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Game Theory

The Nonemptiness of the *f*-Core of a Game Without Side Payments[1]

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Abstract: We prove the nonemptiness of the core of a continuum game without side payments where only small coalitions – ones bounded in absolute size of finite cardinality – are permitted. This result covers assignment games with a continuum of players and includes combinations of several assignment games, such as housing and automobile markets.

1 Introduction

In this paper we prove the nonemptiness of the core, called the f-core, of a continuum game with finite coalitions. The formulation covers both games with and without side payments. Two conditions are required: (1) the sizes of permissible coalitions must be bounded and (2) Pareto-frontiers for permissible coalitions must have slopes bounded above zero. The second condition is automatically satisfied by games with side payments and excludes, for example, cases where payoff sets consist of isolated points. Since the bound on permissible coalition sizes can be arbitrarily large, the first condition imposes virtually no restriction.

The framework of a game with a continuum of players and finite coalitions, and the concept of the f-core, were introduced in Kaneko–Wooders (1986). In the context of an exchange economy with widespread externalities, Hammond–Kaneko–Wooders (1989) proved the equivalence of the f-core and competitive outcomes. From this equivalence, together with the existence of a competitive equilibrium, they also obtained the nonemptiness of the f-core of an economy. In the same context, Kaneko–Wooders (1989) discussed a finite analogue of the continuum case. In the context of a game without sidepayments, Kaneko–Wooders (1986) demonstrated the nonemptiness of the f-core with a finite number of player types. For some applications, the finite types assumption may be cumbersome. This motivates the current paper.

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The nonemptiness result of this paper is suitable particularly for economic models with small coalitions. Examples include continuum extensions of the assignment market models of Shapley–Shubik (1971) and Kaneko (1982) and some combinations of these markets, such as housing and automobiles. It is known that a combination of finite assignment markets may lose the nonemptiness of the core. After stating our main result, we illustrate why this emptiness result occurs in a finite world and why nonemptiness is obtained in a continuum world.

Relative to the results of Hammond–Kaneko–Wooders (1989), and Kaneko–Wooders (1989), the essential restriction of the result of the current paper is that the continuum game does not allow widespread externalities. The incorporation of widespread externalities into a continuum game without side payments and the nonemptiness of the f-core are open problems.

2 The f-Core of a Continuum Game

Let $(N, \mathbf{B}, \mu)$ be a measure space, where N is a Borel subset of a complete separable metric space, $\mathbf{B}$ is the σ-algebra of all Borel subsets of N, and μ is a nonatomic measure with $0 < \mu(N) < +\infty$. Each element in N is called a *player* and N is the *player set*. The σ-algebra $\mathbf{B}$ is necessary for measurability arguments but does not play a game-theoretic role.

Let n, a positive integer, be a *bound on coalition sizes*. Let $\mathbf{F}$ be the set of all finite subsets of N containing no more than n members. Each element S in $\mathbf{F}$ is called simply a *coalition.*[2]

We consider a close of games where the payoffs attainable by a coalition of players depend on the attributes of the members of the coalition. The *set of attributes* $\mathbf{A}$ is given as a compact metric space with metric d. Let $A^* = \bigcup_{t=1}^{n} A^t$, where A^t is the t-fold Cartesian product of $\mathbf{A}$. An element α in A^t is a list of attributes of a t-member coalition.

A *characteristic function* V^* is a function on A^*, which assign to each t-vector $\alpha = (\alpha_1, \ldots, \alpha_t) \in A^t$ $(t = 1, \ldots, n)$ a nonempty closed subset, $V^*(\alpha)$, of R^t with the following properties:

[2] The mixture of a continuum of players and finite coalitions may raise the question of how we should interpret the individual player relative to the total player set. In the present approach the individual player remains the same as in finite models while the total player set is approximated by a continuum. Mathematically, outcomes of cooperation of finite numbers of players are aggregated into outcomes for the total player set by measurement-consistent partitions, to be defined presently. In the traditional approach to a continuum game, where coalitions are of positive measure, the notion of the individual player becomes vague. For more discussion of these issues, see Kaneko–Wooders (1986, 1989) and Hammond–Kaneko–Wooders (1989).

(Comprehensiveness): if $x \in V^*(\alpha)$ and $y \in R^t$ with $y \leq x$, then $y \in V^*(\alpha)$;
(Nontriviality): $\{x \in V^*(\alpha_1, \ldots, \alpha_t) : x_k \geq \sup V^*(\alpha_k) \text{ for } k = 1, \ldots, t\}$ is a nonempty bounded subset of R^t;
(Anonymity): $V^*(\alpha_{\theta(1)}, \ldots, \alpha_{\theta(t)}) = \{(x_{\theta(1)}, \ldots, x_{\theta(t)}) : (x_1, \ldots, x_t) \in V^*(\alpha_1, \ldots, \alpha_t)\}$ for all permutations θ of $\{1, 2, \ldots, t\}$.

The characteristic function V^* assigns a set of attainable payoffs to each list of attributes. The only unfamiliar condition on our context is anonymity, which means that the characteristic function V^* is invariant with respect to permutations of attributes.

From closedness, comprehensiveness and nontriviality, it follows that there is a real-valued function $M(\alpha)$ with domain **A** such that $V^*(\alpha)$ can be written as $V^*(\alpha) = (-\infty, M(\alpha)]$ for any $\alpha \in A$. We define

$$V^*_+(\alpha_1, \ldots, \alpha_1) = \{x \in V^*(\alpha_1, \ldots, \alpha_t) : x_i \geq M(\alpha_i) \text{ for } i = 1, \ldots, t\}$$
$$\text{for all } (\alpha_1, \ldots, \alpha_t) \in A^t \text{ and } t = 1, \ldots, n.$$

The function $V^*_+(\alpha_1, \ldots, \alpha_1)$ gives the set of individually rational payoff vectors.

We require an additional continuity condition. First, let d^* denote the sup metric on each A^t $(1 \leq t \leq n)$, given by $d^*((\alpha_1, \ldots, \alpha_t), (\beta_1, \ldots, \beta_t)) = \max_k d(\alpha_k, \beta_k)$ for any $(\alpha_1, \ldots, \alpha_t), (\beta_1, \ldots, \beta_t) \in A^t$. The t-dimensional Euclidean space R^t is also endowed with the sup metric, denoted by d^t. Let d^t_H be the Hausdorff metric for compact subsets of R^t, that is, for compact subsets T, W of R^t,

$$d^t_H(T, W) = \max\left[\max_{x \in T} d^t(x, W), \max_{y \in W} d^t(T, y)\right],$$

where $d^t(x, W) = \min_{y \in W} d(x, y)$ and $d^t(T, y) = \min_{x \in T} d^t(x, y)$. We make the assumption:

(Continuity): V^*_+ is a continuous function on each A^t $(t = 1, \ldots, n)$.

Since the attribute space A is compact and $V^*_+(\alpha) = \{M(\alpha)\}$ is continuous on A, $M(\alpha)$ has a minimum. Thus we have

$$\min_{\alpha \in A} M(\alpha) = \min_{\alpha \in A} \max V^*(\alpha) > -\infty. \tag{2.1}$$

A game with a continuum of players and finite coalitions is determined by an attribute function, which ascribes an attribute (a point on attribute space) to each player in N. The payoff set of a coalition is defined as the value of the characteristic function determined by the attributes of the members of that coalition. Specifically, an *attribute function* γ is a Borel measurable function from N to A. The *game* V *determined by the attribute function* γ is defined by

$$V(S) = V^*(\gamma(i_1), \ldots, \gamma(i_s)) \text{ for all } S = \{i_1, \ldots, i_s\} \in F. \tag{2.2}$$

Note that by anonymity $V(S)$ does not depend upon the choice of ordering of the members of S.

The continuum player set is consistently connected to finite coalitions through measurement-consistent partitions. A partition p of N into coalitions is *measurement-consistent* iff for any positive integer $k \leq n$,

$$N_k \equiv \bigcup_{\substack{S \in p \\ |S|=k}} S \text{ is a measurable subset of } N\text{; and}^3 \text{ each } N_k \text{ has a}$$
partition, consisting of measurable subsets $\{N_{k1}, \ldots, N_{kk}\}$, with the following property: there are measure-preserving isomorphisms $\psi_{k1}, \psi_{k2}, \ldots, \psi_{kk}$ from N_{k1} to $N_{k1}, \ldots, N_{kk}$, respectively, such that $\psi_{k1}(i)$ is the identity map and $\{\psi_{k1}(i), \ldots, \psi_{kk}(i)\} \in p$ for all $i \in N_{k1}$.[4] (2.3)

Let II denote *the set of measurement-consistent partitions.*

Note that (2.3) implies that for any $S \in p$ with $|S| = k$, we have $S = \{\psi_{k1}(i), \ldots, \psi_{kk}(i)\}$ for some $i \in N_{k1}$. Thus, for each integer k, the set N_k consists of all the members of k-player coalitions and N_{kt} consists of the t^{th} members of these coalitions. The measure-preserving property of the isomorphisms from N_{k1} to N_{kt} $(t = 1, \ldots, k)$ expresses the idea that coalitions of size k have as "many" (i.e. the same measure) first members as second members, as many second members as third members, etc. Figure 1 provides a schematic illustration.[5]

An outcome for the entire continuum game is defined as follows. First we consider a measurement-consistent partition p of the entire player set N. A payoff h for N is feasible iff each coalition in p can achieve its part $(h_i)_{i \in S}$ of h. Thus we define the *outcome set* $H(p)$ *relative to* p by

$$H(p) = \{h \in L(N, R)\colon (h(j))_{j \in S} \in V(S) \text{ for all } S \in p\}, \tag{2.4}$$

where $L(N, R)$ is the set of measurable functions from N to R. Note that $H(p) \neq \emptyset$ for any partition p since, from comprehensiveness, the constant function h, given by $h(i) = \min_{\alpha \in A} M(\alpha)$ for all $i \in N$, is in $H(p)$. The entire outcome space is denoted by $\bigcup_{p \in II} H(p)$ by H.

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[3] $|S|$ denotes the number of players in the set S.

[4] A function ψ from a set A in **B** to a set B in **B** is called a *measure-preserving isomorphism* from A to B iff (i) ψ is 1 to 1, onto, and measurable in both directions, and (ii) $\mu(C) = \mu(\psi(C))$ for all $C \subset A$ with $C \in$ **B**.

[5] The above definition of a measurement-consistent partition may appear to play a minor or negligible role. Indeed, the measure-preserving isomorphisms ψ_{kt} from N_{kt} to N_{kt} will not explicitly appear in this paper. Nevertheless, measure-preserving isomorphisms and measurement-consistency are hidden in Lemma 4.1. Moreover, the general existence of a measurement-consistent partition and the further application of the f-core theory to market economies crucially depends upon the above definition (see Kaneko–Wooders (1986, Lemmas A.2 and 31)).

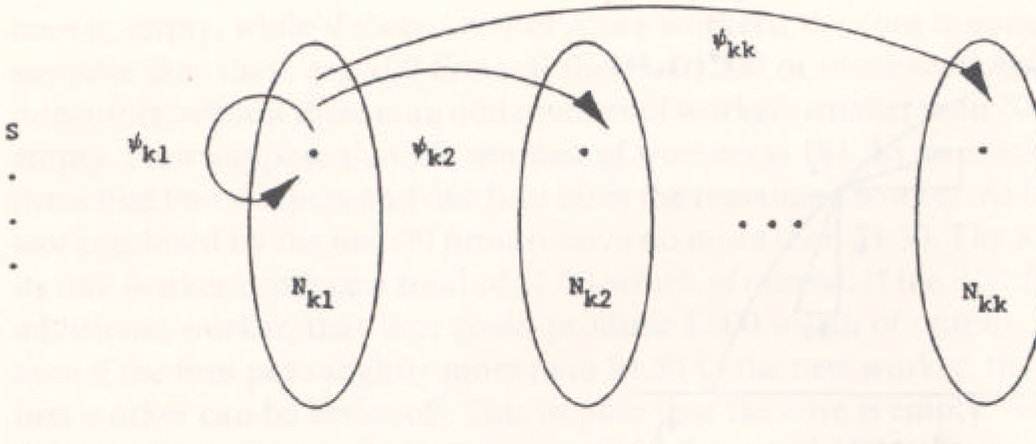

Fig. 1.

The outcome set H is not necessarily closed; limits of sequences in H are not necessarily in H.[6] We extend the outcome space H by adding some idealized outcomes to the space H so that the new space is closed with respect to a suitable concept of convergence. Here we define the extended *outcome set* H^* by

$$H^* = \{h \in L(N,R)\text{: for some sequence } \{h^\nu\} \text{ in } H, \{h^\nu\} \text{ converges in measure to } h\}, \tag{2.5}$$

where "convergence in measure to h" means that for any $\varepsilon > 0$, $\mu(\{i \in N: |h^\nu(i) - h(i)| > \varepsilon\}) \to 0$ as $\nu \to \infty$. Note that $H(p) \subset H \subset H^*$ for all $p \in \Pi$. We call an element in H^* simply an *outcome*.

Let h be a function in $L(N,R)$. We say that a coalition S in $\mathbf{F}$ *can improve upon* h if for some $y \in V(S)$, $y_i > h(i)$ for all $i \in S$. The *f-core* of the game V is defined to be the set C_f:

$$C_f = \{h \in H^*\text{: no coalition in } \mathbf{F} \text{ can improve upon } h\}. \tag{2.6}$$

An outcome h in the f-core C_f is stable in the sense that no coalition can improve upon h, and it is approximately feasible in the sense that h is almost sustained by feasible outcomes in H. Except for this approximate feasibility, the core notion is the same as in finite games.

3 The Nonemptiness of the f-Core

To state the main result of this paper, one more condition is required. The condition states that if a player receives a larger payoff in a coalition than his individually rational payoff, then he can transfer some part of his payoff to other

[6] An example for such nonclosedness is given in Hammond–Kaneko–Wooders (1989).

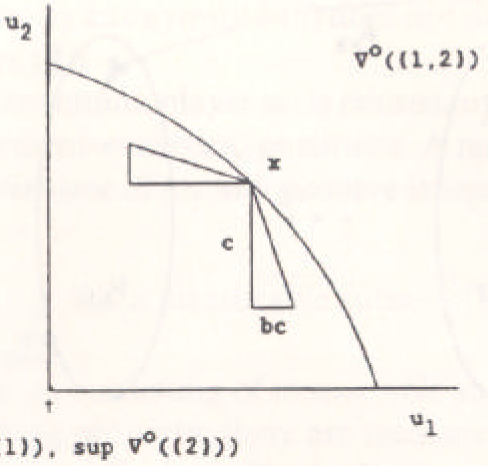

Fig. 2.

players at a given constant rate (which may be very small). Formally, the characteristic function V^* is said to be *strongly comprehensive* iff there is a $b>0$ such that for any $(\alpha_1,\ldots,\alpha_t)\in A^t$ $(1\le t\le n)$ and any k $(1\le k\le t)$, if $x\in V^*_+(\alpha_1,\ldots,\alpha_t)$ with $x_k>\max V^*(\alpha_k)=M(\alpha_k)$, then, for any c with $0<c<x_k-\max V^*(\alpha_k)$, $y\in R^t$, defined by its components

$$y_i=\begin{cases} x_k-c & \text{if } i=k \\ x_i+bc & \text{otherwise,}\end{cases} \tag{3.1}$$

belongs to $V^*(\alpha_1,\ldots,\alpha_t)$.

In the figure above, from point x, point y can be reached by the transfer, at rate b, of a part of the payoff of player 1 to player 2.

We now state our main result, which will be proved in Section 4.

Theorem: Assume that the characteristic function V^* satisfies Strong Comprehensiveness. Then any game (N, V) determined from V^* and an attribute function γ has a nonempty f-core.

The following two examples illustrate applications of this Theorem, and how we obtain the nonemptiness of the f-core in a continuum game.

Example 3.1: (A labour market).
Our first example concerns firms and workers –a labor market. All firms have the same production possibilities and all workers are substitutes for each other. A firm with zero workers can produce nothing. A firm with 1 worker can produce \$1.00 worth of output, and a firm with 2 workers can produce \$3.00 worth of output, that is, there are increasing returns to scale up to two workers. A firm cannot gain from having more than two workers. An unemployed worker can produce nothing.

If there is only a finite number of firms and workers, the core of the game may be empty. For example, consider the case of 2 firms. If there are 3 workers, the core is empty, while if there are 4 or more workers, the core is nonempty. Now suppose that there are 100 firms. If there are 200 or more workers, the core is nonempty, while if there is an odd number of workers smaller than 200, the core is empty. Now suppose that the number of workers is 181. In an efficient state, 90 firms hire two workers and one firm hires the remaining worker. At least half the workers hired by the first 90 firms receive no more than \$1.50. The 91st firm and its one worker produce a total of \$1.00-worth of output. If the 91st firm hires an additional worker, that firm could produce \$3.00 worth of output. In this case, even if the firm pays slightly more than \$1.50 to the new worker, the firm and its first worker can be better off. This implies that the core is empty. Note that this argument does not apply to the case of 100 firms and 180 workers.

To illustrate our formalism and the independence of the core of a large game from the exact numbers of players, we describe the example more precisely. The attribute space consists of two points, say $A = \{f, w\}$. The bound on coalition sizes can be taken as $n = 3$. The characteristic function V^* is defined on lists contained in $\bigcup_{t=1}^{3} A^t$ and is given by

$$V^*(f, w, w) = \{x \in R^3\colon x_1 + x_2 + x_3 \leq 3\}; \text{ and } V^*(f, w) = \{x \in R^2\colon x_1 + x_2 \leq 1\}.$$

All other (essentially different) lists of attributes have a total sum of values equal to zero. Letting the total player set N be the interval $[0, 2.81)$, an attribute function γ is given by

$$\gamma(i) = f \text{ for } i \in [0, 1) \quad \text{and} \quad \gamma(i) = w \quad \text{for } i \in [1, 2.81).$$

The proportional distribution of firms and workers remains the same as that in the above finite example. There is a measurement-consistent partition assigning two workers to each of the firms in $[0, .905)$ and zero workers to the other firms. Such a partition supports a core payoff giving each of the workers \$1.50 and each of the firms \$0.00. (In this example the approximation of taking the closure H^* of H is not necessary.)

In the finite case, the emptiness of the core is derived from the behaviour of the firm having one worker. Such a firm disappears in the continuum game, because there is no distinction of odd and even numbers of workers. Even in the finite case, the effect of the last firm becomes less significant as the economy becomes large, suggesting the nonemptiness of approximate cores of large finite games[7] shown in Wooders (1983) and other papers. The f-core is the limit case of such finite approximate cores.

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[7] Shapley and Shubik (1966) showed the nonemptiness of approximate cores of large economies with quasilinear utilities. Wooders (1983) introduced the idea that large games with many players and effective small coalitions have nonempty approximate cores; this idea has now been developed in a number of papers.

Example 3.2: (Assignment games)
An assignment game with a continuum of players is formulated as follows. Let $\{A_1, A_2\}$ be a partition of A; A_1 and A_2 are the sets of attributes of the buyers and the sellers respectively. Correspondingly, the total player set is divided into the set N_1 of players with attributes in A_1 and the set N_2 of players with attributes in A_2, that is, an attribute function γ satisfies $\gamma(N_1) \subset A_1$ and $\gamma(N_2) \subset A_2$. Here the bound on essential coalition sizes is 2. Let V^* be a characteristic function on $A^* = A \cup (A \times A)$ with the property that

$$V^*(\alpha_1, \alpha_2) = V^*(\alpha_1) \times V^*(\alpha_2) \text{ if } \alpha_1, \alpha_2 \in A_1 \text{ or } \alpha_1, \alpha_2 \in A_2.$$

This states that a coalition consisting of a pair of players on the same side of the market can do no better than each of the players separately. The game determined by V^* and the attribute function γ is given by (2.2).[8] We now have an extension of finite assignment games, as in Shapley–Shubik (1972) and Kaneko (1982), to ones with a continuum of players. Strong comprehensiveness, however, does not allow us to apply the above nonemptiness result to the direct continuum extension of Gale–Shapley (1962).

This example illustrates the application of our framework to assignment games. The ideas inherent in the example apply to generalizations of the assignment games, such as combinations of assignment games, as long as the sizes or permissible coalitions are bounded.

4 Proof of the Theorem

To prove the nonemptiness of the f-core of a game (N, V), we approximate the game by sequences of games with types. For each of the games with types, from Kaneko–Wooders (1986) there is an outcome in the f-core. We take the limit of these f-core outcomes to obtain an outcome in the f-core of the original game. The existence of the limit is proven by a variation of Ascoli's Theorem.

We will use the following facts. Since the attribute space A is compact, continuity implies that the individually rational part V^*_+ of the characteristic function V^* is uniformly continuous on each A^t. Also, the function $\psi(\alpha_1, \dots \alpha_t) \equiv \max\{\max_{1 \le i \le t} x_i \colon (x_1, \dots, x_t) \in V^*_+(\alpha_1, \dots, \alpha_t)\}$ is a continuous function on A^t $(t = 1, \dots, n)$. Since A is compact, ψ is bounded for each $t = 1, \dots, n$. This and (2.1) imply that individually rational payoffs are uniformly bounded. Let K denote an upper bound.

First we will refer to the nonemptiness result for type games obtained by Kaneko–Wooders (1986). The game V determined from the characteristic function V^* and an attribute function τ is said to have *finite types* iff $\tau(N)$ is a finite set and $\mu(\tau^{-1}(\alpha)) > 0$ for all $\alpha \in \tau(N)$.

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[8] For a more specific example, see Kaneko–Wooders (1986, Example 2.2).

Lemma 4.1 (Kaneko–Wooders (1986, Theorem 2)): Assume that the game V determined by V^* and τ has finite types. Then the f-core of the game V is nonempty. Furthermore, there is an outcome h with the equal treatment property in the f-core, i.e. $\tau(i) = \tau(j)$ implies $h(i) = h(j)$.

We will approximate the game (N, V) by a sequence of type games. First we approximate the attribute function γ by a sequence of attribute functions taking only finite values. Let $\{\delta^\nu\}$ be a decreasing sequence of positive numbers with $\lim \delta^\nu = 0$. Since the attribute space A is a compact metric space, for each ν there is a finite (not necessarily open) covering $\{U^\nu(\alpha_t^\nu)\}_{t=1}^{l_\nu}$ of A such that the covering is a partition of A and, for all t, $U^\nu(\alpha_t^\nu)$ contains α_t^ν and is contained in a ball around α_t^ν of radius less than δ^ν.

We define the νth player set $N^\nu = \bigcup_{t\in T^\nu} \gamma^{-1}(U^\nu(\alpha_t^\nu))$, where $T^\nu := \{t: \mu(\gamma^{-1}(U^\nu(\alpha_t^\nu))) > 0\}$. Then $\mu(N^\nu) = \mu(N)$ for all $\nu \geq 1$. Let $N^\infty = \bigcap_\nu N^\nu$. Then $\mu(N^\infty) = \mu(N)$. Define the *νth attribute function* γ^ν: $N^\infty \to A$ by

$$\gamma^\nu(i) = \alpha_t^\nu \text{ if } \gamma(i) \in U^\nu(\alpha_t^\nu). \tag{4.1}$$

We now have a sequence of games $\{(N^\infty, V^\nu)\}_{\nu=1}^\infty$, where V^ν is the game determined by V^* and γ^ν for all $\nu \geq 1$. It follows from Lemma 4.1 that each game (N^∞, V^ν) has an f-core outcome h^ν with the equal-treatment property. It holds that the sequence $\{\gamma^\nu\}$ converges uniformly to the original attribute function γ on N^∞.

By the remark at the beginning of this section, V_+^* is bounded with an upper bound K. Since h^ν is an outcome in (N^∞, V^ν), h^ν is almost-everywhere bounded by K. If $\{h^\nu\}$ is not uniformly bounded, we change each h^ν to $\bar{h}^\nu$, defined $\bar{h}^\nu(i) = h^\nu(i)$ if $h^\nu(i) \leq K$ and $\bar{h}^\nu(i) = K$ otherwise. Then $\bar{h}^\nu$ belongs to the f-core of (N^∞, V^ν) for all ν and $\{\bar{h}^\nu\}$ is uniformly bounded. We assume that $\{h^\nu\}$ itself is uniformly bounded.

The sequence $\{(N^\infty, V^\nu)\}$ approximates the original game (N, V). We will show that there exists a convergent subsequence of $\{h^\nu\}$. For this purpose we will use a variation of Ascoli's Theorem, stating that if a sequence of functions on a compact set satisfies equi-continuity and uniform boundedness, then there exists a uniformly convergent subsequence of the sequence. Thus we first show that the sequence $\{h^\nu\}$ has a kind of equi-continuity property.

Lemma 4.2: The sequence $\{h^\nu\}$ satisfies the following property:

for any $\varepsilon > 0$, there is a $\delta > 0$ and an integer ν_0 such that for any $i, j \in N^\infty$ and $\nu \geq \nu_0$, $d(\gamma(i), \gamma(j)) < \delta$ implies $|h^\nu(i) - h^\nu(j)| < \varepsilon$. (4.2)

Proof: We show the following:

for any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $i, j \in N^\infty$ and for all ν, $d(\gamma^\nu(i), \gamma^\nu(j)) < \delta$ implies $|h^\nu(i) - h^\nu(j)| < \varepsilon$. (4.3)

Once this is proved, (4.2) follows: For any $\delta > 0$, there is a ν_0 such that $\delta_\nu < \delta/3$ for all $\nu \geq \nu_0$, which implies that for all $\nu \geq \nu_0$,

$$d(\gamma(i), \alpha_t^\nu) < \delta/3 \text{ for all } \gamma(i) \in U^\nu(\alpha_t^\nu) \quad (t = 1, \ldots, l_\nu).$$

Then it follows that for all $v \geq v_0$,

$$d(\gamma(i), \gamma(j)) < \delta/3 \text{ implies } d(\gamma^v(i), \gamma^v(j)) < \delta.$$

Indeed, $d(\gamma^v(i), \gamma^v(j)) \leq d(\gamma^v(i), \gamma(i)) + d(\gamma(i), \gamma(j)) + d(\gamma(j), \gamma^v(j)) \leq 3(\delta/3) = \delta$. From (4.3), we have (4.2).

Let us return to the proof of (4.3). On the contrary, suppose that the sequence $\{h^v\}$ does not satisfy (4.3). Then we can assume without loss of generality that there are sequences $\{i_v\}$ and $\{j_v\}$ in N^∞ such that

$d(\gamma^v(i_v), \gamma^v(j_v))$ converges to 0 as $v \to \infty$ and;
for some $c > 0$, $h^v(i_v) - h^v(j_v) > c$ and $h^v(i_v) > \max V^v(\{i_v\}) + c$ for all v.

Since V^v is a type game, this supposition holds for all players of the same types as i_v and j_v. We will derive a contradiction from this supposition.

Let ε be a fixed number with $0 < \varepsilon < bc/12$, where b is a positive constant given by Strong comprehensiveness. We can assume without loss of generality that $0 < b < 1$.

From continuity there is a v such that for all $(\alpha_1, \ldots, \alpha_{t-1})$ in A^{t-1} $(t \leq n)$ and for all x in $V^*_+(\alpha_1, \ldots, \alpha_{t-1}, \gamma^v(i_v))$,

$$V^*_+(\alpha_1, \ldots, \alpha_{t-1}, \gamma^v(j_v)) \text{ has a vector } y \text{ with } |x_k - y_k| < \varepsilon \text{ for all } k = 1, \ldots, t. \tag{4.4}$$

Let such a v be fixed in the following.

Since h^v is in the f-core of the game (N^∞, V^v), there is a sequence $\{g^\lambda\}$ of outcomes, feasible in the sense of (2.4), such that $\{g^\lambda\}$ converges to h^v in measure. For sufficiently large λ, the measure $\mu(\{i \in N^\infty : |h^v(i) - g^\lambda(i)| \geq \varepsilon\})$ is sufficiently small relative to the measure of each type in N^∞. We take λ so that $\mu(\{i \in N^\infty : |h^v(i) - g^\lambda(i)| \geq \varepsilon\}) \leq \frac{1}{2} \min\{\mu(U^v(\alpha^v_t)) : \mu(U^v(\alpha^v_t)) > 0 \text{ and } t = 1, \ldots, l_v\}$.

Since g^λ is feasible, there is a measurement-consistent partition p of N^∞ such that $(g^\lambda(i))_{i \in S} \in V^v(S)$ for all S in p. By the choice of λ, we can find a coalition S in p so that $|h^v(i) - g^\lambda(i)| < \varepsilon$ for all $i \in S$ and so that some member i^* in S is of the same type as i^v. We can also find j^*, not in S, of the type of j_v with $|h^v(j^*) - g^\lambda(j^*)| < \varepsilon$.

Now consider the new coalition $T = (S - \{i^*\}) \cup \{j^*\}$. By (4.4) there is a vector $(y_i)_{i \in T}$ in $V^v_+(T)$ such that

$$|g^\lambda(i) - y_i| < \varepsilon \text{ for all } i \in T - \{j^*\} \text{ and } |g^\lambda(i^*) - y_{j^*}| < \varepsilon.$$

Define a vector $(z_i)_{i \in T}$ by

$$z_i = \begin{cases} y_{j^*} - c/3 & \text{if } i = j^* \\ y_i + bc/3 & \text{otherwise.} \end{cases}$$

Since $(y_i)_{i\in T}$ is in $V^\nu(T)$, by strong comprehensiveness the vector $(z_i)_{i\in T}$ is in $V^\nu(T)$. Now we show that this $(z_i)_{i\in T}$ dominates the original $(h^\nu(i))_{i\in T}$, which means that h^ν is not in the f-core of the game (N^ν, V^ν), a contradiction. Thus we have (4.3). The dominance is shown as follows: For $i\in T$ with $i\neq j^*$,

$$\begin{aligned} z_i &= y_i + bc/3 > g^\lambda(i) + bc/3 - \varepsilon \\ &> h^\nu(i) + bc/3 - 2\varepsilon = h^\nu(i) + bc/3 - 2bc/12 > h^\nu(i); \end{aligned}$$

and for $i = j^*$,

$$\begin{aligned} z_{j^*} &= y_{j^*} - c/3 > g^\lambda(i^*) - c/3 - \varepsilon \\ &> h^\nu(i^*) - c/3 - 2\varepsilon > h^\nu(j^*) + c - c/3 - 2\varepsilon \quad (\text{by } h^\nu(i_\nu) - h^\nu(j_\nu) > c) \\ &= h^\nu(j^*) + 2c/3 - 2bc/12 > h^\nu(j^*). \end{aligned}$$

The following lemma, a variation of Ascoli's Theorem, will be proven after the completion of the proof of the Theorem.

Lemma 4.3: The sequence $\{h^\nu\}$ has a uniformly convergent subsequence.

For simplicity we assume the sequence $\{h^\nu\}$ itself converges uniformly. Denote the limit function of the sequence $\{h^\nu\}$ on N^∞ by h^*, i.e. $h^*(i) = \lim h^\nu(i)$ for all i in N^∞. We extend the function h^* to the domain N as follows: Define

$$h^{**}(i) = \begin{cases} h^*(i) & \text{if } i\in N^\infty \\ K & \text{if } i\in N - N^\infty, \end{cases} \tag{4.5}$$

where K is the upper bound of the individually rational payoffs given at the beginning of this section. For the nonemptiness of the core of the game (N, V), it suffices to prove that 1) no coalition S in $\mathbf{F}$ can improve upon the function h^{**}; and 2) h^{**} is an outcome of (N, V), i.e. $h^{**}\in H^*$.

Claim 1: No coalition $S\in\mathbf{F}$ can improve upon h^{**}.

Proof: On the contrary, suppose that some $S\in\mathbf{F}$ can improve upon h^{**} with $y\in V(S)$. Then $S\subset N^\infty$ by (4.5). Since $\{\gamma^\nu\}$ converges uniformly on N^∞ to γ, it follows from continuity that there is a ν_1 such that for any $\nu\geq\nu_1$, $V^\nu(S)$ contains a vector z with $z_i > (y_i + h^*(i))/2$ for all $i\in S$. Since h^ν converges uniformly to h^* on N^∞, there is a ν_2 such that for all $\nu\geq\nu_2$, $(y_i + h^*(i))/2 > h^\nu(i)$ for all $i\in S$. Therefore we have, for any $\nu\geq\max(\nu_1,\nu_2)$, $z_i > h^\nu(i)$ for all $i\in S$. This contradicts the supposition that h^ν is in the f-core of (N^ν, V^ν).

Claim 2: The function h^{**} is an outcome of the game (N, V), i.e. $h^{**}\in H^*$.

Proof: Since $\mu(N)=\mu(N^\infty)$, it suffices to show that h^* is an outcome of the game (N^∞, V).

Since $\{h^\nu\}$ converges uniformly to h^* on N^∞, it converges also in measure to h^*, i.e. for all $\varepsilon>0$,

$$\mu(\{i\in N^\infty : |h^\nu(i)-h^*(i)|>\varepsilon\})\to 0 \ (\nu\to\infty). \tag{4.6}$$

Since each h^ν is in the f-core of (N^∞, V^ν), there is a sequence $\{h^{\nu\lambda}\}_{\lambda=1}^\infty$ such that $h^{\nu\lambda}\in H^\nu(p^{\nu\lambda})$, i.e. $h^{\nu\lambda}$ is feasible with respect to a measurement-consistent partition $p^{\nu\lambda}$ for all $\lambda=1,\dots,$ and $\{h^{\nu\lambda}\}$ converges in measure to h^ν. Hence for each ν, there is a λ_ν such that for all $\lambda\geq\lambda_\nu$,

$$\mu(\{i\in N^\infty : |h^{\nu\lambda}(i)-h^\nu(i)|>1/2^\nu\})\leq 1/2^\nu. \tag{4.7}$$

Since the number λ_ν is determined for each ν, we have the "diagonal" sequence $\{g^\nu\}=\{h^{\nu\lambda_\nu}\}$. It follows from (4.6) and (4.7) that the sequence $\{g^\nu\}$ converges in measure to h^* on N^∞. Indeed, we have, for all $\varepsilon>0$ and all ν with $1/2^\nu<\varepsilon/2$,

$$\begin{aligned}
&\mu(\{i\in N^\infty : |g^\nu(i)-h^*(i)|>\varepsilon\})\\
&\quad\leq(\{i\in N^\infty : |g^\nu(i)-h^\nu(i)|+|h^\nu(i)-h^*(i)|>\varepsilon\})\\
&\quad\leq\mu(\{i\in N^\infty : |g^\nu(i)-h^\nu(i)|>\varepsilon/2\})+\mu(\{i\in N^\infty : |h^\nu(i)-h^*(i)|>\varepsilon/2\})\\
&\quad\leq\mu(\{i\in N^\infty : |g^\nu(i)-h^\nu(i)|>1/2^\nu\})+\mu(\{i\in N^\infty : |h^\nu(i)-h^*(i)|>\varepsilon/2\})\\
&\quad\leq 1/2^\nu+\mu(\{i\in N^\infty : |h^\nu(i)-h^*(i)|>\varepsilon/2\})\to 0 \ (\nu\to\infty).
\end{aligned}$$

We denote a measurement-consistent partition $\{p^{\nu\lambda_\nu}\}$ corresponding to $\{h^{\nu\lambda_\nu}\}$ by q^ν.

Each $g^\nu=h^{\nu\lambda_\nu}$ is feasible in the type game (N^∞, V^ν) but not necessarily feasible in the game (N^∞, V). Thus we prove that there is a sequence $\{f_s\}$ such that f_s is feasible in the game (N^∞, V) for all $s=1,2,\dots$ and $\{f_s\}$ converges in measure to h^*. Let $\{\varepsilon^s\}$ be a sequence of positive numbers with $\lim_s \varepsilon^s=0$. Take an arbitrary ε^s. Define f_s^ν by

$$f_s^\nu(i)=g^\nu(i)-\varepsilon^s \text{ for all } i\in N^\infty.$$

Since the sequence $\{\gamma^\nu\}$ converges uniformly to γ on N^∞, from continuity of V_+^* and comprehensiveness there is a ν_s such that for all $\nu\geq\nu_s$

$$x\in V_+^\nu(S) \text{ and } S\in F \text{ imply } (x_i-\varepsilon^s)_{i\in S}\in V(S). \tag{4.8}$$

It follows from (4.8) that for all $\nu\geq\nu_s$ and S in q^ν,

$$(g^\nu(j))_{j\in S}\in V^\nu(S) \text{ implies } (f_s^\nu(j))_{j\in S}\in V(S),$$

that is, $f_s^\nu\in H(q^\nu)$.

Since we can assume that $v_s \to \infty$ $(s \to \infty)$, the sequence $\{f_s^{v_s}\}$ converges in measure to h^* by definition. This implies that h^* is an outcome of the game (N^∞, V). □

Finally we prove Lemma 4.3.

Proof of Lemma 4.3: Put $\Gamma = \gamma(N^\infty)$. Since A is a compact metric and since Γ is a subset of A, the space Γ is seperable. (See Royden [1963, p. 130, Proposition 6 and p. 163, Proposition 13].) That is, there is a countable subset $\Gamma^o = \{\alpha_1, \alpha_2, \dots\}$ of Γ such that the relative closure of Γ^o is Γ itself.

Since $\gamma(i) = \gamma(j)$ implies $h^v(i) = h^v(j)$, we can define functions g^v on Γ by $g^v(\alpha) = h^v(i)$ for $i \in \gamma^{-1}(\alpha)$ and $v \geq 1$. Then we prove that there is a uniformly converging subsequence of the sequence $\{g^v\}$.

Consider the sequence $\{g^v(\alpha_1)\}$ of real numbers. Since the sequence $\{g^v(\alpha_1)\}$ is bounded, $\{g^v(\alpha_1)\}$ has a convergent subsequence $\{g^{1v}(\alpha_1)\}$. Similarly, we can find a convergent subsequence $\{g^{2v}(\alpha_2)\}$ of $\{g^{1v}(\alpha_2)\}$. Repeating the same argument, we can construct a sequence of subsequences:

$$\{g^{1v}(\alpha_1)\} = \{g^{11}(\alpha_1), g^{12}(\alpha_1), \dots\}$$
$$\{g^{2v}(\alpha_2)\} = \{g^{21}(\alpha_2), g^{22}(\alpha_2), \dots\}$$
$$\{g^{3v}(\alpha_3)\} = \{g^{31}(\alpha_3), g^{32}(\alpha_3), \dots\}$$

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These subsequences are chosen sequentially so that $\{g^{kv}(\alpha_k)\}$ is a convergent subsequence of $\{g^{(k-1)v}(\alpha_k)\}$ $(k = 2, 3 \dots)$.

Consider the "diagonal" subsequence $\{f^v\} = \{g^{vv}\}$. Then, by the construction of the sequences, at each point $\alpha_t \in \Gamma^o$, $\{f^v(\alpha_1)\}$ is a convergent sequence of reals.

We now show that $\{f^v\}$ is a convergent sequence of function in the sup norm. Let $\varepsilon > 0$ be given. Condition (4.2) states that for some $\delta > 0$ and v_o, $v \geq v_0$ and $d(\gamma(i), \gamma(j)) < \delta$ imply $|h^v(i) - h^v(j)| < \varepsilon/3$. This together with the definition of $\{f^v\}$ implies that for some v_1,

$$v \geq v_1 \text{ and } d^*(\alpha, \alpha') < \delta \text{ imply } |f^v(\alpha) - f^v(\alpha')| < \varepsilon/3. \tag{4.9}$$

Consider the family $\{B_\delta(\alpha_t)\}_{t=1}^\infty$ of open balls in A with radius δ centered on $\alpha_t \in \Gamma^o$. Since Γ^o is a dense in Γ, this family $\{B_\delta(\alpha_t)\}_{t=1}^\infty$ is an open covering of Γ. In fact, $\{B_\delta(\alpha_t)\}_{t=1}^\infty$ forms an open covering of the closure of Γ relative to A. Since the closure of Γ is compact, the family $\{B_\delta(\alpha_t)\}_{t=1}^\infty$ has a finite subcovering, say $\{B_\delta(\beta_t)\}_{t=1}^k$, of the closure of Γ.

Since $\{f^v(\beta_t)\}$ converges for each $t = 1, \dots, k$, there is a v_2 such that for all $v, \lambda, \geq v_2$

$$|f^v(\beta_t) - f^\lambda(\beta_t)| < \varepsilon/3 \text{ for all } t = 1, \dots, k. \tag{4.10}$$

Let $\nu_3 = \max(\nu_1, \nu_2)$. For each $\alpha \in \Gamma$, we can find a β_t in $\{\beta_1, \ldots, \beta_k\}$ with $d(\alpha, \beta_t) < \delta$. Then, from (4.9) and (4.10), for any $\alpha \in \Gamma$,

$$\nu, \lambda \geq \nu_3 \Rightarrow |f^\nu(\alpha) - f^\lambda(\alpha)| \leq |f^\nu(\alpha) - f^\nu(\beta_t)| \\ + |f^\nu(\beta_t) - f^\lambda(\beta_t)| + |f^\lambda(\beta_t) - f^\lambda(\alpha)| \leq 3(\varepsilon/3) = \varepsilon. \quad (4.11)$$

Therefore the sequence $\{f^\nu\}$ is a Cauchy sequence in the sup norm.

For each point $\alpha \in \Gamma$, $\{f^\nu(\alpha)\}$ is also a Cauchy sequence of real numbers, which implies that $\{f^\nu(\alpha)\}$ converges to some real number $f^*(\alpha)$. Letting $\lambda \to \infty$ in (4.11), we have,

$$\text{there is a } \nu_o \text{ such that for any } \alpha \in \Gamma,\ \nu \geq \nu_o \text{ implies } |f^\nu(\alpha) - f^*(\alpha)| \leq \varepsilon. \quad (4.12)$$

This means that the sequence $\{f^\nu\}$ converges uniformly to f^*.

Since the sequence $\{f^\nu\}$ is a subsequence of $\{g^\nu\}$, we can denote $\{f^\nu\}$ by $\{g^{\nu_s}\}$, and consider the corresponding subsequence $\{h^{\nu_s}\}$ of $\{h^\nu\}$. Then the sequence $\{h^{\nu_s}\}$ converges uniformly on N^∞ to h^* defined by $h^*(i) = f^*(\gamma(i))$ for all $i \in N^\infty$. Indeed, it follows from the definition of g^ν and (4.12) that for any $\varepsilon > 0$, there is an s_o such that for any $i \in N^\infty$ with $\gamma(i) = \alpha$ and $s \geq s_o$,

$$|h^{\nu_s}(i) - h^*(i)| = |g^{\nu_s}(\alpha) - f^*(\alpha)| \leq \varepsilon. \quad \square$$

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