THE CORE OF A GAME WITH A CONTINUUM OF PLAYERS AND FINITE COALITIONS:THE MODEL AND SOME RESULTS

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Abstract

In this paper we develop a new model of a cooperative game with a continuum of players. In our model, only finite coalitions – ones containing only finite numbers of players – are permitted to form. Outcomes of cooperative behavior are attainable by partitions of the players into finite coalitions. This is appropriate in view of our restrictions on coalition formation. Once feasible outcomes fare properly defined, the core concept is standard – no permissible coalition can improve upon its outcome. We provide a sufficient condition for the nonemptiness of the core in the case where the players can be divided into a finite number of types, This result is applied to a market game and the nonemptiness of the core of the market game is stated under considerably weaker conditions (but with finite types). In addition, it is illustrated that the framework applies to assignment games with a continuum of players.

Key words: continuum of players; finite coalitions; measure-consistent partitions; game in characteristic function form; f-core.

1 Introduction

Models of games with a continuum of players have become widely used in game and economic theory. Nevertheless there are still subtle questions that arise in the interpretation of these models, in particular, the interpretation of the individual player and the role of the individual player in cooperation. Also, the existing framework is not easily adapted to model some important interesting situations, for example assignment games. These considerations lead us to the conception of a new model of a cooperative game with a continuum of players. We develop and explore our model in this and subsequent papers. In this first paper, we discuss our motivation, introduce our framework and provide some basic results.

Our model differs from those currently in the literature in that only subsets containing finite numbers of players are permitted to form coalitions. Consistent with our restriction on coalition formation, outcomes of cooperative behaviour must be attainable by partitions of the players into finite coalitions. The partitions are required to be compatible with the distribution of players (described by a measure on the player set). Outcomes are then defined relative to feasible partitions. [1]Once feasible outcomes are defined, the core concept is standard - no permissible (i.e. finite) coalition can improve upon an outcome in the core, called the *f* -core.

The motivation for our treatment of cooperative games with a continuum of players arises, as suggested above, in part from the interpretations of the extant models initiated by Aumann (1964). One interpretation is that only portions of the total set of players are relevant; the individual player is some arbitrarily small set of positive measure. Here it is the *player* that is approximated in the continuum model. Once we have interpreted the player as an arbitrarily small set of positive measure it is a consequence that the total set is interpreted as an arbitrarily large but finite set (since the total set of players is given as a continuum is means to approximate a situation with a large but finite player set. Of course both interpretations are intended to approximate a situation with a large but finite player set, but in the first, the approximation is via the player and in the second, the player set.

The first interpretation comes from the analogy to mechanics, e.g. fluid mechanics, where a continuum is used to approximate a situation with a large but finite number of particles. In fluid mechanics this sort of treatment is appropriate when it is not necessary to treat each particle as a distinct entity.

[1]*Note added June 16, 2001*. In hindsight, this paragraph is not as clearly written as it could be. There are other models in the literature where essential coalitions are restricted to be finite. We note particularly papers by Yakar Kannai (1969) and Hans Keiding (1974). The difficulty with these papers is that no measurement-consistency requirments are imposed. Thus, the relative scarcities that are embodied in well-known models of economies with a continuum of players are lost. The notions of, for example, "twice as much" of one commodity as another disappear. A paper clarifying this comment is available from Myrna Wooders, the author of this note. Keiding, H. (1976) "Cores and Equilibria in an Infinite Economy," in *Computing Equilibrium: How and Why*, J. Los and M.W. Los, eds. Noth Holland, Amsterdam/Oxford/New Tork, p. 65-73.

In game theory and in microeconomics, by definitions of the subjects each player is a distinct entity - a separate decision making unit; thus the concept of the player must be precise, as it is in the second interpretation.

If we take the view that cooperation arises from the actions of individual players then it seems reasonable that individual players should be able to be effective (i.e. non-negligible in its non-technical sense) members of coalitions. Furthermore, if we take coalition formation costs into account, then individual players are less effective in larger coalitions. The first interpretation of the individual player as a set of 'small' but positive measure may not violate the requirement that individuals be non-negligible as members of coalitions (of positive measure). However if each point in a continuum of players is a distinct player, an individual player can be regarded as negligible or non-effective in coalitions of positive measure. In this sense standard core theory in continuum models needs the first interpretation. (See also the discussion in Aumann and Shapley, 1974, pp.176-178.)

In our model each point in the continuum represents an individual player and coalitions are finite. Therefore the second interpretation fits and our approach is consistent with cooperation arising from the actions of individuals (the effectiveness of an individual in coalition formation). Our approach allows us to maintain the advantages of both the first and the second interpretation; the individual is not negligible in finite coalitions and it is the total player set that is an approximation. Our treatment of coalitions can also be compatible with the postulate of coalition formation costs because we can restrict essential coalitions to be bounded, e.g. in an assignment game with a continuum of players, sizes of essential coalitions are no more than two. In this fashion, we attempt to maintain a precise concept of the player while capturing the idea of the individual player as effective in facilitating cooperation.

The preservation of the precise concept of the individual player and the restriction to finite coalitions allow us to view our model as the direct limit version of the Debreu-Scarf (1963) and Shapley-Shubik (1966) models with the total set of players approximated by a continuum. Our model can also be viewed as a limit version of a model of large games with nonempty approximate cores initiated by Wooders (1983) (see also Kaneko and Wooders, 1982; Shubik and Wooders, 1983; and Wooders and Zame, 1984). This can be observed in part in the proof of Theorem 1 of this paper. In Kaneko and Wooders (1985), we further explore the connection between the f-core of a continuum game and the approximate (epsilon) cores of large finite games.

In the next section of the paper we develop our model and the concept of the f-core. Simple examples of assignment games are used to demonstrate several of the concepts. Section 3 consists of an application of our model to a market game. A proof of the non-emptiness of the f-core for a game with types is provided in Section 4.

In a subsequent paper, Kaneko and Wooders (1984b), we consider the nonemptiness of the f-core without the types assumption. The price we pay for removing this assumption is that we need uniformly bounded coalition sizes and strong comprehensiveness. In another paper, Hammond, Kaneko and Wooders (1985), in the contexts of private goods exchange economies with and without externalities, equivalence of the f-core and the A-core (Aumann's) is further explored. Finally in Kaneko and Wooders (1985), we obtain the result of convergence of approximate cores of large finite games to the f-core of the continuum game. In each paper the motivation is further developed.

We conclude this introduction with some ideas about future research. At this point in our work we believe we have established and motivated our model and provided results justifying our approach. Also we have carried out some natural applications to assignment games and private goods exchange economies. It would be desirable to have other applications, for example, to housing markets, more generally, urban economics. We believe our framework may be fruitfully applied to these and related situations. On a deeper, conceptual level, our framework may be useful in investigating the noncooperative foundations of cooperation, in particular carrying out Nash's program (1951) to formulate cooperative behaviour as moves in an extensive game. The second interpretation of the individual player (and our approach) is consistent with approaches in non-cooperative game theory, especially recent developments in continuum games in extensive form (cf., Dubey and Kaneko, 1984). Also the second interpretation seems compatible with Nash's programme. Thus we hope our model may facilitate a bridge between cooperative and non-cooperative theories in continuum models.

2 The Model of Continuum Games and the f-core

2.1 The player set and feasible partitions

Let $(N, \mathcal{B}, \mu)$ be a measure space, where N is a Borel subset of a complete separable metric space; $\mathcal{B}$ the σ-algebra of all Borel subset of N ; and μ, a non-atomic measure with $0 < \mu + \infty$.[2] Each element in N is called a **player** and N is the **player set**. The measure μ represents the distribution of players. The σ-algebra $\mathcal{B}$ is necessary for measurability arguments but does not play any important game-theoretic role.

Let F be the set of all finite subsets of N. Each element S in F is called a **finite coalition** or simply a **coalition**. As motivated in Section 1, only finite subsets of players can form coalitions.

Remark 2.1. Since a singleton set is closed in N , every coalition is measurable.

Since only finite subsets of players are allowed to form coalitions, cooperative outcomes are attained by partitions of the player set into finite coalitions. Some

[2] Under our assumptions, (N, B) is measure-theoretically isomorphic to $([0,1], E)$, where E is the σ-algegra of all Borel subsets of the interval $[0,1]$ (see Parthasarathy, 1967, pp.12-14).

conditions must be imposed to ensure that these partitions are compatible with the distribution of players described by the measure μ. For example, consider a marriage (assignment) game where the girls are the points in the interval [0, 1] and the boys, those in the interval $[1,3]$, endowed with Lebesgue measure μ. The set of coalitions $\{\{i, 1+2i\} : i \in [0,1]\}$ constitutes a partition of the set of players into coalitions. See Figure 2. However it is reasonable to require that the measure of girls who are married equals the measure of the boys to whom they are married. The above example violates this requirement. We formulate this requirement in a general manner in the following definition, which will rule out the above example.

Let A and B be sets in $\mathcal{B}$. A function Ψ from A to B is called a **measure-preserving isomorphism from A to B** iff

(i) Ψ is a measure-theoretic isomorphism, i.e. Ψ is 1 to 1, onto, and measurable in both directions, and

(ii) $\mu(C) = \mu(C) = \mu(\Psi(C))$ for all $C \subset A$ with $C \in \mathcal{B}$. A partition p of N is measure-consistent iff for any positive integer k,

$$N_k^p \equiv \bigcup_{\substack{S\in p \\ |S|=k}} S \text{ is a measurable subset of } N\ ;$$

and each N_k^p $(k = 1, 2, ...)$ has a partition $\{N_{kt}^p\}_{t=1}^k$, where each N_{kt}^p is measurable, with the following property: there are measure-preserving isomorphisms $\Psi_{k1}^p, \Psi_{k2}^p, ..., \Psi_{kk}^p$ from N_{k1}^p to N_{k12}^p, ..., N_{kk}^p, respectively, such that $\{\Psi_{k1}^p(i), ..., \Psi_{kk}^p(i)\} \in p$ for all $i \in N_{k1}^p$. (1)

Figure 1

Without loss of generality, we can assume that Ψ_{k1}^p is the identity map from N_{k1}^p to N_{k1}^p. In the following, we assume this without further remark.

Note that (1) implies that for any $S \in p$ with $|S| = k$ we have $S = \{\Psi_{k1}^p(i), ..., \Psi_{k1}^p(i)\}$ for some $i \in N_{k1}^p$. Therefore, for each integer k, N_k^p consists of all the members of k-player coalitions and N_{kt}^p consists of the t^{th} members of these coalitions. The requirement that all the sets N_{kt}^p have equal measure then captures the idea that coalition of size k should have as 'many' (i.e. the same measure) first members as second members, as many second members as third members, etc.

In Figure 1 we provide a schematic illustration. The set of players in k-player coalitions is the union of the sets $N_{k1}^p, ..., N_{kk}^p$, all of equal measure. The isomorphism $\Psi_{k1}^p(i)$ maps i to himself; he is the 'first member' of the coalition $S = \{\Psi_{k1}^p(i), ..., \Psi_{kk}^p(i))\}$. The second member is given by $\Psi_{k2}^p(i)$, etc.

Let Π denote the set of measure-consistent partitions.

We remark that some lemmas on measure-consistent partitions and outcome space are contained in the Appendix. It follows from those lemmas that measure-consistent partitions can be constructed.

Example 2.1. Consider the situation described above where the girls are the points in $[0, 1)$ and the boys, those in $[1, 3)$ with Lebesgue measure. The partition $\{\{i, 1 + 2i\} : i \in [0, 1)\}$ violates (1). An example of a measure-consistent partition is $p = \{\{i, 2 + i\} : \in [0, 1)\} \cup \{\{i\}\} : i \in [1, 2)\}$. For this partition p, we have

$$N_i^p = \bigcup_{\substack{S \in p \\ |S|=1}} S = [1, 2),$$

Figure 2

Figure 3

$$N_2^p = \bigcup_{\substack{S \in p \\ |S|=2}} S = [0, 1) \cup [2, 3), \; N_{21}^p = [0, 1) \text{ and } N_{22}^p = [2, 3);$$

then measure-preserving isomorphisms satisfying (1) are given by

$\Psi_{11}^p(i) = i$, $\Psi_{21}^p(i) = i$ and $\Psi_{22}^p(i) = i$.

Figures 2 and 3 illustrate the partitions described above.

2.1.1 A Characteristic Function Game without Side Payments

A characteristic function game V without side payments is a correspondence on F which assigns to each coalition $S \in F$ a subset $V(S)$ of R^s with the following properties:

$$V(S) \text{ is a nonempty, closed subset of } R^s \text{ for all } \; S \in F \tag{2}$$

$$V(S) \times V(T) \; \subset V(S \cup T) \text{ for any } S, T \in F \text{ with } S \cap T = \phi \; ; \tag{3}$$

$$\inf \sup V(\{i\}) > -\infty; \tag{4}$$

$$\text{for any } S \in F, x \in V(S) \text{ and } y \in R^2 \; \text{ with } y \leq x \text{ imply } y \in V(S); \tag{5}$$

$$\text{for any } S \in F, V(S) - \bigcup_{i \in S} [(interior \;\; V(\{i\})) \times R^{s-\{i\}}]$$

is nonempty and bounded.

These assumptions are all innocuous. We note only that (4) can always ben obtained for a translation of a game, and that conditions (3) and (5) are called, respectively, super-additivity and comprehensiveness.

2.2 Remark 2.2

Remark 2.2. Conditions (2.2) - (2.6) are not sufficient for a full formulation, from a mathematical viewpoint, of our model. Since we are dealing with measurable structures, a full formulation requires some measurability assumption on V. In fact without the following condition one can develop counterexamples, due to non-measurability, to the nonemptiness of the f-core. However, for this paper the following condition is not required since for our nonemptiness theorem we use a 'types' assumption and for market games, the structure of the economy ensures measurability requirements are satisfied.

The condition is as follows. Let p be an arbitrary measure-consistent partition of N. For any positive integer k, we can regard $(\{\psi^p_{k1}(i), \psi^p_{k2}(i), ..., \psi^p_{kk}(i)\})$ as a correspondence from N^p_{k1} to R^k, i.e.,

$$V : N^p_{k1} \rightarrow 2^{R^k}$$

$$i \rightarrow V(\{\psi^p_{k1}(i), \, ... \, , \, \psi^p_{kk}(i)\}).$$

In this sense, we assume that $V(\{\psi^p_{k1}(i)\ ,..., \psi^p_{kk}(i)\})$ has an analytic graph[3], i.e. the set

$$\{(i,x) \in N^p_{k1} \times R^k : x \in V(\{\psi^p_{k1}(i),\ ...,\ \psi^p_{kk}(i)\})\} \tag{7}$$

is analytic in $N^p_{k1} \times R^k$.

We next define the set of possible outcomes of the game. Some preliminary definitions are required.

Given a measure-consistent partition p, define the set $H(p)$ by

$$H(p) = \{h \in L(N,R) : (h(j))_{j\in p(i)} \in V(p(i)) \text{ for almost all } i \in N\}, \tag{8}$$

where $L(N,R)$ is the set of measurable functions from N to R and $p(i)$ is the element of the partition p containing player i; $H(p)$ is the outcome set relative to p. Note that $H(p) \neq \phi$ for any partition p since h given by

$$h(i) = \inf_{j\in N} \sup V(\{j\}) \text{ for all } i \in N$$

is in $H(p)$. Let

$$H = \bigcup_{p\in\pi} H(p). \tag{9}$$

Let $\bar{R} = R \cup \{+\infty\}$. Define the outcome set H^* by

$$H^* = \{h \in L(N,\bar{R}) : \text{for some sequence } \{h^v\} \text{ in } H, \{h^v\} \text{ converges in measure to} h\}\ , \tag{10}$$

where '$\{h^v\}$ converges in measure to h' means that for any $\varepsilon > 0$,

$$\mu(\{i \in N :\mid h^v(i) - h(i) \mid> \varepsilon\}) \longrightarrow 0 \text{ as } v \longrightarrow \infty\ .$$

Note that $H(p) \subset H \subset H^*$ for all $p \in \pi$.

Note that since we take the closure H^* of H and work on H^* instead of H we can, in fact, remove the 'almost' qualification in definition (8). For convenience, however, we adopt the present definition of H.

Remark 2.3. In outcome space we allow $h(i) = \infty$ but only on a null set since h^v in (10)) cannot take the value $+\infty$. Allowing $h \in H^*$ to be in $L(N,\bar{R})$, rather than $L(N,R)$, is simply for notational and expository convenience. See also Section 3.

[3] Every measurable set in the product σ-algebra $\mathcal{B}(N^p_{k1}) \otimes \mathcal{B}(R^k)$ is analytic, where $\mathcal{B}(N^p_{k1})$ and $\mathcal{B}(R^k)$ are the σ-algebras of Borel subsets of N^p_{k1} and R^k, respectively. For a definition of analytic sets see Parthasarathy (1967) p.15 or Meyer (1966), p.34.

Remark 2.4. If $\{h^v\}$ converges in measure to h, then $\{h^v\}$ has the subsequence which converges pointwise to $h\,a.e.$ and, conversely, if $\{h^v\}$ converges pointwise to $h.a.e.$, then $\{h^v\}$ converges in measure. Therefore H^* can be defined by pointwise convergence, and in our proofs we use whichever definition is convenient. Note that H^* is closed with respect to these convergences, i.e. if $\{h^v\}$ in H^* converges either pointwise or in measure to h, then $h \in H^*$; that is, H^* is the closure of H. An example illustrating that H may not itself be closed appears in Hammond, Kaneko and Wooders (1985). Taking the closure H^* as the set of feasible outcomes is an idealization.

Let h be a function in $L(N, \bar{R})$. We say that a coalition S in F can improve upon h iff for some $y \in V(S)$, $y_i > h(i)$ for all $i \in S$. Now the f-**core** of the game V without side payments is defined to be the set C_f:

$$C_t = \{h \in H^* : \text{no coalition in } F \text{ can improve upon } h\} \tag{11}$$

An outcome h in the f-core C_f is stable in the sense that no coalition can improve upon h. It is approximately feasible in the sense that h is approximated by exactly feasible outcomes, where 'an outcome h' is exactly feasible' means $h' \in H$, i.e., h' is actually achieved by some measure-consistent partition p. Except for this feature that feasibility is approximate, the core notion is the same as in infinite games - no permissible coalition can improve upon an outcome in the core, the f-core. In this sense, the f-core is the limit version of approximate cores, e.g., Shapley and Shubik (1966), Wooders (1983), Kaneko and Wooders (1982), Shubik and Wooders (1983) and Wooders and Zame (1984).

One advantage of our framework is clear when we consider an assignment game with a continuum of players. Assignment games with finite numbers of players have been developed by Gale and Shapley (1962), Shapley and Shubik (1972), Crawford and Knoer (1981) and Kaneko (1982). In these games essential coalitions are **pairs** of players, one player from each side of the market. Since, in the standard (Aumann) approach to modelling continuum games, coalitions are sets of positive measure, this approach cannot naturally treat assignment games. The following example illustrates the treatment of assignment games with our framework.

Remark 2.5. Although we gave the formulation of a characteristic function without sidepayments, the sidepayments case can be treated by our formulation in the standard manner. For more details, see Kaneko and Wooders (1984a).

Example 2.2. We continue our marriage game example and describe the f-core of the game.

Suppose the marriage[4] of the i^{th} girl and the j^{th} boy yields a payoff of j utils to the girl and i utils to the boy (there is no transferability of utility)

[4] Only marriages between boys and girls are allowed.

while remaining single yields a payoff of zero. A finite coalition can realise only those outcomes attainable by marriages between members of the coalition. An outcome in the f-core is given by f where

$$h(i)=\begin{cases} 2+i & \text{if } i\in[0,1) \\ 0 & \text{if } i\in[1,2] \\ i-2 & \text{if } i\in[2,3]; \end{cases} \tag{12}$$

only the boys with high index numbers are married and, the higher one's index number, the higher the index number of one's partner. Figure 4 depicts the graph of h. The dotted lines join partners.

Formally, the characteristic function of the marriage game is defined as follows:

for any $i\in[0,3,\ V(\{i\})=\{x_i : x_i\leq 0\}$;

for any $i\in[0,1)$ and $j\in[1,3), V(\{i,j\})=\{(x_i,\, x_j) : x_i\leq j,\, x_j\leq i\}$; (13)

for any $S\in F,\ V(S)=\bigcup_{P_s\in P(S)}\ \prod_{T\in ps} V(T)$,

where $P(S)=\{p_s : p_s$ is a partition of S such that $|T|=1$ or $|\ T\cap[0,1)\ |=1$ and $|\ T\cap[1,3)\ |=1)$ for all $T\in p_s\}$. Then this function V satisfies (2)-(6) (and also (7)) and the outcome h given by (12) is in the f-core of V. Furthermore the outcome h is the unique f-core outcome (up to sets of measure zero). The outcome h is, of course, in H^* and is, in fact, exactly attainable by the measure-consistent partition $\{\{i,2+i\} : i\in[0,1)\}\cup\{\{i\} : i\in[1,2)\}$.

2.3 Nonemptiness of the f-core in the finite type case

We are now in a position to discuss the main theorem of this paper: if the player set N of a game can be partitioned into a finite number of types, then the game has a nonempty f-core under non-restrictive conditions.

Figure 4

To state the theorem, some preliminary definitions are required.

Let $(N, \mathcal{B}, \mu)$ be as described in Section 2.1 and let V be a game without side payments. Players i and j are called **substitutes** iff for any $S \in F$,

$$\text{if } i, j \notin S, \text{ then } x \in V(S \cup \{i\}) \Leftrightarrow x' \in V(S \cup \{j\}) \text{ where } x_l = x'_l \text{ for all } l \in S \text{ and } x_i = x'_j \text{ ; and} \tag{14}$$

$$\text{if } x \in V(S) \text{ and } i, j \in S, \text{ then } x' \in V(S), \text{ where } x'_l = x_l \text{ for all } l \in S - \{i, j\} \text{ and } x'_j = x_i,\ x'_i = x_j. \tag{15}$$

These conditions simply mean that the players i and j are completely identical with respect to the aspects described by the characteristic function V. The game V has **the r-property with respect** to $\{N_t\}_{t=1}^k$ iff

$$\{N_t\}_{t=1}^k \text{ is a partition of } N \text{ with } \mu(N_t) > 0 \text{ for all } t = 1, ..., k, \text{ and all players in each } (N_t(t = 1, ..., k) \text{ are substitutes.} \tag{16}$$

A game with the r-property is simply one with a finite number of types and a positive measure of players of each type. Assume that V has the r-property with respect to $\{N_t\}_{t=1}^k$. For any $S \in F$, a payoff vector $y \in V(S)$ has the equal-treatment property (the ETP) iff

$$y_i = y_j \text{ for all } i, j \in N_t \cap S \text{ and } t = 1, ..., k. \tag{17}$$

The game V is **per-capita bounded with respect** to $\{N_t\}_{t=1}^k$ iff there is a positive real number $\delta(0 < \delta < 1)$ and a K such that

$$\left.\begin{array}{l} S \in F \\ (1 + \delta\frac{\mu(N_t)}{\mu(N)} \geq (1 - \delta)\frac{\mu(N_t)}{\mu(N)} \forall t = 1, ..., k \\ x \in V(S) \text{ has the ETP} \end{array}\right] \Longrightarrow x_i < K \text{ for all } i \in S.^5 \tag{18}$$

That is, there is a constant K such that given any coalition S with approximately the same percentage of players of each type as N and any payoff x with the ETP in $V(S)$, we have $x_i < K$ for all $i \in S$.

Theorem 1 (Nonemptiness Theorem). Let V be a game without side payments. Assume V has the r-property and is per-capita bounded with respect to a partition $\{N_t\}$. Then the f-core of the game is nonempty.

[5] If (18) is true for some δ, then it is also true for any $\delta^*(0 < \delta^* < \delta)$.

The proof of Theorem 1 will be given in Section 4.

The r-property is intuitively clear, but the per-capita boundedness assumption of Theorem 1 might need further explanation. If $\mu(N_1) = \cdots = \mu(N_k) = \mu(N)/k$, then (18) can be replaced by a weaker form:

$$\left.\begin{array}{c} S \in F, \\ | S \cap N_1 | = \cdots = | S \cap N_k |, \\ x \in V(S) \text{ has the ETP} \end{array}\right] \Longrightarrow x_i < K \text{ for all } i \in S. \tag{18'}$$

When the size of the coalition increases uniformly with respect to the percentage of agents of each type, per-capita payoffs are bounded. Even though condition (18') is in the same spirit as (18), it cannot be applied directly because the proportions of different types may vary.

Remark 2.6. As previously noted, our model provides a limit version of models of large games for which nonemptiness of approximate cores is obtained. As will be seen, our Theorem can be viewed as an extension of theorems on approximate cores in Wooders (1983) and Shubik and Wooders (1983).

Remark 2.7. Scarf (1967) has shown that under certain balancedness assumptions finite games have nonempty cores. For sidepayment games, balancedness is a necessary and sufficient condition (see Shapley, 1967) for nonemptiness. In fact, even without sidepayments, balancedness is 'close' to necessary and sufficient. In our continuum framework, the assumptions of per-capita boundedness and of the continuum ensure the nonemptiness, replacing balancedness.

In the next section we give a corollary to the theorem for market games.

3 Market Games

3.1 The f-core of a market game

We formulated our model in terms of an abstract characteristic function. Since market games have been intensively studied as example of cooperative games, it is desirable to show that our formulation can accommodate market games. Specifically, we show that the f-core of a game derived from a market economy coincides with the 'f-core of the economy', i.e. the f-core in allocation space. We also apply Theorem 1 to a market game to obtain nonemptiness of the f-core. The investigation of the f-core of a market game is continued in a subsequent paper (Hammond, Kaneko and Wooders, 1985).

We consider a model of an exchange economy with m commodities where the player set $(N, \mathcal{B}, \mu)$ is as defined in the preceding section. Each player has the same consumption set Ω where

Ω is closed in R^m_+ (endowed in the sup norm, $\|\cdot\cdot\|$); and for any nonempty subset $T \subset \Omega$, the vector a^T , defined by its coordinates $a^T_c = \inf\{a_c : a \in T\}(c = 1, ..., m)$, is in Ω . (19)

For simplicity, we assume that $0 \in \Omega$. For example, one choice for Ω is $\Omega = I^m_+ = I_+ \times \cdots \times I_+$ where I_+ is the set of all-nonnegative integers; all commodities are indivisible. Another choice is $\Omega = \{0, e^1, ..., e^{m-1}\} \times R_+$, where e^c is the c^{th} unit vector; this is the assignment market situation where a player wants at most one unit of one of the first $m-1$ indivisible commodities and the m^{th} commodity is divisible. (See, for example, Kaneko, 1982, for an assignment market of this sort with a finite number of players).

Each player $i \in N$ has an initial endowment vector $\omega(i) \in \Omega$ and the function $\omega : N \longrightarrow \Omega$ is measurable and integrable. Also, each player i has a utility function $U(i, \cdot)$ on Ω and we assume that $inf_{i \in N} U(i, \omega(i)) > -\infty$.

Assumption A. The function $U : N \times \Omega \longrightarrow R$ is measurable with respect to the product σ-algebra $\mathcal{B} \otimes B(\Omega)$;

Assumption B. For each $i \in N$, $U(i, \cdot)$ is continuous.

For the same purpose as the use of $\bar{R}$ rather than R in the preceding section (see Remark 2.3) and to simplify arguments, we extend Ω to

$$\Omega = \Omega \cup \{(\infty, ..., \infty)\}. \quad (20)$$

Then we assign $+\infty$ to $U(i, (\infty, ..., \infty))$, i.e.,

$$U(i, (\infty, ..., \infty)) = \infty \text{ for all } i \in N. \quad (21)$$

A function f in $L(N, \Omega)$ is called an allocation with respect to a partition $p(p \in \Pi)$ iff

$$\sum_{j \in p(i)} f(j) \leq \sum_{j \in p(i)} \omega(j) \text{ for almost all } i \in N, \quad (22)$$

where $p(i)$ is the element of the partition p containing player i. An allocation f with respect to p can be attained by trading commodities only within coalitions in p. Define the sets $F(p)(p \in \Pi)$, F and $\mathcal{F}^*$ by

$$F(p) = \{f \in L(N, \Omega) : f \text{ is an allocation with respect to } p\}; \quad (23)$$

$$F = \bigcup_{p \in \pi} F(p); \quad (24)$$

$$F^* = \{f \in L(N, \bar{\Omega}) : \text{for some sequence } \{f^v\} \text{ in } F, \{f^v\} \text{ converges in measure to } f\}. \quad (25)$$

Note that $F(p) \subset F \subset \mathcal{F}^*$ for all $p \in \Pi$ and that $F(p) \neq \phi$ since the initial endowment function is in $F(p)$ for each p. We call a function f in F^* simply an **allocation**. Also, for any allocation f we can find a pointwise convergent sequence $\{f^v\}$ in F where each f^v in the sequence can be attained a.e. by trades within finite coalitions.

Although we defined allocations in a quite different manner than the standard definition due to Aumann (1964), it is a remarkable fact that the sets of allocations are equivalent. The following lemma holds.

Lemma 3.1. $F^* = \{f \in L(N, \bar{\Omega}) : \int_N f(i) \leq \int_n \omega(i))\}$.

Proof. Appendix.

This lemma states that any function $f \in L(N, \bar{\Omega})$ with $\int_N f(i) \leq \int_N \omega(i)$ is an allocation, i.e. for some partition of the players into finite coalitions the allocation f is approximately attainable by trades within coalitions in the partition. This lemma is crucial in that it enables us to compare the f-core, the Walrasian allocations, and the A-core (Aumann's core concept (1964)).[6]

We say that a coalition S in F **can improve upon** a function f in $L(N, \bar{\Omega})$ iff there is a vector $(a^j)_{j \in s}$such that

$$a^j \in \Omega \text{ for all } j \in S; \tag{26}$$

$$\sum_{j \in S} a^j \leq \sum_{j \in S} \omega(j); \tag{27}$$

$$U(j, f(j)) < U(j, a^j) \text{ for all } j \in S. \tag{28}$$

The f-core **of the market game** (in allocation space) is defined to be

$$C_E = \{f \in \mathcal{F}^* : \text{no coalition in } F \text{ can improve upon } f\}.$$

This definition (29) of the f-core of a market game is obviously a translation of the definition of the f-core of a game in characteristic function form to allocation space. In the next section, we show that the f-core of the game in allocation space is equivalent to the f-core of the derived characteristic function game.

[6] One might be tempted to think this lemma provides justification for our approach. On the contrary, this lemma should be interpreted as a justification for Aumann's feasibility condition in the context of market economies. See also the discussion in Hammond, Kaneko and Wooders (1985).

3.2 Characteristic function representation of the market game

Define the characteristic function V_M in the standard manner: for any $S \in F$,

$$V_M(S) = \{x \in R^S : \text{for some } S\text{-allocation } (a^i)_{i \in S},\ x_i \leq U(i, a^i) \qquad (30)$$

where an S-allocation is a vector $(a^i)_{i \in S}$ satisfying (26) and (27). Then the sets $H_M(p)$, H_M and H_M^* are defined by (8), (9) and (10) and the f-core C_f of the game V_M, by (11).

Clearly, the game V_M satisfies (2)-(6) . The game V_M also satisfies the measurability condition (7) in Remark 2.2; a proof is contained in Kaneko and Wooders (1984a).

Lemma 3.2. For any $p \in \Pi$, if $h \in H_M(p)$, then there is an $f \in F(p)$ such that $h(i) \leq U(i, f(i))$ a.e. in N .

Proof. Appendix.

We remark that, since the spaces $H_M(p)$ and $F(p)$ are independently defined, Lemma 3.2, although it seems obvious, requires a proof.

The following theorem ensures that the general formulation of a game in characteristic function form given in Section 2 is adequate for the treatment of a market game. In particular, if we demonstrate nonemptiness of the f-core C_f of V_M , then the f-core C_E in allocation space is nonempty; in the next section we show this non-emptiness.

Theorem 2

(i) For any h in the f-core C_f of the game V_M there is an allocation f in the f-core C_E of the market game such that $U(i, f(i)) \geq h(i)$ for all $i \in N$.

(ii) For any f in the f-core C_E of the market game, the utility representation h of f (i.e., h defined by $h(i) = U(i, f(i))$ for all $i \in N$) belongs to the f-core C_f of the game V_M .

Proof. Since the structure of 'can improve upon' in the market game is fully described by the characteristic function V_M , it suffices to show that (i) an outcome h in C_f can be sustained by an allocation f in C_E in the sense that $h(i) \leq U(i, f(i))$ for all $i \in N$, and (ii) the utility representation of an allocation in C_E belongs to H_M^*.

(i) Suppose $h \in C_f$. Then there is a sequence $\{h^v\}$ in H_M such that $h^v(i) \rightarrow h(i)$ a.e. in N . For each v , there is a $p^v \in \Pi$ such that

$h^v \in H_M(p^v)$. From Lemma 3.2, there is an allocation $f^v \in F(p^v)$ such that $h^v(i) \le U(i, f^v(i))$ a.e. in N. From Lemma 3.1, f^v satisfies

$$\int_N f^v(i) \le \int_N \omega(i). \tag{31}$$

We can assume without loss of generality that $\int_N f^v(i)$ converges. Then we can apply 'Fatou's Lemma in m-dimensions' (hildenbrand, 1974, p.69, Lemma 3) to this sequence and state that there is an integrable function $\bar{f}$ such that

$$\bar{f}(i) \in dLs(f^v(i)) \text{ a.e. in } N; \tag{32}$$

$$\int_N \bar{f}(i) \le \lim \int_N f^v(i), \tag{33}$$

where $Ls(f^v(i))$ is the set of all cluster points of $\{f^v(i))\}$. Hence $\int_N \bar{f}(i) \le \int_N \omega(i)$, i.e., $\bar{f}$ is an allocation by Lemma 3.1. By (32) there is a subsequence $\{f^{v\lambda}(i)\}$ for almost all i in N such that $f^{v\lambda}(i) \to \bar{f}(i)$ $(\lambda \to \infty)$. Since $U(i, f^v(i)) \ge h^v(i)$ and $h^v(i) \to h(i)\,(v \to \infty)$, we have, by Assumption A, $h(i) \le U(i, \bar{f}(i))$. That is, $h(i) \le U(i), \bar{f}(i))$ a.m. in N. If $\bar{f}$ does not satisfy $h(i) \le U(i, \bar{f}(i))$ for all $i \in N$, then we define the function f by

$$f(i) = \begin{cases} \bar{f}(i) \text{ if } U(i, f(i)) \ge h(i) \\ (+\infty, ..., +\infty) \text{ otherwise.} \end{cases}$$

This function f satisfies the condition required.

(ii) Let $f \in C_E$. Then there is a sequence $\{f^v\}$ in F such that $f^v(i) \to f(i)$ a.e. in N. Since $f^v \in F$, there is a $p^v \in \Pi$ such that $f^v \in F(p^v)$. Then $(U(j, f^v(j))_{j \in S} \in V_M(p^v(i))$ for all S in p^v. Since $f^v(i) \to f(i)$ a.e. in N, $U(i,\ f^v(i)) \to U(i,\ f(i))$ a.e. in N by Assumption A. This means the limit function $h(\cdot) = U(\cdot, f(\cdot))$ belongs to H_M^*.

3.3 Nonemptiness of the f-core of a market game with finite types

In this subsection we apply our nonemptiness theorem to market games. The result is striking: nonemptiness of the f-core of a market game is obtained without any restrictions (except that of a finite number of types).

We start with the following lemma which shows that the per-capita boundedness assumption is satisfied.

Lemma 3.3. Consider the market game described in Subsection 3.1. Assume that there is a partition $\{N_t\}_{t=1}^k$ of N with $\mu(N_t) > 0 (t = 1, ..., k)$ such that

$\omega(i) = \omega(j)$ and $U(i,a) = U(j,a)$ for all $i, j \in N_t(t = 1, ..., k)$ and all $a \in \Omega$. Then the derived game V_M (defined by (30)) satisfies the r-property and per-capita boundedness with respect to $\{N_t\}_{t=1}^k$.

Proof. It is clear that V_M satisfies the r-property. Let us prove per-capita boundedness. Suppose the negation, i.e.,

$$\forall\delta(0 < \delta < 1) \forall K \exists S \in F$$

$$\left[(1+\delta)\tfrac{\mu(N_t}{\mu(N)} \geq \tfrac{|S \cap N_t}{|S|} \geq (1-\delta)\tfrac{\mu(N_t)}{\mu(N)},\ t = 1, ..., k\right] \tag{34}$$

$$\exists\, x \in V(S)\ [x \text{ has the } EPT] \exists\, i \in S : x_i \geq K.$$

Let $\{K^v\}$ be an increasing sequence with $\lim_v K^v = +\infty$ and let $\delta(0 < \delta < 1)$ be given. Since the number of types is finite, we can assume without loss of generality that there is some $t_0(1 \leq t_0 \leq k)$ and sequences $\{S^v\}$ and $\{x^v\}$ such that for all v,

$$(1+\delta)\tfrac{\mu(N_g)}{\mu(N)} \geq \tfrac{|S^v \cap N_t|}{|S^v|} \geq (1-\delta)\tfrac{\mu(N_t)}{\mu(N)}\ , ..., k,$$

$$x^v \in V(S^v) \text{ has the } ETP\text{, and}$$

$$x_i^v \geq K^v \text{ for all } i \in S^v \cap N_{t_0}. \tag{35}$$

Since Ω is closed and $U(i,\cdot)(i \in N_{t_0})$ is continuous on Ω, we can find a v_0 such that

$$U(i,a) \geq K^{v_0} \Rightarrow a > \tfrac{\mu(N)}{(1-\delta)\mu(N_{t_0})} \sum_{t=1}^{k} \omega^t\ , \tag{36}$$

where $\omega^t = \omega(i)$ for $i \in N_t(t = 1, ..., k)$. The condition $x^v \in V(S^v)$ implies that for some S^v-allocation $(a^{iv})_{e \in S^v}$, we have $x_i^v \leq U(i, a^{iv})$ for all $i \in S^v$. Then it follows from (35) and (36) that

$$\sum_{i \in S^{v0}} a^{iv_0} >\mid S^{v_0} \cap N_{t_0} \mid \tfrac{\mu(N)}{(1-\delta)\mu(N_{t_0}} \sum_{t=1}^{k} \omega^t$$

$$\geq\mid S^{v_0} \mid \sum_{t=1}^{k} \omega^t$$

$$\geq \sum_{t=1}^{k} \mid S^{v_0} \cap N_t \mid \omega^t$$

$$= \sum_{i \in S^{v_0}} \omega(i).$$

This is a contradiction to the condition that $x^{v_0} \in V(S^{v_0})$.

From Theorems 1, 2 and Lemma 3.3, we have

Corollary. Under the assumptions of Lemma 3.3, the f-core of the market game is non-empty.

Remark 3.1. Lemma 3.3 holds for any utility functions. However, we could transform the utility functions into bounded ones, and per-capita boundedness would be trivially satisfied. The substance of the corollary would be unchanged, since the f-core of the market game (in allocation space) does not depend on the particular choice of utility functions.

For nonemptiness of the f-core of a market game with types, we have essentially used only the existence of utility functions, which can be demonstrated under weak conditions (cf. Debreu, 1959, p.56).

In the corollary, no divisibility of commodities, no monotonicity and no convexity are required. This corollary is indicative of a broad potential of our theory to wide the scope of existing economic theory and increase the range of the subjects which can be studied as part of this theory.

3.4 The relationships between the f-core and the A-core

In the study of continuum economies the approach due to Aumann (1964) has become well-known. In this subsection we briefly compare our concepts to Aumann's; a thorough investigation is carried out in Hammond, Kaneko and Wooders (1985).

The A-core ('A' for Aumann) of the market game is

$C_A = \{f \in L(N, \bar{\Omega}) : \int_N f(i) \leq \int_n \omega(i)$ and no subset of positive measure in

B can improve upon $f\}$,

where a subset S of positive measure is said to be able to improve upon f iff for some $g \in L(S, \Omega)$, $\int_g g(i) \leq \int_s \omega(i)$ and $U(i, g(i)) > U(i, f(i))$ for all $i \in S$.

In Aumann's approach permissible coalitions are subsets of positive measure, while in ours permissible coalitions are finite subsets only. This is the basic difference. Both Aumann's and our definitions of feasibility of allocations are respectively consistent with his and our definitions of permissible coalitions.

These feasibility definitions turn out to yield equivalent sets of feasible allocations, as shown by Lemma 3.1. In fact, in Hammond, Kaneko and Wooders (1985), equivalence of the A-core and the f-core is demonstrated under certain assumptions. However, these results do not deny the fundamental nature of the basic differences in the treatment of permissible coalitions.

As discussed in the introduction, in Aumann's approach there are difficulties with the concept of the individual player. Our approach avoids these difficulties. The advantage of our approach is not only conceptual but also practical.

Continuum games should be idealizations of large finite games. In our approach, the treatment of the individual player is the same in finite games as in the continuum; therefore a natural relationship of finite games to the continuum game can be expected. In the proof of Theorem 1 in this paper, it can be observed that the ε-cores of finite games converge to the f-core, while the basic structure of permissible coalitions is preserved. These ideas are developed and explored in Kaneko and Wooders (1985).

The advantage of the f-core approach in application is illustrated by the following example with widespread externalities, where the f-core coincides with the Walrasian allocations and the definition of the A-core is problematic. This will be discussed in more detail in Hammond, Kaneko and Wooders (1985).

Example 3.1. Let $N = [0,5]$ be endowed with Lebesgue measure. The player set $[0,5)$ is partitioned into three subsets: $[0,1)$, the set of **households**; $[1,3)$, the set of **landlords of type** 1 and $[3,5)$, the set of **landlords of type** 2.

Each landlord has an apartment to rent, with a reservation rent of 1 yen. We imagine that the apartments of landlords of type 1 are located in a different area of the country from those of type 2. Landlord i's consumption set is given by

$$\Omega_1 = \{0, e^1\} \times R_+ \text{ if } i \in [1,3) \text{ and } \Omega_2 = \{0, e^2\} \times R_+ \text{ if } i \in [3,5)$$

and his initial endowment, by

$$\omega(i) = (e^1, 0) \text{ if } i \in [1,3) \text{ and } \omega(i) = (e^2, 0) \text{ if } i \in [3,5).$$

That is, landlord i owns an apartment and no money (a composite commodity). Landlord's i's utility functions $U(i, \cdot)$ is given as, for $i \in [1,3)$

$$U(i, a^i) = \begin{cases} 1 + m & \text{if } a^i = (e^1, m) \\ m & \text{if } a^i = (0, m); \end{cases}$$

and for $i \in [3,5)$

$$U(i, a^i) = \begin{cases} 1 + m & \text{if } a^i = (e^2, m) \\ m & \text{if } a^i = (0, m). \end{cases}$$

Each household $i \in [0,1)$ wants to rent at most one apartment from a landlord in either $[1,3)$ or $[3,5)$. The household's consumption set is given as $\Omega_h = \{0, e^1, e^2\} \times R_+$, and its initial endowment is $\omega(i) = (0,5)$. We allow households' utility functions to depend on external effects so households' utility functions are defined on the outcomes space F^* , rather than the Ω_h^* . Let $U(i,f)(i \in [0,1), f \in L(N,\Omega_h))$ be

$$U(i,f) = \begin{cases} 5 + 5\mu(\{i \in [0,1) : f_h(i) = e^1\}) + m & \text{if } f(i) = (e^1, m) \\ 2 + 10\mu(\{i \in [0,1) : f_h(i) = e^2\}) + m & \text{iff } f(i) = (e^2, m) \\ 0 + m & \text{iff } f(i) = (0, m), \end{cases}$$

where $f(i) = (f_h(i), f_m(i))$ with $f_h(i) \in \{0, e^1, e^2\}$ and $f_m(i) \in R_+$. That is, household i's utility comes from renting an apartment, the measure of players renting apartments in the same area, and the money remaining after paying the rent.

Since we allow widespread externalities, we need to appropriately modify the definition of the f-core. The modified definition is as follows: a function f in $L(A, \bar{\Omega})$ is in the f-core iff f is a feasible allocation; there is no coalition S in F such that for some vector $(a^j)_{j\in S}$,
$\sum_{j\in S} a(j) \le \sum_{j\in S} w(j)$,
$U(i; a^i) > U(i, f(i))$ for all $i \in S \cap [1,5)$, and
$U(i,g) > U(i,f)$ for all $i \in S \cap [0,1)$,

where g is the allocation which agrees with f on the complement of S and agrees with $(a^j)_{j\in S}$ on S . Since f and g are the same except on a finite set S , this is a natural modification of the f-core.

The allocations in the f-core of this game divide into three types:
Type 1. All households rent apartments in $[1,3)$ at 1 yen,

$$f(i) = \begin{cases} (e^1, 4) & \text{if } i \in [0,1) \\ (e^2, 0) & \text{if } i \in [3,5) \end{cases}$$

$$\mu(\{i \in [1,3) : f(i) = (0,1)\}) = \mu(\{i \in [1,3) : f(i) = (e^1, 0)\}) = 1;$$

Type 2. All households rent apartments in $[3,5)$ at 1 yen,

$$f(i) = \begin{cases} (e^2, 4) & \text{if } i \in [0,1) \\ (e^1, 0) & \text{if } i \in [1,3) \end{cases}$$

$$\mu(\{i \in [3,5) : f(i) = (0,1)\}) = \mu(\{i \in [3,5) : f(i) = (e^2, 0)\}) = 1;$$

and

Type 3. Households rent apartments in both areas (at the same rent of 1 yen) and the distribution of households between areas in such that the utilities of households in different areas are equalized.

$\mu(\{i \in [0,1) : f(i) = (e^1, 4)\}) = 7/15,$

$\mu(\{i \in [0,1) : f(i) = (e^2, 4)\}) = 8/15,$

$\mu(\{i \in [1,3) : f(i) = (0,1)\}) = 7/15,$
$\mu(\{i \in [1,3) : f(i) = (e^1, 0)\}) = 23/15,$

$\mu(\{i \in [0,1) : f(i) = (e^1, 4)\}) = 7/15,$

$\mu(\{i \in [3,5) : f(i) = (0,1)\}) = 8/15,$

$\mu(\{i \in [3,5) : f(i) = (e^2, 0)\}) = 22/15,$

For allocations of Type 1, the utility of a household is 14, for Type 2, 16, and for Type 3, $11^{1/3}$. In all cases, rents are 1 yen because the measure of households is smaller than those of landlords of each type. Obviously, the outcomes of Types 1 and 3 are not Pareto-optimal. However, the allocations of Types 1, 2 and 3 are all competitive allocations. (Of course the obvious modification of the definition of the competitive equilibrium is necessary since we allow widespread externalities.)

For this example, coalitions of positive measure influence both the feasibility of allocations achievable by the complementary coalition and the utilities of the members of the complementary coalition. Therefore for this example the A-core is not as naturally defined as the f-core. For more detailed discussion of this problem, see Hammond, Kaneko and Wooders (1985).

We remark that for the situation modelled in the above example the characteristic function defined in Section 2 does not adequately capture the structure of the economy nor determine the f-core of the economy. The definition of the characteristic function needs to be modified to reflect the externalities.

4 Proof of Theorem 1

4.1 Proof of Theorem 1

Without loss of generality, we can assume that sup $V(\{i\}) = 0$ for all $i \in N$. . First consider the case where $\mu(N_1), ..., \mu(N_k)$ are all rational numbers. Then there are positive integers $p_1, q_1, ..., p_k, q_k$ such that $\mu(N_t) = q_t/p_t$ for

$t = 1, ..., k$. Put $m_t = p_1 \times \cdots p_{t-1} \times q_t \times p_{t+1} \times \cdots \times p_t$ for $t = 1, ..., k$. Then N_t can be partitioned into subsets $N_{t1}, ..., N_{tm_t} (t = 1, ..., k)$ such that

$$\mu(N_{tl}) = \frac{1}{p_1 \times \cdots \times p_k} \text{ for all } l = 1, ..., m_t. \quad (37)$$

Let us construct a sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ of finite games as follows:

$$A_r = \bigcup_{t=1}^{k} A_{tr} \text{ and } A_{tr} = \{(t, 1), ..., (t, rm_t)\} \text{ for } t = 1, ..., k; \quad (38)$$

there is a 1 to 1 mapping $g : U_{t=1}^{k}\{(t, 1), (t, 2)...\} \longrightarrow N$
with $g(t, l) \in N_t$ for all $t = 1, ..., k$ and $l = 1, 2, ...$; (39)

and

$$V_r(S) = V(g(S)) \text{ for all } S \subset A_r \text{ and } r = 1, 2, ... \quad (40)$$

Then this sequence $\{(A_t, V_r)_{r=1}^{\infty}$ satisfies the conditions of the following lemma. In Section 4.2, both the terminology of the Lemma and the proof are described.

Lemma 4.1. (A generalization of Shubik and Wooders, 1983, p.34, Theorem 2). Let $(m_1, ..., m_k)$ be a vector or positive integers. Let $(m_1, ..., m_k a)$ be a vector of positive integers. Let $\{(A_r, V_r)\}$ be a sequence of (finite) replica games without side payments, where $A_r = \cup_{t=1}^{k}\{(t, 1), ..., (t, rm_t)\}$. If the sequence satisfies per-capita boundedness, then some subsequence $\{(A_{r_v}, V_{r_v})\}_{v=1}^{\infty}$ of $\{(A_r, V_r)\}_{r=1}^{\infty}$ has a strong approximate core with the strong equal treatment property (the SETP) (43). That is, for any $\varepsilon > 0$, there is a v_o and an $x \in R^k$ with the following properties:

$$\coprod_{t=1}^{k} x_t^{r_v m_t} = \coprod_{t=1}^{k} \overset{r_m m_t}{[x_t \times \cdots \times x_t]} \in V_{r_v}(A_{r_v}) \text{ for all } v \geq v_0; \quad (41)$$

for all $v \geq v_0$ and for any $S \in A_{r_v}$ there does not exist a $y \in V_{r_v}(S)$
such that $y_i > x_t + \varepsilon$ for all $i \in S \cap [t]_{r_v}$ and $t = 1, ..., k$; and (42)

if $i \in [t]_{r_v}$ and $j \in [t']_{r_v}$ are substitutes, then $x_t = x'_t$. (43)

Here $[t]_r$ denotes the set $\{(t, 1), ..., (t, rm_t)\}(t = 1, ..., k)$, that is, the players of type t if the rth game.

Without loss of generality, we can assume that the sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ itself has strong approximate core with the SETP.

Step 1. Let $\{\varepsilon_v\}$ be a sequence of positive number with $\lim_v \varepsilon_v = 0$. From Lemma 4.1, we can choose an r_v for each ε_v such that there is an $x^{r_\mathrm{v}} \in R^k$ satisfying (41), (42) and (43). It follows from sup $V(\{i\}) = 0$ and per-capita boundedness with respect to $\{N_t\}_{t=1}^k$ that $-\varepsilon_v \leq x_t^{r_\mathrm{v}} < K$ for $t = 1, ..., k$ and $v = 1, ...$ Then $\{x^{r_\mathrm{v}}\}$ has a convergent subsequence. Again, without loss of generality, we can assume that $\{x^{r_\mathrm{v}}\}$ itself converges to x^*. Note that $x^* \geq 0$.

Next define sequences $\{h^v\}$ in $L(N, R)$ and a function $h^* \in L(N, R)$ by

$$h^v(i) = x_t^{r_\mathrm{v}} \text{ if } i \in N_t \text{ and } t = 1, ..., k; \text{ and} \tag{44}$$

$$h^*(i) = x_t^* \text{ if } i \in N_t \text{ and } t = 1, ..., k. \tag{45}$$

Since $x^{r_\mathrm{v}} \longrightarrow x^*$, the sequence $\{h^v\}$ converges pointwise (in fact, uniformly) to h^*. Therefore if $h^v \in H$ for all v, then $h^* \in H^*$. Note that each h^v satisfies the SETP, i.e., if i and j are substitutes, then $h^v(i) = h^v(j)$, which implies that h^* also has the SETP.

Step 2. We now prove that $h^v \in H$ for all v . For $t = 1, ..., k$ and $l = 1, ..., m_t$, let $\{N_{tl}^1, ..., N_{tl}^{r_\mathrm{v}}\}$ be a partition of N_{tl} such that

$$\mu(N_{tl}^s) = \tfrac{1}{r_{\mathrm{tv}}}\mu(N_{tl}) = \tfrac{1}{r_\mathrm{r}} \cdot \tfrac{1}{p_1 \times \cdots \times p_\mathrm{k}} \quad \text{for } s = 1, ..., r_v.$$

Then from Lemma A.1 of the Appendix, there is a measure-preserving isomorphism (mod 0), say ψ_{tl}^5, from N_{11}^1 to $N_{tl}^s(t = 1, ..., k, l = 1, ..., m_t$ and $s = 1, ..., r_v)$. Consider the coalition

$$T = \bigcup_{t=1}^{k} \bigcup_{l=t}^{m_t} \{\psi_{tl}^1(i), ..., \psi_{tl}^{r_\mathrm{v}}(i)\}$$

for an arbitrary $i \in \cap_t \cap_l \cap_s$ Dom ψ_{tl}^s. This T has the same proportion of players of each type as A_{r_v}, i.e.,

Therefore, by (39), (40), (41) and (44) we have

$$(h^v(j))_{jeT} \in V(T).$$

Since i is arbitrary and $\mu(\cap_t \cap_l \cap_s \text{Dom } \psi_{tl}^s) = \psi_{11}^s)$, it follows from Lemma A.2 of the Appendix that $h^v \in H$.

Step 3. We now prove that there is no coalition $S \in F$ such that for some $y \in V(S)$, $y_i > h^*(i)$ for all $i \in S$. Suppose the contrary. Since $x^{r_\mathrm{g}} \longrightarrow x^*$ and $h^v(i) = x_t^{r_\mathrm{v}}$ if $i \in N_t(t = 1, ..., k)$, there is a v_0 such that $\varepsilon_v <$ min $\{y_i - h^v(i) : i \in S\}$ for all $v \geq v_0$. That is for all $v \geq v_0$,

$y_i > h^v(i) + \varepsilon_v = x_t^{r_v} + \varepsilon_v$ for all $i \in N_t$
and $t = 1, ..., k.$ (46)

Let S^* be a subset of A_{r_v} such that

$| S^* \cap [t]_{r_v} |=| S \cap N_t |$ for all $t = 1, ..., k;$

and let Θ be a 1 to 1 mapping from S^* to S such that

$\Theta(S^* \cap [t]_{r_v}) = S \cap N_t$ for all $t = 1, ..., k.$

(For sufficiently large v, we can find such an S^*.) Then by the assumption of replication, $(y_i^*)_{i\in S^*} \equiv (y_{\Theta(i)})_{i\in S^*}$ belongs to $V_{r_v}(S^*)$, and by (44) and (46), $y_i^* > x_t^{r_v} + \varepsilon_v$ for all $i \in S^* \cap [t]_{r_v}$ and $t = 1, ..., k$. This is a contradiction to (42).

Let us consider the case where $\mu(N_1), ..., \mu(N_k)$ are not necessarily rational numbers. Let $\{N_1^v\}, \{N_2^v\}, ..., \{N_k^v\}$ be sequences such that

$N_t^1 \subset N_t^2 \subset \cdots$ for all $t = 1, ..., k;$ (47)
$\mu(N_t^v)$ is a rational number for all $t = 1, ..., k$ and $v = 1,$ (48)
$\mu(N_t^v) \longrightarrow \mu(N_t)(v \longrightarrow \infty)$ for all $t = 1, ..., k;$ and

$\left(1 + \frac{\delta}{4}\right) \frac{\mu(N_t)}{\mu(N)} \geq \frac{\mu(N_t^v)}{\mu(N^v)} \geq \left(1 - \frac{\delta}{4}\right) \frac{\mu(N_t)}{\mu(N)}$

for all $t = 1, ..., k$ and $v = 1, 2, ...,$ (50)

where $N^v = \cup_{t=1}^k N_t^v$ for all v and δ is the positive constant given by per-capita boundedness of V. Consider the sequence of games $\{(N^v, V^v)\}$, where V^v is the restriction of V to N^v for all $v = 1, ...$ For all $\delta^v = \delta/4, (N^v, V^v)$ satisfies per-capita boundedness with respect to $\{N_t^v\}_{t=1}^k$. Indeed, $S \in F$ and $S \in N^v$ satisfies

$\left(1 + \frac{\delta}{4}\right) \frac{\mu(N_t^v)}{\mu(N^v)} \geq \frac{|S \cap N_t^v}{|S|} \geq \left(1 - \frac{\delta}{4}\right) \frac{\delta(N_t^v)}{\mu(N^v)},$

then, by (50), we have

$(1 + \delta) \frac{\mu(N_t)}{\mu(N)} \geq \left(1 + \frac{\delta}{4}\right)^2 \frac{\mu(Nt)}{\mu(N)} \geq \left(1 + \frac{\delta}{4}\right) \frac{\mu(N_t^v)}{\mu(N^v)} \geq \frac{|S \cap N_t^v}{|S|}$

$\geq \left(1 - \frac{\delta}{4}\right)^2 \frac{\mu(N_t^v)}{\mu(N^v)} \geq \left(1 - \frac{\delta}{4}\right)^2 \frac{\mu(N_y)}{\mu(N)} \geq (1 - \delta) \frac{\mu(N_t)}{\mu(N)}.$

This together with (18) implies that $x_i < K$ for any $x \in V^v(S) = V(S)$ with the ETP. Therefore the result of Subsection 4.1.1 can be applied to (N^v, V^v) and ensures the existence of a point h^v with the SETP in the f-core of the game (N^v, V^v) for all v. Since $h^v(v = 1, ...)$ has the SETP, we can define a vector x^v in $R^k by$

$x_t^v = h^v(i)$ if $i \in N_t^v$ and $t = 1, ..., k$.

Since sup $V(\{i\}) = 0 \leq x_t^v \leq K$ for all $t = 1, ..., k$, the sequence $\{x^v\}$ has a convergent subsequence. We can assume without loss of generality that $\{x^v\}$ itself converges to x^*. Define h^* by

$h^*(i) = x_t^*$ if $i \in N_t$ and $t = 1, ..., k$.

Claim 1. No coalition in F can improve upon h^*.

On the contrary, suppose that for some $S \in F$ and some $y \in V(S)$ we have $y_i > h^*(i)$ for all $i \in S$. Then $x^v \longrightarrow x^*$ implies that for some v, $y_i > h^v(i) = x_t^v$ if $i \in N_t^v(t = 1, ..., k)$. In this case we can find a coalition $S' \in F$ such that

$\mid S' \cap N_t^v \mid = \mid S \cap N_t \mid$ for all $t = 1, ..., k$.

Then S' can improve upon h^v in the game V^v, which is a contradiction.

Claim 2. $h^* \in H^*$.

Define. $\bar{h}^*$ on N by

$$\bar{h}^*(i) = \begin{cases} h^v(i) & \text{if } i \in N \\ 0 & \text{otherwise,} \end{cases}$$

Then $\bar{h}^v \in H^*$because $h^v \in H^{v*}$, where H^{v*} is the outcome space of (N^v, V^v). It suffices to show that $\bar{h}^v(i) \longrightarrow h^*(i)$ a.e. Since $\mu(\{i \in N : \bar{h}^v(i) = h^v(i)\}) \geq \mu(N^v)$, $\mu(N^v) \longrightarrow \mu(N)$ and $h^v(i) \longrightarrow h^*(i)$ for all $i \in \cup_{v=1}^{\infty} N_v$, we have $\bar{h}^v(i) \longrightarrow h^*(i)$ a.e.

4.2 On Lemma 4.1

We first give the definitions required for Lemma 4.1. Let (A, V) be a finite player game without side payments and with properties (i) $V(S)$ is a closed nonempty subset of R^2 for all $S \subset A$; (ii) $x \in V(S)$ and $y \in R^S$ with $x \geq y$ imply $y \in V(S)$ for all $S \subset A$; (iii) $V(S) \times V(T) \subset V(S \cup T)$ for all $S, T \subset A$ with $S \cap T = \phi$; and (iv) $V(S) - \cup_{i \in N}[(\text{interior } V(\{i\})) \times R^{S-\{i\}}]$

is nonempty and bounded for all $S \subset A$. In this section, by a game, we mean a finite player game without side payments and satisfying conditions (i)-(iv).

Let $\{(A_r, V_v)\}_{r=1}^{\infty}$ be a sequence of games, where $A_r = \cup_{t=1}^{*}\{(t,1),...,(t,rm_t)\}(m = (m_1,...,m_k)$ is an arbitrary given vector of positive integers). 'Substitutes' are defined by (14) and (15). If all players in $\{(t,1),...,(t,rm_t)\}(t = 1,...,k)$ are substitutes, then the game (A_r, V_r) is called a **replica game**. A payoff vector $x \in V_r(S)$ is said to have the **equal-treatment property** (the ETP), iff $x_i = x_j$ for all $i, j \in S \cap [t]_r (t = 1,...,m)$. The sequence $\{(A_r, V_r)\}_{r=1}^{\infty}$ is said to be **per-capita bounded** iff there exists a K such that for any r,

$$\left.\begin{array}{c} S \subset A_r \\ \frac{|S \cap [t]_r|}{|S|} = m_t \text{ for all } t = 1,...,k \\ x \in V_r(S) \text{ has the ETP} \end{array}\right] \Longrightarrow x_i < K \text{ for all } i \in S.$$

In the case where $m = (1,...,1)$, Lemma 4.1 without SETP (43) is the same as Shubik and Wooders (1983, Theorem 2). In their theorem, comprehensiveness of V_r was not assumed and their statement is slightly different from Lemma 4.1. However in the proof of their Theorem 1, they assumed comprehensiveness and demonstrated the conclusion of Lemma 4.1 with the ETP but not necessarily the SETP (Shubik and Wooders, 1983, p.44). Their proof can be used without any substantive change in he case of $m = (m_1,...,m_k)$ arbitrary.

It is now easy to obtain the conclusion of Lemma 4.1. If some players of different types, say t and $t'(t < t')$ are substitutes, then we can treat all players of types t and t' as one type; we consider a new vector of numbers of players of each type.

$$m' = (m_1,...,m_{t'},...,m_{t'+1},...,m_k)$$

and construct, by relabelling players, a sequence of replica games with respect to m' (in this sequence, all players of types t and t' are treated as the same type). By the above generalisation of the Shubik-Wooders theorem, given $\varepsilon > 0$ we have an outcome with the ETP in the ε-core of a subsequence of the sequence of games. Then, again simply by relabelling, we can transform these outcomes into outcomes in a subsequence of the original games. These outcomes treat players of types t and t' identically. We repeat this procedure until we have the conclusion of Lemma 4.1.

Appendix

The following lemma plays a crucial role in constructing measure-consistent partitions.

Let A, B be sets in B with $\mu(A) = \mu(A) = \mu(B) > 0$. We say that ψ is **a measure-preserving isomorphism from** A to B (mod 0) iff for some null sets A_0 and B_0 with $A_0 \subset A$ and $B_0 \subset B, \psi$ is a measure-preserving isomorphism from $A - A_0$ to $B - B_0$.

Lemma A.1. If $\mu(B) > 0$, then there exists a measure-preserving isomorphism ψ(mod 0) from A to B.

Before giving the proof of this lemma, we illustrate the construction of a measure-consistent partition by a simple example. Let $\{N^1, N^2\}$ be a partition of N with $\mu(N^1) = \mu(N^2) = \frac{1}{2}\mu(N)$. Then, by Lemma A.1., there is a measure-preserving isomorphism ψ from N_0^1 to N_0^2, where N_0^1 and N_0^2 are the null subsets of N^1 and N^2. We can then define a measure-consistent partition p by:

$$p = \{\{i, \psi(i)\} : i \in N^1 - N_0^1\} \cup \{\{i\} : i \in N_0^1 \cup N_0^2\}.$$

Proof of Lemma A.1. Let $([0, \mu(A)], \mathcal{B}([0, \mu(A)]), \lambda)$ be the measure space of Borel subsets of the closed internal $[0, \mu(A))]$ endowed with Lebesgue measure λ. Suppose that there exist measure-preserving isomorphisms ψ_1 and ψ_2 (mod 0) from $[0, \mu(A)](= [0, \mu(B)])$ to A and B respectively. That is, there are null sets $L_a \subset [0, \mu(A)]$ (respectively, $L_B \subset [0, \mu(B)]$) and $A_0 \subset A(B_0 \subset B)$ such that $\psi_1(\psi_2)$ is a measure preserving isomorphism from $[0, \mu(A)] - L_A$ to $A - A_0$ (from $[0, \mu(B)] - L_B$ to $B - B_0$). Let $L = [0, \mu(A)] - L_A - L_B$. Then $\lambda(L) = \mu(A)$. Consider the restrictions of ψ_1 and ψ_2 to L. Since the composite function $\psi_2^o \psi_1^{-1}$ is a measure-preserving isomorphism from $\psi_1(L)$ to $\psi_2(L)$, the function $\psi_2^o \psi_1^{-1}$ is a measure-preserving
$[0, \mu(A)]$

isomorphism (mod 0) from A to B. Therefore, it suffices to show the following theorem, where we assume without loss of generality that $\mu(A) = 1$.

Theorem. Let $(A, \mathcal{B}(A), \mu)$ be a measure space, where A is an uncountable Borel sub-set of a complete separable metric space, $\mathcal{B}(A)$, the σ-algebra of all Borel subsets of A, and μ, a nonatomic probability measure on $\mathcal{B}(A)$. Let $(I, \mathcal{B}, \lambda)$ be a measure space, where $I = [0, 1], \beta$, the σ-algebra of Borel subsets of I, and λ, Lebesgue measure. Then there is a measure-preserving isomorphism ψ from $A - A_1$ to $I - I_1$ for some $A_1 \subset N$ and $I_1 \subset 1$ with $\mu(A_1) = \lambda(I_1) = 0$.

This theorem is stated in a slightly stronger form than Royden (1963, p.327, Theorem 9). However, it is sufficient to note that if one replaces the section of Royden's proof dependent upon Royden (1963), p.326, Theorem 8, with the obviously stronger analogue possible using Parthasarathy (1967) p.15, Theorem 2.12, then the proof of the above form of the theorem is identical to Royden's.

Lemma A.2. Let $h \in L(N,R)$. Assume that there is a partition $\{N_t\}_{t=1}^k$ of N and that there are measure-preserving isomorphisms $\psi_1, ..., \psi_k$ (mod 0) from N_1 to

$$(h(\psi_1(i)), h(\psi_2(i)), ..., h(\psi_k(i))) \in V(\{\psi_1(i), ..., \psi_k(i)\})$$

$$\text{for all } i \in \bigcap_{t=1}^{k} \text{Dom } \psi_t, \qquad \text{(A.1)}$$

where Dom ψ_t is the domain of the function ψ_t. Then the function h belongs to H.

Proof of Lemma A.2. We first dispose of the case where $k = 1$. In this case, $N_1 = N$ and $\psi_1(i) = i$ for almost all i in N. Set $p(i) = \{i\}$ for all i in N. Then $h(j)_{j \in p(i)} \in V(p(i))$ for almost all $i \in N$ so $h \in H$.

For the case where $k > 1$, let h' be the function on N defined by

$$h'(i) = \begin{cases} \inf_{j \in N} \max V(\{j\}) & \text{if } i \in N - \bigcup_{t=1}^{k} \psi_t\left(\bigcap_{l=1}^{k} \text{Dom } \psi_1\right) \\ h(i) & \text{otherwise.} \end{cases}$$

Then $h' \in L(N,R)$. Put

$$p = \left\{\{\psi_1(i), ..., \psi_k(i)\} : i \in \bigcap_{t=1}^{k} \text{Dom } \psi_1\right\} \cup \left\{\{j\} : j \in N - \bigcup_{t=1}^{k} \psi_t\left(\bigcap_{l=1}^{k} \text{Dom } \psi_1\right)\right\}$$

Then $p \in \Pi$. If $p(i) = \{i\}$, then $h'(i) \in V(\{i\})$ by (2). Otherwise, there is a $j \in \cap_{t=1}^{k} \text{Dom } \psi_t$ such that $i = \psi_t(j)$ for some t, which together with (A.1) implies

$(h'(l))_{l \in p(i)} = (h(\psi_1(j)), ..., h(\psi_k(j))) \in V(\{\psi_1(j), ..., \psi_1(j), ..., \psi_k(j)\}) = V(p(i))$.

Since $\psi_1, \psi_2, ..., \psi_k$ are measure-preserving isomorphisms (mod 0) from N_1 to $N_1, ..., N_k$, it holds that

$$\mu(N_1) = \mu(N_t) = \mu(\text{Dom}\,\psi_t) = \mu\left(\bigcap_{l=1}^{k} \text{Dom}\,\psi_1\right) = \mu\left(\psi_t\left(\bigcap_{l=1}^{k} \text{Dom}\,\psi_1\right)\right)$$

for all $t = 1, ..., k$ and

$$\mu(\{i \in N : p(i) = \{i\}\}) = \mu\left(\bigcup_{t=1}^{k} \psi_t\left(\bigcap_{l=1}^{k} \text{Dom}\,\psi_l\right)\right)$$

$$= \mu(N) - \sum_{t=1}^{k} \mu(\text{Dom}\,\psi_t) = 0.$$

Therefore $h' \in H$ and $\mu(\{i \in N : h'(i) \neq h(i)\}) \leq \mu\{i \in N : \{i\} \in p\}) = 0$ imply $h \in H$.

Proof of Lemma 3.1. This lemma is critical to our results on market games and we believe will also be critical to future work. Since the later part of the proof (i.e. the proof that $\int_N f(i) \leq \int_N w(i) \Longrightarrow f \in F^*$) is complex, we will provide a preliminary sketch of this part.

First we prove $'f \in F^* \Longrightarrow \int_N f(i) \leq \int_N \omega(i)'$. Since $f \in F^*$, from (25) and Remark 2.4, there is a sequence $\{f^v\}$ such that $f^v \in F$ for all v and f^v converges a.e. on N. For each v, there is a $p^v \in \Pi$ such that $f^v \in F(p^v)$. For each positive integer k, $\cup_{|S|=k, S \in p^v} S \equiv N_k^v$ has a partition $\{N_{k1}^v, ..., N_{kk}^v\}$ with measure-preserving isomorphisms ψ_{kk}^v from N_{k1}^v to $N_{k1}^v, ..., N_{kk}^v$, respectively, satisfying $\{S : S \in p^v$ and $\mid S \mid= k\} = \{\{\psi_{k1}^r(i),...,\psi_{kk}^v(i)\} : i \in N_{k1}^v\}$. Then it follows from (22) that for all v and k,

$$\sum_{t=1}^{k} f^v(\psi_{kt}^v(i)) \leq \sum_{t=1}^{k} \omega(\psi_{kt}^v(i)) \text{ a.e. in } N_{k1}^v.$$

Therefore we have, for any v,

$$\int_N f^v(i) = \sum_{k=1}^{\infty} \sum_{t=1}^{k} \int_{N_{kt}^v} f^v(i) = \sum_{k=1}^{\infty} \int_{N_{kl}^v} \sum_{t=1}^{k} f^v(\psi_{kt}^v(i))$$

$$\leq \sum_{k=1}^{\infty} \int_{N_{k1}^v} \sum_{t=1}^{k} \omega(\psi_{kt}^v(i)) = \sum_{k=1}^{\infty} \sum_{t=1}^{k} \int_{N_{kt}^v} \omega(i) = \int_N \omega(i).$$

Applying Fatou's lemma yields

$$\int_N f(i) \leq \liminf_{v \longrightarrow \infty} \int_N f^v(i) \leq \int_N \omega(i).$$

We now prove that $' \int_N f(i) \leq \int_N \omega(i) \Longrightarrow f \in F^{*\prime}$. As indicated earlier, we first provide a preliminary sketch of the proof.

Sketch. Suppose $\int_N f(i) \leq \int_N \omega(i)$. We first divide $R^m_+ \cap \{x \in R^m : x_c \leq v, c = 1, ..., m\}$ into cubes of the same size $(1/2^v)$. We then form a set of subset $\{N^v_t\}^{l_v}_{t=1}$ such that all subsets have the same measures, and all players i in N^v_t have their allocation $f(i)$ and their endowments $\psi(i)$ in the same cubes. (This is carried out in Claim 1.)

Now form the functions f^v and ω^v by taking $f^v(i)$ equal to the coordinate wise inf of f on the cube containing $f(i)$ and similarly, $\omega^v(i)$ equal to the coordinate-wise sup of ω on the cube containing $\omega(i)$..This has the consequence that for each t and all i and i' in N^v_t, we have $f^v(i) = f^v(i')$ and $\omega^v(i) = \omega^v(i')$.

It may be the case that $\int_N f^v(i)$ is too large for our purpose (because we ignore $\omega(i)$ when $\omega(i)$ is not contained in any of the cubes and because we make $\omega^v(i) \geq \omega(i)$ for all other values $\omega(i)$). Therefore, in Claim 2, we choose a subfamily of the subsets $\{N^v_t\}^{l_v}_{t=1}$ whose total measure tends to zero as $v \longrightarrow \infty$. This subfamily is chosen so that if we redefine $f^v(i) = 0$ for all i in the union of members of the sub-family, then $\int_N f^v(i)$ is no longer too large. We carry this out and call the resulting function $\overset{-v}{f}$.

The next step is simple. The sequence $\{\overset{-v}{f}\}$ converges in measure to f and each member $\overset{-v}{f}$ of the sequence is attained by some feasible partition, where an element of the partition contains one member of N^v_t for each t (and players not in $\cup^{l_v}_{t=1} N^v_t$ are in singleton sets).

The Proof. Now to prove the lemma, without loss of generality we can assume that $\mu(N) = 1$. Let us suppose that $\int_N f_c(i) > 0$ for all $c = 1, ..., m$.

Define

$$C^v(i) = \prod_{c=1}^{m} \left[\tfrac{k_c - 1}{2^v}, \tfrac{k_c}{2^v}\right) \text{ for } k = (k_1, ..., k_m) \in \{1, 2, ..., v2^v\}^m;$$

and

$$B^v = \{a \in R^m_+ : a_c \geq v \text{ for some } c\} \quad (v = 1, 2, ...).$$

Claim 1. There is a sequence $\{N^v\}$ of subsets in $\mathcal{B}$ with the following properties:

Each N^v has a partition into some finite number of elements say $\{N_t^v\}_{t=1}^{l_v}$, with measure-preserving isomorphisms $\psi_1^v, ..., \psi_{l_g}^v$ from N_1^v to $N_1^v, ..., N_{l_v}^v$, respectively, and $\mu(N_1^v) \longrightarrow 0 (v \longrightarrow \infty)$; (A.2)

$\mu(N^v - N^{v+1}) = 0$ for all v and $\mu(N^v) \longrightarrow \mu(N)$; and (A.3)

for any N_t^v, there are $k^1, k^2 \in \{1, ..., v2^v\}^m$ such that $\{F(I) : i \in N_t^v\} \subset C^v(k^1)$ and $\{\omega(i) : i \in N_t^v\} \subset C^v(k^2)$. (A.4)

Note that of course $\mu(N_t^v) = \mu(N_1^v)$ for all t and the measure-preserving isomorphisms are 'exact', i.e. not mod 0.

Proof of Claim 1. Define the sets $\bar{M}^v(k^1, k^2)(k^1, k^2 \in \{1, ..., v2^v\}^m)$ and M^v by

$$\tilde{M}^v(k^1, k^2) = \{i \in N : f(i) \in C^v(k^1) \text{ and } \omega(i) \in C^v(k^2)\}; \text{ and}$$
$$M^v = \{i \in N : f(i) \in B^v \text{ or } \omega(i) \in B^v\}.$$

Then $\cup_{k^1,k^2} \tilde{M}^v(k^1, k^2) \subset \cup_{k^v,k^2} \tilde{M}^{v+1}(k^1, k^2)$ and

$$\mu\left(\bigcup_{k^1,k^2} \tilde{M}^v(k^1, k^2)\right) = \mu(M^v) \longrightarrow \mu(N).$$

If $\mu(\tilde{M}^v(k^1, k^2)) > 0$, then we can choose a subset $M^v(k^1, k^2)$ of $\tilde{M}^v(k^1, k^2)$ such that

$$\mu(M^v(k^1, k^2)) \geq (1 - \tfrac{1}{2^v})\mu(\tilde{M}^v(k^1, k^2)); \quad \text{(A.5)}$$

$$\mu(M^v(k^1, k^2)) \text{ is a rational number.} \quad \text{(A.6)}$$

Let $S_v = \{(k^1, k^2) \in \{1, ..., v2^v\}^m : \mu(M^v(k^1, k^2)) > 0\}$. Without loss of generality, we can assume

$$\bigcup_{(k^1,k^2) \in S_v} M^v(k^1, k^2) \subset \bigcup_{(k^1,k^2) \in S_{v+1}} M^{v+1}(k^1, k^2) \text{ for all } v.$$

Let $\tilde{N}^v = \cup_{(k^1,k^2)\in S_v} M^v(k^1, k^2)(v = 1, ...)$. Then $\tilde{N}^v \subset \tilde{N}^{v+1}$. It follows from (A.5) that

$$\mu(\tilde{N}^v) = \mu\left(\bigcup_{(k^1,k^2) \in S_v} M^v(k^1, k^2)\right) \geq \left(1 - \tfrac{1}{2^v}\right) \mu\left(\bigcup_{k^1,k^2} \tilde{M}(k^1, k^2)\right)$$
$$= \left(1 - \tfrac{1}{2^v}\right)[\mu(N) - \mu(M^v)] \rightarrow \mu(N) \qquad (v \rightarrow \infty).$$

Since $\mu(M^v(k^1,k^2))$ is a rational number for all $(k^1,k^2) \in S_v$ by (A.6), we can find some subpartition, say $\{\tilde{N}_t^v\}_{t=1}^{l_v}$, of $\{M^v(k^1,k^2)\}_{(k^1,k^2)\in S_v}$ such that $\mu(\tilde{N}_1^v) = \cdots = \mu\tilde{N}_{l_v}^v) \leq 1/2^v$.

From Lemma A.1., there are measure-preserving isomorphisms (mod 0) $\psi_1^v, ..., \psi_{l_v}^v$ from $\tilde{N}_l^v$ to $\tilde{N}_l^v, ..., \tilde{N}_{l_v}^v$, respectively. Let $N_t^v = \psi_t^v(\cap_{n=1}^{l_v}\text{dom}\ \psi_n^v)$ for $t = 1,...,l_v$ and let $N^v = \cup_{t=1}^{l_v} N_t^v$. Then these sets $N_1^v, ..., N_{l_v}^v$ and N^v have the same measures as those of $\tilde{N}_l^v, ..., \tilde{N}_{l_v}^v$ and $\tilde{N}^v$, respectively. Therefore the claim holds.

Define the functions f^v and ω^v by their coordinates $(c = 1, ..., m)$

$$f_c^v(i) = \begin{cases} \inf \text{Proj}_c[C^v(k) \cap \Omega] & \text{if } f(i) \in C^v(k),\ k \in \{1, ..., v2^v\}^m \text{ and } i \in N^v \\ 0 & \text{otherwise; and} \end{cases} \tag{A.7}$$

$$\omega_c^v(i) = \begin{cases} \inf \text{Proj}_c[C^v(k) \cap \Omega] & \text{if } \omega(i) \in C^v(k),\ k \in \{1, ..., v2^v\}^m \text{ and } i \in N^v \\ 0 & \text{otherwise; and} \end{cases} \tag{A.8}$$

where $\text{Proj}_c[C^v(k) \cap \Omega] = \{x_c : x \in C^v(k) \cap \Omega\}$. Then $f^v(i) \in \Omega$ for all v and all $i \in N$ by (3.2) (while it may not be the case that $\omega^v(i) \in \Omega$). It follows from (A.3), (A.4) and (A.7) that

$$f^v(i) \leq f^{v+1}(i) \text{ for all } v \text{ and a.e. in } N; \tag{A.9}$$

$$\textstyle\int_N f^v(i) \leq \int_N f(i) \text{ for all } v. \tag{A.10}$$

Since $f^v(i) \to f(i)$ a.e. in N by (A.3), (A.4) and (A.7), we have, by (A.9),

$$\textstyle\int_N f^v(i) \to \int_N f(i). \tag{A.11}$$

It follows from the definition of $C^v(k)$ and (A.8) that for all i in N

$$\omega(i) \geq \omega^v(i) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \text{ for all } v\ . \tag{A.12}$$

Claim 2. There is a sequence $\{F^v\}$ of subsets in B such that for some v_0,

$$\text{for all } v \geq v_0, F^v \bigcup_{t \in T} N_t^v \text{ for some } T_v \subset \{1, ..., l_v\}; \tag{A.13}$$

$$\textstyle\int_{F^v} f^v(i) \geq \int_{N-N^v} \omega(i) + \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \text{ for all } v \geq v_0; \tag{A.14}$$

$$\mu(F^v) \to 0 \qquad (v \to \infty). \tag{A.15}$$

Proof of Claim 2. Since $\int_{N-N^v} \omega(i)+(1/2^v,...,1/2^v) \to 0$ and $\int_{N^v} f^v(i) > 0$ by (A.3) and (A.11), we can find a v_0 such that

$$\int_{N_0^v} f^{v_0}(i) \geq \int_{N-N^{v_0}} \omega(i) + \left(\tfrac{1}{2^{v_0}},...,\tfrac{1}{2^{v_0}}\right).$$

By (A.3), we have

$$\int_{N-N^v} \omega(i) + \left(\tfrac{1}{2^v},...,\tfrac{1}{2^v}\right) \geq \int_{N-N^{v+1}} \omega(i) + \left(\tfrac{1}{2^{v+1}},...,\tfrac{1}{2^{v+1}}\right).$$

for all v.

These inequalities, together with (A.3) and (A.9), imply

$$\int_{N^v} f^v(i) \geq \int_{N^{v_0}} f^{v_0}(i) \geq \int_{N-N_t} \omega(i)+\left(\tfrac{1}{2^v},...,\tfrac{1}{2^v}\right) \text{ for all } v \geq v_0. \tag{A.16}$$

Let c be any element in $\{1, 2, ..., m\}$. It follows from (A.16), (A.9), (A.2) and (A.4) that there is a sequence $\{G_c^v\}_{v=v_0}^{\infty}$ such that for all $v \geq v_0$,

$$G_c^v = \underset{t \in W_v}{\cup} N_t^v \text{ for some } W_v \subset \{1, ..., l_v\};$$

$$\int_{G_c^v} f_c^v(i) \geq \int_{N-N^v} \omega_c(i) + \tfrac{1}{2^v} : \mu(G_c^v) \to 0 \qquad (v \to \infty).$$

Then let $F^v = \cup_{c=1}^m G_c^v$ for all $v \geq v_0$. The sequence $\{F^v\}$ satisfies (A.13)-(A.15). This completes the proof of Claim 2.

We are now in a position to construct a sequence in $F = \cup_{p \in \Pi} F(p)$ which converges in measure to the function f. Define $\{\bar{f}^v\}$ by, for all $v \geq v_0$,

$$\bar{f}(i) = \begin{cases} f^v(i) & \text{if } i \in N^v - F^v, v \geq v_0 \\ 0 & \text{otherwise.} \end{cases} \tag{A.17}$$

For any $\varepsilon > 0$, there is a $v_1 \geq v_0$ such that $1/2^{v_1} < \varepsilon$. Then it follo9ws from (A.17) and (A.7) that for all $v \geq v_1$,

$$\mu(\{i \in N : \| \bar{f}(i) - f(i) \| > \varepsilon\}) \leq \mu(F^v \cup (N - N^v)), \tag{A.18}$$

and from (A.3) and (A.15) that

$$\mu(F^v \cup (N - N^v)) \to (v \to \infty). \tag{A.19}$$

That is, the sequence $\{\bar{f}^v\}$ converges in measure to f .

Next we have to show that for each $\bar{f}^v$, there is a feasible partition p^v of N such that $\bar{f} \in F(p^v)$. Define p^v by

$$p^v = \{\{\psi_1^v(i), ..., \psi_{l_v}^v(i)\} : i \in N_1^v\} \cup \{\{i\} : i \in N - N^v\} \quad (v = 1, 2...) \qquad \text{(A.20)}$$

(where the ψ_1^v's are the isomorphisms given in Claim 1).
Then we have, for any $v \geq v_0$ and any $j \in N_1^v$,

$$\sum_{t=1}^{l_v} \bar{f}^v(\psi_t^v(j)) = \sum_{t=1}^{l_v} f^v(\psi_t^v(j)) - \sum_{\psi_t^v(j) \in F^v} f^v(\psi_t^v(j)) \qquad \text{(by (A.17))}$$

$$= \frac{1}{\mu(N_1^v)} \left(\sum_{t=1}^{l_v} \int_{N_1^v} f^v(\psi_t^v(i)) - \int_{F^v} f^v(i) \right) \quad \text{(by (A.13))}$$

$$\leq \frac{1}{\mu(N_1^v)} \left[\int_{N_v} f^v(i) - \int_{N-N^v} \omega(i) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \right] \quad \text{(by (A.14))}$$

$$\leq \frac{1}{\mu(N_1^v)} \left[\int_{N} \omega(i) - \int_{N-N^v} \omega(i) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \right]$$

$$\leq \frac{1}{\mu(N_1^v)} \left[\int_{N^v} \omega(i) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \right]$$

$$\leq \frac{1}{\mu(N_1^v)} \int_{N^v} \left[\omega^v(i) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \right] \quad \text{(by (A.8) and } \mu(N) = 1)$$

$$= \sum_{t=1}^{l_v} \left[\omega^v(\psi_t^v(j)) - \left(\tfrac{1}{2^v}, ..., \tfrac{1}{2^v}\right) \right] \leq \sum_{t=1}^{l_v} \omega(\psi_t^v(j)) \quad \text{(by (A.12))}.$$

For all $j \in N - N^v$, (i.e., $p^v(j) = \{j\}$) it holds that $\bar{f}^v(j) = 0 \leq \omega(j)$. Therefore $\bar{f}^v \in F(p^v)$. By (A.19), $f \in F^*$.

Finally we remark on the case where $\int_N f_c(i) = 0$ for some $c \in \{1, 2, ..., m\}$. Let $C = \{c : \int_N f_c(i) = 0\}$. If $C = \{1, ..., m\}$, then clearly $f \in F^*$ (in fact, $f \in F^*$ (in fact, $f \in F(p)$ for all $p \in \Pi$). If $C \subset \{1, 2, ..., m\}$ then we define $\bar{f}_c^v (c \notin C)$ in the same manner as above, and put, for $c \in C$,

$\bar{f}_c^v(i) = 0$ for all $i \in N$.

Then $\bar{f}^v \in F$ for all v and $\bar{f}^v$ converges in measure to f.

This completes the proof.

Proof of Lemma 3.2. Let k be an arbitrary positive integer. By (1) any $S \in p$ with $\mid S \mid = k$ can be represented as $S = p(i) = \{\psi^p_{k1}(i), ..., \psi^p_{kk}(i)\}$ for some $i \in N^p_{k1}$. Define $\Phi(i)$ on N^p_{k1} by

$$\Phi(i) = \left\{ (a^1, ..., a^k) \in \Omega^k : \sum_{t=1}^{k} \omega(\psi^p_{kt}(i)) \geq \sum_{t=1}^{k} a^t \text{ and} \right.$$

$$U(\psi^p_{kt}(i), a^t) \geq h(\psi^p_{kt}(i)) \text{ for all } t = 1, ..., k\} .$$

It can be proved in the standard manner the correspondence $\Phi(i)$ has a measurable graph. The nonemptiness of $\Phi(i)$ follows from the assumption that $h \in H_M(p)$ and the definition of V_M. Therefore the Measurable Selection Theorem (Hildenbrand, 1974, p.54, Theorem 1) can be applied to the correspondence and we conclude that there is a measurable function $(a_{k1}(\psi^p_{k1}(i), ..., a_{kk}(\psi^p_{kk}(i)))$ from N^p_{k1} to Ω^k such that $(a_{k1}(\psi^p_{k1}(i)), ..., a_{kk}(\psi^p_{kk}(i))) \in \Phi(i)$ a.e. in N^p_{k1}. Since k is an arbitrary positive integer, we have a countable number of selections. Define a function $f : N \rightarrow \Omega$ by teger, we have a countable number of selections. Define a function $f : N \rightarrow \Omega$ by

$$f(j) = a_{kt}(\psi^p_{kt}(i)) \text{ if } j \in N^p_{kt} \text{ and } j = \psi^p_{kt}(i) \qquad (k = 1, 2, ...).$$

Then this f belongs to $F(p)$, because f is measurable and for almost all $i \in N^p_k(i = \psi^p_{kt}(i_0)$ for some $i_0 \in N^p_{k1})$.

$$\sum_{j \in p(i)} f(j) = \sum_{j \in p(i_0)} f(j) = \sum_{t=1}^{k} a_{kt}(\psi^p_{kt}(i)) = \sum_{j \in p(i)} \omega(j).$$

It follows from the definition of Φ that $U(i, f(i)) \leq h(i)$ a.e. in N.

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