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THE EPSILON CORE OF A LARGE REPLICA GAME

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Sufficient conditions are given for large replica games without side payments to have non-empty approximate cores for all sufficiently large replications. No 'balancedness' assumptions are required. The conditions are superadditivity, a boundedness condition, and convexity of the payoff sets.

1. Introduction

Cooperative behavior lies at the very heart of economics and the fundamental concept of a cooperative social equilibrium is the core. However, the power of the core concept is limited by the fact that the non-emptiness of the core can be assured only in certain ideal environments. In this paper, it is shown that all members of a class of games with many players and relatively few types of players have non-empty approximate cores and the approximation can be made better as the number of players increases.

Since Shapley and Shubik (1966) first introduced concepts of approximate cores, a number of authors have demonstrated sufficient conditions for non-emptiness of approximate cores of large economies, cf. Kannai (1969, 1970), Hildenbrand, Schmeidler and Zamir (1973), and Khan and Rashid (1975). With the exception of Khan and Rashid's work, all these results deal only with the exchange of private goods; Khan and Rashid consider production with the firms exogenously given. Recently, this author obtained sufficient conditions for non-emptiness of approximate cores of large replica economies with a local public good and endogenous jurisdiction formation [Wooders

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(1980)]. All these results are, of course, dependent on the particular formulations of the economies considered.

In this paper we use the framework of n-person game theory. This framework is sufficiently general to accommodate a variety of departures from the classical model of a private goods exchange economy, including increasing returns, coalition production, and the presence of local and pure public goods. The following example illustrates a simple economic model to which the results of this paper can be applied.

Suppose $A=\{1,\dots,n\}$ is the set of agents in the economy and there are two goods, say x and y. Each agent is initially endowed with 1 unit of good x. A unit of good y is produced from a unit of good x, but the production of y requires input of x and also the joint effort of two agents. One agent, by himself, cannot produce any positive quantity of good y; he can only dispose of the initially endowed good. If three agents work together, some one of the agents only impedes the work of the other two. To formally define production technologies which are consistent with this description, let S be a non-empty subset of agents and let $|S|$ denote the number of agents in S. Then define

$$Y[S]=\{(x,y)\in \boldsymbol{R}^2 : x\leqq 0,\ y\leqq -x\} \qquad \text{if } |S| \text{ is even,}$$

$$=\{(x,y)\in \boldsymbol{R}^2 : x\leqq 0,\ y\leqq -x-1\} \quad \text{if } |S| \text{ is odd.}$$

In fig. 1, the production technology sets for both cases are depicted. The utility function of agent i is $u^i(x,y)=y$ for each i; agents do not derive any utility from the initially endowed good. From this economic data, for each coalition of agents S we can define a set $V(S)\subset \boldsymbol{R}^A$ where $V(S)$ represents the utility levels achievable by the members of S using their own initial endowments. Define

$$V(S)=\left\{\bar{u}\in \boldsymbol{R}^A_+ : \sum_{i\in S}\bar{u}^i\leqq |S|\right\} \qquad \text{if } |S| \text{ is even,}$$

$$=\left\{\bar{u}\in \boldsymbol{R}^A_+ : \sum_{i\in S}\bar{u}^i\leqq |S|-1\right\} \quad \text{if } |S| \text{ is odd.}$$

[We follow the convention that coordinates of $V(S)$ not associated with members of S are unrestricted.] The pair (A, V) is a game (a formal definition of a game is provided later). It is easy to see that for this simple model, if $\bar{u}$ is

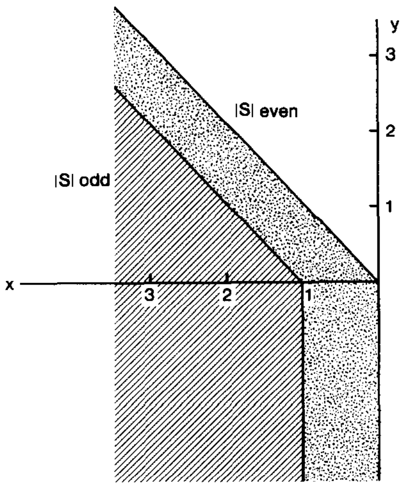

Fig. 1

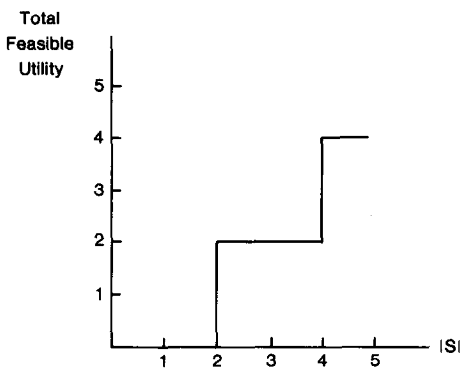

Fig. 2

in $V(S)$, then the members of S, using only their own initial endowments, can produce $\{y^i : i \in S\}$ such that $u^i(y^i) = \bar{u}^i$ for each $i \in S$. In fig. 2, the maximal sum of the utilities achievable by the members of a coalition S, using their own resources is sketched (ignoring indivisibilities of agents). Some features of this model to note are: (1) $V(S)$ is convex for each coalition S, and (2) $V(S) \cap V(S') \subset V(S \cup S')$ for any disjoint coalitions S and S', i.e., V is superadditive. Observe that even though preferences are convex, production sets for each coalition S are convex, and $V(S)$ is convex for all coalitions S of agents, for $n > 2$ the core of the game is non-empty if and only if n is an even number. As our main theorem states, however, given any $\varepsilon > 0$, for all sufficiently large n, an approximate ε-core is non-empty.

A simplifying feature of the above example is that the game (A, V) is one with side-payments, i.e., for each non-empty subset of agents S there is a real number, say $v(S)$, such that

$$\bar{u} \in V(S) \quad \text{if and only if} \quad \sum_{i \in S} \bar{u}^i \leqq v(S).$$

It is easy to generate examples of games without side-payments to which our results apply. For example, for each non-empty subset S of A, define

$$\begin{aligned} V(S) &= \left\{ \bar{u} \in \boldsymbol{R}^A : \sum_{i \in S} (\bar{u}^i)^2 \leqq K|S| \right\} && \text{when } |S| \text{ is even,} \\ &= \left\{ \bar{u} \in \boldsymbol{R}^A : \sum_{i \in S} (\bar{u}^i) \leqq K(|S| - 1) \right\} && \text{when } |S| \text{ is odd,} \end{aligned}$$

where K is some positive constant. In Shubik and Wooders (1982a, b), we provide additional examples of games, including ones generated by models of economies, to which our results can be applied. We remark that Scarf (1967) has shown that 'convex' economies have non-empty cores but this result depends on the economy being a private goods exchange economy.

In this paper, we develop the concept of a sequence of replica games without side-payments (i.e., with not-necessarily-transferable utilities). More specifically, we consider a sequence of games $(A_r, V_r)_{r=1}^{\infty}$ where A_r is the set of players of the rth game, consisting of r players of each of T 'types', and V_r is a correspondence from subsets of A_r to $\boldsymbol{R}^{rT}$. Given any coalition S contained in A_r, the set $V_r(S)$ describes the utility vectors achievable by the members of S. We assume that $A_r \subset A_{r+1}$ for all r. The sequence is then said to be a sequence of replica games if: (a) all players of the same type are substitutes for each other, and (b) $V_r(S)$ does not 'decrease' as r increases; i.e., if $S \subset A_r$ and $r' \geqq r$, then the projection of $V_r(S)$ on the subspace associated with the members of S is contained in that of $V_{r'}(S)$.[1] Simple and quite general conditions are demonstrated under which, given any $\varepsilon > 0$, there is an r^* such that for all $r \geqq r^*$ the game (A_r, V_r) has a non-empty ε-core.

Given a game (A, V), a payoff $u \in \boldsymbol{R}^A$ is in the ε-core of the game if u is feasible, i.e., $u \in V(A)$, and u cannot be 'ε-improved upon' by any coalition of

[1]This definition of a sequence of replica games is sufficiently general to include games derived from sequences of private goods economies as in Shubik (1959) and Debreu and Scarf (1963), of coalition production economies as in Boehm (1974), of economies with local public goods as in Wooders (1980). It does not include the games derived from sequences of economies with a pure public good, as in, for example, Milleron (1972) because the method used there of 'replicating' the economy is different than that used in the other papers references [this is further discussed in Wooders (1981)].

players, i.e., there does not exist an $S \subset A$ and $u' \in V(S)$ such that $u'^{tq} > u^{tq} + \varepsilon$ for all players (t, q) in S.

The conditions we impose on sequences of replica games to ensure non-emptiness of ε-cores for all sufficiently large replications r are that the games are superadditive, that $V_r(A_r)$ is convex for all r, and a 'per-capita boundedness' condition. For the present, we remark that for games with side-payments the per-capita boundedness assumption puts an upper limit on the *average* utility obtainable by the players of the game.[2] We also remark that our conditions are sufficiently general to include, as a special case, sequences of games derived from sequences of replica economies as in, for example, Shapley and Shubik (1966).

The properties of superadditivity and per-capita boundedness arise in sequences of games generated by a variety of economic models including ones with coalition production as, for example, Boehm (1974) and Bennett and Wooders (1979), ones with local public goods, as in Wooders (1980), and, of course, private goods exchange economies as in Debreu and Scarf (1963). However unless the economies are ones with transferable utility, the payoff sets of the derived games may well not be convex. In spite of this fact, it appears that the techniques and results of this paper can be useful in application when the convexity requirement is not satisfied.

The role of convexity in the proofs is to obtain the result that the equal-treatment payoffs[3] of the games converge to those of the associated balanced cover games.[4] Let $E(r) \subset \boldsymbol{R}^T$ and $\tilde{E}(r) \subset \boldsymbol{R}^T$ represent the set of equal-treatment payoffs for the rth game and its balanced cover, respectively. Although the cores of the balanced cover games may well not contain *any* payoffs with the equal-treatment property, given $\varepsilon > 0$, there is an r^* and a $\bar{u}$ in $\tilde{E}(r)$ such that for all $r \geqq r^*$, u is in the ε-core of the balanced cover game of the rth game were, for each type t and for all q, $u^{tq} = \bar{u}_t$. From this type and the fact that $E(r)$ converges to $\tilde{E}(r)$, our main result follows. Consequently, the property we require to obtain our result is that $E(r)$ converges to $\tilde{E}(r)$; convexity of $V_r(A_r)$ for all r is one way to obtain this convergence.

In considering sequences of games derived from sequences of replica economies, some assumptions which can be used to ensure that $E(r)$ converges to $\tilde{E}(r)$ are those of an infinitely divisible good with 'overriding desirability',[5] with which everyone is initially endowed.[6] Informally, these

[2]In the side-payments case, this is simply the assumption that for all r, $v_r/r \leqq K$ for some constant K, where v_r is a real number such that $V_r(A_r) = \{\bar{u} \in \boldsymbol{R}^{A_r} : \sum_{(t,q) \in A_r} \bar{u}^{tq} \leqq v_r\}$.

[3]A payoff $u \in V_r(A_r)$ is one with the equal-treatment property if, for each t and all q and q', $u^{tq} = u^{tq'}$ — players of the same type receive the same payoff.

[4]With each game, we associate a balanced cover game. For the present, we note that the balanced cover game has a non-empty core.

[5]This term is taken from Broome (1972).

[6]These assumptions have become standard in a variety of contexts, cf., Mas-Colell (1977) and Kaneko (1983).

assumptions ensure a certain degree of 'side-paymentness' of the derived games. For example, these assumptions are made in Wooders (1980), and, for the sequence of games derived from that model, the required convergence can be obtained even though the payoff sets are in general not convex.

To prove our main theorems, we require another theorem and several lemmas which are themselves of interest. The theorem is that if there is a 'minimum efficient scale' for coalitions and 'quasi-transferable utility' (informally, a certain degree of 'side-paymentness'), then for all sufficiently large replications r, every payoff in the core of the rth replica game has the equal-treatment property. The lemmas which appear particularly useful are ones relating the payoff sets of the games to each other and to those of the balanced cover games (Lemmas 3 and 5) and one showing that the sequence of equal-treatment payoffs of the balanced cover games, $\tilde{E}(r)$, has a limit (Lemma 8). These results have already been applied in other contexts. In Shubik and Wooders (1982a, b), they are used in showing non-emptiness of certain approximate cores when the convexity assumption used in this paper may not hold.[7] In Kaneko and Wooders (1982), they are used to obtain strong forms of non-emptiness of approximate cores of sequences of partitioning games (generalizations of assignment games). We remark that none of these lemmas require convexity.

Before concluding this introduction, we briefly relate our results to other results concerning non-emptiness of cores and approximate cores of games.

Bondareva (1962, 1963) and, independently, Shapley (1967) introduced the concept of 'balancedness' for games with side-payments and showed that a game with side-payments is balanced if and only if it has a non-empty core. Scarf (1967) extended the concept of balancedness to games without side-payments and showed that if such a game is balanced, it has a non-empty core. Other authors have demonstrated other conditions sufficient to ensure non-emptiness of the core of a game. In particular, some variations of the concept of balancedness have been studied and shown to ensure non-emptiness of the core; cf. Billera (1970, 1971). Shapley (1971) introduced the concept of a convex game and showed that convex games have non-empty cores; these results have been extended to games without side-payments by Vilkov (1977).[8] Shapley and Scarf in (1974) showed that games derived from a certain class of economies with indivisibilities are balanced. Numerous other results have been obtained showing that games derived from particular classes of economies have non-empty cores; however, numerous results have shown that games derived from economic models may well have empty cores, cf. Shubik (1959, 1979), Shapley and Shubik (1966), Shapley and Scarf (1974), and Greenberg (1978).

[7]These results are discussed further in the final section of this paper.

[8]Convexity of $V(A)$ does not imply that the game (A, V) is a convex game.

We remark that other authors, in particular Weber (1979) and Ichiishi and Schäffer (1979) have shown conditions under which games without side-payments and with measure spaces of agents have non-empty approximate cores. These authors have, however, initially assumed the games were balanced, using extensions of the concept of balancedness introduced by Kannai (1969) and Schmeidler (1967). In contrast, we require no assumptions of balancedness.

The paper is divided into several sections. In the next section, we introduce some notation. The third section consists of a statement of the model and results. An informal sketch of the proofs is contained in section 4, and the proofs are carried out in section 5. Section 6 concludes the paper. In the appendix, a technical result used in the paper is developed.

2. Notation

The following notation will be used: $\boldsymbol{R}^n$ is the n-fold Cartesian product of the reals, $\boldsymbol{R}$; $\boldsymbol{R}^n_+$ is the non-negative orthant of $\boldsymbol{R}^n$; given $K \subset \boldsymbol{R}^n$, $\operatorname{int} K$ denotes the interior of the set K; given a finite set S, $|S|$ denotes the cardinal number of S; and $\boldsymbol{R}^S$ is $\boldsymbol{R}^n$ where $|S|=n$. Define $\mathbf{1}=(1,1,\dots,1)\in\boldsymbol{R}^n$. Given $x\in\boldsymbol{R}^n$, we denote the (sup) norm of x by $\|x\|$ where $\|x\|=\max_i ab(x_i)$, and $ab(x_i)$ denotes the absolute value of $x_i\in\boldsymbol{R}^1$.

Given x and y in $\boldsymbol{R}^n$, we write $x\geqq y$ if $x_i\geqq y_i$ for all i; $x>y$ if $x\geqq y$ and $x\neq y$; and $x\gg y$ if $x_i>y_i$ for all i.

3. The model and the results

A *game without side-payments* (or simply a *game*) is an ordered pair (A, V), where A, called the set of *players*, is a finite set and V is a correspondence from the set of non-empty subsets of A into subsets of $\boldsymbol{R}^A$ such that:

(i) for every non-empty $S\subset A$, $V(S)$ is a non-empty, closed subset of $\boldsymbol{R}^A$ containing some member, say x, where $x\gg 0$;
(ii) if $x\in V(S)$ and $y\in\boldsymbol{R}^A$ with $x^i=y^i$ for all $i\in S$, then $y\in V(S)$;
(iii) $V(S)$ is bounded relative to $\boldsymbol{R}^S_+$, i.e., for each S, there is a vector $k(S)\in\boldsymbol{R}^A$, where, for all $x\in V(S)$, $x^i\leqq k^i(S)$ for all $i\in S$.
(iv) if $x\in V(S)$ then there is a $y\in V(S)\cap\boldsymbol{R}^A_+$ such that $y\geqq x$.

The above definition differs from the usual definition of a game in that we've required each payoff set $V(S)$ to contain a strictly positive member and in that we've imposed property (iv). Both these requirements are simply for technical convenience.[9]

[9]The property that there is an $x\in V(S)$ with $x\gg 0$ can be obtained by a parallel transformation

Let (A, V) be a game. A vector $x \in \boldsymbol{R}^A$, where the coordinates of x are superscripted by the members of A, is called a *payoff* for the game. A payoff x is feasible if $x \in V(A)$. Given a payoff x and players i and j, let $\sigma[x; i, j]$ denote the payoff formed from x by permuting the values of the coordinates associated with i and j. Players i and j are *substitutes* if: for all $S \subset A$, where $i \notin S$ and $j \notin S$, given any $x \in V(S \cup \{i\})$, we have $\sigma[x; i, j] \in V(S \cup \{j\})$; and, for all $S \subset A$, where $i \in S$ and $j \in S$, given any $x \in V(S)$, we have $\sigma[x; i, j] \in V(S)$. The game is *superadditive* if whenever S and S' are disjoint, non-empty subsets of A, we have $V(S) \cap V(S') \subset V(S \cup S')$. It is *comprehensive* if, for any non-empty subset S of A, $x \in V(S)$ and $y \leqq x$, then $y \in V(S)$.

Given a game (A, V) and $\varepsilon \geqq 0$, a payoff x is in the *ε-core* of (A, V) if: (a) x is feasible, and (b) for all non-empty subsets S of A, there does not exist an $x' \in V(S)$ such that $x' \gg x + \varepsilon \mathbf{1}$. When $\varepsilon = 0$, we call the ε-core simply the *core*. When (A, V) is comprehensive, condition (b) is equivalent to the condition that $x + \varepsilon \mathbf{1} \notin \operatorname{int} V(S)$ and our definition of the ε-core corresponds to that used by other authors, such as Weber (1979). When (A, V) is comprehensive and, in addition, $\varepsilon = 0$, the ε-core is equivalent to the (exact) core in Scarf (1967).

Given a game (A, V), define $V^p(S)$ by $V^p(S) = \{x \in \boldsymbol{R}^S$: for some $x' \in V(S)$, x is the projection of x' on $\boldsymbol{R}^S\}$, where $\boldsymbol{R}^S$ is the subspace of $\boldsymbol{R}^A$ associated with the members of S.

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of games where, for each r, $A_r \subset A_{r+1}$ and $A_r = \{(t, q) : t \in \{1, \ldots, T\},\ q \in \{1, \ldots, r\}\}$. Write $x = (x_1, \ldots, x_q, \ldots, x_r)$ for a payoff for the rth game, where $x_q = (x^{1q}, \ldots, x^{tq}, \ldots, x^{Tq})$ and x^{tq} is the component of the payoff associated with the (t, q)th player. Given r and t, define $[t]_r$ by $[t]_r = \{(t, q) \in A_r : q \in \{1, \ldots, r\}\}$; the set $[t]_r$ consists of *the players of type t of the rth game*. The sequence $(A_r, V_r)_{r=1}^{\infty}$ is a *sequence of replica games* if:

(a) for each r and each $t = 1, \ldots, T$, all players of type t of the rth game are substitutes for each other;[10]
(b) for any r' and r'' where $r' < r''$ and any $S \subset A_{r'}$, we have $V^p_{r'}(S) \subset V^p_{r''}(S)$ (i.e., the set of utility vectors achievable by the coalition S does not decrease as r increases).[11]

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. A payoff x for the game

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of the sets $V(S)$ for a game without this property. Given that there is an $x \in V(\{i\})$ with $x \gg 0$, for the study of the core members of $V(S)$ for any non-empty subset S of A with negative coordinates are irrelevant.

[10]Players of different types might also be substitutes for each other; thus the requirement that (A_1, V_1) has one player of each type is not as restrictive as it might at first appear.

[11]In an initial draft of this paper, we required that $V^p_{r'}(S) = V^p_{r''}(S)$ — what a coalition S can ensure for its members is independent of the size of the game containing that coalition. This property is common for games derived from sequences of replica economies; cf. Debreu and Scarf (1963) and Wooders (1980). The weaker restriction, that when $r' < r''$ we have $V^p_{r'}(S) \subset V^p_{r''}(S)$, permits some 'positive eternalities' to benefit the coalition as the set of players is replicated. This is sufficient to permit our results.

(A_r, V_r) is said to have the *equal-treatment property* if, for each t, we have $x^{tq'} = x^{tq''}$ for all q' and q''; players of the same type are allocated the same amount. The sequence of games is *superadditive* if (A_r, V_r) is a superadditive game for all r. The sequence is *per-capita bounded* if there is a constant K such that, for all r and for all equal-treatment payoffs x in $V_r(A_r)$, we have $x^{tq} \leqq K$.[12]

Theorem 1. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive, per-capita bounded replica games where $V_r(A_r)$ is convex for all r. Then, given $\varepsilon > 0$. there is an r^* such that, for all $r \geqq r^*$, the ε-core of (A_r, V_r) is non-empty.*

When a sequence of replica games satisfies the condition of Theorem 1, and, in addition, the games are comprehensive, we have equal-treatment payoffs in the ε-core (where $\varepsilon > 0$) for all sufficiently large replications. Our next theorem provides a stronger result concerning equal-treatment payoffs in the ε-cores.

Theorem 2. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games satisfying the conditions of Theorem 1 and, in addition, assume the games are comprehensive. Then, given $\varepsilon > 0$, there is a $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_T) \in \boldsymbol{R}_+^T$ such that x_r is in the ε-core of (A_r, V_r) for all sufficiently large r, where x_r is defined by its coordinates $x_r^{tq} = \bar{x}_t$ for each $(t,q) \in A_r$.*

Part of the strategy of the proof of Theorems 1 and 2 is to construct other sequences of games with additional properties and to approximate the games in the original sequence by the constructed games. Since games having these additional properties are of some interest themselves, we introduce these properties here and state an additional result.

We first review the concepts of balancedness and the balanced cover of a game. Let (A, V) be a game. Consider a family β subsets of A and let $\beta_i = \{S \in \beta : i \in S\}$. A family β of subsets of A is *balanced* if there exist positive 'balanced weights' w_S for S in β with $\sum_{S \in \beta_i} w_S = 1$ for all $i \in A$. Let $\boldsymbol{B}(A)$ denote the collection of all balanced families of subsets of A. Define

$$\tilde{V}(A) = \bigcup_{\beta \in \boldsymbol{B}(A)} \bigcap_{S \in \beta} V(S).$$

Define

$$\tilde{V}(S) = V(S) \quad \text{for all} \quad S \subset A \quad \text{with} \quad S \neq A.$$

Then $\tilde{V}$ maps subsets of A into $\boldsymbol{R}^A$ and is called the *balanced cover of V*. The

[12]We note that the per-capita boundedness assumption does not rule out the possibility that the sequence $(V_r(A_r))_{r=1}^{\infty}$ is unbounded from above.

game $(A, \tilde{V})$ is called the *balanced cover of* (A, V). If the game (A, V) has the property that $\tilde{V}(A) = V(A)$, the game (A, V) is *balanced*, and from Scarf's theorem (1967) the core of the game is non-empty.

A game (A, V) satisfies the assumption of *quasi-transferable utility*, QTU, if, given any subset S of A, when $x \gg 0$ and x is in the boundary of $V^P(S)$, then

$$V^P(S) \cap \{x' \in \boldsymbol{R}^S : x' \geqq x\} = x.$$

This is called the assumption of quasi-transferable utility because it has the implication that for any $x \gg 0$, if $x \in V^P(S)$ for some subset S and $x' \in V^P(S)$ where $x' > x$, then there is an $x'' \in V^P(S)$ where $x'' \gg x$ — a property of games with transferable utility. The QTU assumption is equivalent to assuming that no segment of the boundary of $V^P(S)$ in $\boldsymbol{R}^S_+$ is parallel to the coordinate axes.

Given a game (A, V) and any positive number δ, let (A, V^δ) be a game with the QTU property where, for all non-empty subsets S of A, we have $V(S) \subset V^\delta(S)$ and the Hausdorff distance (with respect to the sup norm) between $V(S)$ and $V^\delta(S)$ is less than δ.[13] Then (A, V^δ) is called a *δ-QTU cover of* (A, V). In the appendix, we show that if (A, V) is a comprehensive game, then there is a comprehensive δ-QTU cover of (A, V).

Let $(A_r, V_r)_{r=1}^\infty$ be a sequence of replica games, and let S be a non-empty subset of A_r for some r. Define the vector $s \in \boldsymbol{R}^T$ by the coordinates $s_t = |S \cap [t]_r|$ for each $t \in \{1, \ldots, T\}$; the vector s is called the *profile* of S. Define $\rho(S) = s$ so $\rho(\cdot)$ maps subsets into their profiles.

A sequence of replica games $(A_r, V_r)_{r=1}^\infty$ is said to satisfy the assumption of *minimum efficient scale* (for coalitions), MES, if there is an r^* such that, for all $r \geqq r^*$, given $x \in \tilde{V}_r(A_r)$, there is a balanced collection β of subsets of A_r with the properties that: (1) $\rho(S) \leqq \rho(A_{r^*})$ for all $S \in \beta$ and (2) $x \in \bigcap_{S \in \beta} V_r(S)$. We call r^* an *MES bound*. Informally, a sequence of games satisfies the MES property if all 'increasing returns to coalition size' are eventually exhausted.

Theorem 3. Let $(A_r, V_r)_{r=1}^\infty$ be a sequence of superadditive replica games satisfying the assumptions of QTU and MES with MES bound r^*. For any $r > r^*$, the core of the game $(A_r, \tilde{V}_r)$ is non-empty and, if x is a payoff in the core, then x has the equal treatment property.

The non-emptiness of the core of the game $(A_r, \tilde{V}_r)$ is Scarf's result (1967). It is well-known that for games with side-payments the core is non-empty if and only if it contains a payoff with the equal-treatment property. The result that the MES and QTU properties ensure that all payoffs in the core of a

[13]The definition of the Hausdorff distance can be found in Hildenbrand (1974, p. 16).

balanced game have the equal-treatment property is new. We remark that if y is a payoff in the core of the game (A_r, V_r) for $r > r^*$, then y will also have the equal-treatment property.

4. An introduction to the proofs

The purpose of this section is to provide an overall view of the strategy of the proofs and provide a preview of how the several lemmas will be used. Also, we introduce a number of definitions and notation used in the proofs.

Neither the proofs of the lemmas nor of the theorems involve particularly sophisticated mathematical techniques; it is the length and complexity of the totality of the arguments that may make it difficult to perceive how the arguments 'work'. Consequently, this section seems warranted.

Throughout this and the following sections, we let $(A_r, V_r)_{r=1}^{\infty}$ denote a sequence of superadditive replica games with T types of players and let $(A_r, \tilde{V}_r)_{r=1}^{\infty}$ be the associated sequence of balanced cover games. We continue to let $\mathbf{1}$ denote the vector of ones, and the reader is to infer from the context the dimension of the space in which $\mathbf{1}$ is contained. Given r and a positive integer n, we write (A_{nr}, V_{nr}) for the game $(A_{r'}, V_{r'})$ where $r' = nr$.

Given a payoff x for the rth game, (A_r, V_r), when we write $y = \prod_{i=1}^{n} x$ it is to be understood that the coordinates of y are superscripted so that y is a payoff for the nrth game. The same remark applies to the Cartesian product of sets, so, for example, when we write $\prod_{i=1}^{n} V_r(S) \subset V_{nr}(S')$ the coordinates of $y \in \prod_{i=1}^{n} V_r(S)$ are to be appropriately superscripted. More generally, given a vector $x \in \boldsymbol{R}^{rT}$, when we write $x \in V_r(S)$ for some r and S, again it is to be understood that the coordinates of x are appropriately superscripted (or re-superscripted).

To avoid constant repetition of 'non-empty', given any $S \subset A_r$ it is to be understood that S is non-empty.

We begin by discussing Theorem 3 since this theorem is used in the proof of the other theorems. For games with side-payments — especially ones with only one type of player — the proof of Theorem 3 is obvious and the same ideas are used in the general case. Assume x is a payoff in the core of the rth game where $r > r^*$, the MES bound. Suppose some player, say (t', q') is being treated worse than another player of the same type, say (t', q''), i.e., $x^{t'q'} < x^{t'q''}$. The MES assumption and the fact that players of the same type are substitutes ensure that there is a coalition S containing (t', q') and not (t', q'') and an x' in $V_r(S)$, such that $x'^{tq} \geq x^{tq}$ for all (t, q) in S and $x'^{t'q'} > x^{t'q'}$. From the QTU property, it follows that there is an x'' in $V_r(S)$ such that $x''^{tq} > x^{tq}$ for all (t, q) in S. Consequently, x must have the equal-treatment property; otherwise we have a contradiction to the assumption that x is in the core.

Recall that $E(r)$ and $\tilde{E}(r)$ represent the equal-treatment payoffs in $V_r(A_r)$

and $\tilde{V}_r(A_r)$, respectively. Formally, we define

$$E(r)=\left\{x\in \boldsymbol{R}^T:\prod_{i=1}^{r} x\in V_r(A_r)\right\},$$

and

$$\tilde{E}(r)=\left\{x\in \boldsymbol{R}^T:\prod_{i=1}^{r} x\in \tilde{V}_r(A_r)\right\}.$$

The equal-treatment payoffs of a sequence of balanced replica games have nice properties; in particular, $\tilde{E}(r)\subset\tilde{E}(r+1)$ for all r and the closed limit of the sequence $(\tilde{E}(r))$, denoted by $L(\tilde{E})$, exists. The relationship $\tilde{E}(r)\subset\tilde{E}(r+1)$ follows from the fact that, given a balanced family β of subsets of A_r, we can construct a related balanced family of subsets of A_{r+1} which yields the same equal-treatment payoffs as β. The existence of the closed limit follows from the inclusion relationship and the per-capita boundedness assumption.

A key step in proving Theorems 1 and 2 is to construct a sequence of comprehensive games with the QTU property from the original sequence. The sequence of games is also constructed so that, informally, the MES property is satisfied by the $(r+1)$th game with MES bound r. We then consider the sequence of balanced covers of these games.

Given r, r', and $S\subset A_r$, let $P(S;r')$ denote the collection of partitions of S, where, given P in $P(S;r')$, for each $S'\in P$, we have $\rho(S')\leqq\rho(A_{r'})$. We define the rth *truncation* of V_r by the correspondence $V_r(\cdot\,;r')$, where, for each subset S of A_r, we have

$$V_r(S;r')=\bigcup_{P\in P(S;r)}\bigcap_{S'\in P} V_r(S').$$

It is easily verified that $(A_r, V_r(\cdot;r'))_{r=1}^{\infty}$ is a sequence of superadditive replica games and satisfies the MES property with MES bound r'. Let $(A_r, \tilde{V}_r(\cdot;r'))$ be the balanced cover of $(A_r, V_r(\cdot;r'))$.

Given $\varepsilon>0$, select a positive number δ such that $\delta<\varepsilon/4$. Let $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$ be the balanced cover of a comprehensive, δ-QTU cover of $(A_{r+1}, V_{r+1}(\cdot;r))$ for each r. Let

$$\tilde{E}^{\delta}(r+1;r)=\left\{x\in \boldsymbol{R}^T:\prod_{i=1}^{r+1} x\in \tilde{V}^{\delta}_{r+1}(A_{r+1};r)\right\}.$$

From Theorem 3, each game $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$ has an equal-treatment payoff in its core; let $y^r\in\tilde{E}^{\delta}(r+1;r)$ represent such a payoff. From per-capita boundedness and the assumption that $V_r(\{(t,q)\})$ contains a strictly positive element, (y^r) has a convergent subsequence. Let $\bar{y}$ denote the limit of some such subsequence. It is shown that given $\varepsilon>0$, for all r sufficiently large $\bar{y}-(\varepsilon/2)\mathbf{1}$ represents an equal-treatment payoff in the ε-core of $(A_r, \tilde{V}^{\delta}_{r+1}(\cdot;r))$. The remainder of the proof of Theorem 2 involves showing that for all r sufficiently large, $E(r)$ is 'close' to $\tilde{E}^{\delta}(r+1;r)$ so $(\bar{y}-\varepsilon\mathbf{1})$ is in $E(r)$ and represents an equal-treatment payoff in the ε-core of (A_r, V_r). For Theorem 1, we then use the fact that $\bar{y}-\varepsilon\mathbf{1}$ represents an equal-treatment payoff in the comprehensive cover of (A_r, V_r) and 'uncover' a payoff y_r in $V_r(A_r)$ where, for each t and all q, $y_r^{tq} \geqq \bar{y}_t-\varepsilon$ for each r; this payoff is in the ε-core of (A_r, V_r).

For Theorem 2, three lemmas play key roles in showing that, for sufficiently large r, there are payoffs in $E(r)$ close to $\bar{y}$. The first is that, given any r and any positive integer n, we have $E(r)\subset E(nr)$; this follows from superadditivity — A_{nr} can be partitioned into n subsets with the same profile as A_r. The second is that given any r' there is an integer n such that $\tilde{E}(r')\subset E(nr)$; this follows from properties of 'minimal' balanced families — in particular, they have rational weights — and superadditivity. Since the relevant weights are all rational, we can select n so that each weight times n is an integer; given x in $\tilde{E}(r')$, this allows us to construct a partition P of A_{nr} so that $\prod_{i=1}^{nr} x$ is in $\bigcap_{S\in P} V_{nr}(S)$. The third key result, the only one using convexity, enables us to show that, given any r', there is an r'' such that for all $r\geqq r''$ we have $\tilde{E}(r')\subset E(r)$. Since the closed limit $L(\tilde{E})$ exists, $\tilde{E}(r)\subset\tilde{E}(r+1)$ and $E(r)\subset\tilde{E}(r)$, for all $r\geqq r''$, we have $\tilde{E}(r')\subset E(r)\subset\tilde{E}(r)\subset L(\tilde{E})$ which yields the desired result.

5. Proofs of the theorems

We first prove a number of lemmas. The first three lemmas are results of superadditivity and the fact that players of the same type are substitutes for each other.

Lemma 1. *Given r, let $y\in V_r(S)$ for some $S\subset A_r$, where y has the equal-treatment property. Let $S'\subset A_r$, where S' has the same profile as S. Then $y\in V_r(S')$.*

Proof. Since S and S' have the same profile we can define a one to one mapping, say ψ, of S onto S' such that, if $\psi((t,q))=(t',q')$, then $t=t'$. Since for each player (t,q) in S, the player $\psi((t,q))$ is a substitute for (t,q), the payoff y' is in $V_r(S')$, where y' is constructed from y by permuting the values of the coordinates of y associated with (t,q) and $\psi((t,q))$ for each (t,q) in S. Since y has the equal treatment property, $y'=y$ and, since y' is in $V_r(S')$, we have y in $V_r(S')$. Q.E.D.

Lemma 2. *Given r and a positive integer n, let $S\subset A_r$ and let $S'\subset A_{nr}$, where, for some $j\in\{1,\dots,n\}$, we have $S'=\{(t,q)\in A_{nr}$: for some $(t',q')\in S$, $t=t'$ and $q=(j-1)r+q'\}$. Then $\prod_{i=1}^{n} V_r(S)\subset V_{nr}(S')$.*

Proof. Let $B_j = \{(t,q) \in A_{nr} : t = 1, \ldots, T \text{ and } (j-1)r < q \leqq jr\}$.

Let $x' \in \prod_{i=1}^{n} V_r(S)$ and let x denote the projection of x' on the coordinates associated with members of B_j. Observe that $\prod_{i=1}^{n} x \in V_{nr}(S)$ since $V_r^P(S) \subset V_{nr}^P(S)$ and from (ii) of the definition of a game. Let $y = \prod_{i=1}^{n} x$. From the construction of y, given any (t,q') in A_r, we have $y^{tq} = y^{tq'}$ where $q = (j-1)r + q'$; therefore $\sigma[y;(t,q'),(t,q)] = y$. Since each player (t,q') in A_r is a substitute for the player (t,q) in A_{nr} it follows that $y \in V_{nr}(S')$. Since $y^{tq} = x'^{tq}$ for all (t,q) in S', we have $x' \in V_{nr}(S')$ from (ii) of the definition of a game. Therefore, $\prod_{i=1}^{n} V_r(S) \subset V_{nr}(S')$. Q.E.D.

Lemma 3. Given any r and any positive integer n, we have $\prod_{i=1}^{n} V_r(A_r) \subset V_{nr}(A_{nr})$.

Proof. Given $j \in \{1, \ldots, n\}$, let $B_j = \{(t,q) \in A_{nr} : t = 1, \ldots, T \text{ and } (j-1)r < q \leqq jr\}$. From Lemma 2, $\prod_{i=1}^{n} V_r(A_r) \subset V_{nr}(B_j)$ for each $j = 1, \ldots, n$; therefore, $\prod_{i=1}^{n} V_r(A_r) \subset \bigcap_{j=1}^{n} V_{nr}(B_j)$. Since $\{B_j : j = 1, \ldots, n\}$ is a partition of A_{nr}, from superadditivity $\bigcap_{j=1}^{n} V_{nr}(B_j) \subset V_{nr}(A_{nr})$. Q.E.D.

It is immediate from Lemma 3 that, given any r and any positive integer n, we have $E(r) \subset E(nr)$.

Given a finite set A, a balanced family β of subsets of A is a *minimal balanced family* of subsets if no proper subset of β is balanced. Our next lemma is a restatement and an easy extension of a result due to Shapley (1967, corollary to lemma 2), and is stated without proof.

*Lemma 4. Let A denote a finite set and let **B** denote the collection of all balanced families of subsets of A. Let* $\{\beta^1, \ldots, \beta^l, \ldots, \beta^L\}$ *denote the minimal balanced families of subsets of A. Then:* (1) $\beta \in \boldsymbol{B}$ *if and only if, for some subset* $L' \subset \{1, \ldots, L\}$, *we have* $\beta = \bigcup_{l \in L'} \beta^l$, *and* (2) *for each l, there is a unique set of balancing weights,* w_S^l *for* $S \in \beta^l$, *and each* w_S^l *is a rational number.*

The next lemma is a key lemma since it relates payoff sets of the balanced cover games to the payoff sets of members of the underlying sequence of games. It is an immediate consequence of this lemma that given $x \in \tilde{E}(r)$, there is a positive integer n such that $x \in E(nr)$.

Lemma 5. Given any r, there is a positive integer n such that if $x \in \tilde{V}_r(A_r)$, *then* $\prod_{i=1}^{n} x \in V_{nr}(A_{nr})$.

Proof. Given r, let $\{\beta^1, \ldots, \beta^l, \ldots, \beta^L\}$ be the set of all minimal balanced families of subsets of A_r. From Lemma 4, for each l there is a unique set of balancing weights, w_S^l for $S \in \beta^l$, and each w_S^l is a rational number. Since all the weights w_S^l are rational we can choose a positive integer n such that nw_S^l is an integer for each $S \in \beta^l$ and for all l. We claim that this n satisfies the requirements of the lemma. More specifically, we claim that given any l, there is a partition of A_{nr}, say P, such that given any $x \in \bigcap_{S \in \beta^l} V_r(S)$, we have $\prod_{i=1}^n x \in \bigcap_{S \in P} V_{nr}(S)$. We next prove this claim.

Given l, let $\beta^l = \{S_1, \ldots, S_k, \ldots, S_K\}$ and, for ease in notation, for each k let w_k denote the associated (rational) balancing weights for $S_k \in \beta^l$. We now construct a partition P of A_{nr} such that P contains nw_k members with the same profile as S_k for each $S_k \in \beta^l$. For each $(t,q) \in A_r$, let $[(t,q);n] = \{(t',q') \in A_{nr}$: $t'=t$ and, for some $j \in \{1,\ldots,n\}$, $q'=(j-1)r+q\}$. Observe that $[(t,q);n]$ contains n players, all of whom are substitutes for each other. For each k, choose nw_k subsets, say $D_k^1, \ldots, D_k^m, \ldots, D_k^{nw_k}$, such that

(1) for each m, if $(t,q) \in S_k$, then $|D_k^m \cap [(t,q);n]| = 1$, and if $(t,q) \notin S_k$, then $|D_k^m \cap [(t,q);n]| = 0$;
(2) for each k', each $m' \leq nw_{k'}$, and each $m \leq nw_k$, we have $D_k^m \cap D_{k'}^{m'} = \phi$ whenever $k \neq k'$ or $m \neq m'$ (or both).

Less formally, the sets D_k^m are selected so that each set D_k^m contains one and only one member of $[(t,q);n]$ for each $(t,q) \in S_k$ and no player appears in any two of the sets $\{D_k^m: 1 \leq k \leq K,\ 1 \leq m \leq nw_k\}$. We observe that from (1) each set D_k^m has the same profile as S_k. We are now going to show this selection is possible. For each $(t,q) \in A_r$, let $K(t,q) \subset \{1,\ldots,K\}$ be such that $k \in K(t,q)$ if and only if $(t,q) \in S_k$. Observe that

$$\left| [(t,q);n] \cap \left(A_{nr} - \bigcup_{k'=1}^{k-1} \bigcup_{m=1}^{nw_{k'}} D_{k'}^m \right) \right| = n - \sum_{\substack{k'<k \\ k' \in K(t,q)}} nw_{k'}.$$

Since β^l is balanced, $\sum_{k' \in K(t,q)} w_{k'} = 1$, and we have

$$n - \sum_{\substack{k'<k \\ k' \in K(t,q)}} nw_{k'} = n \sum_{\substack{k' \geq k \\ k' \in K(t,q)}} w_{k'}.$$

It follows that it is possible to select subsets $D_k^1, \ldots, D_k^{nw_k}$ satisfying the requirements (1) and (2), i.e., for each k and each $(t,q) \in S_k$, there are enough players in $[(t,q);n]$ and not in $\bigcup_{k'=1}^{k-1} \bigcup_{m=1}^{nw_{k'}} D_{k'}^m$, so that nw_k players can be selected from those remaining players who have not previously been selected. Moreover, since $n - \sum_{k \in K(t,q)} nw_k = 0$ for each $(t,q) \in A_r$, all agents in each set $[(t,q);n]$ are eventually taken in the construction of the sets D_k^m. Therefore the collection $P = \{D_k^m: 1 \leq k \leq K,\ 1 \leq m \leq nw_k\}$ is a partition of A_{nr}.

Given $\beta^l = \{S_1, \ldots, S_k, \ldots, S_K\}$ and P as above, let $x \in \bigcap_{k=1}^K V_r(S_k)$. Define $y = \prod_{i=1}^n x$. Observe that $y^{t'q'} = x^{tq}$ for all $(t',q') \in [(t,q);n]$ for each $(t,q) \in A_r$. For each D_k^m, the fact that D_k^m consists of one member of $[(t,q);n]$ for each $(t,q) \in S_k$ defines a one-to-one mapping, say ψ, of the set of agents in S_k onto the set of agents in D_k^m such that $\psi((t,q)) = (t,q')$ for some (t,q') in D_k^m and $y^{tq} = y^{tq'}$. Since for each player (t,q) in S_k the player $\psi((t,q))$ is a substitute for (t,q), the payoff y' is in $V_{nr}(D_k^m)$, where y' is constructed from y by permuting the values for the coordinates of y associated with (t,q) and $\psi((t,q))$ for each $(t,q)\in S_k$ However, since $y^{tq}=y^{tq'}$ when $\psi((t,q))=(t,q')$, we have $y'=y$. Therefore $y\in V_{nr}(D_k^m)$. From superadditivity, since $P=\{D_k^m\}$ is a partition, $y\in V_{nr}(A_{nr})$.

We have shown in the above that given any minimal balanced family of subsets of A_r, say β^l, if $x\in\bigcap_{S\in\beta^l} V_r(S)$, then $\prod_{i=1}^n x\in\bigcap_{S\in P} V_{nr}(S)$ for some partition P of A_{nr}. From superadditivity, $\prod_{i=1}^n x\in V_{nr}(A_{nr})$. Now let β be any balanced family of subsets of A_r and let $x\in\bigcap_{S\in\beta} V_r(S)$. From Lemma 4, for some subset $L'\subset\{1,\dots,L\}$, we have $\beta=\bigcup_{l\in L'}\beta^l$. Therefore $x\in\bigcap_{S\in\beta} V_r(S)$ $=\bigcap_{l\in L'}\bigcap_{S\in\beta^l} V_r(S)$ so, for $l\in L'$, $x\in\bigcap_{S\in\beta^l} V_r(S)$. From the above result, $\prod_{i=1}^n x\in V_{nr}(A_{nr})$. Q.E.D.

We remark that none of the preceding lemmas required comprehensiveness and therefore can be applied to not-necessarily-comprehensive games.

Define $E_+(r)=\boldsymbol{R}_+^T\cap E(r)$ and $\tilde{E}_+(r)=\boldsymbol{R}_+^T\cap\tilde{E}(r)$.

We remark that when (A_r,V_r) is comprehensive and, consequently, $E(r)$ is comprehensive, the set $E(r)+\varepsilon\{\mathbf{1}\}$ used in the following lemma is the ε-neighborhood of $E(r)$; i.e., $E(r)+\varepsilon\{\mathbf{1}\}=\{x\in\boldsymbol{R}^T\colon \text{for some } y\in E(r),\ \|x-y\|\leqq\varepsilon\}$.

Lemma 6. Suppose $V_r(A_r)$ is a convex and comprehensive set for all r. Then, given any $\varepsilon>0$ and any r', there is an r^* such that for all $r\geqq r^*$, we have $\tilde{E}(r')\subset E(r)+\varepsilon\{\mathbf{1}\}$.

Proof. Let r' and $\varepsilon>0$ be given.

The following observations will be relevant. From Lemma 5, we can select a positive integer n' such that $\tilde{E}(r')\subset E(n'r')$. Let $r''=n'r'$. From Lemma 3, for any positive integer n, we have $E(r'')\subset E(nr'')$; therefore $\tilde{E}(r')\subset E(nr'')$ for all positive integers n. Given any $r>r''$, let n and j be non-negative integers such that $r=nr''+j$ where $j\in\{1,\dots,r''\}$.

Since $V_r(S)$ is closed and bounded relative to $\boldsymbol{R}_+^S$ for all S, we have $\tilde{V}_r(A_r)\cap\boldsymbol{R}_+^{rT}$ compact. It follows that $\tilde{E}_+(r')$ is compact. Therefore there are a finite number of points, say $x^1,\dots,x^l,\dots,x^L$, in $\tilde{E}_+(r')$ such that $\tilde{E}_+(r')\subset\bigcup_{l=1}^L\{x\in\boldsymbol{R}^T\colon\|x-x^l\|<\varepsilon/2\}$. Arbitrarily select $z\in V_1(A_1)$. Now given r'' and $r=nr''+j$ as above, given any $l\in\{1,\dots,L\}$, we have $(\prod_{i=1}^{nr''}x^l,\prod_{i=1}^{j}z)$ $\in V_r(A_r)$ from superadditivity. Since players of the same type are substitutes, it follows that any vector with nr'' components (in $\boldsymbol{R}^T$) equal to x^l and any j components (also in $\boldsymbol{R}^T$) equal to z is also in $V_r(A_r)$, and there are C such vectors where $C=(nr''+j)!/(nr'')!j!$. In this collection of vectors, given $p\in\{1,\dots,nr''+j\}$, we have x in the pth position in $(nr''/(nr''+j))C$ of the vectors and z in the pth position in $(j/(nr''+j))C$ of the vectors. From the convexity assumption, a convex combination of these vectors is in $V_r(A_r)$. In particular, the convex combination formed by taking the sum of these vectors times $1/C$ is in $V_r(A_r)$. The vector thus formed has the equal-treatment property and each component (in $\boldsymbol{R}^T$) of the vector is $(nr''x^l/(nr''+j))+(jz/(nr''+j))$. Let $z_{lj}(n)=(nr''x^l/(nr''+j))+(jz/(nr''+j))$. It follows that $z_{lj}(n)$ is in $E(r)$. Also, it is obvious that given any $j\in\{1,\dots,r''\}$, $z_{lj}(n)$ converges to x^l as n becomes large. Since $j\leqq r''$ for all n, we can select r, and therefore n, sufficiently large so that $\|z_{lj}(n)-x^l\|<\varepsilon/2$. Let n^* be sufficiently large so that for all l, all j, and all $n\geqq n^*$, we have $\|z_{lj}(n)-x^l\|<\varepsilon/2$. Suppose $r>n^*r''$ so $n\geqq n^*$, where, as above, $r=nr''+j$ for some $j\in\{1,\dots,r''\}$. Let x be an arbitrary element of $\tilde{E}_+(r')$. Then there is an x^l such that $\|x^l-x\|<\varepsilon/2$. Since $n\geqq n^*$, $\|z_{lj}(n)-x^l\|<\varepsilon/2$, so $\|z_{lj}(n)-x\|<\varepsilon$. Therefore $\tilde{E}_+(r')\subset\{x\in\boldsymbol{R}^T\colon$ for some $x'\in E(r)$, $\|x-x'\|\leqq\varepsilon\}=E(r)+\varepsilon\{\mathbf{1}\}$ and from property (iv) of the definition of a game and comprehensiveness, $\tilde{E}(r')\subset E(r)+\varepsilon\{\mathbf{1}\}$. Q.E.D.

Lemma 7. For all r, we have $\tilde{E}(r)\subset\tilde{E}(r+1)$.

Proof. Given any r, let $x\in\tilde{E}(r)$. Let β be a balanced family of subsets of A_r such that $\prod_{i=1}^r x\in\bigcap_{S\in\beta}V_r(S)$.

Let $\beta=\{S_1,\dots,S_k,\dots,S_K\}$ and let w_k denote the weights for $S_k\in\beta$. Given $q\in\{1,\dots,r+1\}$, let $B_q=\{(t,q)\in A_{r+1}\colon\ t=1,\dots,T\}$. Given $l\in\{1,\dots,r+1\}$, we construct a balanced family of K subsets of $\bigcup_{q=1,q\neq l}^{r+1}B_q$, say $\beta^l=\{S_1^l,\dots,S_k^l,\dots,S_K^l\}$, where $(t,q)\in S_k^l$ if and only if: (*a*) $(t,q)\in S_k$ with $q\neq l$, or (b) $q=r+1$ and $(t,l)\in S_k$. Informally, β and β^l are the same when $l\leqq r$ except that, for each type t, each player (t,l) is replaced by $(t,r+1)$ and $\beta^{r+1}=\beta$. Note that the weights w_k for $S_k^l\in\beta^l$ balance β^l i.e.,

$$\sum_{\{k:(t,q)\in S_k^l\}}w_k=1\quad\text{for each}\quad(t,q)\in\bigcup_{\substack{q'=1\\q'\neq l}}^{r+1}B_{q'}.$$

In this manner we construct $r+1$ balanced families, $\beta^1,\dots,\beta^{r+1}$, of subsets of $\bigcup_{q=2}^{r+1}B_q,\dots,\bigcup_{q=1}^{r}B_q$, respectively. Let $\beta^*=\bigcup_{l=1}^{r+1}\beta^l$. We claim that this is a balanced family of subsets of A_{r+1}. Clearly,

$$A_{r+1}\subset\bigcup_{l=1}^{r+1}\bigcup_{k=1}^{K}S_k^l=\bigcup_{S\in\beta^*}S.$$

Note that given $(t,q)\in A_{r+1}$, there are r members of $\{\bigcup_{q'=1,q'\neq l}^{r+1}B_{q'}\colon$ $l=1,\dots,r+1\}$ containing (t,q). Since, if $(t,q)\in\bigcup_{q'=1,q'\neq l}^{r+1}B_{q'}$, the sum of the weights $\sum_{\{k:(t,q)\in S_k^l\}}w_k$ is one, we have $\sum_{\{k,l:(t,q)\in S_k^l\}}w_k=r$ and the weights w_k/r for $S_k^l\in\beta^*$ balance β^*. From the definition of a sequence of replica games, $\prod_{i=1}^{r+1}x\in V_{r+1}(S_k)$ for each $k=1,\dots,K$. From Lemma 1, $\prod_{i=1}^{r+1}x\in V_{r+1}(S_k^l)$ for all k and l, so

$$\prod_{i=1}^{r+1}x\in\bigcap_{l=1}^{L}\bigcap_{k=1}^{K}V_{r+1}(S_k^l)=\bigcap_{S\in\beta^*}V_{r+1}(S).$$

Since β^* is balanced, we have

$$\prod_{i=1}^{r+1} x \in \tilde{V}_{r+1}(A_{r+1}) \quad \text{and, therefore,} \quad x \in \tilde{E}(r+1). \quad \text{Q.E.D.}$$

In the following, we use the concept of the closed limit of a sequence of sets. A definition of this concept and some properties can be found in Hildenbrand (1974, pp. 15–18). We also employ the theorem that a sequence of subsets (F_n) of a compact metric space converges to a subset F with respect to the Hausdorff distance if and only if the closed limit of the sequence exists and equals F [see Hildenbrand (1974, p. 17)]. The closed limits, which we will show exist, of $(\tilde{E}(r))$ and $(E(r))$ are denoted by $L(\tilde{E})$ and $L(E)$, respectively. We denote the Hausdorff distance between two sets, say F and G (with respect to the sup norm metric) by $\|F, G\|$.[14]

Lemma 8. Assume that $(A_r, V_r)_{r=1}^{\infty}$ is per-capita bounded and that $V_r(A_r)$ is convex and comprehensive for all r. Then the closed limits $L(\tilde{E})$ and $L(E)$ exist and are equal and $\|\tilde{E}(r), E(r)\|$ converges to zero as r goes to infinity.

Proof. Since $\tilde{E}(r) \subset \tilde{E}(r+1)$ for all r and from the per-capita boundedness assumption, it can easily be shown that the closed limit, $L(\tilde{E})$, exists. Also, from the per-capita boundedness assumption there is a compact set, say $\mathscr{K}$, such that $\tilde{E}_+(r) \subset \mathscr{K}$ for all r. It follows that, given $\varepsilon > 0$, there is an r' sufficiently large so that for all $r \geqq r'$, $\|\tilde{E}_+(r), L(\tilde{E}_+)\| < \varepsilon/2$. From property (iv) of a game and comprehensiveness, we have $\|\tilde{E}(r), L(\tilde{E})\| < \varepsilon/2$ for all $r \geqq r'$.

Given $\varepsilon > 0$, let r' be sufficiently large so that, for all $r > r'$, $\|\tilde{E}(r), L(\tilde{E})\| < \varepsilon/2$. From Lemma 6, there is an $r^* \geqq r'$ so that, for all $r \geqq r^*$, $\tilde{E}(r') \subset E(r) + (\varepsilon/2)\{\mathbf{1}\}$. Since $E(r) \subset \tilde{E}(r)$, for all $r \geqq r^*$, we have

$$\tilde{E}(r') \subset E(r) + \frac{\varepsilon}{2}\{\mathbf{1}\} \subset \tilde{E}(r) + \frac{\varepsilon}{2}\{\mathbf{1}\} \subset L(\tilde{E}) + \frac{\varepsilon}{2}\{\mathbf{1}\}.$$

Since $\|\tilde{E}(r'), L(\tilde{E})\| < \varepsilon/2$, it follows that $\|\tilde{E}(r), E(r)\| < \varepsilon$ for all $r \geqq r^*$. [We've squeezed $E(r)$ and $\tilde{E}(r)$ between $\tilde{E}(r')$ and $L(\tilde{E}) + (\varepsilon/2)\{\mathbf{1}\}$.] It follows that $L(\tilde{E}) = L(E)$. Q.E.D.

We remark that nothing we have done so far depends on the assumptions of QTU and MES. Therefore it is possible to use these lemmas in situations where some or all of these assumptions are not made.

The theorems are now proven — in reverse order to their statements since each proof uses the preceding one.

[14]We've used $\|\cdot\|$ to denote the sup norm and $\|\cdot, \cdot\|$ to denote the Hausdorff distance. This should create no confusion, however, since in the first case the variable is a vector and, in the second, two sets.

Proof of Theorem 3

Select $r > r^*$ and let x be a payoff in the core of $(A_r, \tilde{V}_r)$; from Scarf's theorem there is such a payoff. Note that $x \gg 0$ from the assumption that x is in the core and (i) of the definition of a game. Let β be a balanced family of subsets of A_r such that $x \in \bigcap_{S \in \beta} V_r(S)$ and such that $\rho(S) \leqq \rho(A_{r^*})$ for all $S \in \beta$; from the MES assumption such a balanced family β exists. Suppose x does not treat all players of type t equally, i.e., for some q' and q'', we have $x^{tq'} \neq x^{tq''}$. We consider two cases which cover all possibilities and in both cases obtain a contradiction.

Case 1: Suppose for some S' in β we have $(t,q') \in S'$ and $(t,q'') \notin S'$. Since β is balanced, it follows that there must be an S'' in β such that $(t,q'') \in S''$ and $(t,q') \notin S''$; otherwise β could not be a balanced family since no set of 'balancing' weights could sum to one over both these members of the family containing (t,q') and those containing (t,q''). Suppose $x^{tq'} > x^{tq''}$. Let $S^* = (S' - \{(t,q')\}) \cup \{(t,q'')\}$. Then, since (t,q') and (t,q'') are substitutes, the payoff $x' = \sigma[x; (t,q'), (t,q'')]$ is in $V_r(S^*)$. Since $x'^{t'q} \geqq x^{t'q}$ for all (t',q) in S^* and $x'^{tq''} > x^{tq''}$, from the QTU assumption there is a payoff x'' in $V_r(S^*)$ where $x'' \gg x$. This contradicts the assumption that x is in the core. Suppose $x^{tq'} < x^{tq''}$. However, since $(t,q'') \in S''$ and $(t,q') \notin S''$, by reasoning as above, we can again obtain a contradiction to the assumption that x is in the core. Therefore, for Case 1, we have the result that $x^{tq'} = x^{tq''}$.

Case 2: Suppose for some S in β we have both (t,q') and (t,q'') in S. Since $|S \cap [t]_r| \leqq r^*$ there is a player of type t, say (t,q), not in S. From Case 1, we have $x^{tq} = x^{tq'}$ and $x^{tq} = x^{tq''}$, so $x^{tq'} = x^{tq''}$. Q.E.D.

Recall the definition in the preceding section of the r'th truncation of (A_r, V_r), denoted by $(A_r, V_r(\cdot, r'))$, that $(A_{r+1}, \tilde{V}_{r+1}(\cdot; r))$ denotes the balanced cover of the rth truncation of (A_{r+1}, V_{r+1}), and

$$\tilde{E}(r+1; r) = \left\{ x \in \boldsymbol{R}^T : \prod_{i=1}^{r+1} x \in \tilde{V}_{r+1}(A_{r+1}; r) \right\}.$$

Observe that $\tilde{E}(r) \subset \tilde{E}(r+1; r)$; this is an application of Lemma 7. It is immediate that $E(r) \subset \tilde{E}(r+1; r)$ and $\tilde{E}(r+1; r) \subset \tilde{E}(r+1)$.

We note that from Theorem 4 in the appendix, given $\delta > 0$ and a sequence of comprehensive replica games $(A_r, V_r)_{r=1}^{\infty}$, we can select a sequence of comprehensive games $(A_r, V_r^{\delta})_{r=1}^{\infty}$ where each game (A_r, V_r^{δ}) is a δ-QTU cover of (A_r, V_r).

For ease in exposition, since comprehensiveness is assumed for Theorem 2, we use the fact that a feasible payoff x is in the ε-core of a comprehensive game (A, V) if and only if $x + \varepsilon \mathbf{1} \notin \operatorname{int} V(S)$ for all $S \subset A$.

Proof of Theorem 2

Given $\varepsilon>0$, select a positive number δ such that $\delta<\varepsilon/4$. Let $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$ be the balanced cover of a comprehensive, δ-QTU cover of $(A_{r+1}, V_{r+1}(\cdot;r))$ for each r. Let $\tilde{E}^{\delta}(r+1;r)=\{x\in \boldsymbol{R}^T: \prod_{i=1}^{r+1} x\in \tilde{V}^{\delta}_{r+1}(A_{r+1};r)\}$ and let $L(\tilde{E}^{\delta})$ denote the closed limit of the sequence $(\tilde{E}^{\delta}(r+1;r))$; from Lemma 8 this limit exists. It is easily verified that $\|\tilde{V}_{r+1}(A_{r+1};r), \tilde{V}^{\delta}_{r+1}(A_{r+1};r)\|<\varepsilon/4$ for each r, so $\|\tilde{E}(r+1;r), \tilde{E}^{\delta}(r+1;r)\|<\varepsilon/4$ for each r, and $\|L(\tilde{E}^{\delta}), L(\tilde{E})\|<\varepsilon/4$. Since $L(\tilde{E})=L(E)$ from Lemma 8, we have $\|L(E), L(\tilde{E}^{\delta})\|<\varepsilon/4$.

From Theorem 3, for each r we can select $y^r\in\tilde{E}^{\delta}(r+1;r)$ such that $\prod_{i=1}^{r+1} y^r$ is in the core of $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$. Since $\prod_{i=1}^{r+1} y^r$ is in the core and from (i) of the definition of a game, $y^r>0$, and, since $L(\tilde{E}^{\delta})\cap \boldsymbol{R}^T_+$ is compact, (y^r) has a convergent subsequence. Let $\bar{y}$ denote the limit of some such subsequence. Define $\bar{x}=(\bar{y}-(\varepsilon/2)\mathbf{1})$.

We now show that for all r sufficiently large $\prod_{i=1}^{r+1}\bar{x}$ is in the ε-core of $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$. Since $\tilde{E}^{\delta}(r+1;r)$ converges to $L(\tilde{E}^{\delta})$, for all r sufficiently large, say $r\geqq r^*$, $\bar{x}\in\tilde{E}^{\delta}(r+1;r)$. Therefore, if $\prod_{i=1}^{r'+1}\bar{x}$ is not in the ε-core of $(A_{r'+1}, \tilde{V}^{\delta}_{r'+1}(\cdot;r'))$ for some $r'\geqq r^*$, for some $S\subset A_{r'+1}$ we have $\prod_{i=1}^{r'+1}\bar{x}+\varepsilon\mathbf{1}\in\operatorname{int}\tilde{V}^{\delta}_{r'+1}(S;r')$. Consequently, $\prod_{i=1}^{r'+1}(\bar{y}+(\varepsilon/2)\mathbf{1}\in\operatorname{int}\tilde{V}^{\delta}_{r'+1}(S;r')$. Select a member of the sequence (y^r), say $y^{r''}$, such that $r''\geqq r'$ and such that $\|\bar{y}-y^{r''}\|<\varepsilon/2$. From (b) of the definition of a sequence of replica games, and since $\|\bar{y}-y^{r''}\|<\varepsilon/2$ and $\prod_{i=1}^{r'+1}(\bar{y}+(\varepsilon/2)\mathbf{1})\in\operatorname{int}\tilde{V}^{\delta}_{r'+1}(S;r')$, we have $\prod_{i=1}^{r''+1} y^{r''}\in\operatorname{int}\tilde{V}^{\delta}_{r''+1}(S;r'')$. This is a contradiction.

Now select r sufficiently large, say r^{**}, such that $r^{**}\geqq r^*$ and, for all $r\geqq r^{**}$, $\bar{x}\in E(r)$. From comprehensiveness, the fact that $E(r)\subset\tilde{E}^{\delta}(r+1;r)$ and $\|L(E), L(\tilde{E}^{\delta})\|<\varepsilon/4$, it is immediate that there is such an r^{**}. Then for all $r\geqq r^{**}$, $\bar{x}\in E(r)$. Suppose for some $r\geqq r^{**}$ and some $S\subset A_r$, we have $\prod_{i=1}^{r}\bar{x}\in\operatorname{int} V_r(S)-\varepsilon\{\mathbf{1}\}$. But then

$$\prod_{i=1}^{r+1}\bar{x}\in\operatorname{int}\tilde{V}_{r+1}(S;r)-\varepsilon\{\mathbf{1}\}\subset\operatorname{int}\tilde{V}^{\delta}_{r+1}(S;r)-\varepsilon\{\mathbf{1}\},$$

which is a contradiction to the result of the preceding paragraph. Therefore $\prod_{i=1}^{r}\bar{x}$ *is in the ε-core of (A_r, V_r) for all sufficiently large r.* Q.E.D.

Our final result is now easy.

Proof of Theorem 1

Let (A_r, V^c_r) denote the 'comprehensive cover' of each game (A_r, V_r) in the sequence, i.e., $V^c_r(S)=\{x\in\boldsymbol{R}^{A_r}$: for some $y\in V_r(S)$, $x\leqq y\}$. Note that since $V_r(A_r)$ is convex, $V^c_r(A_r)$ is convex. From Theorem 2, given $\varepsilon>0$, we can select r^* such that, for all $r\geqq r^*$, the ε-core of (A_r, V^c_r) is non-empty. Given $r>r^*$ and a payoff x in the ε-core of (A_r, V^c_r), let $x'\in V_r(A_r)$ where $x'\geqq x$; from the definition of the comprehensive cover, there is such an x'. Since x is in the ε-core of (A_r, V_r^c), it is immediate that for all $S \subset A_r$, there does not exist an $x'' \in V_r(S)$ where $x'' \gg x + \varepsilon\mathbf{1}$. Therefore x' is in the ε-core of (A_r, V_r). Q.E.D.

6. Conclusions

In this paper, we have shown that quite simple conditions — convexity of the payoff set for the entire set of players, superadditivity, and per-capita boundedness — ensure that large replica games have non-empty approximate cores. Of these conditions, we view the superadditivity and per-capita boundedness ones as particularly non-restrictive. We view the convexity assumption as more troublesome. As Shapley (1973) has pointed out, although convexity of the sets $V(S)$ often arises in application, convexity is not an ordinal concept since it depends on the topological structure of $\boldsymbol{R}^n$. Moreover, it is easy to generate examples of economic models whose derived games do not satisfy the convexity requirement; cf. Shapley and Scarf (1974). On the other hand, the notion of an approximate core used in this paper is quite restrictive. To illustrate this, observe that from the proof of Theorem 2 it follows that, given any $\varepsilon > 0$, there is an r sufficiently large so that for some equal-treatment imputation x^* in the ε-core of the balanced cover game there is a feasible payoff x for the game, such that $x^{tq} = x^{*tq}$ for *all* players (t, q). An obvious choice for an approximate core concept is one where some feasible payoff x is in the approximate core if for 'most' agents, for some x^* in the ε-core of the balanced cover game $x^{tq} = x^{*tq}$. In Shubik and Wooders (1982a, b), such an approximate core concept is introduced and it is shown that non-emptiness of the approximate core obtains for all sufficiently large replications. It would be of interest, however, to determine less restrictive conditions than convexity under which the approximate core concept considered in this paper is non-empty; at this point we have no results of this nature for sequences of games (except for ones derived from sequences of economies) but we conjecture that a certain degree of 'side-paymentness' or quasi-transferable utility would suffice.

In conclusion, we remark that since this paper was completed, Wooders and Zame (1983) have obtained results analogous to the ones herein for large games with side-payments but without the restriction to replica games.

Appendix

From the following theorem it is immediate that given any $\delta > 0$ and any comprehensive game (A, V), there is a comprehensive δ-QTU cover of the game.

Theorem 4. *Let $K \subset \boldsymbol{R}^n_+$ be compact and comprehensive in $\boldsymbol{R}^n_+$, i.e., if $x \in K$, then $y \in K$ for all $y \in \boldsymbol{R}^n_+$ with $y \leq x$. Then there is a compact, comprehensive* *subset $K' \subset \mathbf{R}^n_+$ such that $K \subset K'$, $\|K, K'\| < \delta$, and K' satisfies the property that if x is in the boundary of the set $\{y \in \mathbf{R}^n$: for some $z \in K'$, $y \leqq z\}$ and $x \gg 0$, then*

$$K' \cap \{x' : x' \geqq x\} = \{x\}.$$

In other words, the upper boundary of K' contains no line segments parallel to the coordinate axis.

Proof. There is no loss of generality in assuming that K is a subset of the unit ball in $\mathbf{R}^n$. Let f be any continuous, real-valued function on K such that

$$1 < f(x) < 1 + \delta \quad \text{for all} \quad x \in K,$$

$$\text{where} \quad x > 0 \quad \text{and} \quad x < y \Rightarrow f(x) > f(y).$$

[For example, we could use the function $f(x) = 1 + (\delta/2)/(1 + \sum_{i=1}^n x_i)$.] Define $T: K \to \mathbf{R}^n_+$ by $T(x) = f(x)x$; this is continuous, so $T(K)$ is compact. Let $K' = T(K)$. Since $\|\{x\}, T(x)\| < \delta$ for $x \in K$ (because we have assumed K is a subset of the unit ball), we have $\|K, K'\| < \delta$.

We note that for $x \in K$, $T(x)$ lies on the ray from 0 through x and (unless $x = 0$) is further out than x. Thus T maps each ray to itself, continuously.

To see that K' is comprehensive, it suffices (because K is comprehensive) to prove (*) if $x \in K$ and $a \in \mathbf{R}^n_+$ where $a < T(x)$, then there is a y in K, $y < x$, with $T(y) = a$. To see this, write $T(x) = f(x)x$ and set $z = (1/f(x))a$. Then $z \in \mathbf{R}^n_+$ and $z < x$, so $z \in K$ and $f(z) > f(x)$. Hence $T(z) = f(z)z > f(x)z = a$; since the points 0, a, and $T(z)$ all lie on the same ray, and in that order, we can apply the Intermediate Value theorem to T on this ray to conclude that there is a y on the ray with $T(y) = a$ and $y < z < x$, as desired.

It remains to prove the assertion about the boundary of K'. If there were a line segment in the boundary of K' parallel to a coordinate axis and not lying in a coordinate plane, its endpoints, say a and b, would give two points in K', with, say $b < a$ and no coordinate of a equal to zero. Since we can replace b by the midpoint of this segment, we assume also that no coordinate of b is equal to zero. Let $x \in K$ such that $a = T(x)$; use (*) to find $y \in K$ such that $T(y) = b$ and $y < x$.

Case 1: $y < x$ but $y \not\ll x$; that is, some coordinate of y is equal to the corresponding coordinate of x, say $y_i = x_i$. Then $b_i = f(y)y_i > f(x)x_i = a_i$, since $f(y) > f(x)$ and no coordinate of a is zero (so no coordinate of x is zero). This is a contradiction.

Case 2. $y \ll x$. On the ray from zero through y there is a unique point w with $w \leqq x$ but $w \not\ll x$; this is the first point on the ray with some coordinate equal to the corresponding coordinate of x; say $w_i = x_i$. If in fact $w = x$, then y and x lie on the same ray, so b and a lie on this ray with $b < a$ which guarantees $b \ll a$. But then b belongs to the open set $\{x' \in \boldsymbol{R}^n_+ : x' \ll a\}$ which is in K' (since $a \in K'$ and K' is comprehensive); hence b would not be in the boundary of K'. Therefore we may assume $w \neq x$. Thus $f(w) > f(x)$; set $c = T(w)$ so, as in Case 1, $c_i = f(w)w_i > f(x)x_i = a_i \geqq b_i$. Since c and b lie on the same ray, we conclude that $b \ll c$ so b lies in the open set $\{x' \in \boldsymbol{R}^n_+ : x' \ll c\}$, which, again, lies in K', contradicting our assumption that b is in the boundary of K'. Q.E.D.

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