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THE TIEBOUT HYPOTHESIS: NEAR OPTIMALITY IN LOCAL PUBLIC GOOD ECONOMIES

BY MYRNA WOODERS[1]

Over two decades ago, Charles Tiebout conjectured that in economies with local public goods, consumers "vote with their feet" and that this "voting" creates an approximate "market-type" equilibrium. He hypothesized that this approximate equilibrium is "nearly" optimal and, the smaller the moving costs, the closer the equilibrium is to an optimum.

This paper provides a formal model of an economy with a local public good and endogenous jurisdiction structures (partitions of the set of agents into jurisdictions) which permits proofs of Tiebout's conjectures. Analogues of classical results pertaining to private-good economies, such as existence of equilibrium and convergence of the core to equilibrium states of the economy, are obtained for the approximate equilibrium and approximate cores.

IN HIS SEMINAL PAPER Charles Tiebout [**12**] argued that the movement of consumers to jurisdictions where their wants are best satisfied, subject to their budget constraints and given (lump sum) taxes within jurisdictions, would lead to "near optimal" provision of a local public good. In addition, he hypothesized that the larger the economy and the number of jurisdictions and the smaller the moving costs, the nearer the equilibrium would be to an optimal state of the economy.

This paper provides proofs of Tiebout's conjectures for local public good economies with endogenous jurisdiction structures (partitions of the set of agents into jurisdictions). An ε-core, similar to the Shapley-Shubik weak ε-core (in [**10**]), is defined and it is shown that the ε-core is non-empty for all sufficiently large economies. For small ε and large economies, points in the ε-core have the property that the utilities of most consumers are nearly equal to the utilities they would realize at a point in the core of an associated economy with a non-empty core (an economy where agents can be appropriately partitioned into jurisdictions); informally, the ε-core "shrinks" to the core. An ε-equilibrium is defined and shown to be in the ε-core; therefore, for small ε, the ε-equilibrium states of the economy are "nearly" Pareto optimal. In addition, for small ε, an ε-equilibrium has the properties that: the utilities of most consumers are nearly equal to their utilities in a local public equilibrium allocation;[2] and the lump sum taxes paid by "most" consumers are "nearly" equal to the Lindahl prices times the quantities consumed of the local public good minus profit shares (the profit shares most consumers nearly receive are the per capita profits in local public good

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[2] The local public equilibrium is defined in [**14**], where it is called a competitive local public equilibrium since prices of private goods are determined competitively. It is a Lindahl equilibrium relative to an appropriate jurisdiction structure. For the purposes of this paper, the most important feature of the equilibrium is that the equilibrium states of the economy are in the core.

production for the jurisdictions of which they are members).[3] Also, when the production set for the local public good (for each jurisdiction) is a closed, convex cone, given ε there is an r sufficiently large so that the ε-equilibrium exists for all replications of the economy greater than r.

In this paper, it is assumed that there is only one private good and one local public good. In a later section, methods of generalizing the model to economies with more than one private and/or local public good will be indicated. Limiting the number of goods serves to simplify the arguments and highlights the distinctive properties of local public good economies. Since the techniques used are relatively novel and already complex, the simplifying assumptions on the number of goods seem justified.

Although this model deals only with a special class of coalition economies, the methods seem to have some general applicability. Similar theorems and results have been obtained by this author in [**16**] for economies where the coalition formation aspects of the economy can be represented by a game in characteristic function form. In [**17** and **18**], the methods are extended to coalition production economies.

The paper contains five sections, including this one. In Section 2, a model of an economy with one private good, one local public good, and crowding is developed. In Section 3 properties of the ε-core as the number of agents becomes large are investigated and in Section 4 the ε-core is related to the ε-equilibrium. Section 5 is a discussion of methods to generalize the results to economies with more than one local public and/or private good. In Section 6, "Conclusions," some related literature and some additional aspects of this paper are discussed.

Notation

The economies to be studied will be described with the help of the following notation and terminology: E: Euclidean one dimensional space; Ω: the nonnegative line, $\Omega \subset E$; $-\Omega$: the nonpositive line, $-\Omega \subset E$; I: the set of positive integers; R: the (finite) set of consumers; $S \subseteq R$: a nonempty subset of R; $|S|$: the cardinal number of a set S, called the *size* of S; $E^S = E \times \ldots \times E$ ($|S|$ times): the $|S|$-fold Cartesian product of E whose coordinates have as superscripts the elements of S; $I^S = I \times \ldots \times I$ ($|S|$ times): the $|S|$-fold Cartesian product of I whose coordinates have as superscripts the elements of S; $\mathcal{R} = \{J_1, \ldots, J_k, \ldots, J_K\}$: a partition of R, called a *jurisdiction structure*; the elements of the partition are called *jurisdictions*; $\mathcal{S} = \{S_1, \ldots, S_g, \ldots, S_G\}$: a partition of S, called a *jurisdiction structure of* S; the elements of the partition of S are also called *jurisdictions*; $J[s; \mathcal{S}]$: the element of the jurisdiction structure, $\mathcal{S}$ of S, containing $s \in S$; $\|x\|$ the absolute value of a scalar x.

[3] The type of convergence used in this paper is similar to that used by Hildenbrand in [**6**, Chapter III], where he shows that "in a simple exchange economy with 'sufficiently' many participants every allocation in the core can be 'approximately' decentralized by a suitably chosen price system."

2. THE MODEL AND ASSUMPTIONS

A. *Consumers*

The economy is composed of m types of consumers with r consumers of each type, indexed by the pair (i, q) where $i \in \{1, \ldots, m\}$ and $q \in \{1, \ldots, r\}$ so that $R = \{(i, q): i \in \{1, \ldots, m\}, q \in \{1, \ldots, r\}\}$ is the set of consumers.

This description implies that there are equal numbers of consumers of each type. However, it will *not* be assumed that different types actually differ in tastes and/or initial endowments; it will be assumed only that all consumers of the same type have the same initial endowments and preferences. Therefore the assumption of equal numbers of each type is not as restrictive as it might at first seem.

B. *Goods and Crowding*

It is assumed that the economy has only 1 private good, 1 local public good, and crowding. A vector of the crowding parameter and the goods is written (n, x, y) where $n \in I$ is the crowding parameter, $x \in E$ is the local public good, and $y \in E$ is the private good.

In the model it will be required that all members of a jurisdiction consume the same amount of the local public good; therefore allocations of goods to members of any subset of consumers are defined relative to a jurisdiction structure of that subset.

DEFINITION 1: *An allocation for $\mathscr{S}$ of S*, where $S \subseteq R$, is a vector $(n^S, x^S, y^S; \mathscr{S})$ where $n^S \in I^S$, $x^S \in \Omega^S$, $y^S \in \Omega^S$, and $\mathscr{S} = \{S_1, \ldots, S_g, \ldots, S_G\}$ is a jurisdiction structure of S, such that: (i) when $J[(i'; q'); \mathscr{S}] = J[(i''; q''); \mathscr{S}]$, then $x^{i'q'} = x^{i''q''}$ for all (i', q'), $(i'', q'') \in S$ (consumers in the same jurisdiction are allocated the same amount of the local public good); (ii) $n^{iq} = |J[(i; q); \mathscr{S}]|$ for all $(i, q) \in S$ (the crowding effect a consumer experiences is determined by the size of the jurisdiction containing that consumer).

It is assumed that each consumer has a positive initial endowment of the private good and that there are no initial endowments of the local public good.

Write w^{iq} for the initial endowment of the (i, q)th consumer.

C. *Consumption*

The utility function of the (i, q)th consumer is denoted by $u^{iq}(\cdot, \cdot, \cdot)$ and maps $I \times \Omega^2$ into E.

The utility function and initial endowment for an arbitrary consumer of type i is denoted by $u^i(\cdot, \cdot, \cdot) = u^{iq}(\cdot, \cdot, \cdot)$ and by $w^i = w^{iq}$, respectively, for any $q \in \{1, \ldots, r\}$. Write $u_n^{iq}(\cdot, \cdot) = u^i(n, \cdot, \cdot)$ where $n \in I$.

ASSUMPTION C.1: For any $n \in I$, $u_n^{iq}(\cdot, \cdot)$ is a continuous function.

ASSUMPTION C.2: If $(x', y'), (x'', y'') \in \Omega \times \Omega, (x', y') > (x'', y'')$, then $u_n^{iq}(x', y') > u_n^{iq}(x'', y'')$ for all $n \in I$ (monotonicity).[4]

ASSUMPTION C.3: Given any $n', n'' \in I$ and $x, y \in \Omega$, there exists $y' \in \Omega$ so that $u^{iq}(n', x, y) = u^{iq}(n'', x, y')$ for any $(i, q) \in R$.

The third assumption above states that given any two jurisdiction sizes, n' and n'', and some quantities of the goods, x and y, there exists y' so that the (i, q)th consumer is indifferent between (n', x, y) and (n'', x, y'). Informally, this assumption will be used to permit the compensation of consumers living in "less-preferred" jurisdiction sizes by increased allocations of the private good.

As an example of the kind of situation being ruled out by C.3., suppose that $u^{iq}(2, x, y) \geq 2$ and $u^{iq}(n, x, y) \leq 1$ for all $x, y \geq 0$ and for all $n \neq 2$. It will later become apparent that if there is an odd number of consumers the ε-core of this economy would be empty for all ε sufficiently small; the "odd consumer" cannot be compensated by increased allocations of goods.

D. *Production*

The production possibilities set for the local public good for any jurisdiction is determined by the size of that jurisdiction. A correspondence, Y, from the set of positive integers I to $\Omega \times -\Omega$, which maps each integer into a closed set, is exogenously given. The production possibilities set for the local public good for a jurisdiction of size n will be $Y[n]$ and an element of $Y[n]$ is denoted by (x, z) where $x \in \Omega$ represents the output of the local public good and $z \in -\Omega$ represents the input of the private good.[5]

The production possibilities set for the local public good for the jurisdiction structure $\mathcal{S} = \{S_1, \ldots, S_g, \ldots, S_G\}$ of $S \subseteq R$ will be denoted by $Y[\mathcal{S}]$.[6] It is assumed that $Y[\mathcal{S}] = Y[|S_1|] \times \ldots \times Y[|S_g|] \times \ldots \times Y[|S_G|]$. An element of $Y[\mathcal{S}]$ is denoted by $(x, z; \mathcal{S}) = ((x_1, z_1), \ldots, (x_g, z_g), \ldots, (x_G, z_G)) \in (\Omega \times -\Omega)^G$ and is called a *state of production for $\mathcal{S}$ of S.*

ASSUMPTION D.1: Given any $z \in -\Omega$, there exists $c \in \Omega$ so that $x \leq c$ for all $(x, z) \in Y[n]$ (a finite amount of input yields at most a finite amount of output).

ASSUMPTION D.2: $Y[n] \cap \Omega \times \Omega = \{(0, 0)\}$.

ASSUMPTION D.3: When $(x', z') \in Y[n]$, then $(x, z) \in Y[n]$ for all $(x, z) \in \Omega \times -\Omega$ such that $(x, z) \leq (x', z')$ (free disposal of inputs and limited free disposal of output).

[4] $(x', y') > (x'', y'')$ means that $x' \geq x''$, $y' \geq y''$ and either $x' > x''$ or $y' > y''$ or both.

[5] Observe that x has been used to denote a consumption and an output of the local public good. However, the meaning of the variable x should be clear from the context.

[6] Observe that $Y[\cdot]$ has been used to denote the production possibilities set for both a jurisdiction size and a jurisdiction structure. This should create no confusion since, in one case, the variable is an integer, and in the other, a jurisdiction structure.

Assumption D.4: Given $n', n'' \in I$, when $(x, z') \in Y[n']$ there exists z'' so that $(x, z'') \in Y[n'']$ (an output level which is possible for one jurisdiction size is possible for every other jurisdiction size, with a (possibly) different input level, i.e., the projection of $Y[n]$ on its first coordinate is independent of n).

Assumption D.5: Let $\varphi: -\Omega \to \Omega \times -\Omega$ where $\varphi(z) = \{(x, z): (x, z) \in Y[n]\}$. Then for every open set G in $\Omega \times -\Omega$, if $\varphi(z') \cap G \neq \varnothing$, there exists a neighborhood V of z' such that $\varphi(z) \cap G \neq \varnothing$ for every $z \in V$ (the correspondence between inputs and the production set is lower hemicontinuous).

Assumption D.6: There exists $(x, z) \in Y[n]$ where $x > 0$ (possibility of positive production).

Assumption D.4 is used in connection with Assumption C.3 so that x (in the statement of Assumption C.3) can actually be produced by a jurisdiction of size n''.

Assumption D.5 is a smoothness condition. If the production set $Y[n]$ is convex then Assumption D.5 is satisfied but the opposite implication does not hold.

E. *States of the Economy*

Definition 2: *A state of the economy for $\mathcal{S}$ of S*, where $S \subseteq R$ and $\mathcal{S} = \{S_1, \ldots, S_G\}$ is a jurisdiction structure of S, is an ordered pair $s(\mathcal{S}) = \langle (n^S, x^S, y^S; \mathcal{S}), (x, z; \mathcal{S})\rangle$, consisting of an allocation for $\mathcal{S}$ of S, $(n^S, x^S, y^S; \mathcal{S})$, and a state of production for $\mathcal{S}$ of S, $(x, z; \mathcal{S})$, such that for all $(i, q) \in R$, $x^{iq} = x_g$ whenever $J[(i, q); \mathcal{S}] = S_g$ (the amount of the local public good allocated to each consumer must equal the amount produced for the jurisdiction containing that consumer).

Definition 3: $s(\mathcal{S}) = \langle (n^S, x^S, y^S; \mathcal{S}), (x, z; \mathcal{S})\rangle$ is a *feasible state of the economy for $\mathcal{S}$ of S* if: (i) $s(\mathcal{S})$ is a state of the economy for $\mathcal{S}$ of S; (ii) $\Sigma_{g=1}^{G} z_g \geqslant \Sigma_{iq \in S} (y^{iq} - w^{iq})$.

Definition 4: A feasible state of the economy for $\mathcal{R}$ of R, $s(\mathcal{R}) = \langle (n^R, x^R, y^R; \mathcal{R}), (x, z; \mathcal{R})\rangle$ is *ε-blocked* by S, where $\varepsilon \geqslant 0$, if there exists some jurisdiction structure, $\mathcal{S}$ of S, $\mathcal{S} = \{S_1, \ldots, S_g, \ldots, S_G\}$, and a state of the economy for $\mathcal{S}$ of S, $s(\mathcal{S}) = \langle (n'^S, x'^S, y'^S; \mathcal{S}), (x', z'; \mathcal{S})\rangle$, such that

(i) $$\sum_{g=1}^{G} z'_g \geqslant \sum_{iq \in S} (y'^{iq} - w^{iq}) + \varepsilon |S|,$$

(ii) $$u^{iq}(n'^{iq}, x'^{iq}, y'^{iq}) > u^{iq}(n^{iq}, x^{iq}, y^{iq}) \quad \text{for all } (i, q) \in S.$$

(To ε-block a state of the economy, a coalition of consumers must be able to make each member of the coalition better-off using only the initial endowments of the members of the coalition, and in addition, give up ε times the number of members of the coalition units of the private good.)

DEFINITION 5: A state of the economy for $\mathcal{R}$ of R is in the *ε-core* if it is feasible and if it is not ε-blocked by any subset of R. If $\varepsilon = 0$, the ε-core is called simply the *core.*

It is easily verified that if a state of the economy is in the ε-core for all $\varepsilon > 0$, then it is in the core.

Notice that in the definitions of feasibility and ε-blocking the private good can be transferred between jurisdictions. Also, if S can ε-block a state of the economy with $s(\mathcal{S})$, where $\mathcal{S} = \{S_1, \ldots, S_g, \ldots, S_G\}$, then there exists $S_{g'} \in \mathcal{S}$ so that $S_{g'}$ can ε-block using the jurisdiction structure $\{S_{g'}\}$.

F. *The Equilibrium*

To define the equilibrium, definitions of prices, profits, and tax systems are required.

Recall that $\mathcal{R} = \{J_1, \ldots, J_k, \ldots, J_K\}$ denotes a jurisdiction structure of R.

Write $V[\mathcal{R}] = \{V_{K+1}, \ldots, V_{K+v}, \ldots, V_{K+V}$: $V_{K+v} \in I$, $V_{K+v} \leq mr$, $V_{K+v} \neq |J_k|$ for any $J_k \in \mathcal{R}\}$. $V[\mathcal{R}]$ denotes the set of positive integers which are not the same as the size of any jurisdiction in $\mathcal{R}$, and which are all less than or equal to the number of consumers (mr). An element of $V[\mathcal{R}]$ is called a *quasi-jurisdiction.*

Write $U[\mathcal{R}] = (|J_1|, \ldots, |J_K|, V_{K+1}, \ldots, V_{K+V}) = (V_1, \ldots, V_u, \ldots, V_{K+V})$. $U[\mathcal{R}]$ is simply a vector whose coordinates are the sizes of jurisdictions in $\mathcal{R}$ and all other possible sizes of jurisdictions.

DEFINITION 6: (a) *An $\mathcal{R}$-extended price system for the local public good* is a vector $\gamma(\mathcal{R}) = (\gamma_1, \ldots, \gamma_u, \ldots, \gamma_{K+V}) \in E^{K+V}$. (b) *An $\mathcal{R}$-extended profit system for the local public good* is a vector $\pi(\mathcal{R}) = (\pi_1, \ldots, \pi_u, \ldots, \pi_{K+V}) \in E^{K+V}$. (c) A *tax system* is a vector $\tau = (\tau^{11}, \ldots, \tau^{mr}) \in E^R$.

An $\mathcal{R}$-extended price system for the local public good is simply a vector of (producer) prices for the local public good whose coordinates are the prices for the local public good for the jurisdictions in $\mathcal{R}$ and for all other possible jurisdiction sizes. An $\mathcal{R}$-extended profit system has the same interpretation as an $\mathcal{R}$-extended price system when "prices" is replaced by "profits".

In the following definition, the private good can be interpreted as having a price equal to one.

DEFINITION 7: *A Tiebout ε-equilibrium* is an ordered quadruple $e(\varepsilon) = \langle s(\mathcal{R}), \gamma(\mathcal{R})\pi(\mathcal{R}), \tau\rangle$ consisting of a state of the economy $s(\mathcal{R}) = \langle (n^R, x^R, y^R; \mathcal{R}), (x, z; \mathcal{R})\rangle$; $\mathcal{R}$-extended price and profit systems, $\gamma(\mathcal{R})$ and $\pi(\mathcal{R})$; and a tax system τ; such that:

(i) $\Sigma_{iq \in R} (y^{iq} - w^{iq}) \leq \Sigma_{k=1}^{K} z_k$ ($s(\mathcal{R})$ is feasible).

(ii) $\pi_u = \gamma_u x_u + z_u$ for all $u \in \{1, \ldots, K\}$ and $\pi_u = \max\{\gamma_u x' + z' : (x', z') \in Y[V_u]\}$ for all $u \in \{1, \ldots, K+V\}$.

(iii) For all $(i, q) \in R$, (a) $y^{iq} - w^{iq} + \tau^{iq} = 0$ (the budget constraint is satisfied, given the lump sum taxes); (b) if $u^{iq}(V_u, x', y') > u^{iq}(n^{iq}, x^{iq}, y^{iq})$ for all (i, q) contained in some subset S, where $|S| = V_u$, then $\Sigma_{iq \in S}(y' - w^{iq}) + \gamma_u x' > \pi_u - \varepsilon|S|$ (no subset of consumers can buy a preferred allocation in another jurisdiction consisting of members of that subset, after paying jurisdiction formation costs of ε per capita, given prices and given that profits would be returned to members of that jurisdiction).

(iv) $\Sigma_{iq \in R} \tau^{iq} - \Sigma_{k=1}^{K} \gamma_k x_k = 0$ (the government balances its budget).

The idea behind subtracting off $\varepsilon|S|$ in (iii)(b) of the definition is that cooperating to form a new jurisdiction costs ε per capita. Also, since (iii)(b) is required to hold for all $S \in \mathscr{R}$, it is possible that in equilibrium for some J_k, $\Sigma_{iq \in J_k}(y^{iq} - w^{iq}) < z_k$; the agents in J_k are paying more in taxes than the cost of the inputs used in J_k. The idea behind this is that cooperating to avoid a tax imposed by the central government also costs ε per capita.

The most unusual features of the Tiebout ε-equilibrium are the prices and profits for quasi-jurisdictions and condition (iii)(b). Informally, the situation imagined is that there are entrepreneurs who are cooperativists in the sense that they are willing to return profits to the members of a jurisdiction and who engage in market research to see if they can profitably form a new jurisdiction. If they cannot, (iii)(b) is satisfied.

It can be demonstrated that, even for the case of $\varepsilon = 0$, a Tiebout ε-equilibrium is not necessarily a local public equilibrium. Also, observe that no restrictions are placed on taxes; τ^{iq} could even be negative for some (i, q). In addition, no profit distribution is specified since it is not necessary to do so, although "in the limit" it becomes apparent that "most" consumers receive "nearly" the average (per capita) profits of the jurisdictions of which they are members.

G. *The Distinguished Numbers*

The intuitive idea underlying much of the interest in local public good economies is that public goods can become crowded or congested so that it is not desirable, in some sense, to have all consumers in one jurisdiction. Some assumptions are required so that the crowding effects are sufficiently strong to differentiate the economy from a pure public good economy.

To ensure that for all sufficiently large economies, the economy has *local* public good properties, and to describe these properties precisely, particular integers, called distinguished numbers, are defined and assumed to exist for each consumer type. A distinguished number for a consumer type is a jurisdiction size which maximizes the utility of a representative consumer of that type under the assumption of equal payment of costs of the local public good by the members of the jurisdiction, given some after tax initial endowment $(w^{iq} - t^{iq})$. Consider the problem:

(*) Maximize $u_n^i(x, y)$ subject to

(1) $x, y \geqslant 0,$

(2) $y + t - w^i \leqslant 0,$

(3) $(x, n(y + t - w^i)) \in Y[n],$

with respect to x and y where $t \in (-\infty, w^i]$.

Let

$$\mathscr{K}(t) = \{(x, y): x, y \geqslant 0, (y + t - w^i) \leqslant 0, (x, n(y + t - w^i)) \in Y[n]\}.$$

Observe that $\mathscr{K}(t)$ is a compact set. Since $u_n^i(x, y)$ is a continuous function, a solution exists to problem (*).

Let

$$\bar{u}(i, t; n) = \max_{x,y} \{u_n^i(x, y) | (x, y) \in \mathscr{K}(t)\}.$$

Now consider the problem:

(**) Maximize $\bar{u}(i, t; n)$ subject to $n \in I$

with respect to n, given t.

It is assumed that for any $t \in (-\infty, w^i]$, *problem* (**) *has a solution.* This is the fundamental assumption used to ensure, for sufficiently large economies, that the public good is sufficiently different than a pure public good.

Let

$$D(i; t) = \{n' : \bar{u}(i, t; n') \geqslant \bar{u}(i, t; n) \text{ for all } n \in I\}.$$

$D(i; t)$ is the *set of distinguished numbers for consumer type i relative to the transfer t.*

Let $u^*(i; t) = \max_n \{\bar{u}(i, t, n)\}$. $u^*(i; t)$ is called the *maximal equal-treatment utility of a consumer of type i relative to the transfer t.* As shown in [**14**], $u^*(i; 0)$ is the local public equilibrium utility of a consumer type i.

3. THE ε-CORE AS THE NUMBER OF CONSUMERS BECOMES LARGE

In this section it will be shown that given any $\varepsilon > 0$, the ε-core is nonempty for all sufficiently large economies. Also,.as $\varepsilon \to 0$, the utilities of "most" consumers of type i converge to $u^*(i; 0)$. It is interesting to note that the convergence result, unlike related results in [**14**], does not depend on the core having a state of the economy with a jurisdiction structure where every consumer type has members in at least two jurisdictions; in the limit *any* state of the economy in the ε-core is "nearly" an equal-treatment state.

Before proceeding with the theorems, it is necessary to show that the functions $\bar{u}^i(i, t; n)$ and $u^*(i; \cdot)$ are continuous. The proof proceeds by showing that $\mathscr{K}(\cdot)$, the correspondence between the domain of t and the constraint set of problem (*), is both upper and lower hemi-continuous and therefore continuous.[7] Since $u_n^i(x, y)$ is a continuous function and $\mathcal{K}(\cdot)$ is a continuous and compact-valued correspondence, it follows from well-known results (cf. Hildenbrand [**6**, p. 30]) that $\bar{u}(i, t; n)$ is a continuous function of t. It then follows that $u^*(i; t)$ is a continuous function of t.

LEMMA 1: *$\bar{u}(i, t; n)$ is a continuous function of t.*

PROOF: First, observe that $\mathcal{K}(t_0) \neq \emptyset$ for any $t_0 \in (-\infty, w^i]$.

To prove that $\mathcal{K}$ is lower hemi-continuous, let $t_0 \in (-\infty, w^i]$ and let G be an open set in E^2 so that $(x_0, y_0) \in \mathcal{K}(t_0) \cap G$. Suppose without loss of generality that $G = \{(x, y): d((x, y), (x_0, y_0)) < \delta\}$ where $d((x, y), (x_0, y_0)) = \max\{\|x - x_0\|, \|y - y_0\|\}$. Let $z_0 = n(y_0 + t_0 - w^i)$ and let $G' = \{(x, z): d((x, z), (x_0, z_0)) < \delta\}$. Note that $(x_0, z_0) \in Y[n]$. By Assumption D.5 there exists an interval, open in $-\Omega$, say $V = \{z : \|z - z_0\| < \gamma, z \leq 0\}$ for some $\gamma > 0$, $\gamma \leq \delta$, so that for all $z \in V$, $\varphi(z) \cap G' \neq 0$ where $\varphi(z) = \{(x', z): (x', z) \in Y[n]\}$. Let $V' = \{t: \|n(y_0 + t - w^i) - z_0\| < \gamma, n(y_0 + t - w^i) \in V, t \in (-\infty, w^i]\}$. Clearly, V' is open in $(-\infty, w^i]$ and if $t \in V'$, then $z = n(y_0 + t - w^i) \in V$ and (from Assumption D.5) there exists $x \in \Omega$ so that $(x, z) \in G' \cap Y[n]$. Then $(x, y_0) \in \mathcal{K}(t) \cap G$ since $x, y_0 \geq 0$, $z \leq 0$, and $(x, y_0) \in \mathcal{K}(t)$ and $\|x - x_0\| < \delta$.

To show that $\mathcal{K}$ is upper hemi-continuous, suppose (t_l) is a sequence in $(-\infty, w^i]$ which converges to t_0 and consider any sequence (ξ_l) with $\xi_l \in \mathcal{K}(t_l)$. Let $\underline{t} = \lim\text{imf}_l\, t_l$. Observe that $\underline{t} \in (-\infty, w^i]$ and that $\mathcal{K}(\underline{t}) \supseteq \mathcal{K}(t_l)$ for all l. $\mathcal{K}(\underline{t})$ is compact; therefore (ξ_l) has a cluster point and we can suppose, without loss of generality, that (ξ_l) has a limit. Let $\xi_l = (x_l, y_l)$ and let $\xi = (x, y)$ denote the limit of (ξ_l). Observe that $\lim_{l\to\infty} y_l + t_l - w^i = y + t_0 - w^i$ and $(x, n(y + t_0 - w^i)) \in Y[n]$ since $Y[n]$ is closed. Also $y \geq 0$ since $y_l \geq 0$ for all l. Consequently, $(x, y) \in \mathcal{K}(t_0)$ and $\mathcal{K}$ is upper hemi-continuous. $\mathcal{K}$ is therefore a continuous correspondence. Since $u_n^i(\cdot, \cdot)$ is continuous, $\bar{u}(i, \cdot\,; n)$ is a continuous function of t. *Q.E.D.*

LEMMA 2: *$u^*(i; t)$ is a continuous function of t.*

PROOF: Let $t_l \in (-\infty, w^i]$; assume that $t_l \to t_0$ and $u^*(i; t_l) \to L$ where $L \neq u^*(i; t_0)$. Let $n_l \in I$ satisfy $u^*(i; t_l) = \bar{u}(i, t_l; n_l)$. Suppose $L > u^*(i; t_0)$. Then, for some $\varepsilon > 0$ and for all l sufficiently large, $\bar{u}(i, t_l; n_l) \geq u^*(i; t_0) + \varepsilon$. From Lemma 1, we have $\bar{u}(i, t_0; n_l) \geq u^*(i; t_0) + \varepsilon$ which is a contradiction. If $L < u^*(i; t_0)$ another contradiction is similarly obtained. Therefore $u^*(i; t)$ is a continuous function of t.

Since $u_n^i(x, y)$ is increasing in x and y, it is immediate that $\bar{u}(i, t; n)$ and $u^*(i; t)$ are decreasing in t.

THEOREM 1: *Given any $\varepsilon_0 > 0$, there exists an r_0 sufficiently large so that if $r \geq r_0$, the ε_0-core of the economy is nonempty.*

[7] The definitions of hemi-continuity are taken from Hildenbrand [**6**, pages 21 and 26].

PROOF: If $\varepsilon_0 > w^{iq}$ for any (i, q), select ε so that $0 < \varepsilon \leqslant w^{iq}$ for all (i, q) and hereafter this ε will be denoted by ε_0.

Given any r, let $\mathscr{R}^* = \{J_1, \ldots, J_k, \ldots, J_K\}$ denote a jurisdiction structure which has the properties that for each k,

$$J_k \subset \{(i', q): q \in \{1, \ldots, r\}\}$$

for some i' (each jurisdiction contains only one type of consumer); and

$$|\{(i, q) \in R: |J[(i, q); \mathscr{R}^*]| = D(i; \varepsilon_0)\}|$$

is maximized (the number of consumers contained in jurisdictions of distinguished number size for their type relative to ε_0 is maximized). Let

$$D = \{(i, q): |J[(i, q); \mathscr{R}^*]| \in D(i; \varepsilon_0)\}$$

and let $L = R \cap \sim D$ (L is the set of all consumers not in D. Intuitively L is the set of "left-over" consumers.)

Let $d_i = \min\{d: d \in D(i, \varepsilon_0)\}$ and observe that $|L| < \Sigma_{i=1}^m d_i$ for any r. Let $t' = -|D|\varepsilon_0/|L|$. Since $|D|$ can be made arbitrarily large, and $|L|$ is bounded independently of r, t' can be made arbitrarily small (i.e., arbitrarily large in absolute value).

Select r_0 sufficiently large so that $\bar{u}(i, t'; n) \geqslant u^*(i; \varepsilon_0)$ for any $n \in I$, $n < d_i$ and for all i (from Assumptions C.3 and D.4 this is possible).

Now, given any $r \geqslant r_0$, a state of the economy in the ε_0-core will be constructed. Construct a jurisdiction structure $\mathscr{R}^*$ as above. Let

$$\langle (n^R, x^R, y^R; \mathscr{R}^*), (x, z; \mathscr{R}^*) \rangle = s(\mathscr{R}^*)$$

be a state of the economy which has the properties: (a) when $(i, q'), (i, q'') \in J_k$, then $n^{iq'} = n^{iq''} = |J_k|$, $x^{iq'} = x^{iq''} = x_k$, and $y^{iq'} = y^{iq''}$; (b) when

$$J_k \subset \{(i, q): q \in \{1, \ldots, r\}\},$$

$|J_k| \in D(i, \varepsilon_0)$, and $(i, q) \in J_k$, then x^{iq} and y^{iq} are solution values for problem (*) with $t = \varepsilon_0$ and $n = |J_k|$, and

$$z_k = |J_k|(y^{iq} + \varepsilon_0 - w^{iq});$$

(c) when

$$J_k \subset \{(i, q): q \in \{1, \ldots, r\}\},$$

$|J_k| \notin D(i; \varepsilon_0)$, and $(i, q) \in J_k$, then x^{iq} and y^{iq} are solution values for problem (*) with $t = t'$ and $n = |J_k|$, and

$$z_k = |J_k|(y^{iq} + t' - w^{iq}).$$

It is immediate that $s(\mathscr{R}^*)$ is feasible, and from the construction of t', that

$$u^{iq}(n^{iq}, x^{iq}, y^{iq}) \geqslant u^*(i; \varepsilon_0)$$

for all (i, q).

Suppose $s(\mathcal{R}^*)$ is not in the ε_0-core. Then there exists $S \subseteq R$, and a state of the economy for $\{S\}$ of S, say

$$s(\{S\}) = \langle (n'^S, x'^S, y'^S; \{S\}), (x', z'; \{S\})\rangle$$

so that S can ε_0-block $s(\mathcal{R}^*)$ with $s(\{S\})$. For at least one $(i', q') \in S$,

$$y'^{i'q'} - w^{i'q'} + \varepsilon_0 \leqslant z'/|S|$$

since

$$z' \geqslant \sum_{iq \in S} (y'^{iq} - w^{iq}) + \varepsilon_0 |S|$$

(i.e., since $s(\{S\})$ is feasible). But then by the construction of $u^*(i'; \varepsilon_0)$,

$$u^{i'q'}(n'^{i'q'}, x'^{i'q'}, y'^{i'q'}) \leqslant u^*(i'; \varepsilon_0)$$

which is a contradiction to the assumption that S can ε_0-block $s(\mathcal{R}^*)$. *Q.E.D.*

In interpreting the following theorem, it is not necessarily the case that any particular replica economy has a nonempty core containing an equal-treatment allocation where the value of the utility function of the (i,q)th consumer is $u^*(i; 0)$. For example, as discussed in [**14**], if there is only one type of consumer and the economy can be represented as a real-valued game in characteristic function form, and n^* is the unique distinguished number, then for all $mr > n^*$, the core is nonempty if and only if $mr = ln^*$ for some integer l. Also, for this example, if $mr > n^*$, the core contains only equal-treatment states of the economy where the utility of (i, q) is equal to $u^*(i; 0)$. However, it follows easily from results in [**14**] that, for the model of this paper, if $r = l_1 d_1 = \ldots = l_m d_m$ where $d_i \in D(i; 0)$ and l_i is an integer, then the core is nonempty and contains an equal-treatment state of the economy. Consequently, the following theorem can be interpreted as showing that for all replications of the economy the ε-core converges to the core of the subsequence of replica economies where consumers can be partitioned into jurisdictions of distinguished sizes for the appropriate types.

THEOREM 2: *Given any $\delta > 0$ and any $\lambda > 0$, there exists an ε^* sufficiently small and an r^* sufficiently large so that for all $r \geqslant r^*$ and for all $\varepsilon \in [0, \varepsilon^*]$, if*

$$s(\mathcal{R}) = \langle (n^R, x^R, y^R; \mathcal{R}), (x, z; \mathcal{R})\rangle$$

is in the ε-core then

$$\frac{1}{mr} |\{(i, q) \in R : \|u^{iq}(n^{iq}, x^{iq}, y^{iq}) - u^*(i; 0)\| > \delta\}| < \lambda$$

(given any $\delta > 0$ and any $\lambda > 0$, there exists some ε^* and r^* so that if $r \geqslant r^*$ and $\varepsilon \leqslant \varepsilon^*$, every state of the economy in the ε-core has the property that the percentage of consumers whose utility differs by more than δ from their utility of an equal-treatment core allocation is less than λ).

PROOF: Arbitrarily select $\varepsilon_0 > 0$ so that $\varepsilon_0 < \min_i w^i$. Let

$$t(i) = \{t : u^*(i; t) = u^*(i; 0) + \delta\}$$

if such a $t(i) > -\infty$ exists, and let $t^* = \max_i t(i)$ (for the $t(i)$'s that exist). Note that $t(i) < 0$ (since preferences are monotonic increasing in x and y). Now select an $\varepsilon^* \in (0, \varepsilon_0]$ sufficiently small so that if $\|t\| \leq \varepsilon^*$, then $\|u^*(i; t) - u^*(i; 0)\| < \delta$ for all i and $-\varepsilon^*/t^* < \lambda/3$ if t^* exists. Let $d_i = \min\{d: d \in D(i; \varepsilon^*)\}$ and let $C = \Sigma_{i=1}^m d_i w^i$. Select r^* so that $(1/mr^*) \Sigma_{i=1}^m d_i < \lambda/3$ and $(-C/mr^*t^*) < \lambda/3$ (if t^* exists).

Throughout the remainder of the proof, let

$$s(\mathscr{R}) = \langle (n^R, x^R, y^R; \mathscr{R}), (x, z; \mathscr{R}) \rangle$$

denote a state of the economy in the ε^*-core for some $r \geq r^*$, and for all $(i, q) \in R$ let $t^{iq} = w^{iq} - y^{iq} + z_k/n^{iq}$ when $(i, q) \in J_k$. We first show that

$$|\{(i, q) \in R: u^*(i; 0) - u^{iq}(n^{iq}, x^{iq}, y^{iq}) \geq \delta\}| \leq \sum_{i=1}^m d_i.$$

Next, an intermediate result, which permits us to bound $|T|$, where

$$T = \{(i, q) \in R: u^{iq}(n^{iq}, x^{iq}, y^{iq}) - u^*(i, 0) \geq \delta\},$$

is obtained. We then show that $|T| \leq -(1/t^*)(C + \varepsilon^* mr)$ if t^* exists. If t^* does not exist, it follows that $T = \varnothing$ and $|T| = 0$.

Define $S(i)$ by

$$S(i) = \{(i, q) \in R: u^*(i; 0) - u^{iq}(n^{iq}, x^{iq}, y^{iq}) \geq \delta, q \in \{1, \dots, r\}\}.$$

Note that since $\|u^*(i; 0) - u^*(i; \varepsilon^*)\| < \delta$ and since $u^*(i; t)$ is decreasing in t, $u^*(i, 0) < u^*(i; \varepsilon^*) + \delta$. Consequently, $S(i) \subseteq S'(i)$ where $S'(i)$ is defined by

$$S'(i) = \{(i, q) \in R: u^*(i; \varepsilon^*) - u^{iq}(n^{iq}, x^{iq}, y^{iq}) > 0, q \in \{1, \dots, r\}\}.$$

If $|S'(i)| \geq d_i$ for any i, a subset of $S'(i)$ containing d_i members could ε^*-block $s(\mathscr{R})$. Therefore, $|S(i)| \leq |S'(i)| < d_i$ for all i, and $\Sigma_{i=1}^m |S(i)| < \Sigma_{i=1}^m d_i$.

For each i, let

$$S''(i) = \{(i, q) \in R: t^{iq} > \varepsilon^*, q \in \{1, \dots, r\}\}.$$

Note that $u^{iq}(n^{iq}, x^{iq}, y^{iq}) < u^*(i; \varepsilon^*)$ for all $(i, q) \in S''(i)$ from the construction of $u^*(i; \cdot)$. Consequently, if $|S''(i)| > d_i$ for any i, a subset of $S''(i)$ containing d_i members could ε^*-block $s(\mathscr{R})$. It follows that $|S''(i)| < d_i$ for all i. Let $S'' = \bigcup_{i=1}^m S''(i)$. Observe that $\Sigma_{iq \in S''} t^{iq} \leq C$; otherwise $y^{i'q'} < 0$ for some $(i', q') \in S''$, which is impossible since $s(\mathscr{R})$ is feasible.

Let

$$T = \{(i, q) \in R: u^{iq}(n^{iq}, x^{iq}, y^{iq}) - u^*(i, 0) \geq \delta\}.$$

Assume that t^* exists. Observe that if $(i, q) \in T$, $t^{iq} \leq t^*$ from the definitions of t^* and of $u^*(i, 0)$. Also, observe that $\Sigma_{iq \in R} t^{iq} \geq 0$ since $s(\mathscr{R})$ is feasible. Consequently,

$$\sum_{iq \in S''} t^{iq} + \sum_{iq \in \sim(S'' \cup T)} t^{iq} \geq -\sum_{iq \in T} t^{iq}.$$

From the observations that $\Sigma_{iq \in S''} t^{iq} \leq C$ and that when $(i, q) \notin S''$, $t^{iq} \leq \varepsilon^*$, we obtain

$$C+\varepsilon^*mr \geqslant \sum_{iq\in S''} t^{iq} + \sum_{iq\in\sim(S''\cup T)} t^{iq} \geqslant -\sum_{iq\in T} t^{iq} \geqslant -|T|t^*$$

so

$$|T| \leqslant -(1/t^*)(C+\varepsilon^*mr).$$

If t^* does not exist, $T=\varnothing$ and $|T|=0$. Consequently,

$$\left|\bigcup_{i=1}^{m} S(i)\cup T\right|/mr \leqslant \sum_{i=1}^{m} d_i/mr-(C+\varepsilon^*mr)/t^*mr<\lambda$$

if t^* exists; otherwise

$$\left|\bigcup_{i=1}^{m} S(i)\cup T\right|/mr \leqslant \sum_{i=1}^{m} d_i/mr<\lambda.$$

The conclusion of the theorem follows immediately from the observation that if $\varepsilon\in[0,\varepsilon^*]$ and $s'(\mathscr{R}')$ is in the ε-core, than $s'(\mathscr{R}')$ is in the ε^*-core. *Q.E.D.*

4. THE ε-CORE AND THE TIEBOUT ε-EQUILIBRIUM

In this section it will be shown that the ε-equilibrium states of the economy are in the ε-core (for the same ε). It will be apparent that the ε'-equilibrium states of the economy are in the ε-core for all $\varepsilon'\leqslant\varepsilon$. Also, when $Y[n]$ is a convex cone for all n, it is shown that for all sufficiently large replications of the economy the ε-equilibrium exists. Finally, it is shown that the Tiebout ε-equilibrium taxes paid by most consumers converge to their local public equilibrium prices times quantities of the local public good minus their local public equilibrium profit shares.

When a state of the economy in the ε-core is interpreted as being close to the core for small ε, where "close" is in the sense of Theorem 2, then the first theorem in this section can be interpreted as showing that an ε-equilibrium state of the economy is close to the core and thus nearly Pareto optimal for small ε (the Tiebout Hypothesis).

THEOREM 3: *If* $e(\varepsilon)=\langle s(\mathscr{R}),\gamma(\mathscr{R}),\pi(\mathscr{R}),\tau\rangle$ *is an* ε*-equilibrium, then* $s(\mathscr{R})$ *is in the* ε*-core.*

PROOF: Suppose

$$e(\varepsilon)=\langle s(\mathscr{R}),\gamma(\mathscr{R}),\pi(\mathscr{R}),\tau\rangle,$$

where

$$s(\mathscr{R})=\langle(n^R,x^R,y^R;\mathscr{R}),(x,z;\mathscr{R})\rangle,$$

is an ε-equilibrium state of the economy and $s(\mathscr{R})$ is not in the ε-core. By the definition of the ε-core, there exists a subset of consumers S, a jurisdiction structure $\mathscr{S}=\{S_1, \dots, S_g, \dots, S_G\}$ of S, and a state of the economy for $\mathscr{S}$ of S,

$$s(\mathscr{S})=\langle (n'^S, x'^S, y'^S; \mathscr{S}), (x', z'; \mathscr{S})\rangle$$

so that

$$\sum_{g=1}^{G} z'_g \geqslant \sum_{iq\in S} (y'^{iq}-w^{iq})+\varepsilon|S| \quad \text{and}$$

$$u^{iq}(n'^{iq}, x'^{iq}, y'^{iq})>u^{iq}(n^{iq}, x^{iq}, y^{iq}) \quad \text{for all } (i, q)\in S.$$

It follows that there is some $g'\in\{1, \dots, G\}$ so that

$$z'_{g'} \geqslant \sum_{iq\in S_{g'}} (y'^{iq}-w^{iq})+\varepsilon|S_{g'}|.$$

By the definition of the ε-equilibrium, there exists some $V_{u'}=|S_{g'}|$, $\gamma_{u'}$, and $\pi_{u'}$ so that $\pi_{u'}\geqslant \gamma_{u'}x'_{g'}+z'_{g'}$. Substitution for $z'_{g'}$ from the feasibility condition yields

$$\pi_{u'}-\gamma_{u'}x'_{g'} \geqslant \sum_{iq\in S_{g'}} (y'^{iq}-w^{iq})+\varepsilon|S_{g'}|$$

which contradicts property (iii)(b) of Definition 7 (the ε-equilibrium). *Q.E.D.*

THEOREM 4: *Assume that $Y[n]$ is a convex cone (in addition to the previous assumptions). Then, given any $\varepsilon_0>0$, there exists an r_0 sufficiently large so that if $r\geqslant r_0$, the ε_0-equilibrium exists.*

PROOF: The proof of this theorem is a continuation of the proof of Theorem 1. Define $\mathscr{R}^*$, D. L, d_i, t', r_0, and $s(\mathscr{R}^*)$ as in Theorem 1. Let $r\geqslant r_0$.

Given $s(\mathscr{R}^*)$, if $(i, q)\in J_k$ and $|J_k|=D(i; \varepsilon_0)$, let $\tau^{iq}=\varepsilon_0+(z_k/n^{iq})$. If $(i, q)\in J_k$ and $|J_k|\neq D(i; \varepsilon_0)$, let $\tau^{iq}=t'+(z_k/n^{iq})$. Recall that $t'=-|D|\varepsilon_0/|L|$ so $|D|\varepsilon_0+|L|t'=0$ and $\Sigma_{iq\in R}\,\tau^{iq}+\Sigma_{k=1}^{K} z_k=0$. Given (x_k, z_k) (as in Theorem 1), note that $(x_k, z_k)\in$ boundary $Y[|J_k|]$ since x_k is a solution value for x in problem (*) and since preferences are monotonic. Since $Y[|J_k|]$ is a convex cone, it follows from the Minkowski Separating Hyperplane Theorem that there exists γ_k so that $\gamma_k x_k+z_k=0$ and so that the hyperplane $\gamma_k x+z=0$ is bounding for $Y[|J_k|]$. From the free-disposal assumption and Assumption D.6 (possibility of positive output), we can select $\gamma_k>0$ so that, given the price γ_k, (x_k, z_k) is a profit-maximizing state of production for the jurisdiction J_k. Define $\pi_k=0$.

For all $V_u\notin\{|J_1|, \dots, |J_k|, \dots, |J_K|\}$, $1\leqslant V_u\leqslant mr$, and $V_u\in I$, select $(x_u, z_u)\in$ boundary of $Y[V_u]$ and determine γ_u as γ_k was determined. Define $\pi_u=0$.

Suppose there exists $S\subseteq R$, x', and y'^{iq} for all $(i, q)\in S$ so that

$$u^{iq}(|S|, x', y'^{iq})>u^{iq}(n^{iq}, x^{iq}, y^{iq})$$

for all $(i, q)\in S$ and suppose that

$$\sum_{iq\in S} (y'^{iq}-w^{iq})+\gamma_u x'\leqslant \pi_u-\varepsilon_0|S|$$

where $|S|=V_u$. Since $Y[|S|]$ is a closed, convex cone, it is possible to select

$(x'', z'') \in Y[|S|]$ where $x'' \geqslant x'$ so that $\pi_u = \gamma_u x'' + z''$. It follows that

$$u^{iq}(|S|, x'', y'^{iq}) > u^{iq}(n^{iq}, x^{iq}, y^{iq}) \qquad \text{for all } (i, q) \in S \quad \text{and}$$

$$\sum_{iq \in S} (y'^{iq} - w^{iq}) \leqslant z'' - \varepsilon_0 |S|$$

and S can ε_0-block $s(\mathcal{R}^*)$ which is a contradiction to the conclusion of Theorem 1. *Q.E.D.*

The next and final theorem shows the convergence of the Tiebout ε-equilibrium taxes to the local public equilibrium prices times the quantities of the local public good minus profit shares. The interpretation of this theorem is similar to that of Theorem 2; the ε-equilibrium taxes converge to the local public equilibrium per capita "net-costs" for the subsequence of replica economies where the local public equilibrium exists.

THEOREM 5: *Given any $\delta > 0$ and any $\lambda > 0$, there exists an ε^* sufficiently small and an r^* sufficiently large so that for all $r \geqslant r^*$ and for all $\varepsilon \in [0, \varepsilon^*]$, if*

$$e(\varepsilon) = \langle s(\mathcal{R}), \gamma(\mathcal{R}), \pi(\mathcal{R}), \tau) \rangle$$

is a Tiebout ε-equilibrium, then

$$\frac{1}{mr} \left| \bigcup_{k=1}^{K} \left\{ (i, q) \in J_k : \left\| \tau^{iq} - \frac{\gamma_k}{n^{iq}} x^{iq} + \frac{\pi_k}{n^{iq}} \right\| > \delta \right\} \right| < \lambda.$$

PROOF: Select $\varepsilon^* < \min\{w^i, \delta : i \in \{1, \ldots, m\}\}$ and so that $\varepsilon^*/\delta < \lambda/2$. Let $d_i = \min\{d : d \in D(i, \varepsilon^*)\}$ and let $C = \Sigma_{i=1}^m d_i w^i$. Now select r^* so that

$$\frac{C}{mr^*\delta} + \frac{1}{mr^*} \sum_{i=1}^{m} d_i < \lambda/2.$$

Let

$$s(\mathcal{R}) = \langle (n^R, x^R, y^R), (x, z; \mathcal{R}) \rangle$$

be an ε^*-equilibrium state of the economy for any $r \geqslant r^*$. Let $t^{iq} = w^{iq} - y^{iq} + (z_k/n^{iq})$ when $(i, q) \in J_k$. Let $S' = \{(i, q) \in R : t^{iq} > \delta\}$. Since $\varepsilon^* < \delta$, $S' \subset S = \{(i, q) \in R : t^{iq} > \varepsilon^*\}$. As shown in the proof of Theorem 2, $|S| < \Sigma_{i=1}^m d_i$ and $\Sigma_{iq \in S} t^{iq} \leqslant C$. Let $U = \{(i, q) \in R : t^{iq} \leqslant -\delta\}$. Observe that $-\delta|U| \geqslant \Sigma_{iq \in U} t^{iq} \geqslant -\varepsilon^* mr - C$. Consequently, $|U| \leqslant (\varepsilon^* mr + C)/\delta$. From the definition of the ε-equilibrium, when $(i, q) \in J_k$, $t^{iq} = \tau^{iq} + (z_k/n^{iq}) = \tau^{iq} + (1/n^{iq})(\pi_k - \gamma_k x_k)$. Consequently,

$$\frac{1}{mr} \left| \bigcup_{k=1}^{K} \left\{ (i, q) \in J_k : \left\| \tau^{iq} - \frac{\gamma_k}{n^{iq}} x^{iq} + \frac{\pi_k}{n^{iq}} \right\| > \delta \right\} \right| \leqslant \frac{1}{mr} (|S| + |U|) < \lambda.$$

To complete the proof, observe that if $e(\varepsilon)$ is an ε-equilibrium for any $\varepsilon \in [0, \varepsilon^*]$, then it is an ε^*-equilibrium. *Q.E.D.*

Since the local public equilibrium prices are Lindahl prices relative to the existing jurisdiction structure, the taxes paid by "most" consumers are "nearly" their Lindahl prices times the quantities consumed of the local public good minus average profits in their jurisdictions.

5. THE ONE PRIVATE GOOD, ONE LOCAL PUBLIC GOOD RESTRICTIONS

Since it is assumed that there is only one private good and one local public good, the model and results of this paper are illustrative rather than comprehensive.

The key construct used in this paper is the distinguished number problem (*), and the generalization of the results can be conceptualized by the generalization of problem (*). In [**14**], for economies with more than one private good, the distinguished numbers were defined relative to a price system for private goods; a similar approach could be used here. In the case where a consumer can belong to only one jurisdiction and there are several local public goods, in problem (*) x can be interpreted as a vector of local public goods. For the case where a consumer can belong to different jurisdictions for different local public goods, problem (*) can be solved for each local public good.

The major apparent difficulty in generalizing the results to more than one private good lies in obtaining upper hemi-continuity, and, simultaneously, convex-valuedness of the appropriate demand correspondence (see [**15**]). While it seems that the assumptions of [**15**] (without the assumptions ensuring that consumers of each type can be partitioned into jurisdictions of distinguished number size), could be utilized to obtain existence of the Tiebout ε-equilibrium when there is more than one private good, these assumptions are quite restrictive. Nevertheless, the less restrictive nature of the ε-equilibrium, in comparison with the (competitive) local public equilibrium, and the properties of local public good economies motivate the conjecture that the existence of the ε-equilibrium would obtain without the assumptions of [**15**].

6. CONCLUSIONS

In this section, the work in this paper will be discussed and further related to Tiebout's conjectures and other literature on local public good economies and on coalition economies. Next, the model and results will be related to Roberts' paper [**9**] on the Lindahl equilibrium. The results and techniques will then be briefly related to those of Shapley and Shubik in [**10**].

As stated in the introduction, the results of this paper provide proofs of Tiebout's conjectures. As Tiebout writes:

> "(in pure public good economies) there is no mechanism to force the consumer-voter to state his true preferences; in fact, the 'rational' consumer will understate his preferences and hope to enjoy the goods while avoiding the tax. (In local public good economies, however) the consumer-voter may be viewed as picking that community which best satisfies his preference pattern for public goods. . . . Moving or failing to move replaces the usual market test of willingness to buy a (private) good and (approximately) reveals the consumer-voter's demand for public goods (his marginal rate of substitution between the

public and private good). Thus each locality has a revenue and expenditure pattern that (approximately) reflects the desires of its residents.... These models ... have mobility as a cost of registering demand. The higher this cost, ceteris paribus, the less optimal the allocation of resources (and the less close the approximations)."[8]

It would have been relatively easy to design an ε-equilibrium so that, in equilibrium, each consumer paid a Lindahl price for the local public good (or equivalently, a local public equilibrium price). The results, however, would have been much less striking. Although in the ε-equilibrium (as defined in this paper) consumers pay lump sum taxes, for small ε and large economies the taxes paid by "most" consumers are nearly their Lindahl price times the quantity consumed of the local public good minus the average profits in their jurisdiction; "most" consumers reveal "nearly" their marginal rates of substitution!

A problem with both this paper and Tiebout's is the size of the consumer's choice set of jurisdictions. In his more general model, Tiebout assumes that there are a "large" number of jurisdictions; in his "severe" model, he assumes the existence of an infinite number of communities (an infinite number of which might contain no residents). The ε-equilibrium of this paper requires that no subset of consumers would all benefit from the formation of a new jurisdiction. Both Tiebout and this author require a "large" choice set of jurisdictions. Although it is beyond the scope of this paper to investigate further equilibrium concepts, the results suggest that it might be possible to develop equilibrium concepts which have the "near" optimality properties of the Tiebout ε-equilibrium without requiring a large number of quasi or virtual jurisdictions. For example, reading of the proofs indicates that only quasi-jurisdictions less than or equal to the maximum (over types) of the distinguished numbers are required so some sort of "zoning" mechanism possibly could be built into an equilibrium to limit the number of quasi-jurisdictions.

In addition to Tiebout's seminal paper, other literature on local public good economies with endogenous jurisdiction structures is discussed in [**14**].

There are a number of other papers dealing with local public good economies where the jurisdiction structure or the number of jurisdictions is exogenous, such as [**4**, **8**, and **13**]. Since optimality in these papers is relative to the exogenous restrictions on jurisdiction structures, they will not be discussed further here.

Since local public good economies are special types of coalition economies, this paper is related to recent ones by Boehm [**1**], Greenburg [**5**], and Ichiishi [**7**] on coalition production economies. To prove existence of equilibria, these authors rely on "balancedness" assumptions. In [**15** and **16**], this author has shown that for coalition production economies, it follows from the balancedness assumption that the agents can be partitioned into "distinguished sets." Consequently, their existence proofs are more closely related to the author's existence proof in [**15**], which relied on a number-theoretical assumption on the distinguished numbers to obtain existence of equilibria.

In [**9**], Roberts investigates possibilities for an Edgeworth type equivalence theorem in public good economies. To eliminate the strict superadditivity prop-

[8] See [**12**]. Additions in parentheses are due to this author.

erty of pure public good economies, he introduces crowding and assumes that there are constant returns to group size. In terms of the notation of this paper, the assumption of constant returns to group size is equivalent to the assumption that $D(i; 0) = I$; the set of jurisdiction sizes which maximize the equal-treatment utility of a consumer of type i (relative to a zero transfer) is the set of all non-negative integers (these are also the assumptions of Tiebout's "severe" model). Roberts shows, and it also follows immediately from results in [**14**], that when all consumers are identical, the core and the set of Lindahl allocations (relative to *any* jurisdiction structure) are equivalent. However, if there is more than one consumer type and types "differ," then the core might be empty and the set of Lindahl allocations (relative to the jurisdiction structure $\{R\}$; all consumers in one jurisdiction) is not necessarily in the core. The problem in Roberts two-type example is that the two types might have different solution values for x in problem (*) (where $t = 0$) so a Lindahl equilibrium with all consumers in one jurisdiction can be blocked by any subset containing consumers of only one type. This problem also appears in [**3**].

The ε-core concept used by this author is similar to the (weak) ε-core developed by Shapley and Shubik for private good exchange economies in [**10**]. Also, the results of this paper, although for local public good economies, are in the spirit of their conjectures in [**10**] and provide conditions under which the ε-core shrinks to the set of equilibrium allocations.[9] In [**16**], the distinguished numbers are generalized to "distinguished sets." The distinguished set concept is quite general and, in fact, for economies with transferable utility, agents can be partitioned into distinguished sets if and only if the core is nonempty. The results of [**16**] are almost directly applicable to the type of economies considered by Shapley and Shubik in [**10**]. In addition, there is reason to believe, that the distinguished set concept can be generalized so that it, and the techniques of this paper, could be fruitfully applied to economies with non-convex preferences, such as Starr's in [**11**] or economies with indivisibilities, such as Broome's in [**2**].

State University of New York at Stony Brook

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