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The Core of a Continuum Economy with Widespread Externalities and Finite Coalitions: From Finite to Continuum Economies

MAMORU KANEKO

Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

AND

MYRNA HOLTZ WOODERS

Department of Economics, University of Toronto, Toronto, Ontario, Canada M5S 1A1

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When permissible coalitions in finite economies are constrained to be small relative to the player set, the continuum model with finite coalitions and its f-core are the limits of large finite economies and their ε-cores. We show convergence both of game-theoretic structures—relatively small coalitions in the finite economies converge to finite coalitions–and of solutions—ε-cores converge to the f-core. Our convergence is carried out in the context of exchange economies with widespread externalities where the requirement that coalitions be small is critical. *Journal of Economic Literature* Classification Numbers: 021, 022, 026.

1. INTRODUCTION

1.1. *General Motivation*

The notion of a perfectly competitive economy with recontracting is of one with a very large number of participants, each of whom actively pursues his own interests and each of whom has negligible influence on economic aggregates.[1] In recontracting, agents meet face to face and, in

[1] By perfect competition with recontracting we mean to suggest a story of individuals meeting in relatively small groups or market places and engaging in trade. Each individual can possibly frequent many different market places. These activities (rather than the actions of a central auctioneer) are envisioned as leading to competitive prices. Thus perfect competition and recontracting are tied together.

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pursuit of their own interests, engage in trade. This suggests that a model describing these notions needs two special aspects: individuals should be effective in pursuit of their own interests through recontracting and simultaneously individuals should be ineffective in influencing any broad economic totals. In other words, in his own recontracting behavior, at the individual level an individual player is nonnegligible to other players, but at the total economy level an individual player is negligible.

To describe these seemingly contradictory aspects of perfect competition with recontracting, we are led to a new model (Kaneko and Wooders [6, 7] and Hammond, Kaneko, and Wooders [4]) with a continuum of players, distinct in several respects from the models initiated by Aumann [2]. The main difference between our model and the standard model is the treatment of permissible coalitions. In both models the total player set is a continuum. In our model, coalitions—recontracting groups—are restricted to be finite sets (ones containing only finite numbers of players) while in the standard model, coalitions are restricted to be subsets of positive measure. With coalitions of positive measure we are unable to simulaneously incorporate the two special aspects of perfect competition with recontracting. To discuss this, we need to consider the subtle interpretation of the concept of the individual player.

With coalitions of positive measure, two interpretations of the concept of the individual player might be possible. If we adopt the straightforward interpretation of the individual player as a point in the continuum, then the individual player has negligible influence on economic aggregates—he has no effect on them at all. However, for the same reason, the player is also negligible in the formation of any recontracting group. On the other hand, if we adopt an alternative interpretation of the individual player as arbitrarily "small," but bigger than a point, so that individuals can be interpreted as having nonnegligible power in forming recontracting groups, then individuals also become nonnegligible relative to the total. Neither of the above interpretations simultaneously captures the two aspects of perfect competition with recontracting—the individual as nonnegligible relative to recontracting groups while negligible relative to the total.

For our approach, since coalitions—finite sets—are groups of individual players, we can adopt only the straightforward interpretation of the individual player as a point in the continuum. Since recontracting groups are finite, players are nonnegligible within these groups. For individual players to be negligible relative to the total economy we need a continuum of players. Thus, we are compelled to use our approach to describe the seemingly contradictory aspects of perfect competition with recontracting.

Our continuum approach is special in that we use absolute magnitudes (the cardinal numbers) and finite sums to describe finite coalitions and their behaviors while we use proportional magnitudes (the measure) to describe economic aggregates. Our approach requires these two separate, noncomparable, notions of sizes within the same model. This phenomenon does not (exactly) appear in a finite world. In a finite world proportional and absolute sizes are always comparable, that is, the absolute magnitudes described by the cardinal numbers are mutually replaceable with the proportional magnitudes described by the counting measure. Now our questions are: does there exist an interpretation of perfect competition with recontracting in a finite world and then, if it exists, what is it? This paper attempts to provide answers to these questions.

We argue that there are affirmative answers; we can give a finite analogue to the continuum model. The seemingly contradictory aspects of perfect competition with recontracting can be approximately realized in a large finite setting. However, whether or not a finite analogue is close to the continuum economy depends on how we measure closeness. Therefore, one finite analogue does not suffice as a finitistic interpretation of perfect competition with recontracting unless a specific norm, or standard, for the approximation is given. If we cannot specify a natural standard for an approximation then one possible approach is to consider sequences of finite analogues instead of a specific one. Our answer is in demonstrating the convergence of a sequence of finite economies to the continuum. We provide this answer in the context of exchange economies with widespread externalities.

A continuum economy is approximated arbitrarily closely by a large finite economy. The restriction of coalitions in the continuum model to finite sets of players is approximated by the restriction of (permissible) coalitions to relatively small sets of players in the finite economies. At the individual level, the effectiveness of the individual is essentially invariant with the size of the economy, while, relative to the entire economy, the individual becomes negligible. The measurements at the individual level correspond to absolute magnitudes and those at the total economy level to proportional magnitudes.

1.2. *The Model and Result*

When we look at the totality of individual behavior in recontracting, the two separate notions of measurement, absolute magnitudes and proportional magnitudes, must be consistently connected. Mathematically, in the continuum this connection is made through a "measurement-consistent" partition of the player set. Such a partition has members (coalitions) each containing only finite numbers of players. It also has the property that the proportional magnitudes of aggregates of finite coalitions in the partition are consistent with the proportional magnitudes given by the measure. The finite analogue of a measurement-consistent partition is a partition of the total player set into permissible small coalitions; the measurement-consistency requirement is not a constraint since it is automatically satisfied. In the continuum model, measurement-consistent partitions make a natural connection between the cardinal numbers and the nonatomic measure.

Feasible allocations for the entire economy must be defined consistently with the postulate that cooperation is allowed only in finite coalitions. To do this, we use the measurement-consistent partitions of the continuum of players into finite coalitions. A feasible allocation is defined so that trades are allowed only within coalitions in some measurement-consistent partition. Analogously, in a large finite economy, a feasible allocation must be attainable by cooperation within permissible small coalitions in some partition of the player set into such coalitions. Now we have a concrete picture of the structures to be connected, from the individual player level to the total economy level.

The solution concept used to capture the outcome of recontracting is the core. More precisely, in this paper we adopt the f-core for the continuum economy and the ε-core for the finite economies. An allocation is in the f-core if it is feasible for the continuum economy and if it cannot be improved upon by any finite coalition. An allocation is in the ε-core if it is feasible for the finite economy and if it cannot be significantly improved upon by any small coalition. Aggregate outcomes of recontracting are allocations in the f-core for the continuum economy and in the ε-cores for the finite economies.

We consider convergence not only of the economic environments and of solutions but also of game structures—permissible coalitions. The continuum economy with finite coalitions and the f-core are approximated by finite economies with small permissible coalitions and by approximate cores.

The convergence is obtained in the context of a private goods exchange economy with widespread externalities. By widespread externalities, we mean that the utilities of players depend on the average actions of other players. In this paper we model these externalities by making the preference of a player depend on his own consumption of commodities and also on the distribution of the entire allocation of commodities. The treatment of coalitions as finite allows a coalition to form and to change the allocation for its members without significantly affecting the distribution of the initially given allocation.

The motivation described in Subsection 1.1 might appear just philosophical in the sense that without externalities the equivalence of the core and the Walrasian allocations in a continuum model is obtained either by Aumann's [2] model (with either or both interpretations of the individual player) or by our model (Hammond, Kaneko and Wooders [4]). However, the situation is quite different in the presence of widespread externalities. Then the equivalence result holds for the f-core and the Walrasian allocations but fails for the standard model; more precisely, the definition of the core itself (with coalitions of positive measure) becomes problematic. See the discussion in Hammond, Kaneko, and Wooders [4, Subsection 2.3]. The substantive differences between our approach and the standard approach become especially apparent in consideration of the convergence from finite economies to the continuum.

The main theorem of this paper is the convergence of finite economies to the continuum economy and of approximate cores to the f-core. In addition, as a corollary to our convergence result and the equivalence result of Hammond, Kaneko, and Wooders [4], we have convergence of approximate cores of the finite economies to the Walrasian allocations of the continuum economy.

The reader will observe that our converging sequences are more restrictively constructed than those in the extant literature, for example Kannai [8]. The reason for the restrictiveness is that we require convergence of economic and game-theoretic structures, while in Kannai (and others) only the convergence of the economic environments is considered. The convergence of game-theoretical structures necessitates the restrictiveness of the converging sequences (see Remark 4.1).

Finally we point out that some modification of the definition of "can improve upon" becomes necessary in finite economies in the presence of widespread externalities. This modification is unnecessary in the continuum economy, even with widespread externalities, since the formation of a new coalition affects only a negligible set of players. In finite economies, even though coalitions are small, the formation of a new coalition will affect other players both directly and indirectly. When a coalition forms, some players in some coalitions in the initial partition will find that their coalitions have lost some members; these players are *directly affected* by the coalition formation. The other players are *indirectly affected*, via their utilities, by the change in the distribution of the allocation. We define "can improve upon" under the assumption that those players directly affected may arbitrarily form small coalitions among themselves, and under the Nash assumption that those players only indirectly affected do not change their behavior. In the convergence process the percentage of players in a coalition and those directly affected by the formation of that coalition is required to tend to zero; smallness of coalitions is essential for convergence to the f-core when we have widespread externalities.

The next two sections of this paper develop our framework. The fourth section contains the formulation of convergence and our results. Finally, all proofs are in the fifth section.

2. Commodities and Preferences

We begin by describing a commodity space permitting indivisible commodities. Then preferences, defined over commodities and distributions of allocations of commodities, are described.

We denote the *commodity space* by Ω and assume $\Omega = Z_+^I \times R_+^D$ where Z_+ is the set of nonnegative integers, I is a finite index set for indivisible commodities, and D is a finite index set for divisible commodities. Either I or D may be empty. We endow Ω with the metric $|x - y| = \max|x_c - y_c|$.

The preference relation of a player depends on his own consumption of commodities and also on the distribution of the allocation of commodities. Distributions of allocations are probability measures on commodity space Ω. Without any loss of generality, we actually consider the space of probability measures on $\Omega^* := \Omega \cup \{x_\infty\}$, the Alexandroff compactification of Ω. Note that, since Ω is a separable and locally compact metric space, the space Ω^* is also metrizable (see Hildenbrand [5, p. 15]).

Let $B(\Omega^*)$ be the σ-algebra of all Borel subsets of Ω^* and let M be the set of all probability measures on $B(\Omega^*)$. Note that the space M is a compact metrizable space with the topology of weak convergence (cf. Hildenbrand [5, p. 49, (30)]). Let ρ_M be a metric for the space M.

The space P of preferences is defined to be the set of all open, irreflexive, and transitive binary relations on $\Omega \times M$. That is, $P = \{\succ : \succ \text{ is an open, irreflexive, and transitive subset of } (\Omega \times M)^2\}$. The expression $([x^1, \nu^1], [x^2, \nu^2]) \in \succ$ means that the commodities represented by x^1 and the distribution represented by ν^1 are preferred to those represented by x^2 and ν^2. As is standard, we will simply write $[x^1, \nu^1] \succ [x^2, \nu^2]$.

Since Ω is a locally compact separable metric space and M is a compact metric space, the product space $\Omega \times M$ is also a locally compact separable metric space with the metric

$$d([x^1, \nu^1], [x^2, \nu^2]) = \max(|x^1 - x^2|, \rho_M(\nu^1, \nu^2)).$$

Therefore the space C of all closed subsets of $\Omega \times M$ endowed with the topology of closed convergence is a compact metrizable space (cf. Hildenbrand [5, p. 19, Theorem 2]).

Since $\succ$ is an open subset of $(\Omega \times M)^2$, the complement $\succ^c$ of $\succ$ is a closed set. It can be proved in the same way as in Hildenbrand [5, p. 96, Theorem 1] that $P^c = \{\succ^c : \succ \in P\}$ is a closed subset of C, which implies that P^c is a compact metrizable space.[2] Let ρ_P be a metric for P^c. Because of the natural bijection $\phi(\succ) = \succ^c$ between P and P^c, we can give the topology and metric of P^c to P. This means that a sequence of preferences $\{\succ^\nu\}$ converges to $\succ^0$ in this topology iff $\rho_P(\phi(\succ^\nu), \phi(\succ^0)) \to 0$ as $\nu \to \infty$.

[2] Since we fix the commodity space Ω we do not need convexity of Ω as in Hildenbrand [5].

3. Economies and Cores

In this section we describe, in a parallel manner, a finite economy with small coalitions and the ε-core, and then a continuum economy with finite coalitions and the f-core.

3.1. *Finite Economies with Small Coalitions and the ε-Core*

Let N be the *player set*, a finite set; β, an *attribute function*,[3] a mapping from N to $\Omega \times P$; and B, a *bound* on the size of *permissible coalitions, a* positive integer. We denote $\beta(a)$ by $(\omega^{\beta}(a), \succ_a^{\beta})$, or $(\omega(a), \succ_a)$ when the meaning is clear.

A *finite economy* is a triple (N, β, B), simply a specification of the player set N, a description given by β of the endowment and preference relation of each player, and a bound B on permissible coalition sizes (only coalitions containing no more than B members will be permitted to form).

We allow trades only within permissible coalitions containing, say, fewer than B players and define feasible allocations arising from cooperation consistently. More precisely, we define feasible allocations for the total player set by allowing trades only in coalitions in partitions of the player set into B-bounded coalitions. We call a partition of N with the property that each element in the partition contains no more than B members, a *B-partition of N*.

For comparison with the analogous concept for a continuum economy, we now provide another description of a B-partition p of N:

> for any positive integer $k \leqslant B$, let $N_k^p = \bigcup_{S \in p, |S| = k} S$. Then N_k^p has a partition $\{N_{kt}^p\}_{t=1}^k$ with the properties that there are bijections $\phi_{k1}^p, ..., \phi_{kk}^p$ from N_{k1}^p to $N_{k1}^p, ..., N_{kk}^p$, respectively, and that $\{\phi_{k1}^p(a), ..., \phi_{kk}^p(a)\} \in p$ for all $a \in N_{k1}^p$.

In a B-partition of N, the set of all members of k-member coalitions, N_k^p, is divided into k subsets, $N_{k1}^p, ..., N_{kk}^p$, and the members of each subset N_{kt}^p are labelled as the tth members of these coalitions. Every k-member coalition has one and only one member in N_{kt}^p for each t. The bijection ϕ_{kt}^p associates the tth member of a k-member coalition with the first member of that coalition. The partition $\{N_{kt}^p\}_{t=1}^k$ is called the *partition associated with p* and the mappings $\{\phi_{kt}^p\}$, the *bijection associated with p.*

Let p be a B-partition. Denote by $X(p)$ the set of all *feasible allocations relative to p*, i.e.,

$$X(p) = \left\{ x \in \Omega^N \colon \sum_{a \in S} x_a \leqslant \sum_{a \in S} \omega(a) \qquad \text{for all} \quad S \in p \right\}.$$

[3] This is sometimes called "an economy".

Then the set of *feasible allocations* X is the union of the feasible allocations $X(p)$ relative to p over all B-partitions p, i.e., $X=\bigcup_p X(p)$.

It will be convenient to define allocations which are feasible for a subset of players S. We say $(x_a)_{a\in S}$ is a *S-allocation* iff $x_a\in\Omega$ for all $a\in S$ and $\sum_{a\in S}x_a\leqslant\sum_{a\in S}\omega(a)$. (In the continuum case, S-allocations are defined only for finite subsets of players.)

For each allocation $x\in\Omega^N$, we define the *distribution* $D[x]$ induced from x by

$$D[x](T)=\frac{|\{a\in N\colon x_a\in T\}|}{|N|} \qquad \text{for all Borel subsets } T \text{ of } \Omega^*.$$

Then the expression $[x_a, D[x]]\succ_a[y_a, D[y]]$ means that a player a prefers the consumption x_a in the allocation x to consumption y_a in the allocation y.

Because new behavior of a coalition affects the feasibility of the allocations of some players, to define the ε-core we must specify the admissible actions of the remaining players when a new coalition forms. For those players who are only *indirectly affected* (by the change in the distribution) since the feasibility of their allocations is maintained, we can make the Nash assumption that these players make no changes in their actions. In contrast, since the allocations of players who are *directly affected* by the formation of a new coalition (find that the coalitions they were in have been broken up) may be infeasible, we cannot make the Nash assumption. Instead, we allow all possible feasible actions (reactions) within B-bounded coalitions of directly affected players.

Let p be a B-partition, and let S be a coalition with $|S|\leqslant B$. We call $\bar{p}$ a *(p, S)-partition* iff

$$\bar{p} \text{ is a } B\text{-partition} \qquad \text{and} \qquad S\in\bar{p};$$

and

$$R\in p \qquad \text{and} \qquad R\cap S=\varnothing \text{ imply } R\in\bar{p}.$$

The partition $\bar{p}$ consists of the coalition S, the coalitions R in p that do not intersect S, and arbitrary B-bounded coalitions of directly affected players. Figure 1 provides a simple illustration of two (p, S) partitions. In each case, the indirectly affected players are members of R and the directly affected players are the remaining members of $R\cup S$.

Let x be in $X(p)$ for a B-bounded partition p and let $(y_a)_{a\in S}$ be an S-allocation for a coalition S. We say that an allocation $\bar{x}$ is *compatible with* x, p, *and* $(y_a)_{a\in S}$ iff $\bar{x}$ is in $X(\bar{p})$ for some (p, S)-partition $\bar{p}$; for all $R\in\bar{p}\cap p$, $\bar{x}_a=x_a$ for all $a\in R$; and $\bar{x}_a=y_a$ for all $a\in S$.

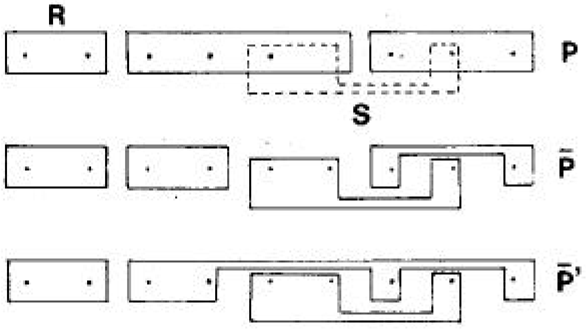

FIGURE 1 ($B=4$).

The allocation $\bar{x}$ agrees with x on the coalitions that do not intersect the coalition S, agrees with $(y_a)_{a\in S}$ on S, and is feasible for some partition of the remaining players. (For the example $\bar{p}$ of Fig. 1, $\bar{p}\cap p$ has the unique element R. Then $\bar{x}$ agrees with x on R, agrees with $(y_a)_{a\in S}$ on S, and is arbitrary on the remaining players.)

Let x be a feasible allocation, and let ε be a positive number. We say that a coalition S ($|S|\leqslant B$) can *ε-improve upon* x iff there is an S-allocation $(y_a)_{a\in S}$ such that for any B-partition p under which x is feasible and any allocation $\bar{x}$ compatible with x, p, and $(y_a)_{a\in S}$,

$$[\bar{x}_a, D[\bar{x}]]\succ_a U[x_a, D[x];\varepsilon] \qquad \text{for all} \quad a\in S,$$

where $U(x_a, D[x];\varepsilon)=\{[z,v]\in\Omega\times M\colon d([z,v],[x_a,D[x]])<\varepsilon\}$ is the ε-ball around $[x_a, D[x]]$ and "$[\bar{x}_a, D[\bar{x}]]\succ_a U(x_a, D[x];\varepsilon)$" means that $[\bar{x}_a, D[\bar{x}]]\succ_a [z,v]$ for all $[z,v]\in U(x_a, D[x];\varepsilon)$. Figure 2 and the following discussion illustrate the structure behind "ε-improve upon".

In Fig. 2, as in the definition of "ε-improve upon," the allocation x is given. Then, there may be many partitions, say p, p', p'', under which x is feasible. Each one of these partitions determines a set of possible (p, S)-partitions, say $\bar{p}$, $\bar{p}'$, $\bar{p}''$ for p, and each of these (p, S)-partitions determines

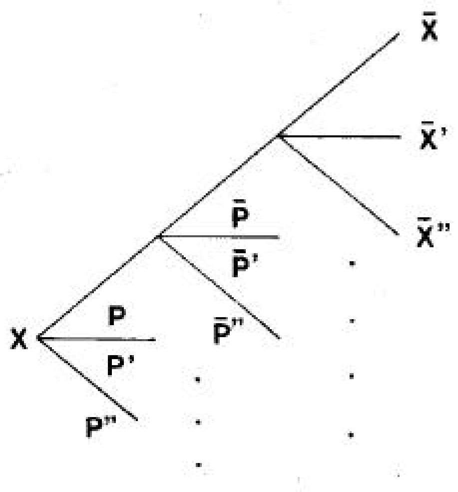

FIGURE 2.

a possible set of compatible allocations, say $\bar{x}$, $\bar{x}'$, $\bar{x}''$ for $\bar{p}$. The above definition of "ε-improve upon" ensures the improvement in the utility of each player in the coalition S for any combination of possible partitions p, (p, S)-partitions $\bar{p}$, and compatible allocations $\bar{x}$. Thus, our definition of "ε-improve upon" is a conservative one in that the improvement must be for all admissible alternative cases[4] (though we make the Nash assumption that the indirectly affected players keep the same allocations of commodities).

The *ε-core* of the finite market (N, β, B) is the set of all feasible allocations that cannot be ε-improved upon by any coalition S with $|S| \leqslant B$.

When there are no widespread externalities and $\varepsilon = 0$, our definition of the ε-core reduces to the usual notion of the core (except for our feasibility condition). When $\varepsilon > 0$ our definition suggests that a coalition will not attempt to improve upon an allocation unless it can be "significantly" (measured in commodities) better off, no matter what admissible actions the directly affected players undertake.

Observe that the ε-core depends on the choice of metric. However, we will consider an approximate core property (defined in Section 4) instead of the ε-core itself and this property is a topological, rather than a metric, notion. Therefore the dependence of the ε-core on the metric creates no substantive problems in our approach.

3.2. *Continuum Economies with Finite Coalitions and the f-Core*

Let $(A, \mathscr{A}, \mu)$ be a measure space, where A is a Borel subset of a complete separable metric space; $\mathscr{A}$, the σ-algebra of all Borel subsets of A; and μ, a nonatomic measure with $0 < \mu(A) < +\infty$. Each element in A is called a *player* and A is the *player set.* The measure μ represents the distribution of players. The σ-algebra is necessary for measurability arguments but does not play any important game-theoretic role.

An *attribute function* γ is a function from A to $\Omega \times P$ which is measurable in the sense that $\gamma^{-1}(T) \in \mathscr{A}$ for any Borel subset T of $\Omega \times P$. We write $\gamma(a) = (\omega^{\gamma}(a), \succ_a^{\gamma})$ or simply $(\omega(a), \succ_a)$ when the meaning is clear.

Let F be the set of all finite subsets of A. Each element S in F is called a *finite coalition* or simply a *coalition.* Let F_B be the subset of F consisting of finite coalitions with no more than B members. The family F_B is the set of *permissible coalitions.* When B is infinite $(\aleph_0)$, F_B is simply F.

[4] We think of a feasible allocation as actually a pair, x and p, with $x \in X(p)$, but we focus only on allocations. Therefore, for the idea of the model, it suffices to consider some given partition p where $x \in X(p)$. For our result, either definition may be used; their differences become irrelevant in large economies. For economy of mathematical definitions, we choose the current presention.

A *continuum economy* is a triple (A, γ, B), comprising a specification of the player set A; a description, given by γ, of the endowment and preference relation of each player; and a bound B on permissible coalition sizes. If $B = +\infty$, then the continuum economy is denoted by (A, γ, F).

Recall that only coalitions with no more than B members can form, and feasible allocations must be achieved by trades in coalitions in some partition of the set of players into B-bounded coalitions. These partitions are required to be "consistent" with the two notions of measurement described by the cardinal numbers for finite coalitions and by the measure μ for broad economic totals.

A partition q of A in F_B is *measurement-consistent* iff for any positive integer $k \leqslant B$

$$A_k^q = \bigcup_{\substack{S \in q \\ |S| = k}} S \text{ is a measurable subset of } A;$$

and

> each A_k^q has a partition $\{A_{kt}^q\}_{t=1}^k$, where each A_{kt}^q is measurable, with the following property: there are measure-preserving isomorphisms[5] $\psi_{k1}^q, \psi_{k2}^q, ..., \psi_{kk}^q$ from A_{k1}^q to $A_{k1}^q, ..., A_{kk}^q$, respectively, such that $\{\psi_{k1}^q(a), ..., \psi_{kk}^q(a)\} \in q$ for each $a \in A_{k1}^q$.

Let Π_B denote the set of measurement-consistent partitions. When B is infinite, we denote Π_B simply by Π. An element in Π_B will be called a *B-partition*.

Almost as in the preceeding subsection, for each positive integer k we define A_k^q as the set of members of k-member coalitions. For the partition q to be measurement-consistent, we must be able to partition A_k^q into k subsets $\{A_{kt}^q\}_{t=1}^k$ so that the tth subset A_{kt}^q can be viewed as the set of tth members of k-member coalitions. The isomorphisms associate the first members of k-member coalitions with the tth members for $t = 1, ..., k$. The measurement-consistency requirement inherent in the measure-preserving property of the isomorphisms ensures that the set of first members has the same measure as the set of tth members for each t. Without loss of generality we assume throughout the paper that ψ_{k1}^q is the identity map.

Paralleling the definition of feasible allocations for a finite economy, we define the set of *feasible allocations* $F(q)$ for each $q \in \Pi_B$ by

$$F(q) = \left\{ f \in L(A, \Omega) \colon \sum_{a \in S} f(a) \leqslant \sum_{a \in S} \omega(a) \text{ for all } S \in q \right\},$$

[5] Recall that a function ψ from a set G in $\mathscr{A}$ to a set C in is a *measure-preserving isomorphism* from G to C iff (i) ψ is a measure-theoretic isomorphism, i.e., ψ is 1 to 1, onto, and measurable in both directions, and (ii) $\mu(T) = \mu(\psi(T))$ for all $T \subset A$ with $T \in \mathscr{A}$.

where $L(A, \Omega)$ is the set of measurable functions from A to Ω. An allocation in $F(q)$ will be called a *q-feasible* allocation. Let

$$F = \bigcup_{q \in \Pi_B} F(q)$$

and let F^* be the set of *feasible allocations* where

$$F^* = \{f \in L(A, \Omega)\text{: for some sequence } \{f^\nu\} \text{ in } F, \{f^\nu\} \text{ converges in measure to } f\}.$$

Recall that "$\{f^\nu\}$ converges in measure to F" means that for any $\varepsilon > 0$, there is a $\bar{\nu}$ such that for all $\nu \geqslant \bar{\nu}$, we have $\mu(\{a \in A: |f^\nu(a) - f(a)| > \varepsilon\}) < \varepsilon$.

In a finite economy the set of feasible allocations $X = \bigcup_p X(p)$ is already closed, but this is not necessarily true for the set of (exactly feasible allocations F for a continuum economy, as illustrated via an example in Hammond, Kaneko, and Wooders [4]. The set F^* of feasible allocations is an idealization of the exactly feasible set. A member of F^* is to be interpreted as approximately feasible for arbitrarily close approximation. It is approximately feasible in the sense that for some partition q^ν of A into permissible (finite) coalitions, there is an $f^\nu \in F(q^\nu)$ such that f is "close" to f^ν for most players.

Finally we note the rather surprising result that although our game structure is quite difference from that of Aumann [2], our set of feasible allocations F^* of the continuum economy (A, γ, F) coincides with the Aumann-feasible allocations, i.e.,

$$F^* = \left\{f \in L(A, \Omega)\text{: } \int f \leqslant \int \omega\right\}.$$

(Kaneko and Wooders [7, Lemma 3.1, p. 130]).[6]

For each function f in $L(A, \Omega)$, we define the distribution $D[f]$ induced from f by

$$D[f](T) = \frac{1}{\mu(A)} \mu(\{a \in A: f(a) \in T\})$$

for all Borel subsets T of Ω^*.

Let f be a function in $L(A, \Omega)$. We say that a coalition S with $|S| \leqslant B$ *can improve upon* f iff for some S-allocation $(x_a)_{a \in S}$, we have $[x_a, D[f]] \succ_a$

[6] This result is essential for the equivalence of the f-core and the Walrasian allocations but is not used in the main theorem of this paper. It is used indirectly in the proof of the corollary to the main theorem.

$[f(a), D[f]]$ for all $a \in S$. Now the *f-core* of the economy (A, γ, B) is defined to be the set

$$C = \{f \in F^*: \text{for some full subset } \tilde{A} \subset A, \text{ no coalition } S \text{ in } \tilde{A} \text{ with } |S| \leqslant B \text{ can improve upon } f\}.$$ [7]

The definition of "can improve upon" is much simpler for the continuum than "can ε-improve upon" for large finite economies. This is because a coalition S, since it is finite, does not change the distribution of the allocation when it changes its own part of the allocation. Therefore, in our definition of "can improve upon" we can take the distribution $D[f]$ as unchanged by the change from $(f(a))_{a \in S}$ to $(x_a)_{a \in S}$. As in the finite economy case there will be some players directly affected by the formation of the coalition S. However, these directly affected players are negligible in the continuum case. These aspects enable us to use the standard (game-theoretic) notion of the core in the context of a continuum of players and finite coalitions, even with widespread externalities.

4. Convergence of Structure and Solution from Finite to Continuum Economies

In this section we introduce the concepts required for the statement of our results, state our main theorem and a variation of the theorem for the case of no widespread externalities, and apply our main theorem to the Walrasian equilibrium.

4.1. *Approximation Sequences and the Approximate Core Property*

To define the concept of convergence from finite economies to a continuum, we first embed finite economies within "equivalent" continuum economies, called *nonatomic representations*. With this embedding, attributes and allocations for the finite economies are represented in the same spaces as those of the continuum economy. The nonatomic representation of a finite economy has a continuum of players and has the same percentage of players with each attribute as the finite economy. Our convergence is defined in terms of the convergence of nonatomic representations of finite economies.

We now fix a limit continuum economy (A^0, γ^0, F) for the remainder of the paper.

[7] We could adopt a definition of the f-core requiring that *no* finite coalition could improve upon an allocation in the f-core. However, this complicates the statement of our results without adding any substance to the meaning.

Let N be a finite set and let β be an attribute function on N, i.e., $\beta: N \to \Omega \times P$. A continuum economy (A, γ, B) (where $A \subset A^0$ and $\mu(A) > 0$) is called a *nonatomic representation* of the finite economy (N, β, B) iff there is an onto mapping $\xi: A \to N$ such that

$$\mu(\xi^{-1}(a)) = \frac{\mu(A)}{|N|} \qquad \text{for all} \quad a \in N; \tag{4.1}$$

$$\gamma(a') = \beta(a) \qquad \text{for all} \quad a' \in \xi^{-1}(a) \text{ and } a \in N. \tag{4.2}$$

We call ξ the *representation mapping*.

Let $\{(N^\nu, \beta^\nu, B^\nu)\}$ be a sequence of finite economies. We call a sequence $\{(A^\nu, \gamma^\nu, B^\nu)\}$ of continuum economies a *nonatomic representation* of $\{(N^\nu, \beta^\nu, B^\nu)\}$ if $(A^\nu, \gamma^\nu, B^\nu)$ is a nonatomic representation for each ν. This definition extends each of the finite economies $(N^\nu, \beta^\nu, B^\nu)$ to a continuum economy. Each player in N^ν is replaced by a continuum of players with the same attribute so that the proportion of players having a given attribute remains unchanged. (This is a similar concept to Kannai's [8] "continuous representation" but we allow the player set to be a proper subset of A^0 instead of A^0 itself. If a function on A^0 is unbounded, by using a subset of A^0 as the domain we can approximate the function uniformly by simple functions where, for each simple function, each step has the same size base).

Again let (N, β, B) be a finite economy and let (A, γ, B) be a nonatomic representation with the representation mapping ξ. Let α be a function from N to some metric space Y. We call $\bar{\alpha}: A \to Y$ the *nonatomic extension* of α iff $\bar{\alpha}(a') = \alpha(a)$ when $a' \in \xi^{-1}(a)$. Note that the nonatomic extension is uniquely determined by α and a nonatomic representation (A, γ, B), and that γ is the nonatomic extension of β itself. The nonatomic extension $\bar{\alpha}$ is a simple function and a substitute for α in the nonatomic representation (A, γ, B) of (N, β, B).

Let $\{(N^\nu, \beta^\nu, B^\nu)\}$ be a sequence of finite economies, let $\{\alpha^\nu\}$ be a sequence of functions from N^ν to some metric space Y with metric ρ_Y, and let $\{(A^\nu, \gamma^\nu, B^\nu)\}$ be a nonatomic representation of $\{(N^\nu, \beta^\nu, B^\nu)\}$. We say that $\{\alpha^\nu\}$ *expandingly converges* to a function $\alpha^0: A^0 \to Y$ with respect to $\{(A^\nu, \gamma^\nu, B^\nu)\}$ iff

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^\nu\right) = \mu(A^0); \tag{4.3}$$

the sequence $\{\bar{\alpha}^\nu\}$, where $\bar{\alpha}^\nu$ is the nonatomic extension of α^ν to $(A^\nu, \gamma^\nu, B^\nu)$, converges uniformly to α^0 on the domains, i.e., $\sup_{a \in A^\nu} \rho_Y(\bar{\alpha}^\nu(a), \alpha^0(a)) \to 0$ $(\nu \to \infty)$. (4.4)

Each function α^ν on N^ν is replaced by a function $\bar{\alpha}^\nu$ on A^ν. Then $\bar{\alpha}^\nu$ can be compared with the limit function α^0, though the domain A^ν of $\bar{\alpha}^\nu$ is still different from A^0. Therefore we require convergence both of the domains (4.3) and of the functions (4.4).

A sequence of finite economies $\{(N^\nu, \beta^\nu, B^\nu)\}$ is called an *approximation sequence* for (A^0, γ^0, F) iff there is a nonatomic representation $\{(A^\nu, \gamma^\nu, B^\nu)\}$ with the representation mappings $\{\xi^\nu\}$ such that

$$\beta^\nu(N^\nu)(=\gamma^\nu(A^\nu)) \subset \gamma^0(A^0) \qquad \text{for all } \nu; \tag{4.5}$$

$$B^\nu \to \infty \text{ as } \nu \to \infty; \tag{4.6}$$

$\{\beta^\nu\}$ expandingly converges to γ^0 with respect to $\{(A^\nu, \gamma^\nu, B^\nu)\}$ and with a higher order than B^ν on Ω, i.e., $B^\nu \times \sup_{a \in A^\nu} |\omega^{\gamma^\nu}(a) - \omega^{\gamma^0}(a)| \to 0$ $(\nu \to 0)$; (4.7)

and

$$\min_{e \in \beta^\nu(N^\nu)} |\{a \in N^\nu : \beta^\nu(a) = e\}| \to \infty \qquad (\nu \to \infty). \tag{4.8}$$

The sequence $\{(A^\nu, \gamma^\nu, B^\nu)\}$ will be called an *associated nonatomic representation of* $\{(N^\nu, \beta^\nu, B^\nu)\}$.

Condition (4.5) is that the attributes of the players in the finite economies are subsets of the attributes of the players of the continuum economy; the attributes of the players of the finite economies are converging to those of the continuum economy "from the inside". The next condition (4.6) simply allows coalitions to become arbitrarily large. However, since commodities are transferable, commodities may be accumulated by some small subset of players and nonconvergence of such accumulated commodities may result. We rule this out by assuming, in (4.7), that the rate of convergence of the endowments ω^{γ^ν} is faster than the rate of divergence of B^ν. If sizes of permissible coalitions are growing slowly, this condition would be satisfied. In the case of widespread externalities we will, in fact, require stronger conditions on the rate of divergence of B^ν. The next condition, (4.8), ensures that each player has "many" substitutes; we call this the *thickness assumption* for the approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$. This assumption does *not*, however, necessarily mean that every player has a positive measure of substitutes in the limit economy.

Let $\{(N^\nu, \beta^\nu, B^\nu)\}$ be a sequence of finite economies and let $\{x^\nu\}$ be a sequence of feasible allocations; i.e., x^ν is a feasible allocation for $(N^\nu, \beta^\nu, B^\nu)$. The sequence $\{x^\nu\}$ is said to have the *approximate core property* iff for any $\varepsilon > 0$,

there is a positive integer ν_0 such that for all $\nu \geqslant \nu_0$, x^ν is in the ε-core of $(N^\nu, \beta^\nu, B^\nu)$. (4.9)

The idea is that the sequence itself has a core or the "limit" of the sequence has a core, even though we do not have a limit and even though every economy in the sequence may have an empty core. As indicated earlier, the approximate core property is independent of the particular choice of metric on $\Omega \times M$.

4.2. *The Convergence Theorem*

We now give our theorem, designed to show convergence in structures and convergence in solutions.

The following three conditions are used in the main theorem: Let f be a function in $L(A^0, \Omega)$, let $\{(N^\nu, \beta^\nu, B^\nu)\}$ be an approximation sequence for (A^0, γ^0, F), and let $\{x^\nu\}$ be a sequence of allocations, where x^ν is feasible for $(N^\nu, \beta^\nu, B^\nu)$;

$$\frac{(B^\nu)^2}{|N^\nu|} \to 0 \qquad as \quad \nu \to \infty; \tag{4.10}$$

the sequence $\{x^\nu\}$ expandingly converges to f with respect to some associated nonatomic representation $\{(A^\nu, \gamma^\nu, B^\nu)\}$; (4.11)

the sequence $\{x^\nu\}$ has the approximate core property. (4.12)

MAIN THEOREM (Upper–Lower Hemicontinuity). *The following three statements on a function f in $L(A^0, \Omega)$ are equivalent*:

(1) *f is in the f-core of the continuum economy (A^0, γ^0, F)*;

(2) *There is an approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ and a sequence $\{x^\nu\}$ of allocations (where x^ν is feasible for $(N^\nu, \beta^\nu, B^\nu)$) satisfying conditions* (4.10), (4.11), *and* (4.12);

(3) *For an approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ and sequence $\{x^\nu\}$ of allocations (where x^ν is feasible for $(N^\nu, \beta^\nu, B^\nu)$) with conditions* (4.10) *and* (4.11), *condition* (4.12) *holds.*

The first statement of the theorem is clear. The second statement is that there exists an approximation sequence satisfying (4.10), (4.11), and (4.12). In the third statement, instead of the existence of an approximation sequence, it is claimed that any approximation sequence with (4.10) and (4.11) has the approximate core property. (Since we can construct an approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ satisfying (4.10) and (4.11), the third statement is not vacuous). In the "folk language," $(1) \Rightarrow (2)$ is a form of "lower hemicontinuity" and $(3) \Rightarrow (1)$ is a form of "upper hemicontinuity."[8] The second statement (2) states the existence of an approximation

[8] C. Kannai [8, p. 803]. For his framework, Kannai gave a counterexample to lower hemicontinuity. In our case, we avoid his counterexample by assuming $\beta(N^\nu) \subset \gamma^0(A^0)$ for all ν, and we consider the limit, from the "inside", to the total.

sequence; this existence is quite subtle because of the special properties required of the sequence.

The theorem provides an answer to the questions raised in the introduction. The idea of perfect competition with recontracting in our continuum model can be approximately realized in large finite economies with small permissible coalitions. The two special aspects of perfect competition with recontracting, which make the different treatment of small coalitions and total economies necessary, are approximately captured by the restriction of permissible coalitions (4.10). Thus the theorem provides a finitistic interpretation of the continuum model with finite coalitions.

One of the special properties required of an approximation sequence in statements (2) and (3) of the theorem is condition (4.10) ensuring that relative to N^{ν} the sizes of coalitions become small. With widespread externalities, for the effects of coalition formation to become negligible we need the percentage of directly affected players to become negligible, i.e., we need $(B^{\nu})^2/|N^{\nu}| \to 0$. To see this, let p^{ν} be a B^{ν}-partition and let S be a subset of players, S not in p^{ν}, with $|S| \leqslant B^{\nu}$. If S forms a coalition, this breaks some coalitions in p^{ν}. The number of players influenced directly by the formation of S is at most $|S| \times B^{\nu}$, i.e.,

$$|S| \times B^{\nu} \geqslant \left| \bigcup_{\substack{T \in p^{\nu} \\ T \cap S \neq \varnothing}} T \right|;$$

this leads us to require condition (4.10).

When preferences do not depend on the distributions, i.e., there are no widespread externalities (this is the standard case), we do not need to impose (4.10) in the theorem. In other words, we can delete the references to (4.10) in statements (2) and (3) of the theorem. In fact, in the case of no widespread externalities, this feature that (4.10) can, but does not need to, be imposed is the reason behind the equivalence, shown in Hammond, Kaneko, and Wooders [4], of the f-core and the Aumann core. This can be considered as the reason that the opposing Schmeidler [10]–Vind [11] requirements (Schmeidler requires "small" coalitions of positive measure less than ε, whereas Vind can require "large" coalitions) both lead to Aumann-core–equilibrium equivalence. Nevertheless, when we consider the convergence to the f-core it is natural to restrict coalition sizes, at least to ensure $B^{\nu}/|N^{\nu}| \to 0$.

Remark 4.1. In the third part of the Theorem, an approximation sequence of finite economies depends upon the particular f-core allocation f under consideration. Since we do not assume convexity the f-core allocation f is not necessarily continuous with respect to attributes and does not necessarily satisfy even the equal-treatment property (the property that players with the same attributes receive the same allocation). Thus under our assumptions the f-core allocations have more "variety" than the attribute functions and we cannot define an approximation sequence based only on attribute functions.

4.3. *An Application to Walrasian Equilibrium*

The combination of our main theorem and the equivalence theorem in Hammond, Kaneko, and Wooders [4] gives us convergence of the core to the Walrasian equilibrium allocations of the continuum economy. We note that these allocations might not be Pareto-optimal.

In Hammond, Kaneko, and Wooders [4], under the following conditions the equivalence of the f-core and the Walrasian equilibrium allocations is obtained.

(A.0) There is at least one divisible commodity, i.e., the index set D of divisible commodities is nonempty;

(A.1) $\succ_a$ is strictly monotone on R^D_+;

(A.2) for all $[(x_I, x_D), v] \in \Omega \times M$ there is a $y_D \in R^D_+$ such that $[(0_I, y_D), v] \succ_a [(x_I, x_D), v]$;

(A.3) $[\omega(a), v] \succ_a [(x_I, 0_D), v]$ for all $x_I \in Z^I_+$ and for all probability measures v.

THEOREM (Hammond, Kaneko, and Wooders [4]). *Under assumptions* (A.0) *to* (A.3) *it holds that f is in the f-core of (A^0, γ^0, F) if and only if f is a Walrasian allocation, i.e., for some price vector $p \in R^M_+$,*

$$p \cdot f(a) \leqslant p \cdot \omega(a) \qquad \text{a.e. in } A^0;$$

$$\text{a.e. in } A^0,\ [x, D[f]] \not\succ_a [f(a), D[f]] \text{ for all } x \in \Omega \text{ with } p \cdot x \leqslant p \cdot \omega(a);$$

and

$$\int_{A^0} f \leqslant \int_{A^0} \omega.$$

COROLLARY. *Assume* (A.0) *to* (A.3) *hold. The three statements of the theorem on a function f in $L(A^0, \Omega)$ are equivalent to:*

(1′) *f is a Walrasian allocation for the continuum economy (A^0, γ^0, F).*

There is a vast literature on convergence of cores to equilibrium allocations in the case of no widespread externalities (see Anderson [1] for a recent survey). In many of these papers, the convergence of the core allocations to Walrasian allocations or Walrasian-like allocations is considered in the context of large, finite economies—convergence to the continuum is not investigated. Nevertheless, this is not the main difference between our convergence results and existing works. The main difference is that we require not only convergence of cores but also convergence of the economic structures (moreover, with widespread externalities).

Although we considered convergence of ε-core allocations to Walrasian allocations as a corollary to our main theorem, our primary concern was the interpretation of the continuum model with finite coalitions as the limit of finite economies. Kannai's [8] paper is related to this aspect of our paper. He considered the convergence from core to core and also from economies to economies, and obtained several fundamental results. However, Kannai's paper is devoted to the interpretation of Aumann's model of a continuum with coalitions of positive measure, and convergence of game theoretic structures is not required. With this qualification our approach and Kannai's approach are counterparts to each other for the continuum models with finite coalitions and with coalitions of positive measure.

For large economies with no widespread externalities, Mas-Colell [9] obtained an important result on the convergence of the core to the Walrasian allocations. This result gave an evaluation of a bound on coalition sizes which still permitted a convergence result. His evaluation is sharper than ours but his paper is different in that feasibility does not require trade only within small coalitions. An interesting open problem is what the Mas-Colell result becomes when the same coalition size bound is used in defining feasible allocations as in "can improve upon." This is important in interpretation of the meaning of the result.

Remark 4.2. In Hammond, Kaneko, and Wooders [4] we prove nonemptiness of the f-core under the assumptions (A.0), (A.1), (A.2), and (A.3) for the case where the widespread externalities depend on the integrals of allocations (rather than their distributions). Of course the case of no widespread externalities can also be covered. From this existence result and the above corollary we have the existence of the approximate core for any approximation sequence for f in the f-core.

5. Proofs

In this section, we prove only the equivalence between statements (1) and (2) of the main theorem. When we have existence of an approximation sequence satisfying (4.10) and (4.11), and a sequence $\{x^\nu\}$ of allocations with the required properties, then $(3) \Rightarrow (2)$ is straightforward. The existence of such a sequence can be proved in the same way as the first part of the proof of (1) ⇒ (2). The logical relation (1) ⇒ (3) is almost immediate and thus omitted.

5.1. *Preliminaries*

Before we proceed to the main body of the proofs, we prepare several auxiliary notions and some lemmas. First, we define a typed continuum economy with the rational number property. The rational number property means that the proportional aspects of the economy can be described by rational numbers. This implies that we can find a finite economy with the same proportional aspects.

A continuum economy (A, γ, B) is said to be *typed* iff $\gamma(A)$ is a finite set. The typed continuum economy (A, γ, B) is said to have the *rational number property* iff

$$\frac{\mu(\{a \in A: \gamma(a) = \alpha\})}{\mu(A)} \text{ is a positive rational number for all } \alpha \in \gamma(A). \tag{5.1}$$

Let $g: A \to \Omega$ be a simple function such that $\zeta := (\gamma, g): A \to (\Omega \times P) \times \Omega$ has the rational number property ($\mu(\{a \in A: \zeta(a) = \lambda\})/\mu(A)$ is a positive rational number for any $\lambda \in \zeta(A)$). We denote $(\Omega \times P) \times \Omega$ by Λ.

Let q be a B-partition of A with the associated partition $\{A^q_{kt}: t = 1, ..., k$ and $k = 1, ..., B\}$ and isomorphisms $\{\psi^q_{ht}: t = 1, ..., k$ and $k = 1, ..., B\}$. We say that q has the *rational number property with respect to* ζ iff for all $k = 1, ..., B$ and $\lambda = (\lambda_1, ..., \lambda_k) \in \Lambda^k$,

$$\text{if } (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_k(a)) = \lambda \text{ for some } a \in A^q_{k1},$$

$$\text{then } \frac{\mu(\{a \in A^q_{k1}: (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_{kk}(a)) = \lambda\})}{\mu(A)} \tag{5.2}$$

is a positive rational number.

Note that if some B-partition q of A has the rational number property, then the economy (A, γ, B) also has the rational number property.

Let (N, β, B) be a finite economy, and let (A, γ, B) be a nonatomic representation of (N, β, B) with the representation mapping ξ. Let p be a B-bounded partition of N. We call a B-bounded partition q of A a *nonatomic representation* of p iff

$$\xi(S) \in p \quad \text{and} \quad |\xi(S)| = |S| \quad \text{for all} \quad S \in q. \tag{5.3}$$

Note that since ξ is an onto mapping, (5.3) implies $\xi(q) = p$. A nonatomic representation q is a partition of A and the structure of q mimics that of

p. The partition q, however, contains a continuum of coalitions with the same profile as S for each $S \in p$ since a continuum of players $\xi^{-1}(a)$ corresponds to a for each $a \in N$.

A B-partition of A with the rational number property with respect to (γ, g) can be represented as a B-partition of N in some finite economy (N, β, B). That is,

LEMMA 1. *Let (A, β, B) be a B-bounded typed continuum economy. Suppose that q is a B-bounded partition of A with the rational number property with respect to $\zeta = (\gamma, g)$. Then for any $\varepsilon > 0$ there are a finite economy (N, β, B) $(N \subset A)$ and a B-partition p of N such that*

(i) *(A, γ, B) is a nonatomic representation of (N, β, B) with a representation mapping ξ where $\xi(a) = a$ for all $a \in N$;*

(ii) *q is a nonatomic representation of p;*

(iii) *$(B)^2/|N| < \varepsilon$; and*

(iv) *the representation mapping ξ preserves the values of ζ, i.e., $(\gamma(a'), g(a')) = (\gamma(a), g(a))$ for all $a' \in \xi^{-1}(a)$ and $a \in N$.*

(In the finite market, the proportional aspects of a B-bounded partition q are characterized by a finite number of rational numbers. Since rational numbers are ratios of integers, we can multiply those rationals characterizing the partition by one integer and obtain integers describing the same proportional aspects of q. Then we construct a finite economy described by these integers. For completeness, we give a full proof.) We remark that since ξ is a representation mapping, the restriction of γ to N is β, so the part of (iv) concerning γ is redundant.

Proof. Given the B-partition q of A, let $\{A^q_{kt}: t = 1, ..., k$ and $k = 1, ..., B\}$ and $\{\psi^q_{kt}: t = 1, ..., k$ and $k = 1, ..., B\}$ be the associated partition of A and isomorphisms.

Denote the set $\{\lambda \in \Lambda^k: (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_{kk}(a)) = \lambda$ for some $a \in A^q_{k1}\}$ by $F^k \equiv \{\lambda^{k1}, ..., \lambda^{kl_k}\}$. Note that F^k may be empty.[9] By the rational number property of q, it holds that

$$\frac{\mu(\{a \in A^q_{k1} : (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_{kk}(a)) = \lambda^{ks}\})}{\mu(A)}$$

is a positive rational number for all $\lambda^{ks} \in F^k$

and $k = 1, ..., B$. (5.4)

[9] If $F^k = \varnothing$, then $l_k = 0$.

Therefore there are positive integers i_{ks} ($s=1, ..., l_k$ and $k=1, ..., B$)[10] such that

$$\frac{\mu(\{a \in A^q_{k1} : (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_{kk}(a)) = \lambda^{ks}\})}{\mu(A)} = \frac{i_{ks}}{\sum_{k=1}^{B} \sum_{s=1}^{l_k} k \times i_{ks}} \tag{5.5}$$

for all $s=1, ..., l_k$ and $k=1, ..., B$; and

$$\frac{(B)^2}{\sum_{k=1}^{B} \sum_{s=1}^{l_k} k \times i_{ks}} < \varepsilon. \tag{5.6}$$

Since $F^k \neq \phi$ for some k, the above ratios are well-defined. (This means that the rational numbers describing the proportional aspects of (A, γ, B) are replaced by natural numbers so that the proportional aspects are preserved.)

Now we define a finite economy (N, β, B). Denote $A^q_{k1}(\lambda^{ks}) = \{a \in A^q_{k1} : (\zeta \circ \psi^q_{k1}(a), ..., \zeta \circ \psi^q_{kk}(a)) = \lambda^{ks}\}$. Then we choose a subset N^p_{k1s} from $A^q_{k1}(\lambda^{ks})$ ($s=1, ..., l_k$ and $k=1, ..., B$) so that $|N^p_{k1s}| = i_{ks}$. Let $N^p_{kts} = \psi^q_{kt}(N^p_{k1s})$ for $t=1, ..., k$, $N^p_{kt} = \bigcup_{s=1}^{l_k} N^p_{kts}$, $N^p_k = \bigcup_{t=1}^{k} N^p_{kt}$, and $N = \bigcup_{k=1}^{B} N^p_k$. Then we have, by (5.6),

$$\frac{(B)^2}{|N|} = \frac{(B)^2}{\sum_{k=1}^{B} \sum_{s=1}^{l_k} \sum_{t=1}^{k} |N_{kts}|} = \frac{(B)^2}{\sum_{k=1}^{B} \sum_{s=1}^{l_k} k \times i_{ks}} < \varepsilon,$$

which is condition (iii).

Let β be the restriction of γ to N. We have attached to each $\psi^q_{kt}(A^q_{k1}(\lambda^{ks}))$ the finite set N^p_{kts} so that the proportions of these sets to A and to N, respectively, are the same, i.e., $\mu(\psi^q_{kt}(A^q_{k1}(\lambda^{ks})))/\mu(A) = |N^p_{kts}|/|N|$. Since N^p_{k1s} is a subset of $A^q_{k1}(\lambda^{ks})$ $(= \psi^q_{k1}(A^q_{k1}(\lambda^{ks}))$, we can find an onto mapping $\xi: A^q_{k1}(\lambda^{ks}) \to N^p_{k1s}$ such that

$$\xi(a) = a \qquad \text{for all} \quad a \in N^p_{k1s};$$

and

$$\mu(\xi^{-1}(a)) = \frac{1}{|N^p_{kt1}|} \mu(A^q_{k1}(\lambda^{ks})) = \frac{\mu(A)}{|N|} \qquad \text{for all} \quad a \in N^p_{k1s}. \tag{5.7}$$

Then we define $\xi: \psi^q_{kt}(A^q_{k1}(\lambda^{ks})) \to N^p_{kts}$ by

$$\xi(a) = \psi^q_{kt} \circ \xi \circ \psi^{q^{-1}}_{kt}(a) \qquad \text{for all} \quad a \in \psi^q_{kt}(A^q_{k1}(\lambda^{ks})). \tag{5.8}$$

Since ψ^q_{kt} is a measure-preserving isomorphism, it follows from (5.7) that for each $a \in N^q_{kts}$, $\mu(\xi^{-1}(a)) = \mu(\psi^q_{kt}(\xi^{-1}(\psi^{q^{-1}}_{kt}(a))) = \mu(\xi^{-1}(\psi^{q^{-1}}_{kt}(a))) =$

[10] If $l_k = 0$, then there is no i_{ks} and $\sum_{s=1}^{0} k \times i_{ks} = 0$.

$\mu(A)/|N|$. Thus we have shown that (A, γ, B) is a nonatomic representation of (N, β, B) with the representation mapping ξ.

From the definition of the representation mapping ξ, it is clear that ξ preserves the values of (γ, g).

Since $N^p_{kt} = \psi^q_{kt}(N^p_{k1})$ for all $t = 1, ..., k$ and $s = 1, ..., l_k$, the restrictions of $\psi^q_{k1}, ..., \psi^q_{kk}$ are bijections from N^p_{k1} to $N^p_{k1}, ..., N^p_{kk}$. Denote these restrictions by $\phi^p_{k1}, ..., \phi^p_{kk}$. Then define a partition p as follows:

$$p = \{\{\phi^p_{k1}(a), ..., \phi^p_{kk}(a)\} : a \in N^p_{k1} \text{ and } k = 1, ..., B\}.$$

Then it suffices to prove that q is a nonatomic representation of p with the representation mapping ξ. Pick an arbitrary coalition S from q with $|S| = k$. Then S can be represented as $S = \{\psi^q_{k1}(a), ..., \psi^q_{kk}(a)\}$ for some $a \in A^q_{k1}(\lambda^{ks})$. It follows from (5.8) that

$$\begin{aligned}\xi(S) &= \{\xi \circ \psi^q_{k1}(a), \xi \circ \psi^q_{k2}(a), ..., \xi \circ \psi^q_{kk}(a)\} \\ &= \{\xi(a), \psi^q_{k2} \circ \xi(a), ..., \psi^q_{kk} \circ \xi(a)\} \\ &= \{\xi(a), \phi^p_{k2} \circ \xi(a), ..., \phi^p_{kk} \circ \xi(a)\} \in p. \quad \blacksquare\end{aligned}$$

Since we require an exact form in (5.3) for the definition of a nonatomic representation of a B-partition of N in a finite economy (N, β, B), a nonatomic representation (A, γ, B) of (N, β, B) does not necessarily have a nonatomic representation of a B-partition. However, the following lemma holds.

LEMMA 2. *Let (N, β, B) be a finite economy and let $x = (x_a)_{a \in N}$ be a p-feasible allocation in (N, β, B), where p is a B-partition of N. Let $(\bar{A}, \bar{\gamma}, B)$ be a nonatomic representation of (N, β, B) with the representation mapping ξ, and let $\bar{g}$ be the nonatomic extension of x to $\bar{A}$. Then there is another nonatomic representation (A, γ, B) of (N, β, B) such that*

(i) *A is a full subset of $\bar{A}$ and γ is the restriction of $\bar{\gamma}$ to A;*

(ii) *(A, γ, B) has a nonatomic representation q of p;*

(iii) *the restriction g of $\bar{g}$ to A is q-feasible.*

(The reason $(\bar{A}, \bar{\gamma}, B)$ is replaced by (A, γ, B) is the following: A measurement-consistent partition is constructed by measure-preserving isomorphisms. However, the existence of a measure-preserving isomorphism between two Borel sets with the same measure can be ensured only up to a null set. See Kaneko and Wooders [7, p. 129]. Therefore we have to omit null sets appropriately.)

Proof. Since $(\bar{A}, \bar{\gamma}, B)$ is a nonatomic representation of (N, β, B) with the representation mapping ξ, for any $S := \{a_1, a_2, ..., a_k\} \in p$ (k is a positive integer), we have $\mu(\bar{A}_{a_1}) = \cdots = \mu(\bar{A}_{a_k})$, where $\bar{A}_{a_t} = \xi^{-1}(a_t)$ for $t = 1, ..., k$. Therefore from Lemma A.1 of Kaneko and Wooders [7], there are measure-preserving isomorphisms $\psi^S_{k1}, ..., \psi^S_{kk}$ (mod 0) from $\bar{A}_{a_1}$ to $\bar{A}_{a_1}, ..., \bar{A}_{a_k}$ respectively, that is, there are full subsets $\bar{A}^1_{a_1}, \bar{A}^2_{a_1}, ..., \bar{A}^k_{a_1}$ of $\bar{A}_{a_1}$ such that $\psi^S_{k1}, ..., \psi^S_{kk}$ are measure-preserving isomorphisms from $\bar{A}^1_{a_1}, \bar{A}^2_{a_1}, ..., \bar{A}^k_{a_1}$ to $\psi^S_{k1}(\bar{A}^1_{a_1}), \psi^S_{k2}(\bar{A}^2_{a_1}), ..., \psi^S_{kk}(\bar{A}^k_{a_1})$, respectively. Let $A_{a_1} = \bigcap_{t=1}^k \bar{A}^t_{a_1}$. Of course, $\mu(A_{a_1}) = \mu(\bar{A}_{a_1})$. Here we can assume without loss of generality that $a_t \in \psi^S_{kt}(A_{a_1})$ for all $t = 1, ..., k$.[11] Define

$$q(S) = \{\{\psi^S_{k1}(a), ..., \psi^S_{kk}(a)\} : a \in A_{a_1}\} \qquad \text{and} \qquad q = \bigcup_{S \in p} q(S).$$

Let $A = \bigcup_{T \in q} T$ and let γ be the restriction of $\bar{\gamma}$ to A. It is clear that A is a full subset of $\bar{A}$, and that (A, γ, B) is a nonatomic representation of (N, β, B) with the representation mapping ξ. Then q is also a nonatomic representation of p, since it holds that for any $\{\psi^S_{k1}(a), ..., \psi^S_{kk}(a)\} \in q(S)$ and $S \in p$,

$$\begin{aligned}\xi(\{\psi^S_{k1}(a), ..., \psi^S_{kk}(a)\}) &= \{\xi \circ \psi^S_{k1}(a), ..., \xi \circ \psi^S_{kk}(a)\} \\ &= \{a_1, ..., a_k\} = S.\end{aligned}$$

Finally, we prove that the restriction g of $\bar{g}$ to A is q-feasible. Since g is also a nonatomic extension of $(x_a)_{a \in N}$, and since ξ is the representation mapping from A to N, we have, for all $T \in q$,

$$\sum_{a \in T} g(a) = \sum_{a \in \xi(T)} x_a \leqslant \sum_{a \in \xi(T)} \omega^\beta(a) = \sum_{a \in T} \omega^\gamma(a). \quad \blacksquare$$

The next step is to characterize a feasible allocation $f \in F^*$ in (A^0, γ^0, F) in terms of a sequence of feasible allocations in typed continuum economies with the rational number property. After this step, using the previous lemmas, we will construct an approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ for (A^0, γ^0, F).

A sequence of typed continuum economies $\{(A^\nu, \gamma^\nu, B^\nu)\}$ is called a *nonatomic approximation sequence* for (A^0, γ^0, F) if

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^\nu\right) = \mu(A^0); \tag{5.9}$$

$$\gamma^\nu(A^\nu) \subset \gamma^0(A^0) \qquad \text{for all } \nu; \tag{5.10}$$

$$B^\nu \to \infty \qquad \text{as} \quad \nu \to \infty; \tag{5.11}$$

[11] Indeed, if $a_t \notin \psi^S_{kt}(A_{a_1})$, then we can replace an arbitrary $a \in \psi^S_{kt}(A_{a_1})$ by a_t. The new function plays the same role as the function ψ^S_{kt}.

$\{\gamma^\nu\}$ converges uniformly to γ^0 on the domains and with a higher order than B^ν on Ω, i.e., $\sup_{a \in A^\nu} |\omega^{\gamma^\nu}(a) - \omega^{\gamma^0}(a)| \times B^\nu \to 0$ as $\nu \to \infty$. (5.12)

Remark 5.1. Of course these conditions correspond to (4.3), (4.5), (4.6), and (4.7) in the definition of an associated nonatomic representation of a sequence of finite economies. The existence of an approximation sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ implies the existence of a nonatomic approximation sequence $\{(A^\nu, \gamma^\nu, B^\nu)\}$.

LEMMA 3. *A function $f \in L(A^0, \Omega)$ is feasible, i.e., $f \in F^*$, for the continuum economy (A^0, γ^0, F) if and only if there are a nonatomic approximation sequence $\{(A^\nu, \gamma^\nu, B^\nu)\}$, a sequence $\{q^\nu\}$ of B^ν-partitions of A^ν, and a sequence $\{g^\nu\}$ of q^ν-feasible allocations such that*

(i) *each g^ν is a simple function on A^ν and each q^ν has the rational number property with respect to (γ^ν, g^ν);*

(ii) *the sequence $\{g^\nu\}$ converges uniformly to f on the domains.*

Proof of Sufficiency. Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\lim_n \varepsilon_n = 0$ and $\varepsilon_n < 1$ for all n. Since the sequence $\{\omega^{\gamma^\nu}\}$ converges uniformly with a higher order than B^ν by (5.12), there is a ν_n for each n such that

$$|\omega^{\gamma^{\nu_n}}(a) - \omega^{\gamma^0}(a)| \times B^{\nu_n} < \varepsilon_n \qquad \text{for all } a \in A^{\nu_n}. \tag{5.13}$$

Define $\bar{g}^{\nu_n}: A^0 \to \Omega$ by

$$\text{for all} \quad a \in A^{\nu_n}, \qquad \bar{g}_c^{\nu_n}(a) = \begin{cases} g_c^{\nu_n}(a) & \text{if} \quad c \in I \\ \max(g_c^{\nu_n}(a) - \varepsilon_n, 0) & \text{if} \quad c \in D; \end{cases} \tag{5.14}$$

$$\text{for all} \quad a \in A^0 - A^{\nu_n}, \qquad \bar{g}^{\nu_n}(a) = 0$$

(recall that I and D denote the index sets for the indivisible and divisible commodities respectively); and define $\bar{q}^{\nu_n}$ by

$$\bar{q}^{\nu_n} = q^{\nu_n} \cup \{\{a\} : a \in A^0 - A^{\nu_n}\}.$$

Since $\{g^\nu\}$ converges uniformly to f, it follows from (5.14) that $\{\bar{g}^\nu\}$ converges in measure to f. Therefore it suffices to prove that each $\bar{g}^{\nu_n}$ is $\bar{q}^{\nu_n}$-feasible. For a coalition $\{a\} \in \bar{q}^{\nu_n} - q^{\nu_n}$, it holds that $\bar{g}^{\nu_n}(a) = 0 \leqslant \omega^{\gamma^0}(a)$. Let S be an arbitrary coalition in q^{ν_n}. For an indivisible commodity $c \in I$, we have

$$\sum_{a \in S} \bar{g}_c^{\nu_n}(a) = \sum_{a \in S} g_c^{\nu_n}(a) \leqslant \sum_{a \in S} \omega_c^{\gamma^{\nu_n}}(a) = \sum_{a \in S} \omega_c^{\gamma^0}(a),$$

since $\varepsilon_n < 1$ for all n, so $\omega_c^{\gamma^0}(a) = \omega_{c_v}^{\gamma^{v_n}}(a)$ for all $a \in A^{v_n}$. Let c be a divisible commodity in D. If $\bar{g}_c^{v_n}(a) = 0$ for all $a \in S$,

$$\sum_{a \in S} \bar{g}_c^{v_n}(a) = 0 \leqslant \sum_{a \in S} \omega_c^{\gamma^0}(a).$$

If $\bar{g}_c^{v_n}(a) > 0$ for some $a \in S$, then

$$\begin{aligned}\sum_{a \in S} \bar{g}_c^{v_n}(a) &\leqslant \sum_{a \in S} g_c^{v_n}(a) - \varepsilon_n \\ &\leqslant \sum_{a \in S} g_c^{v_n}(a) - B^{v_n} \times \sup_{a \in A^{v_n}} |\omega_c^{\gamma^v}(a) - \omega_c^{\gamma^0}(a)| \qquad \text{(by (5.13))} \\ &\leqslant \sum_{a \in S} (g_c^{v_n}(a) - |\omega_c^{\gamma^{v_n}}(a) - \omega_c^{\gamma^0}(a)|) \\ &\leqslant \sum_{a \in S} (\omega_c^{\gamma^{v_n}}(a) - |\omega_c^{\gamma^{v_n}}(a) - \omega_c^{\gamma^0}(a)|) \leqslant \sum_{a \in S} \omega_c^{\gamma^0}(a).\end{aligned}$$

Thus we have shown that $\sum_{a \in S} \bar{g}^{v_n}(a) \leqslant \sum_{a \in S} \omega^{\gamma^0}(a)$ for all $S \in \bar{q}^{v_n}$, which means that $\bar{g}^{v_n}$ is $\bar{q}^{v_n}$-feasible. ∎

Proof of Necessity. Since f is a feasible outcome in the continuum economy (A^0, γ^0, F), there are sequences $\{f^v\}$ and $\{\bar{q}^v\}$ such that

> $\bar{q}^v \in \Pi$, f^v is $\bar{q}^v$-feasible for all v, and $\{f^v\}$ converges in measure to f. (5.15)

We can assume without loss of generality that there is a sequence $\{B^v\}$ of integers such that for all v,

$$|S| \leqslant B^v \qquad \text{for all} \quad S \in \bar{q}^v.^{12} \tag{5.16}$$

Here we can assume that $B^v \to \infty$ as $v \to \infty$. Let $\{A_{kt}^v : t = 1, ..., k$ and $k = 1, ..., B^v\}$ and $\{\psi_{kt}^v : t = 1, ..., k$ and $k = 1, ..., B^v\}$ be the partition of A^0 and isomorphisms associated with the partition $\bar{q}^v$.

[12] If there is not such a sequence $\{B^v\}$, then we can find a sequence $\{B^v\}$ such that

$$\mu\Big(\bigcup_{\substack{S \in \bar{q}^v \\ |S| \leqslant B^v}} S\Big) > \mu(A^0) - 2^{-v} \qquad \text{for all } v.$$

Then $\bar{\bar{q}}^v = \{S \in \bar{q}^v : |S| \leqslant B^v\} \cup \{\{a\} : a \in S$ for some $S \in \bar{q}^v$ with $|S| > B^v\}$ and $\bar{\bar{f}}^v$, defined by

$$\bar{\bar{f}}^v(a) = \begin{cases} f^v(a) & \text{if } a \in S \text{ for some } S \in \bar{q}^v \text{ with } |S| \leqslant B^v \\ \omega^{\gamma^0}(a) & \text{otherwise,} \end{cases}$$

satisfy conditions (5.14), (5.15) and (5.16).

Let ν be an arbitrary positive integer. In the following, we will define a B^ν-bounded and typed continuum economy $(A^\nu, \gamma^\nu, B^\nu)$ and a feasible outcome g^ν of the economy $(A^\nu, \gamma^\nu, B^\nu)$. Put $\varepsilon = 1/2^\nu$.

We can assume by (5.15) that

$$\mu\left(\left\{a \in A^0 : |f^\nu(a) - f(a)| > \frac{\varepsilon}{4}\right\}\right) < \frac{\varepsilon}{4}.\text{[13]} \tag{5.17}$$

Since $\Lambda = (\Omega \times P) \times \Omega$ is σ-compact, we can choose a compact subset Λ^ν of Λ so that

$$\mu(\zeta^{\nu^{-1}}(\Lambda^\nu)) > \mu(A^0) - \frac{\varepsilon}{4}, \tag{5.18}$$

where $\zeta^\nu = (\gamma^0, f^\nu)$. Put $\tilde{A}^\nu = \{a \in \zeta^{\nu^{-1}}(\Lambda^\nu) : |f^\nu(a) - f(a)| \leqslant \varepsilon/4\}$. Then it follows from (5.17) and (5.18) that

$$\mu(\tilde{A}^\nu) > \mu(A^0) - \left[\mu(A^0 - \zeta^{\nu^{-1}}(\Lambda^\nu)) + \mu\left(\left\{a \in A^0 : |f^\nu(a) - f(a)| > \frac{\varepsilon}{4}\right\}\right)\right]$$

$$> \mu(A^0) - \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \mu(A^0) - \frac{\varepsilon}{2}. \tag{5.19}$$

Since Λ^ν is a compact metric space, there is a finite partition $\{\Lambda_1^\nu, \ldots, \Lambda_{l_\nu}^\nu\}$ of Λ^ν such that for $s = 1, \ldots, l_\nu$,

$$\text{the radius of } \Lambda_s^\nu \text{ is less than } \frac{1}{B^\nu \times 2^\nu}. \tag{5.20}$$

We are now going to define sequences $\{A^\nu\}$, $\{\gamma^\nu\}$, $\{q^\nu\}$, and $\{g^\nu\}$. Let $L_\nu := \{1, 2, \ldots, l_\nu\}$. For any k $(1 \leqslant k \leqslant B^\nu)$ and

$$i = (i_1, \ldots, i_k) \in (L_\nu)^k := \overbrace{L_\nu \times \cdots \times L_\nu}^{k},$$

choose a subset $C_{k1}^\nu(i)$ of $\tilde{A}_{k1}^\nu := A_{k1}^\nu \cap \tilde{A}^\nu$ so that

$$C_{k1}^\nu(i) \subset \{a \in \tilde{A}_{k1}^\nu : (\zeta^\nu \circ \psi_{k1}^\nu(a), \ldots, \zeta^\nu \circ \psi_{kk}^\nu(a)) \in \Lambda_{i_1}^\nu \times \cdots \times \Lambda_{i_k}^\nu\}; \tag{5.21}$$

$$\mu(C_{k1}^\nu(i)) \text{ is a positive rational number} \quad \text{or} \quad C_{k1}^\nu(i) = \phi; \tag{5.22}$$

$$\mu(C_{k1}^\nu(i)) \geqslant \mu(\{a \in \tilde{A}_{k1}^\nu : (\zeta^\nu \circ \psi_{k1}^\nu(a), \ldots, \zeta^\nu \circ \psi_{kk}^\nu(a)) \in \Lambda_{i_1}^\nu \times \cdots \times \Lambda_{i_k}^\nu\})$$

$$-\frac{1}{4} \cdot \frac{\varepsilon}{k \cdot B^\nu \cdot (l_\nu)^k}. \tag{5.23}$$

[13] Otherwise, we replace the sequence $\{f^\nu\}$ by an appropriate subsequence.

Take one player $a_{k1}^{\nu}(i)$ arbitrarily from each set $C_{k1}^{\nu}(i)$ as the representative of $C_{k1}^{\nu}(i)$. Define the νth player set A^{ν}, the νth function $\gamma^{\nu}\colon A^{\nu} \to \Omega \times P$, an allocation $g^{\nu}\colon A^{\nu} \to \Omega$, and a measurement-consistent partition q^{ν} of A^{ν} by

$$A^{\nu} = \bigcup_{k=1}^{B^{\nu}} \bigcup_{i \in L_{\nu}^{k}} \bigcup_{t=1}^{k} \psi_{kt}^{\nu}(C_{k1}^{\nu}(i)); \tag{5.24}$$

$$\gamma^{\nu}(a) = \gamma^{0}(\psi_{kt}^{\nu}(a_{k1}^{\nu}(i))) \qquad \text{for all} \quad a \in \psi_{kt}^{\nu}(C_{k1}^{\nu}(i)),$$
$$i \in L_{\nu}^{k},\ t = 1, ..., k, \text{ and } k = 1, ..., B^{\nu}; \tag{5.25}$$

$$g^{\nu}(a) = f^{\nu}(\psi_{kt}^{\nu}(a_{k1}^{\nu}(i))) \qquad \text{for all} \quad a \in \psi_{kt}^{\nu}(C_{k1}^{\nu}(i)),$$
$$i \in L_{\nu}^{k},\ t = 1, ..., k, \text{ and } k = 1, ..., B^{\nu}; \tag{5.26}$$

$$q^{\nu} = \bigcup_{k=1}^{B^{\nu}} \bigcup_{i \in L_{\nu}^{k}} \{\{\psi_{k1}^{\nu}(a), ..., \psi_{kk}^{\nu}(a)\} : a \in C_{k1}^{\nu}(i)\}. \tag{5.27}$$

The first claim follows from these definitions and (5.22).

Claim 1. The partition q^{ν} is measurement-consistent and has the rational number property with respect to (γ^{ν}, g^{ν}).

Claim 2. $\mu(\bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^{\nu}) = \mu(A^{0})$.

Proof. It follows from (5.19), (5.23), and $\varepsilon = 1/2^{\nu}$ that $\mu(A^{\nu}) > \mu(A^{0}) - 1/2^{\nu}$ for all ν. Since

$$\mu\left(\bigcap_{\nu=k}^{\infty} A^{\nu}\right) \geqslant \mu(A^{0}) - \sum_{\nu=k}^{\infty} \mu(A^{0} - A^{\nu}) > \mu(A^{0}) - \frac{1}{2^{k}}$$

for all k, we have

$$\mu\left(\bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^{\nu}\right) = \lim_{k} \mu\left(\bigcap_{\nu=k}^{\infty} A^{\nu}\right) \geqslant \lim_{k} \left(\mu(A^{0}) - \frac{1}{2^{k}}\right) = \mu(A^{0}). \quad \blacksquare$$

Claim 3. The sequence $\{\gamma^{\nu}\}$ converges uniformly to γ^{0} with a higher order than B^{ν} on Ω.

Proof. It follows from (5.25) and (5.20) that $\{\gamma^{\nu}\}$ converges uniformly to γ^{0}. It also follows from those conditions that

$$B^{\nu} \times \sup_{a \in A^{\nu}} |\omega^{\gamma^{\nu}}(a) - \omega^{\gamma^{0}}(a)| \leqslant \frac{1}{2^{\nu}} \to 0 \qquad \text{as} \quad \nu \to \infty. \quad \blacksquare$$

Claim 4. The sequence $\{g^{\nu}\}$ converges uniformly to f.

Proof. It follows from the definition of $\tilde{A}^\nu$, (5.26), and (5.20) that for all $a \in \psi^\nu_{kt}(C^\nu_{k1}(i))$,

$$\begin{aligned}|g^\nu(a) - f(a)| &\leqslant |g^\nu(a) - f^\nu(a)| + |f^\nu(a) - f(a)| \\ &\leqslant |f^\nu(\psi^\nu_{kt}(a^\nu_{k1}(i))) - f^\nu(a)| + |f^\nu(a) - f(a)| \\ &\leqslant \frac{1}{\beta^\nu \cdot 2^\nu} + \frac{1}{2^\nu}.\end{aligned}$$

This means that $\{g^\nu\}$ converges uniformly to f. ▮

Finally we can prove

Claim 5. Each g^ν is q^ν-feasible.

Proof. Consider any coalition S in q^ν. It follows from (5.27) that S can be represented as $S = \{\psi^\nu_{k1}(a), \psi^\nu_{k2}(a), ..., \psi^\nu_{kk}(a)\}$ and $a \in C^\nu_{k1}(i)$. From (5.25) and (5.26), we have

$$\begin{aligned}\omega^{\gamma^\nu}(\psi^\nu_{kt}(a)) &= \omega^{\gamma^0}(\psi^\nu_{kt}(a^\nu_{k1}(i))) \qquad \text{for} \quad t = 1, ..., k; \\ g^\nu(\psi^\nu_{kt}(a)) &= f^\nu(\psi^\nu_{kt}(a^\nu_{k1}(i))) \qquad \text{for} \quad t = 1, ..., k.\end{aligned} \tag{5.28}$$

Since f^ν is $\bar{q}^\nu$-feasible by (5.15),

$$\sum_{t=1}^{k} f^\nu(\psi^\nu_{kt}(a^\nu_{k1}(i))) \leqslant \sum_{t=1}^{k} \omega^{\gamma^0}(\psi^\nu_{kt}(a^\nu_{kt}(i))). \tag{5.29}$$

It follows from (5.28) and (5.29) that

$$\begin{aligned}\sum_{t=1}^{k} g^\nu(\psi^\nu_{kt}(a)) = \sum_{t=1}^{k} f^\nu(\psi^\nu_{kt}(a^\nu_{k1}(i))) &\leqslant \sum_{t=1}^{k} \omega^{\gamma^0}(\psi^\nu_{kt}(a^\nu_{k1}(i))) \\ &= \sum_{t=1}^{k} \omega^{\gamma^\nu}(\psi^\nu_{kt}(a)). \quad ▮\end{aligned}$$

5.2. *Proof of* (1) ⇔ (2)

In this subsection, we prove the equivalence of (1) and (2) of the main theorem.

(1) ⇒ (2). We show that if f is an allocation in the f-core of the continuum economy (A^0, γ^0, F), then we can find an approximation sequence of finite economies $\{(N^\nu, \beta^\nu, B^\nu)\}$ with conditions (4.10), (4.11), and (4.12).

From Lemma 3, we have a nonatomic approximation sequence $\{(A^\nu, \gamma^\nu, B^\nu)\}$ and sequences $\{q^\nu\}$ and $\{g^\nu\}$ with conditions (i) and (ii) of Lemma 3. Let ν be an arbitrary integer. Then condition (i) of Lemma 3 implies that we can apply Lemma 1 to each continuum economy

$(A^\nu, \gamma^\nu, B^\nu)$, so we can find a finite economy $(N^\nu, \beta^\nu, B^\nu)$ $(N^\nu \subset A^\nu)$ and a B^ν-partition p^ν of N^ν such that

$$(A^\nu, \gamma^\nu, B^\nu) \text{ is a nonatomic representation of } (N^\nu, \beta^\nu, B^\nu) \text{ with the representation mapping } \xi^\nu \text{ where } \xi^\nu(a) = a \text{ for all } a \in N^\nu; \tag{5.30}$$

$$q^\nu \text{ is a nonatomic representation of } p^\nu; \tag{5.31}$$

$$\frac{(B^\nu)^2}{|N^\nu|} < \frac{1}{2^\nu}; \tag{5.32}$$

and

$$\xi^\nu \text{ preserves the values of } (\gamma^\nu, g^\nu) \text{ (i.e., } (\gamma^\nu(a'), g^\nu(a')) = (\gamma^\nu(a), g^\nu(a)) \text{ for all } a' \in \xi^{\nu^{-1}}(a) \text{ and } a \in N^\nu). \tag{5.33}$$

Since $\{(A^\nu, \gamma^\nu, B^\nu)\}$ is a nonatomic approximation sequence of (A^0, γ^0, F) from Lemma 3, the sequence $\{(N^\nu, \beta^\nu, B^\nu)\}$ is a (finite economy) approximation sequence for (A^0, γ^0, F). It also follows from (5.32) that $(B^\nu)^2/|N^\nu| \to 0$ as $\nu \to \infty$.

Let x^ν be the restriction of g^ν to N^ν for each ν. Since g^ν is q^ν-feasible, we have

$$\sum_{a \in S} g^\nu(a) \leqslant \sum_{a \in S} \omega^{\gamma^\nu}(a) \qquad \text{for all} \quad S \in q^\nu. \tag{5.34}$$

For any $T \in p^\nu$, we can find an $S \in q^\nu$ such that $\xi^\nu(S) = T$. Since ξ^ν preserves values of γ^ν, by (5.13), and x^ν is the restriction of g^ν to N^ν it follows from (5.34) that

$$\sum_{a \in T} x^\nu(a) = \sum_{a \in S} g^\nu(a) \leqslant \sum_{a \in S} \omega^{\gamma^\nu}(a) = \sum_{a \in T} \omega^{\beta^\nu}(a).$$

This means that x^ν is p^ν-feasible.

From Lemma 3 (ii), $\{g^\nu\}$ converges uniformly to f, $\mu(\bigcup_{k=1}^\infty \bigcap_{\nu=k}^\infty A^\nu) = \mu(A^0)$ by (5.9), and each g^ν is the nonatomic extension of x^ν by (5.33). Hence $\{x^\nu\}$ expandingly converges to f and satisfies (4.11).

Finally we show (4.12) that the sequence $\{x^\nu\}$ has the approximate core property. On the contrary, suppose that there is a positive ε such that for any ν_0, we can find a $\nu \geqslant \nu_0$ so that some coalition S^ν with $|S^\nu| \leqslant B^\nu$ can ε-improve upon x^ν, i.e., there is an S^ν-allocation $\{y_a^\nu\}_{a \in S^\nu}$ such that for any partition p^ν under which x^ν is feasible and for any allocation $\bar{x}^\nu$ compatible with x^ν, p^ν, and $\{y_a^\nu\}_{a \in S^\nu}$,

$$[y_a^\nu, D[\bar{x}^\nu]] \succ_a^{\beta^\nu} U([x_a^\nu, D[x^\nu]]; \varepsilon) \qquad \text{for all} \quad a \in S^\nu.$$

Since the sequence $\{g^\nu\}$ of nonatomic extensions of $\{x^\nu\}$ converges uniformly to f and $\mu(\bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^\nu)=\mu(A^0)$, it follows from Hildenbrand [5, p. 51, (39)] that the sequence $\{D[g^\nu]\}=\{D[x^\nu]\}$ converges weakly to $D[f]$. Since $\bar{x}^\nu$ is different from x^ν for at most $(B^\nu)^2$ players (see the discussion after the main theorem), and since $\lim_\nu (B^\nu)^2/|N^\nu|=0$, the sequence $\{\bar{g}^\nu\}$ of nonatomic extensions of $\{\bar{x}^\nu\}$ converges in measure to f. Therefore, from Hildenbrand [5, p. 51, (39)], $\{D[\bar{x}^\nu]\}$ converges weakly to $D[f]$. Thus using the fact that γ^ν converges uniformly to γ^0 on the domains with $\gamma^\nu(a)=\beta^\nu(\xi^\nu(a))$ for all $a\in A^\nu$, we have, for large ν and all $T^\nu\subset A^\nu$ with $\xi^\nu(T^\nu)=S^\nu$ and $|T^\nu|=|S^\nu|$,

$$[z_a^\nu, D[f]]\succ_a^{\gamma^0} U([f(a), D[f]];\varepsilon) \qquad \text{for all} \quad a\in T^\nu,$$

where $(z_a^\nu)_{a\in T^\nu}$ is the T^ν-allocation in the continuum representation (A^ν,γ^ν,B^ν) such that $z_a^\nu=y_{\xi^\nu(a)}^\nu$ for all $a\in T^\nu$. Since γ^ν converges uniformly to γ^0 with a higher order on Ω than B^ν, we can find a T^ν-allocation $(\bar{z}_a^\nu)_{a\in T^\nu}$ for a large ν *in the continuum economy* (A^0,γ^0,F), which is sufficiently close to $(z_a^\nu)_{a\in T^\nu}$ for $[\bar{z}_a^\nu, D[f]]\succ_a^{\gamma^0}[f(a), D[f]]$ for all $a\in T^\nu$. Thus the coalition T^ν can improve upon the allocation f in the economy (A^0,γ^0,F) and the measure of the union of such improving coalitions is positive, i.e., $\mu(\xi^{\nu^{-1}}(S^\nu))=\mu(A^\nu)\times|S^\nu|/|N^\nu|>0$. Therefore f cannot be in the f-core of the continuum economy (A^0,γ^0,F), a contradiction. Thus (4.12) holds.

(2) ⇒ (1). We now show that a function f in $L(A^0,\Omega)$ belongs to the f-core of the continuum economy if we can find an approximation sequence of finite economies $\{(N^\nu,\beta^\nu,B^\nu)\}$ with conditions (4.10), (4.11), and (4.12).

From the supposition and Remark 5.1, there are a nonatomic approximation sequence $\{(\bar{A}^\nu,\bar{\gamma}^\nu,B^\nu)\}$ and the sequence $\{\bar{g}^\nu\}$ of the nonatomic extensions of x^ν's to $\bar{A}^\nu$'s. From Lemma 2, each $(\bar{A}^\nu,\bar{\gamma}^\nu,B^\nu)$ has a full continuum subeconomy (A^ν,γ^ν,B^ν) with a nonatomic representation q^ν of p^ν for each ν. Also, for each economy (A^ν,γ^ν,B^ν), the restriction of $\bar{g}^\nu$ to A^ν is q^ν-feasible. Since $\mu(\bigcup_{k=1}^{\infty}\bigcap_{\nu=k}^{\infty}\bar{A}^\nu)=\mu(\bigcup_{k=1}^{\infty}\bigcap_{\nu=k}^{\infty}A^\nu)=\mu(A^0)$, the allocation f is feasible in the continuum economy (A^0,γ^0,F) from Lemma 3.

Finally we have to prove that there is a full subset A of A^0 such that no coalition in A can improve upon f in the continuum economy (A^0,γ^0,F). On the contrary, suppose that there is no full subset A of A^0 with the property that no coalition in A can improve upon f.

Consider the set $\tilde{A}=\bigcup_{k=1}^{\infty}\bigcap_{\nu=k}^{\infty}A^\nu$. Since $\mu(A^0)=\mu(\tilde{A})$ by (4.3), $\tilde{A}$ is a full subset of A^0. Therefore we can find a finite coalition S in

$\bar{A} = \bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} A^\nu$ such that S can improve upon f, i.e., for some S-allocation $(z_a)_{a \in S}$ in the continuum economy (A^0, γ^0, F),

$$[z_a, D[f]] \succ_a^{\gamma^0} [f(a), D[f]] \qquad \text{for all} \quad a \in S. \tag{5.35}$$

We can assume without loss of generality that S is included in A^ν for all ν.

From (5.35) and the openness of $\succ_a^{\gamma^0}$, we have, for some $\varepsilon > 0$,

$$U([z_a, D[f]]; 3\varepsilon) \succ_a^{\gamma^0} U([f(a), D[f]]; 3\varepsilon) \qquad \text{for all} \quad a \in S. \tag{5.36}$$

Since γ^ν converges uniformly to γ^0 on the domains, there is a ν_1 such that for all $\nu \geqslant \nu_1$,

$$U([z_a, D[f]]; 2\varepsilon) \succ_a^{\gamma^\nu} U([f(a), D[f]], 2\varepsilon) \qquad \text{for all} \quad a \in S. \tag{5.37}$$

Again since γ^ν converges uniformly to γ^0 on the domains, there is a ν_2 such that for all $\nu \geqslant \nu_2$, we can find an S-allocation $(y_a^\nu)_{a \in S}$ with $|y_a^\nu - z_a| < \varepsilon$ for all $a \in S$ in the economy $(A^\nu, \gamma^\nu, B^\nu)$. Since g^ν converges uniformly to f and $\mu(A^\nu) \to \mu(A^0)$, it follows from Hildenbrand [5, p. 51, (39)] that $D[g^\nu]$ converges weakly to f. Therefore we can find a ν_3 such that for all $\nu \geqslant \nu_3$,

$$[g^\nu(a), D[g^\nu]] \in U([f(a), D[f]]; \varepsilon) \qquad \text{for all} \quad a \in S;$$

and

$$[y_a^\nu, D[g^\nu]] \in U(z_a, D[f]]; \varepsilon) \qquad \text{for all} \quad a \in S.$$

Then, by (5.37), we have, for all $\nu \geqslant \max(\nu_1, \nu_2, \nu_3)$,

$$[y_a^\nu, D[g^\nu]] \succ_a^{\gamma^\nu} U([g^\nu(a), D[g^\nu]]; \varepsilon) \qquad \text{for all} \quad a \in S; \tag{5.38}$$

because $[g^\nu(a), D[g^\nu]] \in U([f(a), D[f]]; \varepsilon) \Rightarrow U([g^\nu(a), D[g^\nu]]; \varepsilon) \subset U([f(a), D[f]]; 2\varepsilon)$.

From the thickness assumption (4.8) and the fact that $S \subset A^\nu$ for all ν, we can find a ν_4 such that for all $\nu \geqslant \nu_4$, there is a subset T^ν of N^ν such that some bijection $\psi^\nu: T^\nu \to S$ preserves the attributes. Therefore for notational simplicity, we assume that $T^\nu = S \subset N^\nu$ for all ν. (This does not cause any problems.)

Let $\hat{x}^\nu$ be an allocation compatible with x^ν, p^ν, and $(y_a^\nu)_{a \in S}$ in $(N^\nu, \beta^\nu, B^\nu)$, where p^ν is an arbitrary partition under which x^ν is feasible. The nonatomic extension $\hat{g}^\nu$ of $\hat{x}^\nu$ to $(A^\nu, \gamma^\nu, B^\nu)$ converges in measure to

f because $|T^\nu| \leqslant B^\nu$ for all ν and $\lim_\nu (B^\nu)^2/|N^\nu| = 0$. (Recall the discussion after the main Theorem.) It follows from Hildenbrand [5, p. 51, (39)] that $D[\hat{g}^\nu]$ converges weakly to $D[f]$. Since $\hat{g}^\nu$ is the nonatomic extension of $\hat{x}^\nu$, it holds that $D[\hat{x}^\nu] = D[\hat{g}^\nu]$. Hence $D[\hat{x}^\nu]$ also converges weakly to $D[f]$. Since $D[g^\nu]$ also converges weakly to $D[f]$, there is a $\nu_5 \geqslant \max(\nu_1, \nu_2, \nu_3, \nu_4)$ from (5.38) such that for all $\nu \geqslant \nu_5$,

$$[y_a^\nu, D[\hat{x}^\nu]] \succ_a^{\gamma^\nu} U\left([g^\nu(a), D[g^\nu]]; \frac{\varepsilon}{2}\right) \qquad \text{for all} \quad a \in S.$$

Since g^ν is the nonatomic extension of x^ν to A^ν, we have $g^\nu(a) = x_a^\nu$ and $D[x^\nu] = D[g^\nu]$. Therefore, for all $\nu \geqslant \nu_5$,

$$[y_a^\nu, D[\hat{x}^\nu]] \succ_a^{\beta^\nu} U\left([x_a^\nu, D[x^\nu]]; \frac{\varepsilon}{2}\right) \qquad \text{for all} \quad a \in S.$$

This means that $\{(N^\nu, \beta^\nu, B^\nu)\}$ does not have the approximate core property, which is a contradiction. ∎

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