Indeterminacy in Two Sector Models with Factor Market Distortions:

The Role of VIPIRs

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Abstract: Previous literature has shown that local indeterminacy and local instability can arise in two sector models when factor market distortions create a divergence between capital intensity ranking of the sectors on a physical basis and on a value basis. We identify a previously unnoticed source of indeterminacy that arises when there are such value intensity - physical intensity reversals (VIPIRs), which is the existence of multiple static production equilibria. Specifically, there is a region of the phase plane that is consistent with three production equilibira, one with incomplete specialization in production and two other equilibria with specializaiton in the respective goods. We show how this multiplicity of equilibria can be used to construct "compound paths" in which the economy switches between production patterns over time. We show that in an open economy model with VIPIRs, there will exist compound paths that reach the steady in finite time. We also establish conditions for the existence of cyclical equilibria that alternate forever between specialization in the consumption and specialization in the investment good. Consideration of the compound paths can expand the range of parameter values for which the economy has a multiplicity of equilibrium paths and can generate paths to the steady in examples where the steady state is locally unstable.

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The two sector model growth model with constant returns to scale in production has frequently been utilized to show how dynamic indeterminacy can arise in the presence of factor market distortions. Local indeterminacy has been shown to arise both in endogenous growth models with constant returns to scale in reproducible factors (Bond, Wang, and Yip (1996), Behnabib, Meng, and Nishimura (2000)) and in models with a steady state (Benhabib and Nishimura (1998), Meng and Velasco (2004)). In each of these examples, the possibility of perverse local behavior (i.e. local indeterminacy or local instability) arises when the capital intensity ranking of the two sectors on the basis of their capital labor ratios (the physical factor intensity) differs from the capital intensity ranking of the sectors on the basis of their cost shares (the value intensity ranking).[1]

In this paper we identify a previously unnoticed characteristic of an economy with value intensity - physical intensity reversals (VIPIRs), which is that the economy will have multiple production equilibria. In the presence of VIPIRs, any factor supply and price pair consistent with an equilibrium with incomplete specialization in production in a static two sector production model will also satisfy the conditions for equilibrium with complete specialization in each of the goods. We then use this result to show that in a dynamic model with VIPIRs, there will be a region of the phase plane in which there are three separate autonomous systems of differential equations (i.e. associated with incomplete specialization and complete specialization in each good) that describe the evolution of the system under the respective production pattern. These different systems can be used to construct "compound equilibria" in which the economy switches between different patterns of production specialization over time. This introduces a new form of indeterminacy, in that there are multiple compound paths that are consistent with a

[1]In models with distortions arising from externalities, Benhabib, Meng and Nishimura (2000) define a ranking of sectors on the basis of social returns and private returns that is equivalent to the concepts of value and physical intensity rankings in the case of factor tax distortions. The equivalence between the tax distortions and externalities case is discussed in detail in footnote 3. It should be noted that the reversal of the physical and value intensity ranking is different from a factor intensity reversal. The reversal of value and physical rankings involves a comparison of the value and physical intensities at a given wage/rental ratio, and can only occur in the presence of factor market distortions. A factor intensity reversal occurs when the physical intensity ranking of the sectors differs when compared at different wage/rental ratios.

competitive equilibrium satisfying the transversality condition. These compound paths can have a variety of different forms, and can include paths that reach the steady state in finite time or cycle forever between production patterns.

In an economy without VIPIRs, the set of factor supplies and goods prices for which there is incomplete specialization will be disjoint from the sets for which there is specialization in production in the respective goods. Therefore, it is safe to ignore the possibility of complete specialization when examining behavior of the dynamic system in the neighborhood of a steady state with incomplete specialization in production. As a result, relatively little attention has been paid to the dynamics of two sector models in regions where there is complete specialization. In an economy with VIPIRs, on the other hand, our results show that the complete specialization equilibria assume a new importance because they can be part of a compound equilibria in the neighborhood of the steady state.

In section II, we establish our result on the multiplicity of equilibria in the static two sector production model by showing that when there is a VIPIR, the set of capital labor ratios consistent with factor market equilibrium with incomplete specialization is contained in the set of capital labor ratios that yield factor market equilibrium with complete specialization in each of the two goods. Thus, there will be three equilibria for each of these capital labor ratios. In the absence of a VIPIR, on the other hand, the set of capital labor ratios consistent with incomplete specialization are disjoint from the set of capital labor ratios that yield complete specialization.

The remainder of the paper shows how this result can be used to construct compound equilibria in the neighborhood of the steady state of a two sector model with a consumption good and an investment good.[2] In section III, we consider the case of an open economy in which the consumption good is traded and the investment good is non-traded. This model provides a useful starting point because households

[2]Boldrin and Deneckere (1990) show that oscillatory or chaotic behavior can arise in a discrete time model in which there are factor intensity reversals. The source of their result is quite different than the one we examine here, since their production model has a unique equilibrium for a given factor endowment.

can smooth consumption on international bond markets, so the conditions for local indeterminacy are determined by production side considerations alone. If there is a VIPIR, the steady state is characterized by local indeterminacy (local instability) when the investment good sector is capital (labor) intensive in the value sense but labor (capital) intensive in the physical sense. We also identify two types of compound paths that can arise in the neighborhood of the steady state when there is a VIPIR. One possibility is a complete specialization path that reaches the steady state in finite time, followed by incomplete specialization at the steady state thereafter. The initial period of specialization will involve specialization in the investment (consumption) good if the initial capital stock is below (above) the steady state. We also show that there will be cyclical compound paths, in which the economy alternates forever between specialization in the consumption good and specialization in the investment good. These results show that for the open economy case, compound paths introduce an additional type of indeterminacy in the case where the steady state exhibits local indeterminacy. Furthermore, in the case where the steady state is locally unstable, compound paths provide the possibility of a stable path to the steady state as well as the possibility of paths with cycles.

In section IV we consider the closed-economy version of the model where a VIPIR is not a sufficient condition for local indeterminacy (as shown by Benhabib and Nishimura (1998)). Local indeterminacy in the closed economy model requires both a VIPIR (in which the investment good sector is capital intensive in the value sense) and an intertemporal elasticity of substitution that is sufficiently large. However, we show that compound equilibria will exist if there is a VIPIR, regardless of the magnitude of the intertemporal elasticity of substitution. That is, we show that indeterminacy in the larger sense that includes compound equilibria depends only on the factor market distortions and not on the value of the intertemporal elasticity of substitution parameter.

II. Equilibrium in the Two Sector Production Model with Factor Market Distortions

We begin by analyzing the static equilibrium of a two good, two factor production model in the presence of factor market distortions where the relative price of outputs is taken as given. This analysis is basic to two sector dynamic models with accumulation of physical capital accumulation only (one sector produces a consumption good and the other the investment good) and also to models with both physical and human capital acccumulation (one sector produces a consumption/investment good and the other produces human capital). Factor prices must be such that they satisfy competitive profit conditions and full employment conditions at each point in time along the equilibrium path in each of these types of model, although the models will differ in the constraints imposed on the time path of prices by intertemporal arbitrage conditions. Our main result of this section is to establish the multiplicity of equilibria when there is a VIPIR. Since this result does not depend on the dynamic structure of the particular model, we simply identify the sectors as 1 and 2.

Each good is produced under conditions of constant returns to scale and perfect competition using labor and capital, with factors being exchanged in competitive factor markets. The aggregate stocks of labor (L) and capital (K) will be taken as exogenously given, as are the prices of the goods. We describe the factor market distortions as being due to the imposition of sector-specific taxes on the earnings of the factor of production, with t_{ji} denoting the tax rate on income earned from factor $j \in \{K,L\}$ in sector $i \in \{1,2\}$. Letting w and r denote the after-tax return to labor and capital received by households, the pretax cost of the respective factors in sector i will be $w_i = wT_{Li}$ and $r_i = rT_{Ki}$, where $T_{ji} = 1/(1 - t_{ji})$. The technology in sector i is assumed to be described by a strictly concave production function $f_i(k_i)$, where k_i is the capital labor ratio employed in sector i. The cost minimization problem yields a unit cost function $c^i(w_i, r_i)$, where c^i is strictly concave and homogeneous of degree 1 in factor prices.

The following definitions, which are due to Jones (1971), will be useful in characterizing the technologies and the magnitude of the factor market distortions.

Definition: Let $k_i(w_i/r_i) \equiv c^i_{r_i}/c^i_{w_i}$ and $\theta_{Ki}(w_i/r_i) \equiv r_i c^i_{w_i}/c^i$; Sector i is defined to be capital intensive in a <u>physical</u> sense if $k_i(wT_{Li}/(rT_{Ki})) > k_j(wT_{Lj}/(rT_{Kj}))$, and it is capital intensive in a <u>value</u> sense if

$\theta_{Ki}(wT_{Li}/(rT_{Ki})) > \theta_{Kj}(wT_{Lj}/(rT_{Kj}))$.

If sector i is capital intensive in the physical sense, it will also be capital intensive in the value sense if $k_i/k_j > (T_{Li}/T_{Ki})(T_{Kj}/T_{Lj})$. The rankings on physical and value measures can disagree if the relative tax on labor is sufficiently greater in the sector that is capital intensive in the physical sense.

We will impose the following restriction on the technologies:

<u>*Assumption 1:*</u> *k_i - k_j and θ_{Ki} - θ_{Kj} do not change sign for $w/r \in (0, \infty)$.*

In the absence of distortions, Assumption 1 rules out factor intensity reversals as is commonly done in the literature on two sector models. In the presence of distortions, we are imposing a slightly restriction that prevents reversals of either of the measures as factor prices change. It will hold, for example, if the cost functions have a constant elasticity of substitution that is common across sectors. The commonly assumed case of Cobb-Douglas production functions in each sector is thus a special case of our analysis. It should be emphasized that this assumption is primarily used to simplify the exposition, because it leads to a single value of w/r consistent with incomplete specialization for each p. We note below how the argument can be extended to the case where there are reversals of factor intensities.

The competitive profit conditions in sector i require that

$$p_i \leq c^i(wT_{Li}, rT_{Ki}) \quad \textbf{(1)}$$

We choose good 1 as numeraire, and let p denote the relative price of good 2. The full employment conditions for labor and capital require that

$$\begin{aligned} c_{w_1}(wT_{L1}, rT_{K1})x_1 + c_{w_2}(wT_{L2}, rT_{K2})x_2 &= 1 \\ c_{r_1}(wT_{L1}, rT_{K1})x_1 + c_{r_2}(wT_{L2}, rT_{K2})x_2 &= k \end{aligned} \quad \textbf{(2)}$$

where x_i is output of sector i normalized by the aggregate stock of labor and $k \equiv K/L$. Using these conditions, we can define a competitive equilibrium in the production economy with distortions.[3]

Definition: A static production equilibrium will be values {w, r, x_1, x_2} satisfying (1) and (2) for given {k, p}.

(a) An equilibrium with ***incomplete specialization*** *will exist if there are factor prices {w,r} such that (1) is satisfied with equality in each sector and (2) has a solution with $x_1, x_2 > 0$.*

(b) An equilibrium with ***specialization in good i*** *will exist if there are factor prices {w,r} such that $c^i(wT_{Li}, rT_{Ki}) = p_i$, $c^j(wT_{Lj}, rT_{Kj}) > p_j$ for $j \neq i$, and (2) has a solution with $x_j = 0$.*

A. Factor Endowments and Production Equilibrium

It is well known that in the case without distortions, the endowment space will be divided into 3 (non-overlapping) regions for a given p. For k sufficiently low (high), the economy will be specialized in the labor (capital) intensive goods, and for "intermediate" values of k the economy will produce both goods. This generates a unique production equilibrium for each k. However, what has not been investigated is the extent to which this result relies on agreement of the value intensity and physical intensity rankings of the sectors. The following result establishes that when there is a VIPER, the sets of endowments consistent with the respective equilibria overlap.[4]

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[3]Meng and Velasco (2004) analyze a case in which the sectoral production function takes the form $X_i = L_i^{\alpha_i} K_i^{\beta_i} \bar{K}_i^{1-\alpha_i-\beta_i}$, where $\alpha_i > 0$, $\beta_i > 0$, $1 - \alpha_i - \beta_i \geq 0$, and $\bar{K}_i$ denotes the industry capital stock. The distortion in this case arises from the externality from capital, which firms ignore in making decisions on the level of capital. In this case the loci of factor prices that are consistent with factor market equilibrium (i.e. the equating of private marginal products to the factor price) will have the form $w_i = p_i \alpha_i (r_i/\beta_i)^{(\alpha_i - 1)/\alpha_i}$. These conditions yield equilibrium conditions analogous to those in (1), where the social factor shares α_i and $(1-\alpha_i)$ play the role of the value shares and determine the impact of goods price changes on factor prices. The capital labor ratios, on the other hand, are determined by the private factor shares α_i and β_i.

[4]Although there is an extensive literature on factor market distortions, the possibility of multiple production equilibria when value and physical intensities differ has received relatively little attention in the literature. Magee (1973) discusses the possibility of conflicts between value and physical intensities, but focuses on the comparison between the ranking of industries in the absence of distortions with that which occurs in distorted equilibria. Neary (1978) appends an adjustment process to the static model and shows that the interior equilibria are unstable under this adjustment process when value and physical intensity rankings differ (see the discussion in section III). He then argues that the economy will move either to another equilibrium where value and physical rankings coincide or to an

Let $\omega \equiv w/r$ and define $g(\omega) = c^2(\omega T_{K2}, T_{L2}) / c^1(\omega T_{L1}, T_{K1})$, $\bar{p} = \sup_{\omega} g(\omega)$, and $\underline{p} = \inf_{\omega} g(\omega)$.

Proposition 1: (a) For $p \in (\underline{p}, \bar{p})$*, there will exist unique factor prices* $\{\tilde{w}(p), \tilde{r}(p)\}$ *satisfying (1) with equality in both sectors. Letting* $\tilde{k}_i(p) \equiv k_i(\tilde{\omega}(p))$*, an equilibrium with incomplete specialization and factor prices* $\{\tilde{w}(p), \tilde{r}(p)\}$ *will exist for* $k \in \left(k_{min}(p) \equiv \min_i \tilde{k}_i(p)), k_{max}(p) \equiv \max_i \tilde{k}_i(p) \right)$.

(b) Let κ *denote the sector that is capital intensive in the value sense, and* λ *the sector that is labor intensive in the value sense. For* $k > \tilde{k}_\kappa(p)$*, there will exist a complete specialization equilibrium in good* κ *with factor prices determined by the conditions* $k = k_\kappa(w/r)$ *and* $c^\kappa(wT_{L\kappa}, rT_{K\kappa}) = p_\kappa$*. For* $k < \tilde{k}_\lambda(p)$*, there will exist a complete specialization equilibrium in good* λ *with factor prices determined by the conditions* $k = k_\lambda(w/r)$ *and* $c^\lambda(wT_{L\lambda}, rT_{K\lambda}) = p_\lambda$.

Proof: The function $g(\omega)$ will be continuous and increasing (decreasing) in ω if good 2 is the labor (capital) intensive good in the value sense. It then follows from assumption 1 that for $p \in (\underline{p}, \bar{p})$, there will exist a unique ω satisfying $g(\omega) = p$ which we denote $\tilde{\omega}(p)$. Solutions of (2) with $x_1, x_2 > 0$ will then exist for $k \in \left(k_{min}(p), k_{max}(p)\right)$. For $\omega < \tilde{\omega}(p)$, we have $g(\omega) < (>)\ p$ if good 2 is the labor (capital) intensive good. This means that for $\omega < \tilde{\omega}(p)$, $c^\kappa(wT_{L\kappa}, rT_{K\kappa}) > p_\kappa$ if $c^\lambda(wT_{L\lambda}, rT_{K\lambda}) = p_\lambda$. Production of the capital intensive good will be shut down, which will satisfy (2) for $k = k_\lambda(w/r)$. Since $k_\lambda(\omega)$ is an increasing function, a complete specialization equilibrium in good λ will exist for all $k < \tilde{k}_\lambda(p)$. Repeating this argument for $\omega > \tilde{\omega}(p)$ establishes the existence of a complete specialization equilibrium in good κ for $k > \tilde{k}_\kappa(p)$.||

Proposition 1 can then be used to identify the mapping between k and equilibrium factor prices:

Theorem 1:

(a) If $\tilde{k}_\kappa(p) > \tilde{k}_\lambda(p)$*, the value and physical intensity rankings agree and the equilibrium is unique for each k. For* $k \leq \tilde{k}_\lambda(p)$ *there is specialization in good* λ*. For* $k \geq \tilde{k}_\kappa(p)$ *there is specialization in*

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equilibrium with specialization. Our result shows that the specialization equilibria will exist for any equilibrium with incomplete specialization.

good κ, and for $k \in (k_\lambda(p),\ k_\kappa(p))$ there is incomplete specialization.

(b) If $\tilde{k}_\kappa(p) < \tilde{k}_\lambda(p)$, there is a VIPIR and there will exist 3 equilibria (incomplete specialization, specialization in κ, and specialization in ℓ) for $k \in (k_\kappa(p),\ k_\lambda(p))$. There will be a unique equilibrium with specialization in good λ for $k \leq \tilde{k}_\kappa(p)$ and in good κ for $k \geq \tilde{k}_\lambda(p)$

Figure 1a illustrates Theorem 1 for the case in which good 2 is capital intensive in both the physical and value sense, with the $k_i(\omega)$ loci illustrate factor usage in the respective sectors. The solid lines indicate values which are consistent with an equilibrium: complete specialization equilibria exist for $k < \tilde{k}_1(p)$ and for $k > \tilde{k}_2(p)$ by Proposition 1b. For $k \in (\tilde{k}_1(p),\ \tilde{k}_2(p))$ there is an equilibrium with incomplete specialization by proposition 1a, with factor prices determined by $\tilde{\omega}(p)$. The insensitivity of factor prices to factor supplies in the region of incomplete specialization is associated with the factor price equalization theorem of international trade. Since these intervals are non-intersecting, there is a unique equilibrium relationship between k and ω. Figure 1b shows the case in which good 2 is capital intensive in the value sense but labor intensive in the physical sense. The overlapping of the complete specialization regions for $k \in (\tilde{k}_1(p)\ ,\ \tilde{k}_2(p)\ )$ means that there will be 3 equilibria in this interval. [5]

The multiplicity arises because an increase in w/r from $\tilde{w}/\tilde{r}$ has two conflicting effects on the demand for capital. One effect is to induce substitution of capital for labor in each sector of the economy. The second effect is to result in the shutdown of sector 1 (labor intensive in the value sense), which reduces the demand for capital when $k_1(\tilde{w}/\tilde{r}) > k_2(\tilde{w}/\tilde{r})$. The value of w/r at which these two effects exactly offset will also be an equilibrium of the static model. A similar argument applies for reductions in w/r in the neighborhood of $\tilde{w}/\tilde{r}$.

[5]If assumption 1 does not hold, then there may be multiple solutions to $g(\omega) = p$. Assuming a finite number of solutions, let $\Omega(p) = \{\omega|\ g(\omega) = p\}$. For each $\omega \in \Omega(p)$, the interval $\left(k_{min}(\omega), k_{max}(\omega)\right)$ is consistent with an equilibrium with incomplete specialization. The set of k of for which there is an equilibrium with incomplete specialization is then a finite union of open intervals. For each of these intervals, the arguments of Proposition 1 can be used to show that the equilibrium will be unique iff the value and physical intensities agree.

B. Comparative Statics for the Production Equilibria

We conclude the discussion of the production equilibrium by summarizing the relationship between the exogenous variables (k,p) and the endogenous variables (w, r, x_1, x_2) for the respective equilibria. These results will be useful in characterizing the dynamics of the model. In the region of incomplete specialization, the system (1) - (2) has a block recursive structure and the factor prices can be solved independently of factor supplies. The equilibrium values can be expressed as $\left(\tilde{w}(p),\tilde{r}(p),\tilde{x}_1(p,k),\tilde{x}_2(p,k)\right)$. Total differentiation of the competitive profit conditions yields the well known Stolper-Samuelson relationship between goods prices and factor prices

$$\hat{r} = \left(\frac{\theta_{L1}\,\hat{p}}{\theta_{K2} - \theta_{K1}}\right) \qquad \hat{w} = -\left(\frac{\theta_{K1}\,\hat{p}}{\theta_{K2} - \theta_{K1}}\right) \tag{3}$$

where a "^" over a variable denotes a percentage change. An increase in the price of the investment good raises the real return to the factor used intensively in that sector and reduces the return to the other factor. Note that is the value intensity ranking of the sectors that determines which factor benefits from an increase in the price of good 2. Defining $\sigma_i = \hat{k}_i/(\hat{w} - \hat{r})$ to be the elasticity of substitution between labor and capital in sector i, the impact of changes in output prices on factor usage in sector i to be

$$\frac{\hat{k}_i}{\hat{p}}\Big|_{k_i = \tilde{k}_i(p)} = \frac{\sigma_i}{\theta_{K1} - \theta_{K2}} \tag{4}$$

An increase in the relative price of the labor intensive good raises w/r from (3), which will raise capital usage in sector i. The magnitude of this effect is proportional to the elasticity of substitution.

Utilizing the solutions $\tilde{w}(p)$ and $\tilde{r}(p)$ in (3), we can solve for the output levels as

$$\tilde{x}_1(p,k) = \left(\frac{k - \tilde{k}_2(p)}{\tilde{k}_1(p) - \tilde{k}_2(p)}\right) f_1(\tilde{k}_1(p)) \qquad \tilde{x}_2(p,k) = \left(\frac{\tilde{k}_1(p) - k}{\tilde{k}_1(p) - \tilde{k}_2(p)}\right) f_2(\tilde{k}_2(p)) \tag{5}$$

In the output expressions, output of good i per unit labor is expressed as the product of the share of labor employed in sector i and output per worker employed in the sector. Differentiating this expression and using (4) yields

$$\hat{x}_1 = \left(\frac{k}{k - \tilde{k}_2} \right) \hat{k} - \left[\frac{\left(\theta_{K1} k_2 + \theta_{L1} k_1\right) \sigma_1 + \left(\frac{k_2(k_1 - k)}{k - k_2} \right) \sigma_2}{(\theta_{K1} - \theta_{K2})(k_1 - k_2)} \right] \hat{p}$$

$$\hat{x}_2 = \left(\frac{k}{k - \tilde{k}_1} \right) \hat{k} + \left[\frac{\left(\frac{k_1(k - k_2)}{k_1 - k} \right) \sigma_1 + \left(\theta_{K2} k_1 + \theta_{L2} k_2\right) \sigma_2}{(\theta_{K1} - \theta_{K2})(k_1 - k_2)} \right] \hat{p} \qquad \textbf{(6)}$$

The first term in each expression is the Rybczynski effect, which shows that an increase in the stock of a factor of production will increase the output of the good that uses that factor intensively more than proportionally and decrease the output of the other good. The relevant factor intensity for the Rybczynski effect is the physical factor intensity. The second term in each expression shows that an increase in the relative price of good 2 will raise the output of good 1 and reduce the output of good 2 iff there is not VIPIR.

In the complete specialization equilibria, factor prices will be determined by the requirement that all factors be employed in the operative sector. With competitive factor markets we have $x_i = f_i(k)$, $rT_{Ki} = p_i f_i'(k)$, and $wT_{Li} = p_i\left(f_i(k) - kf_i'(k)\right)$ when the economy is specialized in good i. This yields

$$\hat{r} = \hat{p} - \theta_{Li}\sigma_i\hat{k}, \qquad \hat{w} = \hat{p} + \theta_{Ki}\sigma_i\hat{k}, \qquad \hat{x}_i = \theta_{Ki}\hat{k} \qquad \textbf{(7)}$$

Factor prices are affected by the relative factor supplies when the economy is completely specialized.

III. A Small Open Economy Model with Physical Capital Accumulation

In this section we consider a model of a small open economy that produces a traded consumption good and a non-traded investment good. Households are assumed to be able to lend and borrow on international capital markets at a fixed rate of interest that is equal to the discount rate of households, which provides a particularly simple environment because the intertemporal arbitrage condition is independent of the level of consumption.[6] We first establish conditions for existence of a steady state with incomplete specialization and show that the local dynamics fail to exhibit saddle path behavior iff the value and physical intensity rankings of the sectors differ. We then use multiplicity of production equilibria identified in Theorem 1 for these cases to construct compound equilibria, in which the economy moves between different types of equilibria (i.e. complete and/or incomplete specialization) along the path. Compound equilibria provide an additional source of indeterminacy when the steady state exhibits local indeterminacy, and provide the only paths to the steady state in the case where the steady state exhibits local instability.

Letting good 2 be the investment good, the assumption that investment goods are non-traded yields

$$\dot{k} = x_2 - \delta k \tag{8}$$

where δ is the rate of depreciation on capital. Under the assumption of an open economy with a fixed world interest rate of ρ, the dynamics of the relative price of the investment good (p) will be governed by the intertemporal aribitrage condition[7]

[6]This model has been analyzed by Meng and Velasco (2004) for the case where production functions are Cobb-Douglas

[7] Let household preferences for the consumption good are given by $U = \int_0^\infty u(C_t)e^{-\rho t}dt$. The household budget constraint will be $C_t = w_t L + r_t K_t + \rho B_t - \dot{B}_t - p_t(\dot{K}_t + \delta K_t)$, where B_t is the stock of bonds at time t. It is straightforward to show that under these assumptions households will have a constant level of consumption, with investment decisions in capital made to maximize the present value of income and bond holdings used to smooth consumption.

$$\dot{p} = (\rho + \delta)p - r \qquad \textbf{(9)}$$

Theorem 1 identifies the static equilibrium that is consistent with a particular (p,k) pair, so the corresponding solutions for r and x_2 can be substituted into equations (8) and (9) to yield an autonomous system of differential equations in (p,k).

A. Existence and of a Steady State

We begin by establishing the existence of a steady state with incomplete specialization for this model, and then characterize the local dynamics.

Proposition 2: If $\min_p \tilde{r}(p)/p \leq \rho + \delta \leq \max_p \tilde{r}(p)/p$ *and* $T_{K2} > \delta/(\delta + \rho)$*, then there will be a unique steady state* $k^{SS} < f_2^{-1}(\delta k)$ *with* $(r/p)^{SS} = \rho + \delta$*. The economy is incompletely specialized at this steady state.*

Proof: See Appendix.

Note that two restrictions are required to guarantee the existence of a steady state. The first requires that the after tax returns on capital be sufficiently productive that it be profitable to invest in capital, but not so profitable that growth be unbounded. This restriction imposes conditions both on the technology and on the level of distortions.[8] The second condition imposes a bound on the amount of subsidy to capital in the investment good sector, which is required to prevent steady states in which only the investment good is produced.

Figure 2 illustrates the region of incomplete specialization and the steady values in the phase plane for the case in which $\theta_{K2} > \theta_{K1}$. There will be a negative relationship between p and $\tilde{w}/\tilde{r}$ when good 2 is capital intensive in the value sense, so the $\tilde{k}_i$ loci will be downward-sloping. The $\dot{p} = 0$ locus will be horizontal at p^{SS}, and $\dot{p}$ is decreasing in p from (3) as illustrated by the arrows in Figure 2. It can be shown using (6) that the $\dot{k} = 0$ locus will be downward-sloping regardless of the physical intensity

[8] A necessary condition for this to hold is $\lim_{k \to 0} f_2'(k) > (\rho + \delta)T_{K2} > \lim_{k \to \infty} f_2'(k)$. If there is a finite upper bound on the marginal product of capital, there will also be a finite upper bound on T_{K2}.

ranking. The $\dot{k} = 0$ locus must be contained in the region of incomplete specialization for $k \leq k^{SS}$, and there will exist a $\bar{k}_2 > k^{SS}$ at which it intersects the $\tilde{k}_2$ locus. Figure 3 illustrates the phase plane for the case where $\theta_{K2} < \theta_{K1}$. In this case there is a positive relationship between p and $\tilde{w}/\tilde{r}$, so the slopes of the $\tilde{k}_i$ loci are reversed and $\dot{p}$ is increasing in p . In addition, the $\dot{k} = 0$ locus will be upward sloping.

The physical intensity rankings will have two impacts on the dynamics in the phase plane diagrams in Figures 2 and 3. The first is that the sign of $\partial\dot{k}/\partial k$ and $\partial\dot{k}/\partial p$ depend on the sign of $k_1 - k_2$ when the economy is incompletely specialized from (6), which will affect the dynamics of the system in the neighborhood of the steady state. This point has been studied in the literature, and the results will be briefly summarized below. The second impact of the physical factor intensity rankings, which was identified in Theorem 1, is that the region $\{p,k| \; k_\ell(p) < k < k_\kappa(p)\}$ of incomplete specialization will also be consistent with an equilibrium with specialization in good 1 and 2 if the physical and value intensity rankings differ. We explore the implications of this observation for the dynamics in this region in the following section.

The local dynamics of the system can be established by linearizing the system of differential equations (8) and (9) in the neighborhood of the steady state using (3) and (6). This yields

$$\begin{pmatrix} d\dot{p} \\ d\dot{k} \end{pmatrix} = \begin{pmatrix} A & 0 \\ \partial x_2/\partial p & \Gamma \end{pmatrix} \begin{pmatrix} p - p^{SS} \\ k - k^{SS} \end{pmatrix}$$

$$A \equiv \frac{(\rho+\delta)\,\theta_{L2}}{\theta_{K1}-\theta_{K2}} \qquad \Gamma \equiv \frac{\delta k}{k - \tilde{k}_1} \tag{10}$$

Due to the block recursive structure of the system, the diagonal elements of this matrix will be the eigenvalues. The coefficient $A > 0$ iff sector 1 is capital intensive in a value sense, while $\Gamma > 0$ iff sector 2 is capital intensive in the physical sense.

The following proposition, which is proven by Meng and Velasco for the case with Cobb Douglas technologies, follows immediately from (10).

Proposition 3: If the value and physical factor intensity rankings agree, then the system (8) has a saddle path. If $\theta_{K2} > \theta_{K1}$ *and* $\lambda_{K2} < \lambda_{K1}$*, the system exhibits local indeterminacy at the steady state. If* $\theta_{K2} < \theta_{K1}$ *and* $\lambda_{K2} > \lambda_{K1}$ *the steady state is locally unstable.*

In the case where good 2 is capital intensive according to both measures ($A < 0$, $\Gamma > 0$), the price adjustment process is stable but the capital adjustment process (at fixed p) is unstable. If the capital stock is increasing, then p must be decreasing along the saddle path to discourage the output of good 2 and prevent the capital stock from diverging. The saddle path must thus be negatively sloped and lie above the $\dot{k} = 0$ in the region of incomplete specialization in the neighborhood of the steady state in Figure 2. If good 2 is capital intensive in the value sense ($A < 0$) and labor intensive in the physical sense, then both the price and quantity adjustment process are stable and there will be a continuum of paths in the neighborhood of the steady state.

In the case where sector 1 is capital intensive according to both definitions ($A > 0$, $\Gamma < 0$), the price adjustment process is dynamically unstable and the only possibility for convergence to the steady state is that p jump immediately to the steady state level. At fixed p, the capital adjustment process will be stable due to the Rybczynski effect: an increase in the capital stock will decrease the output of the investment good along the adjustment process. The saddle path will thus coincide with the $\dot{p} = 0$ locus in Figure 4 in the region of complete specialization. If sector 1 is capital intensive in the value sense but labor intensive in the physical sense, the capital adjustment process will be unstable at fixed p. Since both adjustment processes are unstable in this case, the system exhibits dynamic instability in the neighborhood of the steady state.

B. Complete Specialization and Compound Paths

We now turn to the analysis of the dynamics in the regions of the phase plane in which the economy is completely specialized. Our focus in this section will be on cases where the value and physical rankings of the sectors differ, so that the region of incomplete specialization in (p,k) space is also contained in the regions of specialization for both goods 1 and 2.[9] In order to simplify the exposition we will refer to the two cases in which the intensity rankings differ as the "local indeterminacy" (i.e. $\theta_{K2} > \theta_{K1}$ and $\tilde{k}_1 > \tilde{k}_2$) and "local instability" ($\theta_{K2} < \theta_{K1}$ and $\tilde{k}_1 < \tilde{k}_2$) cases, reflecting the dynamics of the local linear approximation to the system of differential equations with incomplete specialization at the steady state. For each of these cases there will be three separate pairs of equations in the region of the phase plane between the $\tilde{k}_1$ and $\tilde{k}_2$, each of which is an autonomous system describing the movement of p and k under one of the production equilibria defined in Theorem 1.

Our purpose in this section is to illustrate how "compound" equilibria can be constructed, in which the economy moves from one pattern of production to another along the path. This introduces another dimension of indeterminacy to the dynamics, since these paths are in addition to any equilibrium paths that may exist that involve incomplete specialization along the entire path.

Definition: A <u>*compound equilibrium*</u> *is described by continuous functions p(t) and k(t) and a set of dates* $\{T_0, ..\}$ *such that*

$$p(T_{i+1}) = \int_{T_i}^{T_{i+1}} \big((\rho + \delta)p(s) - r^i(p(s),k(s))\big)ds + p(T_i), \qquad k(T_{i+1}) = \int_{T_i}^{T_{i+1}} \big(x_2^i(p(s),k(s)) - \delta k(s)\big)ds + k(T_i),$$

and $\lim_{t\to\infty} p(t)k(t)e^{-\rho t} = 0$ *where* r^i *and* x_2^i *denote equilibrium relationships for the type of equilibrium chosen on segment i.*

[9]In cases where the value and physical factor intensity rankings agree, the dynamic adjustment process may involve periods in which the economy is specialized in the investment (consumption) good if the initial level of the capital stock is extremely high (low). This can be seen most simply using Figure 3 where good 1 is capital intensive according to both measures. The saddle path adjustment process in this case is along the $\dot{p} = 0$. For k < $\tilde{k}_2(p^{SS})$, we have $p < p^{SS}$ for the entire range of prices consistent with incomplete specialization and there is no path that can reach the steady state and remain entirely in this region. The adjustment path in this case would involve specialization in the investment good until the economy reaches $\tilde{k}_2(p^{SS})$. A similar argument establishes that there will be specialization in the consumption good for k > $\tilde{k}_1(p^{SS})$.

The compound equilibrium requires that agents have common beliefs that the economy will follow the path associated one type of equilibrium (i.e. incompletely or completely specialized) from T_i to T_{i+1}, and will then switch to another type of equilibrium at $T_{i+1.}$ This requirement clearly involves significant coordination of beliefs, but similar coordination of beliefs arises in situations where there is local indeterminacy and agents must choose from a continuum of paths that converge to the steady state. The requirement that the paths are continuous will ensure that the compound path satisfies the conditions for an equilibrium for the entire path $[T_0, \infty)$. The set of switch dates could be either finite or infinite, as we illustrate in examples below.

If the economy is specialized in the consumption good, which we refer to as regime I, we can substitute from (7) into (8) and (9) to obtain a system of equations in p and k,

$$\begin{aligned} \dot{p} &= p(\rho + \delta) - f_1^{\prime}(k)/T_{K1} \\ \dot{k} &= -\delta k \end{aligned} \qquad \textbf{(11)}$$

Equations (11) comprise an autonomous coupled system of differential equations, so that one and only one integral curve passes through each point in the plane. These non-intersecting integral curves solve the differential equation formed by dividing the $\dot{p}$ equation by the $\dot{k}$ equation:

$$\frac{dp}{dk} = \frac{f_1^{\prime}(k)/T_{K1} - p(\rho + \delta)}{\delta k} \qquad \textbf{(12)}$$

These curves can be described by the implicit function $\Psi_1(p, k, C_1)$, where C_1 is a constant of integration. These curves will be positively (negatively) sloped for values of k that lie to the left (right) of the $\dot{p} = 0$ locus.

The $\dot{p} = 0$ locus will be negatively sloped in the region consistent with incomplete specialization in good 1, and will go through the point $(p^{ss}, \tilde{k}_1(p^{ss}))$. In the local indeterminacy case illustrated in

Figure 4, the $\dot{p} = 0$ locus must be more steeply sloped than the $\tilde{k}_1$ locus at the intersection point.[10] Since $k^{SS} < \tilde{k}_1(p^{SS})$ in this case, the integral curve passing through (p^{SS}, k^{SS}) must be positively sloped in the neighborhood of the steady state. Denoting the constant of integration associated with the integral curve passing through (p^{SS}, k^{SS}) by C_1^{SS}, there will exist a value $k^1 < \tilde{k}_1(p^{SS})$ such that $\Psi_1(p, k, C_1^{SS})$ is contained in the region of incomplete specialization for all $k \in [k^{SS}, k^1]$. Since the $\dot{p}$ and $\dot{k}$ are strictly bounded below zero on this interval, a path starting on $\Psi_1(p, k, C_1^{SS})$ for $k \in [k^{SS}, k^1]$ will reach (p^{SS}, k^{SS}) at some finite time T. This establishes that for $k \in [k^{SS}, k^1]$ the economy has a competitive equilibrium in which agents believe that resources will be specialized in the production of good 1 in the interval [0, T] and then will switch to incomplete specialization with $k = k^{SS}$ from time T onward.

We can obtain a similar result for the local instability case, which is illustrated in Figure 5. The integral curves will be negatively sloped for all points lying to the right of the $\dot{p} = 0$ locus. Since $\tilde{k}_1(p^{SS}) < k^{SS}$, the integral curve passing through (p^{SS}, k^{SS}) will be negatively sloped in this case. Using an argument similar to that above, we obtain an interval $[k^{SS}, k^1]$ such that the economy has a competitive equilibrium in which there is specialization in good 1 for a finite time period followed by incomplete specialization thereafter. Along this path, the capital stock will be falling and the value of a unit of capital rising until the steady state is reached.

We now turn to the case in which the economy is specialized in the investment good, which we label regime II. The movement of p and k will be described by the system

$$\frac{\dot{p}}{p} = (\rho + \delta) - f_2'(k)/T_{K2}$$

$$\dot{k} = f_2(k) - \delta k \qquad \textbf{(13)}$$

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[10] Along the $\dot{p} = 0$ locus, we have $\hat{p} = \hat{r} = -\theta_{L1}\hat{k}/\sigma_1$, where the second equality follows from the definition of the elasticity of substitution and the zero profit condition in sector 1. Comparing this with (10), it can be seen that the $\dot{p} = 0$ locus will be steeper in Figure 3.

In regime II, the $\dot{p} = 0$ locus will be vertical and will intersect the $\tilde{k}_2$ locus at $\tilde{k}_2(p^{SS})$, with $\dot{p} > 0$ ($\dot{p} < 0$) for k $> \tilde{k}_2(p^{SS})$ (k $< \tilde{k}_2(p^{SS})$). The $\dot{k} = 0$ locus will be vertical at $\bar{k}_2 \equiv f_2^{-1}(\delta k)$, with $\dot{k} < 0$ ($\dot{k} > 0$) for k $> \bar{k}_2$ (k $< \bar{k}_2$). .

Equations (13) yield a system of non-intersecting integral curves as in regime I. These curves can be described by the implicit function $\Psi_{II}(p, k, C_2)$, where C_2 is a constant of integration, with the slope of an integral curve given by

$$\frac{dp}{dk} = \frac{p\left(\rho + \delta - f_2'(k)/T_{K2}\right)}{f_2(k) - \delta k} \qquad \textbf{(14)}$$

In the local indeterminacy case, we will have $\tilde{k}_2(p^{SS}) < k^{SS} < \bar{k}_2$, so the numerator and denominator of (14) will be positive for k ∈ $[\tilde{k}_2(p^{SS}), k^{SS}]$. This means that the integral curves for regime II will be positively sloped for all values of (p,k) that are consistent with specialization in good 2 in this interval, which includes the integral curve passing through (p^{SS}, k^{SS}). There will exist a value $k^0 \in [\tilde{k}_2(p^{SS}), k^{SS}]$ such that $\Psi_{II}(p, k, C_2^{SS})$ is contained in the region of incomplete specialization for $k \in [k^0, k^{SS}]$, and a path starting on this integral curve will reach k^{SS} in finite time. This is illustrated in Figure 4. In the local instability case we have $k^{SS} < \tilde{k}_2(p^{SS}) < \bar{k}_2$, so the integral curves for regime II will be negatively sloped for k ∈ $[\tilde{k}_2(p^{SS}), k^{SS}]$. This results in a negatively sloped path as illustrated in Figure 5 that will reach the steady state in finite time.[11]

[11]When the conditions for local instability are met, every non-compound equilibrium path that begins in the state-space region of incomplete specialization (with the exception of the steady state point) must in finite time hit a region of complete specialization. This result is analogous to that found in Ortigueira and Santos (2002), who considered a two-sector endogenous growth model with VIPIRS that satisfied the condition for local instability. They did not consider the possibility of compound equilibrium paths, however.

These results can be summarized as:

Theorem 2: If the value and physical intensities differ, there will exist an interval $[k^0, k^1]$ *containing* k^{SS} *such that there is a compound path that reaches the steady state in finite time. For* $k < k^{SS}$ $(k > k^{SS})$ *in this interval, this path can be constructed with specialization in good 2 (1) for a finite period followed by incomplete specialization at the steady state thereafter. Movements in p and k will be positively (negatively) correlated during the period of specialization in the case of local indeterminacy (local instability).*

C. Cyclical Equilibria

A natural question to ask is whether there exist other types of compound equilibria. The above discussion of Figure 4 shows that in the local indeterminacy case there will exist a region of the phase plane for which prices and capital stocks move in opposite directions in the respective specialization equilibria. This raises the possibility that the economy could forever alternate between specialization in the consumption good and specialization in the investment good. If there exist values (p^A, k^A) and (p^B, k^B) in this region of the phase plane and constants of integration C_I and C_{II} such that $\Psi_I(p^i, k^i, C_I) = \Psi_{II}(p^i, k^i, C_{II})$ for i = A, B, then there will be a cyclical equilibrium as illustrated by the points A and B in Figure 6 for the case of local indeterminacy. A compound equilibrium path for this case could be constructed in which p and k fall from (p^A, k^A) to (p^B, k^B) with specialization in good 1, and then rise from (p^B, k^B) to (p^A, k^A) with specialization in good 2. Such a path would be an equilibrium for any initial capital stock in the interval $[k^A, k^B]$. A similar potential exists for the case where parameter values satisfy the conditions for local instability, because prices and capital stocks will move in opposite directions in the respective specialization equilibria for a region of the phase plane as illustrated in Figure 5.

If the integral curves were respectively concave and convex as illustrated in Figure 6, then it would be straightforward to identify cyclical equilibria. Unfortunately, this property does not hold in general for these integral curves.[12] However, we can establish that the curves will have tangencies in the interior of the region of interest when there is a VIPIR. Equating the slopes of the respective curves from (12) and (14), we obtain the solution function $p = \Omega(k)$, which is the locus of tangencies between the Ψ_I and Ψ_{II} curves in (p,k) space

$$p = \Omega(k) \equiv \frac{f_1'(k)\left(f_2(k) - \delta k\right)}{T_{K1}(\rho + \delta)\left(f_2(k) - \frac{\delta}{\rho + \delta}\frac{k f_2'(k)}{T_{K2}}\right)} \qquad \textbf{(15)}$$

The following Lemma establishes some useful properties of this locus and the integral curves for each of the cases in which there is a VIPIR.

Lemma 1: (a) $\Omega(\tilde{k}_{min}(p^{SS})) > p^{SS} > \Omega(\tilde{k}_{max}(p^{SS}))$, *so there will exist a value* $z \in \left[\tilde{k}_{min}(p^{SS}), \tilde{k}_{max}(p^{SS})\right]$ *such that* $p^{SS} = \Omega(z)$.

(b) If $\tilde{k}_2(p) < \tilde{k}_1(p)$, *the* Ψ_i *(i = I, II) loci are positively sloped for* p^{SS} *and* $k \in \left(\tilde{k}_2(p), \tilde{k}_1(p)\right)$. *For* $p > (<) \Omega(k)$, *the* Ψ_I *locus will be flatter (steeper) than the* Ψ_{II} *locus.*

(c) If $\tilde{k}_1(p) < \tilde{k}_2(p)$, *the the* Ψ_i *loci are negatively sloped at* p^{SS} *and* $k \in \left(\tilde{k}_1(p), \tilde{k}_2(p)\right)$. *For* $p > (<) \Omega(k)$ *the* Ψ_{II} *locus will be flatter (steeper) than the* Ψ_I *locus.*

Proof: See appendix.

Part (a) of the Lemma shows that the $\Omega(k)$ curve is downward-sloping on average in the relevant region. Since $\Omega(k)$ is a continuous function, this guarantees a tangency between the integral curves for a (p^{SS},k) pair consistent with incomplete specialization. Since the integral curves are monotonic on the region of interest by (b) and (c), we can express them as $p = h_i(k, C_i)$ for i = I, II. Defining $H(k, C_I, C_{II}) = h_I(k,C_I) - h_{II}(k,C_{II})$, part (a) then ensures a z in the interior of the region of incomplete specialization such

[12] It can be shown by differentiation of (12) that the Ψ_I loci are concave in the region where they are upward sloping, but the Ψ_{II} curves are not necessarily convex in that region. We have been unable to establish general results on convexity/concavity of either type of integral curve in the region where they are both downward-sloping.

that $H(z, C_I, C_{II}) = 0$ and $\partial H(z, C_I, C_{II})/\partial k = 0$. It follows that if $\partial^2 H(z, C_I, C_{II})/\partial k^2 \neq 0$, there will exist values $k^A > z > k^B$ and C_{II} such that $H(k^i, C_I^0, C_{II}) = 0$ for i = A, B. There will then be a cyclical equilibrium in which the economy cycles between k^A and k^B. It is shown in the Appendix that if Ω(k) is negatively sloped for the case with local indeterminacy, then $\partial^2 H(k, C_I, C_{II})/\partial k^2 < 0$. Since Ω(k) must be negatively sloped for some portion of the region with incomplete specialization by part (a) of the Lemma, then cyclical equilibria must exist in the local indeterminacy case.

Theorem 3: If factor market distortions are such that there is a VIPIR and either

(i) $\partial^2 H(k, C_I, C_{II})/\partial k^2 \neq 0$ *at* $k = \Omega^{-1}(p^{SS})$.

(ii) the conditions for local indeterminacy are satisfied

there will exist a continuum of compound equilibria characterized by alternating periods of complete specialization. These equilibria will be characterized by capital stocks k_A *and* k_B ($k_A < k_B$) *such when the capital stock reaches* k_A (k_B) *the economy specializes in the investment (consumption) good until the capital stock reaches* k_B (k_A).

To provide an example of the existence of cyclical equilibria, we characterize the Ψ_I and Ψ_{II} curves for the case in which the production functions in each sector have the Cobb-Douglas from $x_i = k_i^{\theta_{Ki}}$. It is straightforward to show that Ω(k) must be negatively sloped for $k < \bar{k}_2$ for both the local indeterminacy and local instability choices of parameter values with Cobb Douglas production functions. For regime II, the differential equation described in (14) has a known solution expressed in the following implicit function:[13].

$$\ln\left(-\delta k^{\theta_{L2}} + 1\right) = \frac{-\delta\theta_{K2}\ln\left(k^{\theta_{K2}/T_{K2}}p\right)}{\rho + \delta - \delta\theta_{K2}/T_{K2}} + C_2 \qquad \textbf{(16)}$$

For regime I, the differential equation (12) also has a known solution described by the following implicit

[13] Briefly, the transformation $t = k^{\beta_2}p$ and $z = k^{\alpha_2}$ leads to a seperable differential equation in t and z that can be integrated. See Polyanin and Zaitsev 1995, p. 11, equation 67.

function:[14]

$$\ln(k) \quad - \quad \left(\theta_{L1} - \left(\frac{\rho+\delta}{\delta}\right)\right)^{-1} \ln\left(\left(\theta_{L1} - \left(\frac{\rho+\delta}{\delta}\right)\right) pk^{\theta_{L1}} + \frac{\theta_{K1}}{\delta T_{K2}}\right) + C_1 = 0 \qquad \textbf{(17)}$$

Parameter values $\theta_{L1} = .6$, $\theta_{L2} = .5$, $T_{K1} = T_{L1} = T_{L2} = 1$ and $T_{K2} = 2$ will satisfy the conditions for local indeterminacy, since $\theta_{K1}/\theta_{K2} = .8$ and $k_1/k_2 = 1.6$. This yields steady state values $p^{SS} = 1.1819$ and $k^{SS} = 4.634$, with $\tilde{k}^1(p^{SS}) = 4.931$ and $\tilde{k}^2(p^{SS}) = 3.698$. The integral curves $\Psi_1{}^0$ and $\Psi_2{}^0$ in Figure 7 illustrate this tangency at the steady state capital stock, and are associated with constants of integration $C_1 = 2.469$ and $C_2 = .0195$. These curves can be used to construct a composite path that reaches the steady state in finite time as discussed in the previous section. The $\Psi_{II}{}^1$ curve has a constant of integration $C_2 = .0198$, and intersects the $\Psi_1{}^0$ curve at values of 4.518 and 4.589 for the capital stock. These two curves generate a composite equilibrium in which the capital stock cycles forever between these values once the capital stock enters this region. For initial values above (below) this interval, there will be an initial period in which the economy specializes in the consumption (investment) good and moves along the respective integral curve until it hits the steady state cycle.

A similar argument can be used to establish the existence of cyclical equilibria under parameter values that satisfy the conditions for local instability. Although theorem 3 does not guarantee the existence of cyclical equilibria for this case, the following example illustrates that they will exist for some parameter values. The fact that $\Omega(k)$ is decreasing in k depends only on the requirement that $k < \bar{k}_2$ in the feasible region, which will also apply with local instability. The Ψ_1 and Ψ_{II} integral curves are negatively sloped as illustrated in Figure 5 and Lemma 1 ensures that $\Omega^{-1}(p^{SS}) \in \left(\tilde{k}_1(p), \tilde{k}_2(p)\right)$. Cyclical equilibria can be obtained for this case as well, since the slope of the Ψ_1 curves will be steeper (flatter) than the Ψ_{II}

[14] The transformation of variables $z = pk^{\alpha_1}$ leads to a separable differential equation which can be integrated to yield the above solution. See Polyanin and Zaitsez 1995, p.6, equation 26.

curves below (above) the $\Omega(k)$ locus. Parameter values $\theta_{L2} = .6$, $\theta_{L1} = .5$, $T_{K2} = T_{L1} = T_{L2} = 1$ and $T_{K1} = 2$ will satisfy the conditions for local indeterminacy, since $\theta_{K1}/\theta_{K2} = 1.25$ and $k_1/k_2 = .75$. This yields steady state values $p^{SS} = .8704$ and $k^{SS} = 5.289$, with $\tilde{k}^1(p^{SS}) = 4.882$ and $\tilde{k}^2(p^{SS}) = 6.509$. The steady state values (p^{SS}, k^{SS}) are associated with constants of integration $C_1 = 1.532$ and $C_2 = .0346$ in this case.

The cyclical equilibria present an interesting illustration of these because they show that there are compound equilibria which alternate production patterns forever without reaching the steady state. There are clearly a variety of other types of equilibria that could be constructed, including those that would involve all three types of production patterns being observed along the path. These latter equilibria could include those that reach the reach the steady state in finite time, reach the steady state asymptotically (if the parameter values are consistent with local instability), or never reach the steady state.

IV. A Closed Economy Model

We now extend the analysis to the case of a closed economy. We first show that the conditions for local indeterminacy in the closed economy model are not sufficient to generate local indeterminacy in the closed economy case - it is also required that the curvature of the utility function not be "too large."[15] In this case the conditions for the economy to have multiple equilibria are necessary but not sufficient for non-saddle path behavior (i.e. local indeterminacy/instability) to be exhibited in the neighborhood of the steady state. We then show that when compound paths are considered for capital stocks in the neighborhood of the steady state, there will be a continuum of compound paths that begin with a period of specialization in the consumption good followed by incomplete specialization on the saddle path. The consideration of compound paths expands the set of parameter values for which there are a continuum of paths converging to the steady state to coincide with that in the open economy case. Thus, we can obtain indeterminacy of paths for all values of the intertemporal elasticity of substitution.

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[15]This condition is similar to that obtained by Behnabib and Nishimura (1998) for a two sector model with Cobb Douglas production technologies and an endogenous labor supply decision of households.

In the closed economy case, both the consumption and investment good markets must clear domestically and households will not have access to the international bond market. The intertemporal arbitrage condition for this case will be

$$\dot{p} = p\left(\rho + \delta - \frac{U''(c)}{U'(c)}\dot{c}\right) - r \qquad \textbf{(18)}$$

In the absence of the international bond market, the domestic interest rate will exceed ρ iff domestic consumption is rising. The market clearing condition for the consumption good sector requires $c = x_1$, which yields

$$\dot{c} = \left(\partial x_1/\partial p\right)\dot{p} + \left(\partial x_1/\partial k\right)\dot{k} \qquad \textbf{(19)}$$

Letting $\sigma_C \equiv -U''(c)c/U'(c)$, we can substitute (19) into (18) to describe the evolution of p in terms of p and k for the closed economy model:

$$\dot{p} = \frac{1}{\Delta}\left(p(\rho + \delta) - r + \sigma_C\left(\frac{p}{x_1}\frac{\partial x_1}{\partial k}\right)\left(x_2(p,k) - \delta k\right)\right)$$

$$\Delta \equiv 1 - \sigma_C\left(\frac{p}{x_1}\frac{\partial x_1}{\partial p}\right) \qquad \textbf{(20)}$$

The term Δ captures the effect of an increase in $\dot{p}$ on the return to capital, which includes both the direct effect (i.e. the capital gain) and its impact on the marginal rate of time preference through changes in $\dot{c}$. When $\Delta > 0$, an increase in $\dot{p}$ makes holding capital more attractive. A sufficient condition for this to hold is that the output of the consumption good be decreasing in p, since then an increase in $\dot{p}$ will lower the marginal rate of time preference. If output of the consumption good is increasing in p, then $\Delta > 0$ as long as σ_c is sufficiently small.

Equations (20) and the market clearing condition for the investment good, $\dot{k} = \tilde{x}_2(p,k) - \delta k$, are a system of differential equations in p and k that describe the evolution of the closed economy under incomplete specialization. Since $\tilde{r}(p)/p = \rho + \delta$ and $k = \tilde{x}_2(p,k)/\delta$ imply $\dot{p} = \dot{k} = 0$, the values (p^{SS}, k^{SS}) derived in Proposition 2 will also be a steady state for the closed economy. A steady state will thus exist under the conditions of Proposition 2, with consumption of $c^{SS} = \tilde{x}_1(p^{SS},k^{SS})$. The dynamics in the neighborhood of the steady state can be obtained by linearizing the system in the neighborhood of the steady state.

$$\begin{pmatrix} d\dot{p} \\ d\dot{k} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta}\left(A + \frac{\sigma_c p}{x_1}\frac{\partial x_1}{\partial k}\frac{\partial x_2}{\partial p}\right) & \frac{\sigma_c p}{x_1}\frac{\partial x_1}{\partial k}\frac{\Gamma}{\Delta} \\ \frac{\partial x_2}{\partial p} & \Gamma \end{pmatrix} \begin{pmatrix} p - p^{SS} \\ k - k^{SS} \end{pmatrix} \qquad \textbf{(21)}$$

where A, Γ, and Δ are as defined in (10) and (20) above. The effects of p and k on capital accumulation are identical to those in the closed economy case analyzed in (10), but the dynamics of p are more complicated because the intertemporal arbitrage condition is affected by the growth rate of consumption. In particular, the system in (21) no longer has a block recursive structure because an increase in k affects the marginal rate of time preference through its impact on the growth rate of consumption.

The behavior of the system in the neighborhood of the steady state will be determined by the trace and determinant of the maxtrix in (21), which are given by

$$\text{Trace} = \frac{1}{\Delta}\left(A + \frac{\sigma_c p}{x_1}\frac{\partial x_1}{\partial k}\frac{\partial x_2}{\partial p}\right) \qquad \text{Det} = \frac{A\Gamma}{\Delta} \qquad \textbf{(22)}$$

It is straightforward to show that in the case where the value and physical intensity rankings of the sectors agree, the closed economy will have a saddle path as in the open economy case.[16] We will focus our attention on the case where $\theta_{K2} > \theta_{K1}$ and $k_1 > k_2$, which gave rise to local indeterminacy in the closed economy case. Local indeterminacy will arise in the closed economy case if the trace is negative and the determinant positive. Since $A < 0$, $\Gamma < 0$, $\partial x_1/k > 0$ and $\partial x_2/\partial p < 0$ in this case, the conditions for local indeterminacy will be satisfied if $\Delta > 0$. If $\Delta < 0$, the determinant will be negative and the system will have a saddle path.

These results can be summarized in the following proposition:

Proposition 4: If $\theta_{K2} > \theta_{K1}$ *and* $k_1 > k_2$ *at the steady state, then there will be a critical value* $\bar{\sigma}_c$ *such that*

(a) If $\sigma_c > \bar{\sigma}_c$*, the closed economy will have a saddle path.*

(b) If $\sigma_c < \bar{\sigma}_c$*, the closed economy will exhibit local indeterminacy.*

The dynamics for the case with $\sigma_c > \bar{\sigma}_c$, are illustrated in Figure 8. The $\dot{k} = 0$ and $\dot{p} = 0$ loci will both be negatively sloped, with the latter curve being relatively flatter. Since $\dot{k}$ is decreasing in k and $\dot{p}$ is increasing in p, convergence to the steady state will be along a negatively sloped path as illustrated by the bold line in Figure 8.

Since the value and physical intensity rankings differ in the case under consideration, Theorem 1 ensures that values of (p,k) consistent with incomplete specialization are also consistent with an equilibrium in which the economy specializes in one of the two goods. Equilibria with specialization in the investment good will not arise in the closed economy model if $\lim_{c \to 0} u'(c) = \infty$, so we concentrate on equilibria with specialization in the consumption good. There will be an integral curve through each point in the region below the $\tilde{k}^1(p)$ locus describing the dynamics of the system when the economy is specialized in the consumption good. We first show that the dynamics in this region must be such that p and k are declining when the economy is specialized in the consumption good. With this result, it will then be possible to construct compound paths in which the economy specializes in the consumption good

[16] When value and physical intensity rankings agree, the signs of A and Γ differ and $\partial x_1/\partial p < 0$. This ensures $A\Gamma/\Delta < 0$.

for a finite period of time until it hits a point (p,k) on the saddle path. The economy will then converge along the saddle path to the steady state.

Lemma 2: If the economy is specialized in the consumption good for p > 0 and $k \in (0, \tilde{k}_1(p)]$*, then* $\dot{p} < 0$ *and* $\dot{k} < 0$.

Proof: With specialization in the consumption good, the growth rate of consumption in this case will be $\dot{c}/c = \theta_{K1}\dot{k}/k$ from (7). Substituting this result in the intertemporal arbitrage condition (18), we obtain the dynamics of p and k with specialization in the consumption good to be

$$\dot{p} = p(\rho + \delta - \sigma_c\delta\theta_{K1}) - f_1'(k)/T_{K1}$$
$$\dot{k} = -\delta k \qquad \textbf{(23)}$$

The fact that $\dot{k} < 0$ for all k > 0 with incomplete specialization follows immediately. If $(\rho + \delta - \sigma_c\delta\theta_{K1}) \leq 0$, then $\dot{p} < 0$ for all p,k > 0 consistent with specialization of the consumption also follows immediately from (23). It $(\rho + \delta - \sigma_c\delta\theta_{K1}) > 0$, then the locus of values at which $\dot{p} = 0$ will be given by $p = f_1'(k)/[T_{K1}(\rho + \delta - \sigma_c\delta\theta_{K1})]$. This locus will be negatively sloped with elasticity $-\theta_{L1}/\sigma_1$, so by comparison with (4) it must be steeper than $\tilde{k}_1(p)$ at an intersection. Furthermore, this intersection must occur at $k > \tilde{k}_1(p^{SS})$ since $p^{SS}(\rho + \delta) - f_1'(k)/T_{K1} = 0$. Therefore, we must have $\dot{p} < 0$ for the region of specialization in good 1 for $k < \tilde{k}_1(p^{SS})$. ‖

The construction of compound paths that reach the steady state can be seen by considering an initial capital stock k_0 in the neighborhood of the steady state in Figure 8. An initial value p corresponding to point A will result in convergence to the steady state along the saddle path. For initial values of p on the segment AB lying above the saddle path, there will be associated positively sloped integral curves corresponding to an equilibrium with specialization in the consumption good.

Theorem 4: If $\theta_{K2} > \theta_{K1}$ *and* $k_1 > k_2$*, then there will be a continuum of paths to the steady state for initial values of k in the neighborhood of the steady state.*

V. Conclusions

In this paper we have shown how the presence of VIPIRs will lead to a multiplicity of production equilibria in the region of the phase plane for which incomplete specialization is an equilibrium. We have used this observation to create compound equilibria in which the economy switches between production regimes over time. In the open economy model, where complete specialization in either good can be part of an equilibrium path, we showed how the compound equilibria could lead to a path that reaches the steady state in finite time or to paths that cycle forever between specialization in the two goods. These paths expand the set of equilibrium paths for the case of local indeterminacy. For the case of local instability, compound paths are the only ones with the possibility of reaching the steady state.

In the case of a closed economy, it is possible to have a steady state with a saddle path in the presence of a VIPIR if the intertemporal elasticity of substitution is high enough. Complete specialization in the investment good will not be possible when the consumption good is essential in each period, but compound paths that involve periods of specialization in the consumption good are possible. In particular, we showed that there will be a continuum of paths in the neighborhood of the steady state in which initial periods of specialization in the consumption good are followed by convergence to the steady state on the saddle path.

Our analysis has focused on two sector models in which only capital is accumulated, so that the economy has a steady state. This approach could also be extended to two sector endogenous growth models of the type analyzed by Bond, Wang and Yip (1996) in which there is both physical and human capital accumulation. The dynamics of the endogenous growth model can also be expressed in terms of intertemporal arbitrage conditions and factor accumulations as in the current model, and the phase plane will also have a region with multiple production paths in the presence of a VIPIR. The characterization of compound paths for that model remains an area for future work.

Our focus in this paper has been on identifying equilibria IN (to) the model. The question of which of these equilibria are the most likely for agents to coordinate on is beyond the scope of this paper, and remains an area for future research. Given the distortions in the model, these paths will not be welfare equivalent. Thus, one role for government policy might be to facilitate coordination on the path that yields the highest welfare. A more direct role for policy would be to choose taxes to minimize the costs of the distortions, which would involve choosing taxes to correct externalities (when factor market distortions are due to externalities from factor usage) or to minimize the deadweight loss due to raising government revenue.

References

Benhabib, Jess, and Qinglai Meng and Kazuo Nishimura, “Indeterminacy under Constant Returns to Scale in Multisector Economies,” *Econometrica*, **68**, 2000,

Benhabib, Jess and Kazuo Nishimura, “Indeterminacy and Sunspots with Constant Returns,” *Journal of Economic Theory*, 81, 1998, 58-96.

Boldrin, Michele and Raymond Deneckere, “Sources of Complex Dynamics in Two-Sector Growth Models,” *Journal of Economic Dynamics and Control*, **14**, 1990, 627-53.

Bond, Eric W., and Robert A Driskill, “Distortions in Two Sector Dynamic Models with Incomplete Specialization, manuscript, 2006.

Bond, Eric W., Kathleen Trask, and Ping Wang, “Factor Accumulation and Trade: Dynamic Comparative Advantage with Endogenous Physical and Human Capital,” *International Economic Review*, **44** 2003, 1041-60.

Bond, Eric W, Ping. Wang, and Chong Yip, “A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics,” *Journal of Economic Theory*, 68, 1996, 149-73.

Jones, Ronald W., “Distortions in Factor Markets and the General Equilibrium Model of Production,” *Journal of Political Econ*omy, **79**, 1971, 437-95.

Jones, Ronald W., “The Structure of Simple General Equilibrium Models,” *Journal of Political Economy,* **73**, 1965, 557-572.

Magee, Stephen, “Factor Market Distortions, Production, and Trade: A Survey,” *Oxford Economic Papers*, **25**, 1973, 1-43.

Meng, Qinglai and Andres Velasco, “Market Imperfections and Instability in Open Economies, *Journal of International Economics*, **64**, 2004, 503-19.

Neary, Peter J., “Dynamic Stability and the Theory of Factor Market Distortions,” *American Economic Review*, **68**, 1978, 671-82.

Ortigueira, Salvador and Manuel S. Santos (2002), “Equiliabrium Dynamics in a Two Sector Model with Taxes,” Journal of Eocnomic Theory, 105, 91-119.

Polyanin, Andrei and Valentin Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press (Boca Raton, FL), 1995.

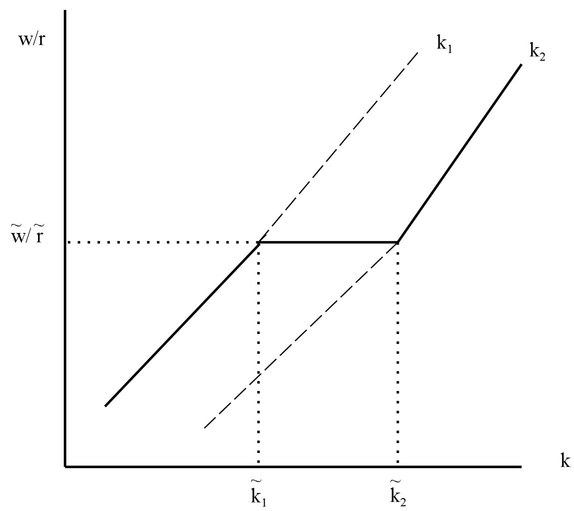

Figure 1a: $\theta_{K2} > \theta_{K1}$ and $k_2 > k_1$

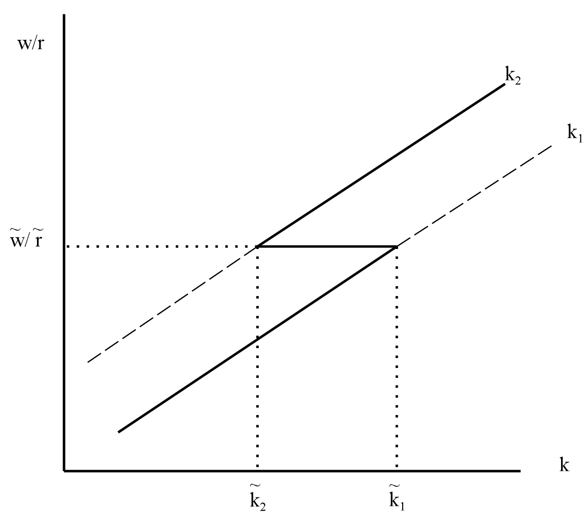

Figure 1b: $\theta_{K2} > \theta_{K1}$, $k_1 > k_2$

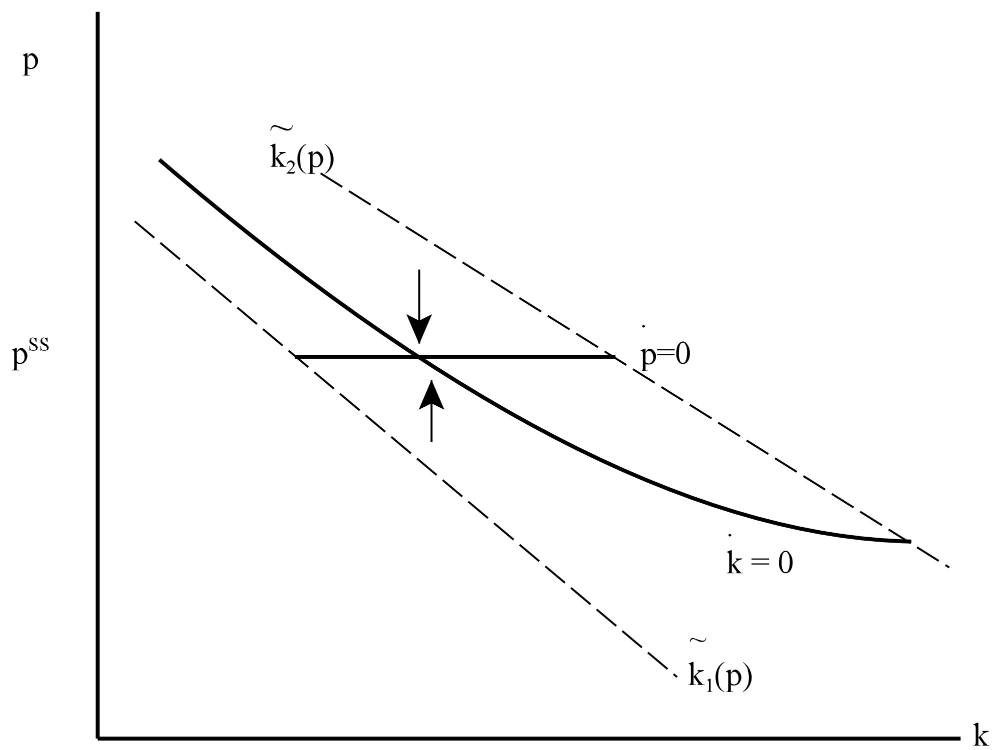

Figure 2 Phase Plane with Dynamics for Incomplete Specialization when $\theta_{K2} > \theta_{K1}$, $k_2 > k_1$

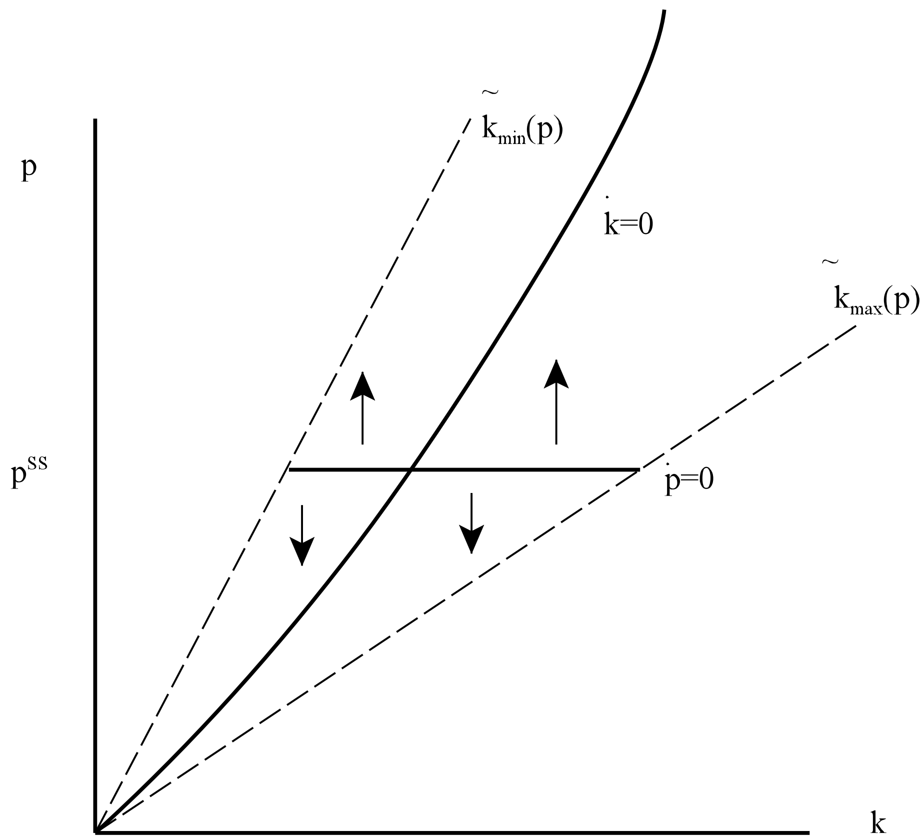

Figure 3 Phase diagram with Incomplete Specialization when $\theta_{K2} < \theta_{K1}$, $k_2 < k_1$

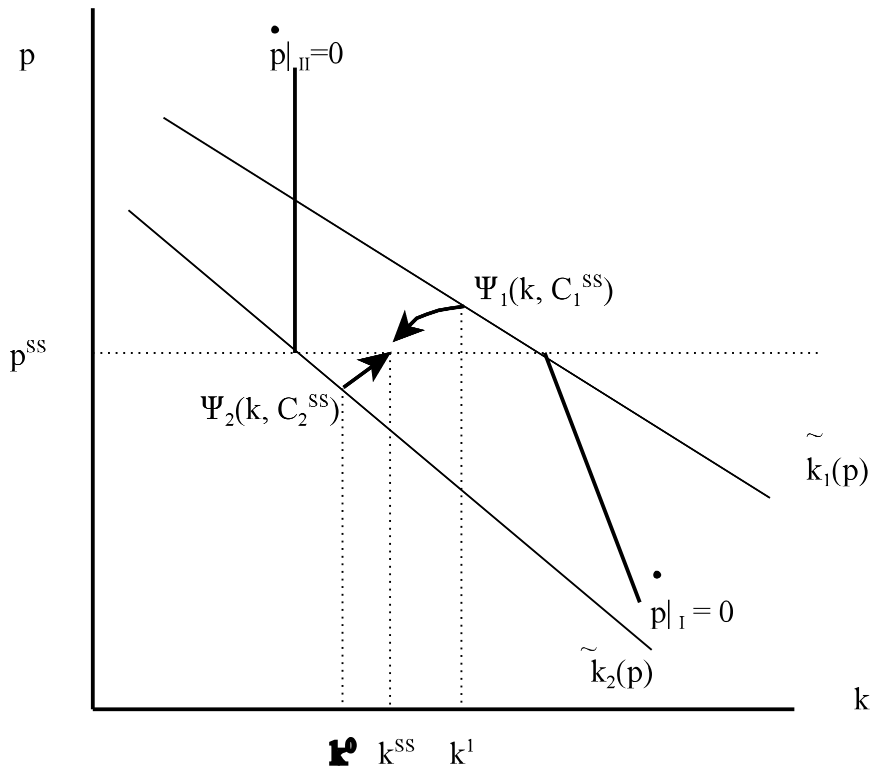

Figure 4 Compound Path to Steady State with Local Indeterminacy

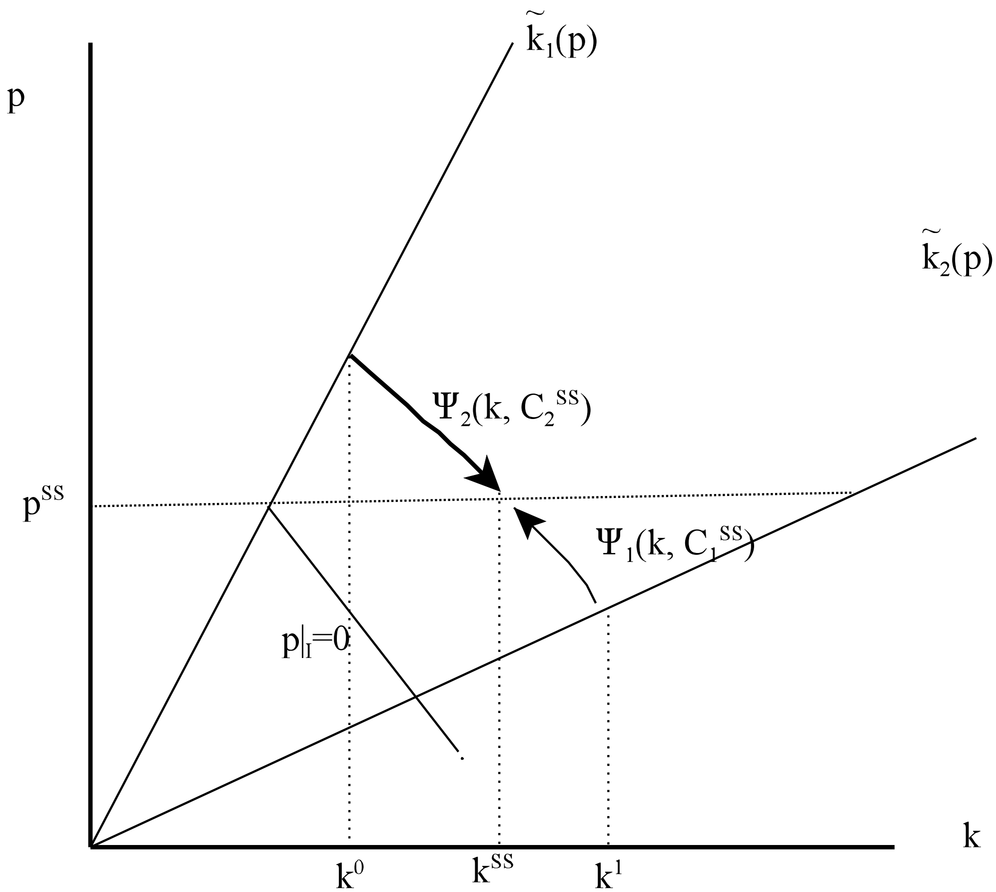

Figure 5 Compound Path to Steady State with Local Instability

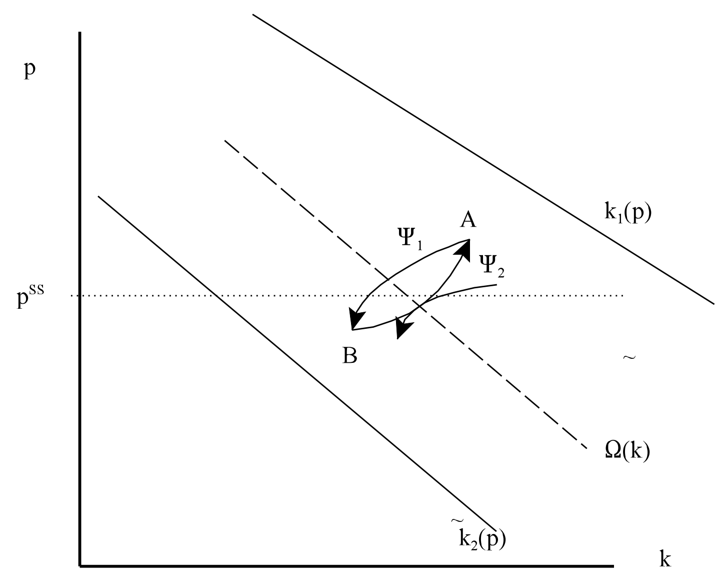

Figure 6 Cyclical Compound Paths with Local Indeterminacy

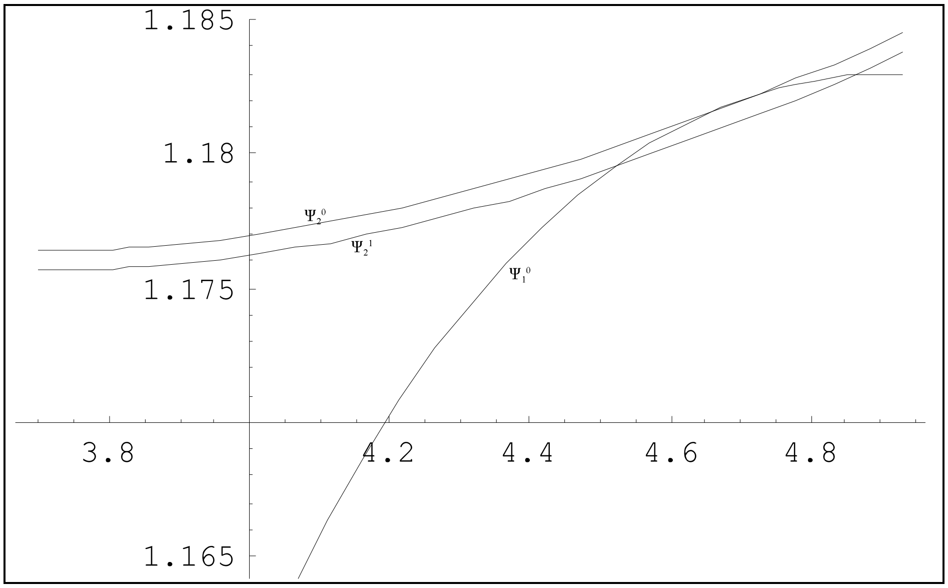

Figure 7 An Example of Cyclical Path with Local Indeterminacy

K

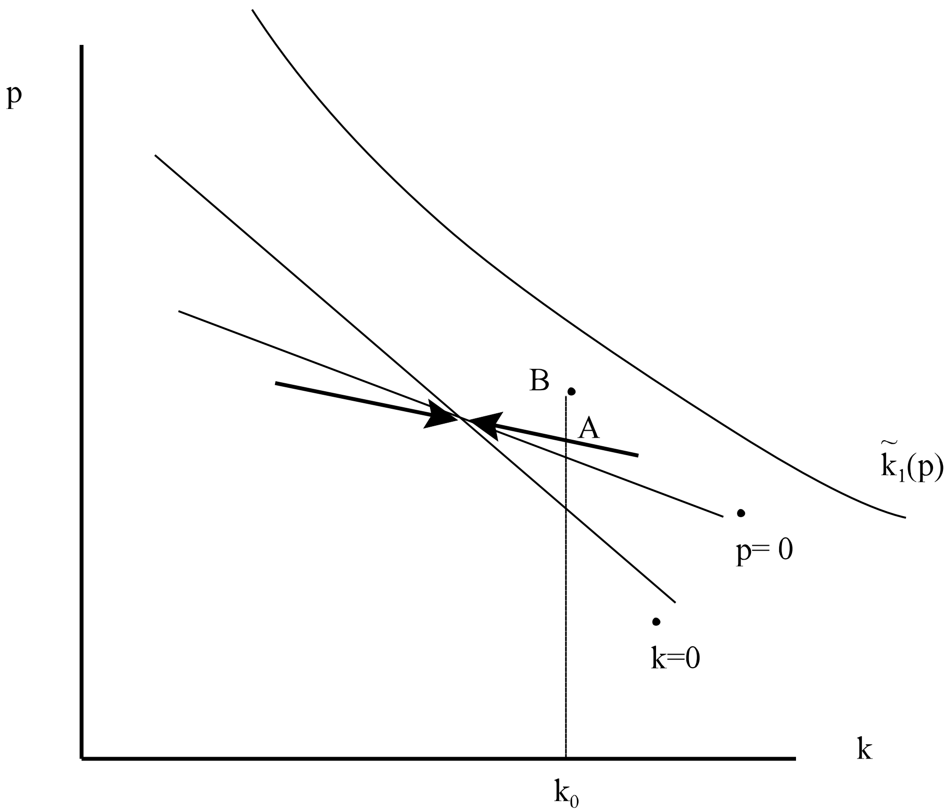

Figure 8: Local Dynamics with $\theta_{K2} > \theta_{K1}$, $k_1 > k_2$ and $\Delta < 0$

Appendix

Proof of Proposition 2: Since $\tilde{r}(p)/p$ is a monotone function of p for all factor intensity rankings by (3), there can be at most one value of p^{ss} that is consistent with incomplete specialization. The existence and uniqueness of a steady state price consistent with incomplete specialization is thus ensured if $\min_p r(p)/p \leq \rho + \delta \leq \max_p r(p)/p$.

The steady state capital labor ratio will be the value k^{SS} solving $E(k) = \tilde{x}_2(p^{SS}, k) - \delta k = 0$. From (6) it can be seen that E(k) will be linear in k with $E(\tilde{k}_1) = -\delta k$, so a unique value of k^{SS} consistent with incomplete specialization will exist if it can be shown that $E(\tilde{k}_2(p^{SS})) > 0$. Utilizing the competitive profit conditions and the fact that $r^{SS}/p^{SS} = \rho + \delta$ we have $E(\tilde{k}_2(p^{SS})) = (\rho + \delta)T_{K2}k + w^{SS}T_{L2}/p^{SS} - \delta\tilde{k}_2(p^{SS})$. A sufficient condition for $E(\tilde{k}_2(p^{SS})) > 0$ is that $T_{K2} > \delta/(\rho + \delta)$, so that a steady state with incomplete specialization will exist as long as the subsidy to capital in the investment good sector is not too large.

We next show that there can be no steady state with complete specialization and k > 0. A steady state with specialization in good 1 and k > 0 is clearly not possible because $\dot{k} = -\delta k$. A steady state with specialization in good 2 requires $f_2'(\bar{k}_2) = (\delta + \rho)T_{K2}$, where $\bar{k}$ is the minimum value of k satisfying $f(k) - \delta k = 0$. However, this marginal product condition cannot occur with for $T_{K2} > \delta/(\rho + \delta)$ because $f_2'(\bar{k}_2) \leq \delta$ by the concavity of f.‖

Proof of Lemma 1: (a) First consider the case where $\tilde{k}_2(p) < \tilde{k}_1(p)$. The respective loci will be positively sloped since $f_1'(\tilde{k}_1(p^{SS}))/(T_{K1}p^{SS}) = f_2'(\tilde{k}_2(p^{SS}))/T_{K2} = (\rho + \delta)$. Using these results in (15) yields

$$\Omega(\tilde{k}_2(p^{SS})) = p^{SS} f_1'(\tilde{k}_2(p^{SS})/f_1'(\tilde{k}_1(p^{SS})) > p^{SS} >$$
$$\Omega(\tilde{k}_1(p^{SS})) = p^{SS}\left(f_2(\tilde{k}_1(p^{SS})) - \delta\tilde{k}_1(p^{SS})\right)\Big/\left(f_2(\tilde{k}_1(p^{SS}) - \delta\tilde{k}_1(p^{SS})f_2'(\tilde{k}_1(p^{SS})/f_2'(\tilde{k}_2(p^{SS}))\right)$$

By the continuity of Ω, there will exist a $z \in \left[\tilde{k}_{min}(p^{SS}), \tilde{k}_{max}(p^{SS})\right]$ such that $p^{SS} = \Omega(z)$. It follows from (12) that the slope of the Ψ_1 schedules are decreasing in p, and from (14) that the slope of the Ψ_2 schedules is increasing in p. Therefore, the Ψ_1 loci will be flatter (steeper) than the Ψ_2 loci at intersections above (below) the Ω(k) locus.

(b) For $\tilde{k}_1(p) < \tilde{k}_2(p)$, evaluation of the expressions in (A.1) yields $\Omega(\tilde{k}_1(k^{SS}) > p^{SS} > \Omega(\tilde{k}_2(k^{SS})$. The existence of z $\in \left[\tilde{k}_{min}(p^{SS}), \tilde{k}_{max}(p^{SS}) \right]$ such that $p^{SS} = \Omega(z)$ follows from the continuity of Ω. An increase in p (at fixed k) will lead to a proportional reduction in the slope of the Ψ_2 loci from (14) and a more than proportional reduction in the slope of the Ψ_1 curve, so the Ψ_2 loci will be flatter (steeper) than the Ψ_1 loci at intersections above (below) the Ω(k) locus. ||

Proof of Proposition 3b: The p = Ω(k) line is determined by the conditions $p - h_I(k, C_I) = 0$, $H(k, C_I, C_{II}) = 0$, and $H_k(k, C_I, C_{II}) = 0$. Totally differentiating these conditions and evaluating at p = Ω(k) yields

$$\begin{pmatrix} 1 & -h_{IC_I} & 0 \\ 0 & H_{C_I} & H_{C_{II}} \\ 0 & H_{kC_I} & H_{kC_{II}} \end{pmatrix} \begin{pmatrix} dp \\ dC_I \\ dC_{II} \end{pmatrix} = \begin{pmatrix} h_{Ik} \\ 0 \\ -H_{kk} \end{pmatrix} dk$$

Solving yields $\frac{dp}{dk} = \frac{\partial h_I}{\partial k} + H_{kk} h_I T_{K1} \delta k / f'$. Since the first term must be positive in the case with inderterminacy, then $H_{kk} < 0$ is necessary for dp/dk < 0.