to zero from some k and some $(t,q)\in S$. More formally, let $\psi_j(S_j)=((x_j^S, y_j^S), (\beta_j(S_j), z_j))$ be a sequence of states of the economy for S and let α_j be a payoff where $\alpha_j^{tq}=u^{tq}(S_j, x_j^{tq}, y_j^{tq})$. If α_j converges to some vector α_0, and if, for some convergent subsequence of (y_j^S), the limit of the subsequence is strictly greater than zero, then $\alpha_0\in\hat{V}_r^\varepsilon(S)$. If the limit of every convergent subsequence of (y_j^S) has a coordinate equal to zero, then α_0 may not be in $\hat{V}_r^\varepsilon(S)$. These observations follow from closedness of production sets and the fact that the consumption set is $R_+^m\times R_{++}^k$ (so, of course, the consumption sets are not closed).

When $\varepsilon=0$, we denote $\hat{V}_r^\varepsilon(S)$ simply by $\hat{V}_r(S)$.

For each r, let V_r^ε be the correspondence mapping subsets S of N_r into subsets of R^{N_r} where $V_r^\varepsilon(S)$ is the closure of the set $\hat{V}_r^\varepsilon(S)$. Similarly, let V_r be the correspondence mapping subsets S of N_r into subsets of R^{N_r} where $V_r(S)$ is the closure of $\hat{V}_r(S)$.

It is straightforward to verify that the sequence of derived games $(N_r, V_r^\varepsilon)_{r=1}^\infty$ is a sequence of superadditive replica games. Moreover, the games are comprehensive.

Given $\varepsilon=0$, we let the rth derived game be denoted by (N_r, V_r). The balanced covers of the games (N_r, V_r^ε) (for any $\varepsilon>0$) and (N_r, V_r) are denoted by $(N_r, \tilde{V}_r^\varepsilon)$ and $(N_r, \tilde{V}_r)$ respectively.

The sets of equal treatment payoffs of (N_r, V_r^ε) and (N_r, V_r), represented as subsets of R^T, are denoted and defined as follows. Given r and $\varepsilon\geq 0$, let $E(r;\varepsilon)=\{\alpha\in R^T: \prod_{i=1}^r \alpha\in V_r^\varepsilon(N_r)\}$ and $\tilde{E}(r;\varepsilon)=\{\alpha\in R^T: \prod_{i=1}^r \alpha\in \tilde{V}_r^\varepsilon(N_r)\}$, and for $\varepsilon=0$, let $E(r)=\{\alpha\in R^T: \prod_{i=1}^T \alpha\in V_r(N_r)\}$, and $\tilde{E}(r)=\{\alpha\in R^T: \prod_{i=1}^T \alpha\in \tilde{V}_r(N_r)\}$. We denote the closed limits of these sets, which are shown to exist, by $L(E(\varepsilon))$, $L(\tilde{E}(\varepsilon))$, $L(E)$ and $L(\tilde{E})$ respectively.

Lemma 1. *Given $\varepsilon\geq 0$, the sequence of games $(N_r, V_r^\varepsilon)_{r=1}^\infty$ is superadditive and per-capita bounded.*

Proof. To verify superadditivity is straightforward so the proof is omitted. To show per-capita boundedness, we need only consider the sequence of games $(N_r, V_r)_{r=1}^\infty$. This is because, from monotonicity, we have $V_r^\varepsilon(N_r)\subset V_r(N_r)$ for all $\varepsilon\geq 0$ so if $(N_r, V_r)_{r=1}^\infty$ is per-capita bounded, then (N_r, V_r^ε) is per-capita bounded for any $\varepsilon\geq 0$.

To show per-capita boundedness of $(N_r, V_r)_{r=1}^\infty$ we construct a related sequence of economies, called the $*$-economies, and consider the sequence of games, denoted by $(N_r, V_r^*)_{r=1}^\infty$, derived from the sequence of $*$-economies. We construct the $*$-economies so that $V_r(N_r)\subset V_r^*(N_r)$ for all r and show that $(N_r, V_r^*)_{r=1}^\infty$ satisfies per-capita boundedness to obtain the conclusion of the lemma. Informally in the $*$-economies, we have the 'most favorable' possible constraints on feasibility in terms of production possibility sets and partitions of agents. The constraints, in this case, are sufficiently similar to standard ones on sequences of private goods economies so that we can easily obtain per-capita boundedness.

Given r, for each $S \subset N_r$ define $\bar{Y}_0^*(S) = \{(x,z) \in R^{M+K} : (|S|x, z) \in Y_0^*\}$ and observe that $Y_0(S) \subset \bar{Y}_0^*(S)$. Given $S \subset N_r$, let $S = \{S_1, \ldots, S_l, \ldots, S_L\}$ be a jurisdiction structure of S where each jurisdiction S_l contains one and only one agent. Let $S' = \{S_1', \ldots, S_l', \ldots, S_L'\}$ be any other jurisdiction structure of S. Observe that if $\prod_{l=1}^{L'} (x', z') \in \prod_{l=1}^{L'} \bar{Y}_0^*(S_l')$, then there is $\prod_{l=1}^{L} (x,z) \in \prod_{l=1}^{L} Y_0(S_l)$ such that if $(t,q) \in S_l \cap S_{l'}'$, then $x_l = x_{l'}'$ for all $(t,q) \in S$, and $\sum_{l=1}^{L'} z_l' = \sum_{l=1}^{L} z_l$; there are 'constant returns' to productive coalitions.

For each r, for the rth $*$-economy, let the private goods production set be Y^*.

For each type t, let the utility function of each agent (t,q) be defined by

$$u^{*tq}(x,y) = \max_{s \in I_{r^*}(t)} u^t(s,x,y)$$

where r^* satisfies assumption 2.3(e) for each $(x,y) \in R_+^M \times R_{++}^K$. Observe that from assumptions 2.3(a) and (e), the utility functions $u^{*tq}(.,.)$ are well defined and are quasi-concave. Also, it is clear that given any $(s,x,y) \in I(t) \times R_+^M \times R_{++}^K$, we have $u^{*tq}(x,y) \geq u^{tq}(s,x,y)$. This completes the construction of the sequence of $*$-economies.

Let A_r^* be the set of allocations (x^{N_r}, y^{N_r}) such that for some feasible state of the rth $*$-economy, (x^{N_r}, y^{N_r}) is the associated allocation and, for each t, $u^{*tq}(x^{tq}, y^{tq}) = u^{*tq'}(x^{tq'}, y^{tq'})$ for all q and q', i.e., the allocations have the equal-treatment property (in terms of utilities). Let K be a real number such that

$$K > \sup_{(x^{N_1}, y^{N_1}) \in A_1^*} \sup_t u^{*t1}(x^{t1}, y^{t1});$$

from the closedness of the production sets and quasi-concavity there is such a real number. We claim K is a per-capita bound, i.e., for all r, if $\alpha \in V_r^*(N_r)$ and α has the equal-treatment property, then $\alpha^{tq} < K$ for all $(t,q) \in N_r$.

Suppose not. Then for some r', we can select an $\alpha' \in V_{r'}^*(A_{r'})$ where (i) α' has the equal-treatment property, (ii) $\alpha'^{t'q} \geq K$ for some t' (and all $q = 1, \ldots, r'$), and (iii) for some feasible state of the r'th $*$-economy, say $\psi(N_{r'}) = ((x^{N_{r'}}, y^{N_{r'}}), (\beta(N_{r'}), z))$ where $N_{r'} = \{\{(t,q)\} : (t,q) \in N_{r'}\}$, we have $u^{*tq}(x^{tq}, y^{tq}) \geq \alpha_t'$ for each t and all q. We denote the components of $\beta(N_{r'})$ by (x^{tq}, z^{tq}) for each $(t,q) \in N_{r'}$. Define $\bar{x}^t = \sum_{q=1}^{r'} x^{tq}/r'$ and $\bar{y}^t = \sum_{q=1}^{r'} y^{tq}/r$. Since $u^{*t}(.,.)$ is quasi-concave, $u^{*tq}(\bar{x}^t, \bar{y}^t) \geq u^{*tq}(x^{tq}, y^{tq})$ for all q and for each t. Since Y^* is a closed convex cone, $(1/r')z \in Y^*$. From the convexity of Y_0^* and construction of $\bar{Y}_0^*(.)$, we have $\prod_{t=1}^{T} (x^t, z^t) \in Y_0^*(N_1)$ where $N_1 = \{(t,q)\} : (t,q) \in N_1\}$ and, for each t, $\bar{z}^t = \sum_{q=1}^{r'} z^{tq}/r'$. For each $(t,q) \in N_1$, let $\bar{x}^{tq} = \bar{x}^t$ and $\bar{y}^{tq} = \bar{y}^t$. Since $\psi(N_{r'})$ is feasible, we have

$$\frac{1}{r'}\left[\sum_{tq \in N_{r'}} (y^{tq} - w^{tq})\right] \leq \frac{1}{r'}\left[z + \sum_{(t,q) \in N_{r'}} z^{tq}\right]$$

and therefore

$$\sum_{(t,q) \in N_1} (\bar{y}^{tq} - w^{tq}) \leq \frac{1}{r'} z + \sum_{t=1}^{T} \bar{z}^t.$$

Consequently, the state of the economy $\psi(N_1)=((\bar{x}^{N_1},\bar{y}^{N_1}),(\beta(N_1),z/r'))$ is feasible where $\beta(N_1)=\prod_{(t,q)\in N_1}(\bar{x}^{tq},\bar{z}^t)$. Also for each $(t,q)\in N_1$, we have $u^{tq}(\bar{x}^{tq},\bar{y}^{tq})\geq\alpha'^{tq}$. This is a contradiction to the assumption that $\alpha'^{tq}>K$. $\square$

The following notation and definitions are used in the proof of the next lemma. We require concepts of replications of a state of the economy and its components. Given $N_{r'}=\{J_1,\dots,J_g,\dots,J_G\}$, a jurisdiction structure of $N_{r'}$, and a positive integer n, let $r=nr'$. Let $n(N_{r'})$ be a jurisdiction structure of N_r containing nG jurisdictions, say $n(N_{r'})=\{J_{gj}:g=1,\dots,G,\ j=1,\dots,n\}$, with $J_{gj}=\{(t,q)\in N_r$: for some $(t,q')\in J_g$, $q=(j-1)r'+q'\}$ for each $g\in\{1,\dots,G\}$ and $j\in\{1,\dots,n\}$. Informally, $n(N_{r'})$ consists of n 'copies' of $N_{r'}$. Note that for each j, the profile of J_{gj} equals the profile of J_g. We call $n(N_{r'})$ the nth *replica of* $N_{r'}$. This definition of a replica of a jurisdiction structure is more restrictive than is actually required. Essentially, we need only that the nth replica of $N_{r'}$ contains n jurisdictions with the same profile as $J_g\in N_{r'}$ for each J_g. The additional restriction, that $(t,q)\in J_{gj}$ when $(t,q')\in J_g$ and $q=(j-1)r+q'$, simplifies notation, subsequent definitions, and proofs.

Given r', let $(x^{N_{r'}},y^{N_{r'}})$ be an allocation for the r'th economy where $r=nr'$, and where, for each $(t,q)\in N_{r'}$, we have $(x'^{tq'},y'^{tq'})=(x^{tq},y^{tq})$ for all $q'=q,2q,\dots,nq$. Then (x'^{N_r},y'^{N_r}) is called the nth *replica* of $(x^{N_{r'}},y^{N_{r'}})$ and is denoted by $n(x^{N_{r'}},y^{N_{r'}})$.

We similarly define replications of productions. Given r', let $N_{r'}=\{J_1,\dots,J_g,\dots,J_G\}$ be a jurisdiction structure and let $(\prod_{g=1}^{G}(x_g,z_g))=\beta(N_{r'})$. Given n, the nth *replica of* $\beta(N_{r'})$ is denoted by $\beta_n(N_{r'})=\prod_{g=1}^{G}\prod_{i=1}^{n}(x_g,z_g)\in Y_0(n(N_{r'}))$. Let $z\in Y_1$; then the nth *replica of* z is nz.

Given r', a positive integer n, and a state of the r'th economy $\psi(N_{r'})=((x^{N_{r'}},y^{N_{r'}}),(\beta(N_{r'}),z))$, let $n(\psi(N_{r'}))$ denote the nth replica of $\psi(N_{r'})$ where $n(\psi(N_{r'}))=(n(x^{N_{r'}},y^{N_{r'}}),(\beta_n(N_{r'}),nz))$.

In the following lemma, we prove that $E(r)$ converges to $L(\tilde{E})$. We remark that in the proof of Theorem 1 in Wooders (1983) it is shown that for sequences of replica games, when the payoff sets for the sets of all players in each game are convex, the equal-treatment payoffs of the games converge to those of the balanced cover games. It was stated in that paper that for games derived from sequences of economies an assumption which would ensure such convergence without convexity is that of an infinitely divisible good with 'over-riding desirability' (i.e., a substitute for every other good), with which everyone is initially endowed. Informally, this assumption ensures a certain degree of 'sidepaymentness' of the derived games. In this paper, it is the case that private goods are substitutes for both public goods and the group with whom an agent produces and consumes the public goods, which leads to the required convergence (see assumption 2.3(c)).

We first note that, as is obvious, $E(r;\varepsilon)\subset\tilde{E}(r;\varepsilon)$ for all r and for all $\varepsilon\geq 0$. Also, from comprehensiveness of the payoff sets, it suffices to prove that for some $\delta>0$, for all r sufficiently large, $L(\tilde{E}_+)\subset E(r)+\delta\underline{1}$ where $L(\tilde{E}_+)=L(\tilde{E})\cap R_+^T$.

Lemma 2. *Given $\delta>0$, there is an r^0 such that for all $r\geq r^0$, $L(\tilde{E})\subset E(r)+\delta\underline{1}$.*

Proof. Choose r'' such that for all $r \geq r''$, we have $L(\tilde{E}) \subset \tilde{E}(r) + \frac{1}{4}\delta\underline{1}$; this is possible from Wooders (1983, Lemma 8).

From Wooders (1983, Lemmas 3 and 5), given r'' we can choose a replication number r' such that $\tilde{E}(r'') \subset E(nr')$ for all positive integers n.

From the per-capita boundedness assumption and closedness of $V_r(A_r)$, we have $\tilde{E}_+(r)$ compact for all r where $\tilde{E}_+(r) = \tilde{E}(r) \cap R^T_+$. Let $\{\alpha^1, \ldots, \alpha^q, \ldots, \alpha^Q\}$ be a set of members of $\tilde{E}(r'')$ such that given any $\alpha \in \tilde{E}_+(r'')$, there is a q such that $\|\alpha - \alpha^q\| < \delta/4$. Given q, we now show that there is a replication number r_q such that for all $r \geq r_q$, we have $\alpha^q - \frac{1}{2}\delta\underline{1}$ in $E(r)$.

Since $V_{r'}(N_{r'})$ is the closure of $\hat{V}_{r'}(N_{r'})$ (where $\varepsilon = 0$) and since $\tilde{E}(r'') \subset E(r')$ there is a jurisdiction structure $N_{r'}$ of $N_{r'}$ and a feasible state of the economy for $N_{r'}$ relative to $N_{r'}$, say $\psi(N_{r'}) = ((x^{N_{r'}}, y^{N_{r'}}), (\beta(N_{r'}), z))$, such that $u^{tq}(N_{r'}, x^{tq}, y^{tq}) > \alpha_t - \delta/4$ for all $(t,q) \in N_{r'}$. Since $\psi(N_{r'})$ is feasible, $y^{tq} \gg 0$ for all $(t,q) \in N_{r'}$. Since utility functions are continuous, there is a $\Delta y \in R^K$ such that $\Delta y \gg 0$, $y^{tq} - \Delta y \gg 0$ for all $(t,q) \in N_{r'}$, and $u^{tq}(N_{r'}, x^{tq}, y^{tq} - \Delta y) > \alpha^q_t - \delta/2$.

Given $r \geq r'$, in the following we let n and j be non-negative integers such that $r = nr' + j$ where $j \in \{1, \ldots, r'-1\}$.

Given $r = nr' + j$, let $B_j = \{(t,q) \in N_{nr'} : nr' < q \leq r\}$. Let $\boldsymbol{B}_j$ be a jurisdiction structure of B_j and let $\psi(\boldsymbol{B}_j) = ((x^{B_j}, y^{B_j}), (\beta'(\boldsymbol{B}_j), z'))$ be a feasible state of the economy for B_j relative to $\boldsymbol{B}_j$. Let $\psi(N_r) = ((x^{*N_r}, y^{*N_r}), (\beta^*(N_r), z^*))$ be a state of the economy for N_r of N_r where

(a) $N_r = n(N_{r'}) \cup \boldsymbol{B}_j$ (agents in $N_{nr'}$ are in a jurisdiction structure which is the nth replica of $N_{r'}$ and agents in B_j are in the jurisdiction structure which is identical to $\boldsymbol{B}_j$);

(b) for all $(t,q) \in N_{nr'}$ we have $x^{*tq} = x^{tq'}$ and $y^{*tq} = y^{tq'} - \delta y$ where $(t,q') \in N_{r'}$ and $q = q', 2q', \ldots, nq'$.

(c) for all $(t,q) \in B_j$, we have $x^{*tq} = x'^{tq}$ and $y^{*tq} = y'^{tq} + (nr'\Delta y/jT)$;

(d) $\beta^*(N_r) = \beta_n(N_{r'}) x \beta(\boldsymbol{B}_j)$ and

(e) $z^* = nz + z'$.

From the free-disposal assumption, and the fact that what has been 'given' to the members of B_j equals what has been 'taken away from' the members of $N_{nr'}$, it follows that $\psi(N_r)$ is feasible. Given j, we can choose r_j such that if $r = nr' + j$ and $r \geq r_j$ then $u^{tq}(N_r, x^{*tq}, y^{*tq}) > \alpha^q_t - \delta/2$ for all $(t,q) \in B_j$, this follows from the assumption that private goods can substitute for public goods and crowding effects (2.3(c)). Let

$$r_q = \max_{j \in \{1, \ldots, r'-1\}} r_j.$$

Thus, for all $r \geq r_q$, we have $(\alpha^q - \frac{1}{2}\delta\underline{1}) \in E(r)$. Let $r^0 = \max_q r_q$. Given any $r \geq r^0$ and $\alpha \in \tilde{E}_+(r'')$, we have $(\alpha - \frac{3}{4}\delta\underline{1}) \in E(r)$; this follows from the facts that given any $\alpha \in \tilde{E}_+(r')$, there is an α^q such that $\|\alpha - \alpha^q\| < \delta/4$ and an $\alpha' \in E(r)$ such that $\|\alpha^q - \alpha'\| < \delta/2$.

For all $r \geq r^0$ we now have

$$\tilde{E}_+(r'') \subset E(r) + \tfrac{3}{4}\delta\underline{1} \subset \tilde{E}(r) + \tfrac{3}{4}\delta\underline{1}$$

(since $E(r) \subset \tilde{E}(r)$ for all r) and $L(\tilde{E}) \subset \tilde{E}(r'') + \frac{1}{4}\delta \subset E(r) + \delta\underline{1}$ from comprehensiveness so, for all $r \geq r^0$, we have $L(\tilde{E}) \subset E(r) + \delta\underline{1}$. □

Note that from Lemma 2, we have $\|E(r), \tilde{E}(r)\| \to 0$ as $r \to \infty$. Also, the same proof as used in Lemma 2 can be used to show that for any $\varepsilon > 0$ we also have $\|E(r; \varepsilon), \tilde{E}(r; \varepsilon)\| \to 0$ as $r \to \infty$.

Proof of Theorem 1. The proof of Theorem 1 is now essentially the proof of Theorem 1 in Wooders (1983). Using the fact that $E(r) \to \tilde{E}(r)$ and Theorem 5, (stated herein), it follows as in Wooders (1983, Theorem 1) that given $\varepsilon > 0$ for all r sufficiently large, the $\varepsilon/2$-core of (N_r, V_r) is non-empty and contains an equal-treatment payoff. Since V_r is the closure of $\hat{V}_r$, there is an $\alpha^* \in E(r)$ where $\prod_{i=1}^{r} \alpha^*$ is in the ε-core of (N_r, V_r) and for some feasible state of the economy, $\psi(N_r) = ((x^{N_r}, y^{N_r}), \beta(N_r), z)$, for each t we have $u^{tq}(N_r, x^{tq}, y^{tq}) \geq \alpha_t^*$ for all q. Therefore $\psi(N_r)$ is $s(\varepsilon)$-stable. □

Throughout the remainder of this section, we assume r^* is an MES bound for the sequence of economies. It is obvious that r^* is also an MES bound for the sequence of derived games. Also, recall that given $\varepsilon > 0$ we assume that ε is sufficiently small so that $-c(S; \varepsilon)/|S| < w^{tq}$ for all $(t,q) \in S$ and for all $S \subset N_{r^*}$.

Our next two lemmas, along with the preceding one, will enable us to show that there are feasible states of the economy which cannot be improved upon by any coalition, given that the coalition faces coalition formation costs.

Lemma 3. *Given $\varepsilon > 0$ there is a $\delta > 0$ such that for all r we have $E(r; \varepsilon) + 2\delta\underline{1} \subset E(r)$.*

Proof. First, note that from the specification of jurisdiction formation costs, given any r and any jurisdiction structure $\mathcal{N}_r$ of N_r, we have $\sum_{S \in \mathcal{N}_r} c(S; \varepsilon) = \varepsilon r T \bar{z}$. Let $\Delta y = -\varepsilon \bar{z}$.

Suppose the claim of the lemma is false. Then there is an α^* in the boundary of $L(E)$ where $\alpha^* > 0$ and a sequence (α^r) where $\alpha^r \in E(r; \varepsilon)$ for each r and $\alpha^r \to \alpha^*$ as $r \to \infty$. We will show that this contradicts the assumption that α^* is in the boundary of $L(E)$.

Since $\alpha^r \in E_+(r; \varepsilon)$ for each r, there is a sequence of $c(\varepsilon)$-feasible states, say $\psi(\mathcal{N}_r))$ with associated allocations $(x_r^{N_r}, y_r^{N_r})$, such that $u^{tq}(\mathcal{N}_r, x_r^{tq}, y_r^{tq}) \geq \alpha_t^r$ for each t and all q. Without loss of generality, we can assume $\varrho(S) \leq \varrho(N_{r^*})$ for all $S \in \mathcal{N}_r$ and for all r. It can be demonstrated using standard arguments and, as in Lemma 1, taking the 'most favourable' possible production possibilities sets, that for all (t,q) in N_r and for all r, $(x_r^{tq}, y_r'^{tq})$ is contained in a compact subset $K \subset R^{M+K}$.

For each r, there is a feasible state of the economy with jurisdiction structure $\mathcal{N}_r$ and associated allocation $(x_r^{N_r}, y'^{N_r})$ where $y_r'^{tq} = y_r^{tq} + \Delta y$ for each $(t,q) \in N_r$.

Now for each t, define

$$\delta_t = \inf u^{tq}(s,x,y+\Delta y) - u^{tq}(s,x,y),$$

where the inf is taken over the set $I_{r^*}(t) \times (K \cap (R_+^M x R_{++}^K))$. From continuity and monotonicity, $\delta_t > 0$ for each t. Let $2\delta = \min \delta_t$. It follows that $\alpha_t^r \le u^{tq}(N_r, x_r^{tq}, y_r^{tq}) + 2\delta \le u^{tq}(N_r, x_r^{tq}, y_r^{tq} + \Delta y)$ for all r. Therefore α^* is in the interior of $L(E)$, which is a contradiction. □

Lemma 4. *Given $\varepsilon > 0$ there is a $\delta > 0$ such that for all r sufficiently large we have $\tilde{E}(r;\varepsilon) + \delta \underline{1} \subset E(r)$.*

Proof. From Lemma 2 we have the closed limit of $(E(r;\varepsilon)), L(E(\varepsilon))$, equal to the closed limit of $(\tilde{E}(r;\varepsilon))$, denoted by $L(\tilde{E}(\varepsilon))$.

Let δ satisfy the requirements of the preceding lemma. Since $E(r;\varepsilon) + 2\delta \underline{1} \subset E(r)$ for all r, we have

$$L(\tilde{E}(\varepsilon)) + 2\delta \underline{1} \subset L(E(r)).$$

Therefore for all r sufficiently large,

$$\tilde{E}(r;\varepsilon) \subset E(r). \quad \square$$

Proof of Theorem 2. From Theorem 4, for all $r > r^*$ the core of $(N_r, \tilde{V}_r^\varepsilon)$ contains payoffs with the equal treatment property. Let r' be sufficiently large so that for all $r \ge r'$ we have

$$\tilde{E}(r;\varepsilon) + \delta \underline{1} \subset E(r) \text{ for some } \delta > 0.$$

Given $r \ge r'$, let $\alpha \in \tilde{E}(r;\varepsilon)$ be such that $\prod_{i=1}^r \alpha$ is in the core of $(N_r, \tilde{V}_r^\varepsilon)$. Now let $\psi(N_r)$ be a feasible state of the economy with associated allocation (x^{N_r}, y^{N_r}) such that $u^{tq}(N_r, x^{tq}, y^{tq}) > \alpha_t$ for all $(t,q) \in N_r$; this is possible from the preceding lemma. Since $\prod_{i=1}^r \alpha$ is in the core of $(N_r, \tilde{V}_r^\varepsilon)$, it follows that $\psi(N_r)$ is $c(\varepsilon)$-stable. □

Proof of Theorem 3. Given any $\varepsilon \ge 0$, from the MES assumption it follows that $\tilde{E}(r^*;\varepsilon) = \tilde{E}(r;\varepsilon)$ for all $r \ge r^*$. From results in Wooders (1983), there is an n^0 such that $\tilde{E}(r^*;\varepsilon) \subset E(ln^0 r^*;\varepsilon)$ for all positive integers l. Therefore each economy in the sequence of economies with agents $N_{ln^0 r^*}$) has a $c(\varepsilon)$-stable state. □

We remark that the $c(\varepsilon)$-stable states in the above theorem can be chosen so that for each l the $c(\varepsilon)$-stable states of the $ln^0 r^*$th economy and the payoffs derived from these $c(\varepsilon)$-stable states have the equal-treatment property. To show this, simply choose a vector in $\tilde{E}(r^*;\varepsilon)$ which represents an equal-treatment payoff in the core of the rth derived game for all r; this can be done by choosing the limit, say α^*, of some subsequence of (α^r) where α^r represents an equal-treatment payoff in the core of the rth game. Then $\alpha^* \in E(n^0 r^*;\varepsilon)$. There is a state of the $n^0 r^*$th economy associated with α^* and the lth replica of this state is a $c(\varepsilon)$-stable state of the ln^0r^*th economy.

5. Some extensions and remarks

Remark 1. We first remark that the Shubik–Wooders (1983a) approximate core theorem can be applied immediately to the class of economies considered. To obtain that result, we need only per-capita boundedness of the derived sequence of games, which was demonstrated in Lemma 1. Restating the result in terms of the sequence of economies underlying the derived sequence of games we have:

Theorem 7. *Given any $\varepsilon>0$ and any $\lambda>0$ there is an r' such that for all $r\geq r'$, for some feasible state of the economy $\psi(N_r)$ and some state of the economy $\psi'(N_r)$, with associated allocations (x^{N_r}, y^{N_r}) and (x'^{N_r}, y'^{N_r}) respectively we have*

(a) $|\{(t,q)\in N_r : u^{tq}(N_r, x^{tq}, y^{tq}) \neq u^{tq}(N_r, x'^{tq}, y'^{tq})\}| < \lambda |N_r|$ *and*

(b) $\psi'(N_r')$ *cannot be ε-improved upon by any coalition S; i.e. there does not exist an $S\subset N_r$ and a feasible state of the economy $\psi''(S)$ with associated allocation (x''^S, y''^S) such that $u^{tq}(S, x''^{tq}, y''^{tq}) > u^{tq}(N_r', x'^{tq}, y'^{tq}) + \varepsilon$ for all $(t,q)\in S$.*

Informally, there is a feasible state of the economy, and a state of the economy which cannot be ε-improved upon by any coalition, with the property that most agents have identical payoffs in both states. Another way of stating Theorem 7 is to say that given $\lambda>0$ there is an r' sufficiently large such that for all $r\geq r'$, there is a state of the economy and an associated payoff α for the derived game such that for some $N_0\subset N_r$, α cannot be ε-improved upon by any coalition S contained in N_0 and $|N_0|/|N_r|>1-\lambda$.

In Shubik and Wooders (1983b), another approximate core theorem was obtained for coalition production economies. A similar theorem can be obtained for the class of economies considered herein.

Theorem 8. *Given any $\varepsilon>0$, there is a $\Delta\in R_+^K$ and an r' such that for all $r\geq r'$ for some state of the economy $\psi(N_r)$ with associated allocation (x^{N_r}, y^{N_r}) we have*

(a) $-\Delta \leq \sum_{tq\in N_r}(y^{tq}-w^{tq}) - \sum_{l=1}^{L} z_l - z \leq \Delta$,

(b) *the allocation cannot be ε-improved upon by any subset S of N_r, i.e.* (b) *of Theorem 7 is satisfied.*

Informally, in Theorem 7 it is, in a sense, the 'improvement' condition that is relaxed and in Theorem 8, it is feasibility which is relaxed (although in per-capita terms, the state is approximately feasible for large r). In the literature on approximate cores of economies, both these types of approximate cores appear.

We note that neither of these theorems, 7 and 8, require the MES property. Also, they do not even require that the equal-treatment payoffs of the derived games converge to those of the balanced cover games.

Remark 2. The results herein depend primarily on three properties: per-capita boundedness, the presence of at least one infinitely divisible good which is a substitute for all other goods ('some sidepaymentness') and, for theorems on $c(\varepsilon)$-stability, minimum efficient scale for jurisdictions.

Many of the specifies of the model can be changed and the same results still obtained. In particular, in the model we require that

(a) all agents in the same jurisdiction consume the same amounts of the public goods;

(b) agents belong to one and only one jurisdiction, i.e., jurisdictions do not overlap in the sense that no agent can consume the public goods available in two or more jurisdictions;

(c) agents are not permitted to be members of different jurisdictions for the consumption of different public goods;

(d) public goods for a jurisdiction are produced by a production set determined by the membership of that jurisdiction.

All these constraints on the model can be relaxed and/or varied without affecting the results.[11] As stated above, the essential features are per-capita boundedness, a certain degree of 'sidepaymentness' or 'quasi-transferable utility', and for some results, a minimum efficient scale for jurisdictions or coalitions. Both per-capita boundedness and some 'sidepaymentness' seem to be quite reasonable conditions. The minimum efficient scale would not hold in models with pure public goods or pure-public-good-like features which result in all increasing returns to coalition size being unrealizable by some finite economy.

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[11] For some possible modifications, particularly to (b) above, see Shubik and Wooders (1986).