Practice Problems Fall 2015
Econ 263

1 Budget constraints

1. Each month, Andy gets the following (exogenous) endowments:

$$\overline{C}_A = 5, \overline{T}_A = 3.$$

Andy can exchange these endowments for currency at market prices, which are exogenous to him, in a central marketplace.

(a) If the price of coffee is 7 units of currency per unit of coffee, and the price of tea is 5 units of currency per unit of tea, show why Andy's income/month measured in currency is \$50.00 .

A:

$$\overbrace{5}^{\overline{C}_A} \times \overbrace{7}^{P_C} + \overbrace{3}^{\overline{T}_A} \times \overbrace{5}^{P_T} = 50.$$

(b) What is his income/month measured in units of coffee? In units of tea?

A:

$$\frac{50}{7} = 49\frac{1}{7} = 7.1429; \qquad \text{(C/month)}$$
$$\frac{50}{5} = 10. \qquad \text{(T/month)}$$

(c) What is the relative price of coffee? What are the units of this price?

A:

$$\frac{P_C}{P_T} = \frac{7}{5} = 1.4$$

The units are units of tea/unit of coffee.

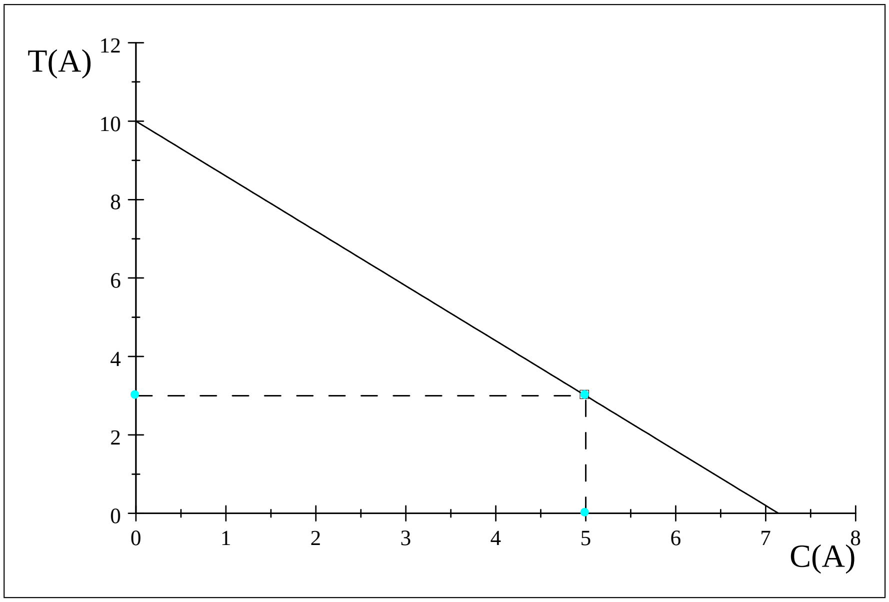
(d) Write his budget constraint in standard slope-intercept form with consumption of tea/month on the left-hand-side of the equality sign.

A:

$$T_A = 10 - 1.4C_A.$$

(e) With tea on the vertical axis and coffee on the horizontal, draw a schematic diagram of his budget constraint, making sure you identify all relevant features, i.e., slope, intercepts, and endowment point.

A:$y = 10 - 1.4x$. Coffee-intercept is 7.1429., endowment point is $(5, 3)$



(f) What would happen to this schematic diagram if both P_C and P_T were to double? Triple? Be cut in half?

A: nothing. Both slope and intercept remain unchanged.

2. Now consider another scenario. Andy grows coffee for a living, and takes his harvest to market once a year. There, he can sell as much of his crop as he wants at a market price of a certain amount of dollars per pound of coffee. While at the market, Andy can use the money he gets from selling his coffee to purchase the only other good he likes to consume, tea, at a market price of a certain amount of dollars per pound of tea.

   (a) Suppose Andy grows 80 pounds of coffee per year, and coffee exchanges in the market place for \$2.00/kilo. Tea exchanges in the market place for \$4.00/kilo. Let C_A symbolize the variable that measures the amount of coffee Andy consumes per year, and T_A symbolize the variable that measures the amount of tea Andy consumes per year. Describe in an equation with only T_A on the left-hand-side of the equality sign all those pairs of kilos of coffee/yr. and kilos of tea/year that Andy could consume at these prices, assuming he spent all of his income.

   Answer: Let me put the budget constraint in parametric form. First I express in an equation the equality of income and expenditure:

$$\overbrace{P_C C_A + P_T T_A}^{Expenditure} = \overbrace{P_T \overline{T}_A + P_C \overline{C}_A}^{Income}.$$

I then rearrange to isolate T_A on the l.h.s. of the equality sign:

$$T_A = \frac{P_C\overline{C}_A}{P_T} + \overline{T}_A - \frac{P_C}{P_T}C_A.$$

Upon substitution of the given values of the exogenous variables:

$$\begin{aligned}
\frac{P_C}{P_T} &= \frac{2}{4} = .5; \overline{C}_A = 80; \\
T_A &= .5 \times 80 - .5C_A; \\
T_A &= 40 - .5C_A
\end{aligned}$$

(b) Again suppose Andy grows 80 pounds of coffee per year, and again suppose coffee exchanges in the market place for \$2.00/pound, and tea exchanges in the market place for \$4.00/pound. Which of the following amounts of coffee and tea can Andy take home with him from the market place?

i. Ten (10) lbs. of coffee and 35 pounds of tea.

yes: $\overbrace{35}^{T_A} = \overbrace{40}^{\frac{P_C\overline{C}_A}{P_T}+\overline{T}_A} - \overbrace{5}^{.5\times \overbrace{10}^{C_A}}$

ii. 20 pounds of coffee and 30 pounds of tea.
yes: $30 = 40 - 10$

iii. 20 pounds of coffee and 35 pounds of tea.
no: $35 > 40 - 10$

iv. 30 pounds of coffee and 30 pounds of tea.
no: $30 > 40 - 15$

v. 40 pounds of coffee and 20 pounds of tea.
yes: $20 = 40 - 20$

vi. 40 pounds of coffee and 30 pounds of tea.
no: $30 > 40 - 20$

(c) What is Andy's income per year measured in units of dollars/year?
A:

$$\overbrace{80}^{\overline{C}_A} \times \overbrace{2}^{P_C} = 160\$/year$$

\$160.

3. Let P_C symbolize the variable that describes the market price of coffee in terms of dollars/pound of coffee, and let P_T symbolize the variable that describes the market price of tea in terms of dollars/pound of tea. Given that Andy produces 80 pounds of coffee per year, describe in an equation using the above symbols all those pairs of pounds of coffee/year (C_A)and pounds of tea/year (T_A) that Andy could consume for arbitrary values of

P_C and P_T , assuming he spent all of his income. In this equation, put T_A as the only variable on the left-hand-side of the equality sign.

Answer:

$$T_A = \frac{80P_C}{P_T} - \frac{P_C}{P_T}C_A.$$

4. Let $\overline{C}_A$ symbolize the variable that describes Andy's production of lbs. of coffee per year. For arbitrary values of Andy's production of lbs. of coffee/yr. and arbitrary values of the price of coffee and the price of tea, describe in an equation all those pairs of pounds of coffee/yr. and pounds of tea/yr. that Andy could consume if he spent all of his income. Again, write this equation with T_A as the only variable on the left-hand-side of the equality sign.

   Answer:

$$T_A = \frac{P_C}{P_T}\overline{C}_A - \frac{P_C}{P_T}C_A.$$

5. Draw a **schematic** diagram of the above equation in the coffee-tea plane. By coffee-tea plane, we mean the standard picture in which tea/year is measured on the vertical axis and coffee/year on the horizontal axis. By **schematic** we mean that the key qualitative features of the equation, namely the slope and the intercepts, are depicted and identified, although not necessarily to scale.

   Answer: slope and intercepts identified, and the line connecting the intercepts arguably a straight line.

6. What would happen to this schematic diagram if both P_C and P_T were to double? Triple? Be cut in half?

   A: nothing. Both slope and intercept remain unchanged.

2 Preferences, Demand, Equilibrium in GE Model

1. 15 points. Now consider an individual (Antoine) who receives an endowment every month of $\frac{3}{2}$ units of coffee and $\frac{1}{2}$ units of tea. He trades in a perfect market in which currency prices of coffee and tea are symbolized as P_C and P_T, respectively. Define the relative price of of coffee as $\frac{P_C}{P_T}$ and symbolize this as p.

2. What is Antoine's demand function when his preferences are represented by the utility functions:

$$U_A = C_A T_A. \quad (1)$$

$$U_A = C_A T_A + (C_A)^2 (T_A)^2. \quad (2)$$

$$U_A = \frac{C_A T_A}{2(C_A + T_A)}. \quad (3)$$

Answer: First, the tangency conditions for (1) and (2): Note they are identical.

$$\begin{aligned}
dU_A &= \left(-\frac{C_A T_A(2)}{(2^2(C_A+T_A)^2} + \frac{C_A(2)(C_A+T_A)}{(2(C_A+T_A))^2}\right)dT_A \\
&\quad +\left(-\frac{2C_A T_A}{(2(C_A+T_A))^2} + \frac{2T_A(C_A+T_A)}{(2(C_A+T_A))^2}\right)dC_A; \\
dU_A &= 0: \ \frac{dT_A}{dC_A} = -\frac{-2C_A T_A + 2T_A(C_A+T_A)}{-2C_A T_A + 2C_A(C_A+T_A)} \\
&= -\frac{-2C_A T_A + 2C_A T_A + 2(T_A)^2}{-2C_A T_A + 2C_A T_A + 2(C_A)^2} = -\frac{(T_A)^2}{(C_A)^2} = -p.
\end{aligned}$$

Now for (3):

$$\begin{aligned}
dU_A &= \left(-\frac{C_A T_A(2)}{(2^2(C_A+T_A)^2} + \frac{C_A(2)(C_A+T_A)}{(2(C_A+T_A))^2}\right)dT_A \\
&\quad +\left(-\frac{2C_A T_A}{(2(C_A+T_A))^2} + \frac{2T_A(C_A+T_A)}{(2(C_A+T_A))^2}\right)dC_A; \\
dU_A &= 0: \ \frac{dT_A}{dC_A} = -\frac{-2C_A T_A + 2T_A(C_A+T_A)}{-2C_A T_A + 2C_A(C_A+T_A)} \\
&= -\frac{-2C_A T_A + 2C_A T_A + 2(T_A)^2}{-2C_A T_A + 2C_A T_A + 2(C_A)^2} = -\frac{(T_A)^2}{(C_A)^2} = -p.
\end{aligned}$$

Or to put it in a form useful for graphing:

$$\begin{aligned}
(T_A)^2 &= p\,(C_A)^2; \\
T_A &= \sqrt{p}C_A.
\end{aligned}$$

Next, use the tangency conditions in the budget constraints. The budget constraint is

$$T_A = \overline{T}_A + p\overline{C}_A - pC_A$$

So, for both (1) and (2),

$$\begin{aligned}
\overbrace{pC_A}^{C_A} &= \overbrace{\frac{1}{2}}^{\overline{T}_A} + \overbrace{\frac{3}{2}}^{\overline{C}_A} p - pC_A; \\
2pC_A &= \frac{1+3p}{2}; \\
C_A^d &= \frac{1+3p}{4p};
\end{aligned}$$

For (3)

$$\begin{aligned}
(p)^{\frac{1}{2}}\,(C_A) &= \frac{1+3p}{2} - pC_A; \\
C_A(p+p^{\frac{1}{2}}) &= \frac{1+3p}{2} \\
C_A^d &= \frac{1+3p}{p+p^{\frac{1}{2}}}
\end{aligned}$$

3. 10 points. Now assume Antoines's endowment is $(3, 25)$, and his preferences are represented by

$$\begin{aligned} U_A &= T_A + 10C_A - (C_A)^2 \; for \; C_A \leq 5; \\ U_A &= T_A + 25 \; for \; C_A \geq 5. \end{aligned}$$

What is his demand curve for coffee? For tea?

A: For coffee:

$$\begin{aligned} T_A &= U_A - 10C_A + (C_A)^2, \; C_A < 5; \\ \frac{dT_A}{dC_A} &= -10 + 2C_A = -p; \\ p &= 10 - 2C_A^d; \\ C_A^d &= -\frac{1}{2}p + 5 \; ; \end{aligned}$$

For tea, we need the budget constraint:

$$\begin{aligned} T_A &= \overbrace{25}^{\overline{T}_A} + \overbrace{3}^{\overline{C}_A} p - p\left(-\frac{1}{2}p + 5\right) \\ &= 25 + (3-5)\,p + \frac{1}{2}p^2 \\ &= 25 - 2p + \frac{1}{2}p^2. \end{aligned}$$

3 Importance of substitutability (a difficult question not likely to be on an exam); but still interesting and fun.

Consider the following model of a two-person two-good endowment economy based on Radford's account of a British POW camp . Each individual has preferences over goods x and y that can be represented by the following utility functions:

$$\begin{aligned} U_1 &= \min\{\frac{x_1}{\frac{3}{4}}, \frac{y_1}{\frac{1}{4}}\}; \\ U_2 &= \min\{\frac{x_2}{\frac{3}{11}}, \frac{y_2}{\frac{8}{11}}\} \end{aligned}$$

In words, for any given permissible value of x_1, y_1, the utility level for individual one (1) associated with that point is the minimum value of the two numbers $\{\frac{x_1}{\frac{3}{4}}, \frac{y_1}{\frac{1}{4}}\}$. For example, if individual one (1) were consuming $\{x_1 = 1, y_1 = 1\}$, then this person's associated level of utility would be given by the minimum of $\{\frac{1}{\frac{3}{4}}, \frac{1}{\frac{1}{4}}\}$ which is $\frac{4}{3}$. The graphs of the families of indifference curves associated with such preferences have L-shaped individual indifference curves (with respect to the origin of the (x, y) plane) which have the right-angle apex of each curve lying along rays through the origin with equations for these rays given by:

$$y_1 = \frac{\frac{1}{4}}{\frac{3}{4}} x_1 = \frac{1}{3} x_1;$$
$$y_2 = \frac{\frac{8}{11}}{\frac{3}{11}} x_2 = \frac{8}{3} x_2.$$

Each POW in this camp receives an endowment from the camp commandant of $(\overline{x}_i, \overline{y}_i),\ i = 1, 2.$

1. (a) Suppose each of the above two individuals consumes his endowment. True (T) or false (F):

   i. _____We can unequivocally conclude that individual one (1) is better off than individual two (2).

   ii. _____We can unequivocally conclude that individual two (2) is better off than individual one (1).

   iii. _____We can unequivocally conclude that both individuals are equally well off.

   iv. _____There is no way to tell which individual is better or worse off.

   Answers: FFFT. No way to make interpersonal comparisons of utility.

   (b) Denote the relative price of x by p, i.e., $p \equiv \frac{P_x}{P_y}$. Write down the competitive budget constraint for individual i.

   Answer: $p(x_i - \overline{x}_i) = \overline{y}_i - y_i$; (any variant will do, even in nominal terms).

   (c) With the above preferences and endowments, the demand curves for each of the above individuals is given by:

$$x_1 = \frac{p\overline{x}_1}{a_1 + p} + \frac{\overline{y}_1}{a_1 + p},\ a_1 = \frac{1}{3}; \qquad (x_1^d)$$
$$x_2 = \frac{p\overline{x}_2}{a_2 + p} + \frac{\overline{y}_2}{a_2 + p},\ a_2 = \frac{8}{3}. \qquad (x_2^d)$$

   We get these by noting from a diagram that any most-preferred pair lies on the straight-line ray through the origin with slope, as given above. Subbing this "tangency condition" into the budget constraint then yields the above demand curves. Assume each prisoner receives endowments of one unit of x and one unit of y. Verify, i.e., show that the equilibrium conditions for this model are satisfied, that an equilibrium price for this competitive autarkic economy is given by:

$$\widehat{p} = \frac{11}{9}$$

Answer: For this to be an equilibrium, aggregate demand must equal aggregate supply:

$$\begin{aligned} x_1 + x_2 &= \overline{x}_1 + \overline{x}_2 = 2; \\ \overbrace{\frac{\overbrace{1}^{\overline{y}_1} + \overbrace{\left(\frac{11}{9}\right)}^{\widehat{p}\overline{x}_1}}{\frac{1}{3} + \frac{11}{9}}}^{x_1} + \overbrace{\frac{\overbrace{1}^{\overline{y}_2} + \overbrace{\left(\frac{11}{9}\right)}^{\widehat{p}\overline{x}_2}}{\frac{8}{3} + \frac{11}{9}}}^{x_2} &= \\ \frac{\frac{20}{9}}{\frac{14}{9}} + \frac{\frac{20}{9}}{\frac{35}{9}} &= \\ \frac{20}{14} + \frac{20}{35} &= \\ \frac{10}{7} + \frac{4}{7} &= 2 \end{aligned}$$

For future reference, note that $\widehat{x}_1 = \frac{10}{7}$ and $\widehat{x}_2 = \frac{4}{7}$.

(d) Compute the equilibrium quantities of y consumed by each individual.

Answer: Substitute $p = \frac{11}{9}$ and $\widehat{x}_1 = \frac{10}{7} = 1.43...$ and $\widehat{x}_2 = \frac{4}{7} = .57...$ into the budget constraint, solve for:

$$\begin{aligned} \widehat{y}_1 &= \frac{10}{21} = .48...; \\ \widehat{y}_2 &= \frac{32}{21} = 1.52.... \end{aligned}$$

For your interest, the demand curves are derived as follows from the following utility functions (known as *constant elasticity of substitution* or *CES* utility functions):

$$\begin{aligned} U_i &= [(\alpha_i x_i)^{-\rho} + \{(1-\alpha_i)y_i\}^{-\rho}]^{-\left(\frac{1}{\rho}\right)}, \\ 0 &< \alpha_i < 1; \ -1 \leq \rho \leq \infty. \end{aligned}$$

The slope of the indifference curves are given by:

$$\frac{dy}{dx} = -\left(\frac{\alpha}{1-\alpha}\right)^{-\rho}\left(\frac{y}{x}\right)^{(1+\rho)}$$

The tangency condition that $\frac{dy}{dx} = -p$ can be rearranged as

$$y = x\left(\frac{\alpha}{1-\alpha}\right)^{\left(\frac{\rho}{1+\rho}\right)} p^{\left(\frac{1}{1+\rho}\right)}$$

Note bene: Take the limit of this tangency condition as $\rho \to \infty$ yields (remember any number raised to the zero is equal to one (1)

$$\lim_{\rho\to\infty} y = \left(\frac{\alpha}{1-\alpha}\right) x.$$

Combining the tangency condition with the budget constraint and rearranging yields:

$$x = \frac{p\overline{x} + \overline{y}}{p + (\frac{\alpha}{1-\alpha})^{\left(\frac{\rho}{1+\rho}\right)} p^{\left(\frac{1}{1+\rho}\right)}}$$

Taking the limit as $\rho \to \infty$ gives the demand curves x_1^d and x_2^d. Alternatively, looking at the diagram of such indifference families shows that any individual will always consume along a ray through the origin with slope given by . Hence, $y = (\frac{\alpha}{1-\alpha})x$. Subbing this into the budget constraint yields

$$px + (\frac{\alpha}{1-\alpha})x = p\overline{x} + \overline{y}$$

or

$$x\{p + \frac{\alpha}{1-\alpha}\} = p\overline{x} + \overline{y}$$

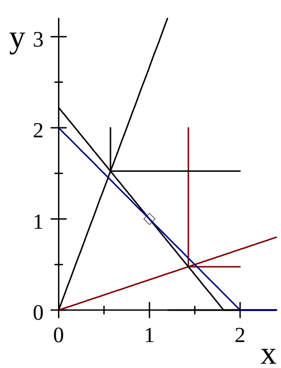
Hence,

$$x^d = \frac{p\overline{x} + \overline{y}}{p + a}; a = \frac{\alpha}{1-\alpha}$$

(e) Now imagine that, following distribution of the endowments but before the two POW's can trade with each other, this camp is integrated with the larger camp, i.e., the camp with many more prisoners from other nations, and that now the two British members can trade at the exogenous price $\widehat{p}_{FT} = 1$. That is, they go to the market carrying their red cross endowments of $(\overline{x}_i = 1, \overline{y}_i = 1, i = 1, 2)$. True (T) or false (F):

i. _____Individual one (1) is now better off than in autarky.
ii. _____Individual two (2) is now better off than in autarky.
iii. _____Individual two (2) is now worse off than in autarky.
iv. _____Both individuals are better off than in autarky.

Answer: Depict autarkic equilibrium to get the answer. Let us depict autarkic equilibrium:



Equilibrium:$y_1 = \frac{1}{3}x_1; y_2 = \frac{8}{3}x_2$

where individual 1 has indifference curves with right-angles and a "tangency condition," i.e., ray through the origin that passes through the right angles of the indifference curves, that are below those of individual 2. In autarky, each person's budget constraint goes through the endwoment point $(1, 1)$ and has slope $-\frac{11}{9}$. Now imagine rotating the budget constraint through the endowment point $(1, 1)$ so that it now has slope -1. This is depicted in the above figure as the flatter budget line. Clearly, individual one is better off and individual two is worse off. So, TFTF.

(f) Now imagine that the camp commandant intercepts the red cross packages and reallocates the two goods so that the endowments for the two POW's are given by:

$$\overline{x}_1 = \frac{10}{7}; \overline{y}_1 = \frac{10}{21}; \quad (1)$$

$$\overline{x}_2 = \frac{4}{7}; \overline{y}_2 = \frac{32}{21} \quad (2)$$

The two British POW's can now trade in the larger camp at the exogenous price $\widehat{\pi}_{FT} = 1$. True (T) or false (F):

i. _____Individual one (1) is now better off than in autarky.
ii. _____Individual two (2) is now better off than in autarky.
iii. _____Neither individual is now worse off than in autarky.
iv. _____Both individuals are better off than in autarky.
Answer: In the above diagram, imagine rotating each individual's budget constraint through the autarkic consumption points. clearly, their most-preferred pairs don't change. Hence, FFTF.

4 Preferences

For each question, be prepared to explain the steps used in how you got your answer, and be prepared to illustrate these steps with appropriate schematic diagrams.

1. Consider three individuals, Alex, Bobby, and Don, who have preferences represented by the following utility functions:

$$U_A = C_A T_A \quad \text{(Alex's)}$$

$$U_B = C_B T_B + (C_B)^2 (T_B)^2. \quad \text{(Bobby's)}$$

$$U_D = (C_D)^{\frac{1}{2}} (T_d)^{\frac{1}{2}}. \quad \text{(Don's)}$$

5 points each. True or false with short explanation:

(a) _____If both Alex and Bobby consume two (2) units of coffee and two (2) units of tea per unit of time, then we can say that Bobby has a higher level of satisfaction than does Alex, and Alex a higher level than Don.

Answer: False. First, though, consider what happens if we simply substitute the values of coffee and tea in each utility function:

$$\begin{aligned} U_A &= C_A T_A = \overbrace{2}^{C_A} \times \overbrace{2}^{T_A} = 4; \\ U_B &= 2 \times 2 + 2^2 \times 2^2 = 20; \\ U_D &= \sqrt{2 \cdot 2} = 2. \end{aligned}$$

One might be tempted to say this means Bob is better off than Andy, and Andy is better off than Don, when they each consume two (2) units of coffee and two(2) units of tea. But utility numbers are ordinal rankings, and we can't compare across individuals.

(b) _____If Alex's, Bobby's, and Don's consumptions of coffee and tea per unit of time increase from two (2) units of coffee and two (2) units of tea to three (3) units of coffee and three (3) units of tea, then we can say that Bobby's increase in well-being is greater than Alex's increase in well-being and Alex's is greatere than Don's.

Answer: false. Again, first consider a naive approach of considering what happens to changes in utility levels:

$$\begin{aligned} U_A &= \overbrace{3}^{C_A} \times \overbrace{3}^{T_A} = 9;\ 9 - 5 = 5\ utils; \\ U_B &= 3 \times 3 + 3^2 \times 3^2 = 90;\ 90 - 20 = 70\ utils; \\ U_D &= \sqrt{3 \cdot 3} = 3;\ 3 - 2 = 1\ util. \end{aligned}$$

But these are not useful for making comparisons: utility numbers provide an ordinal ranking.

(c) _____Bobby would be classified as a tea-lover relative to Alex.

Answer: false, they have the same mrs function, i.e., same slopes of indifference curves at every point in the coffee-tea plane. Calculate the $mrs's$: (for Bob, you need to use the "take total derivative of U and set equal to zero" approach): once without using total derivatives, once with)

$$\begin{aligned}
T_A &= \frac{U_A}{C_A}; \frac{dT_A}{dC_A} = -\frac{U_A}{(C_A)^2} = -\frac{\overbrace{T_A C_A}^{U_A}}{(C_A)^2} = -\frac{T_A}{C_A}; \\
U_B &= C_B T_B + (C_B)^2 (T_B)^2 = T_B C_B (1 + T_B C_B); \\
dU_B &= \left(T_B + 2(C_B)(T_B)^2\right) dC_B + \left(C_B + 2(C_B)^2 (T_B)\right) dT_B = 0; \\
\frac{dT_B}{dC_B} &= -\frac{\left(T_B + 2(C_B)(T_B)^2\right)}{(C_B + 2(C_B)^2 (T_B))} = -\frac{T_B\,(1 + 2(C_B)(T_B))}{C_B\,(1 + 2(C_B)(T_B))} = -\frac{T_B}{C_B}; \\
(T_D)^{\frac{1}{2}} &= \frac{(U_D)^{\frac{1}{2}}}{(C_D)^{\frac{1}{2}}};\ T_D = \frac{(U_D)}{C_D};\ \frac{dT_D}{dC_D} = -\frac{(U_D)}{(C_D)^2} = -\frac{\overbrace{T_D C_D}^{U_D}}{(C_D)^2} = -\frac{T_D}{C_D}.
\end{aligned}$$

2. In New Jersey, exits on the New Jersey Turnpike start at "1" at the first one north of Delaware, and continue "2," then "3," and so forth, no matter the miles between the exits, as one heads north. In Tennessee, exits on I65 are designated by how many miles they are from the border with Kentucky. Which scheme is analogous to how members of an indifference curve are identified?

   A: The NJT. It is an ordinal ranking, whereas the TN one is cardinal.

3. Andy has an indifference curve described as

$$T_A = \frac{1}{C_A}$$

   Bob has an indifference curve described as

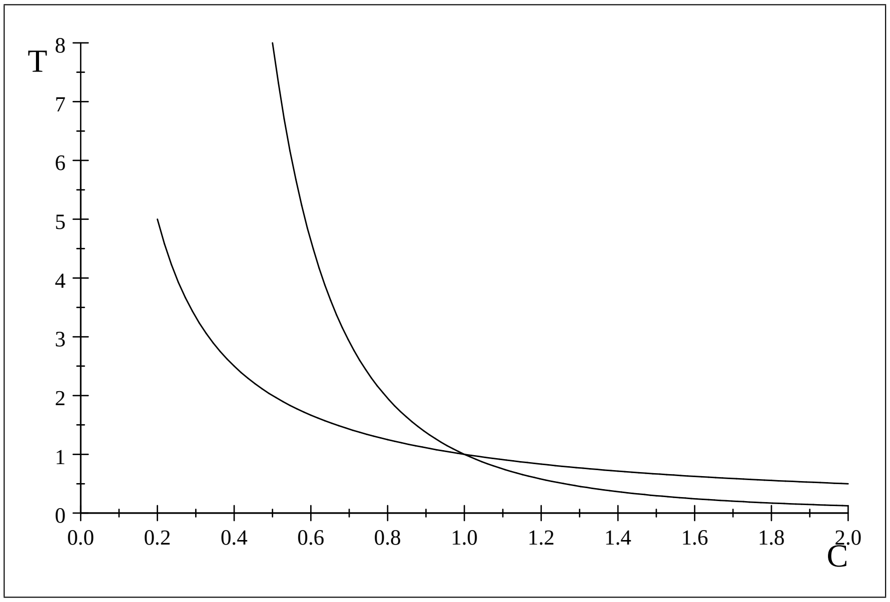
$$T_B = \frac{1}{(C_B)^3}.$$

   Who is the "coffee-lover" between these two, and why?

   A: Consider the slopes of indifference curves evaluated at the point at which they intersect, $(1, 1)$:

$$\begin{aligned}
\frac{dT_A}{dC_A} &= -\frac{1}{(C_A)^2} = -1; \\
\frac{dT_B}{dC_B} &= -3\frac{1}{(C_B)^4} = -3.
\end{aligned}$$

   This says that, at teh point $(1, 1)$, Andy is willing to substitute "as small as possible" a unit of tea for a unit of coffee. That is, if offered one infinitesimal unit of coffee in exchange for one infinitesimal unit of tea, he would be willing to make the exchange, because it would leave him on gthe same indifference curve. Bob, though, if offered one unit infinitesimal unot of coffee in exchange for tea, would require three (3) units of tea to keep him on the same indifference curve. That is, Bob loves coffee so much in comparison to Andy that he needs to be compensated for the loss of one unit of coffee with three units of tea. Bob is the coffee-lover, relatively speaking. These indifference curves are plotted below, with Andy's "steeper" at (1.1):



5 Income distribution as source of trade

1. Andy and Bob are two POW's in an English camp. They have preferences specified as:

$$U_A = (C_A)^{\frac{3}{4}} (T_A)^{\frac{9}{4}};$$
$$U_B = (C_B)^{\frac{1}{2}} (T_B)^{\frac{1}{2}}.$$

which have known MRS functions:

$$MRS_A = \frac{1}{3}\frac{T_A}{C_A};\ MRS_B = \frac{T_B}{C_B}.$$

Nota bene:

$$\begin{aligned}
U_A &= (C_A)^{\frac{3}{4}} (T_A)^{\frac{9}{4}} ; \\
\frac{3}{4} + \frac{9}{4} &= \frac{12}{4} = 3 \\
3\gamma_1 &= \gamma_2 \\
\gamma_1 &= \frac{1}{3}\gamma_2; \\
\gamma_1 + \gamma_2 &= 1;\ \gamma_1 + 3\gamma_1 = 1; \\
\gamma_1 &= \frac{1}{4},\ \gamma_2 = \frac{3}{4} \\
U_A &= \left((C_A)^{\frac{1}{4}} (T_A)^{\frac{3}{4}}\right)^3 \\
U_B &= (C_B)^{\frac{1}{2}} (T_B)^{\frac{1}{2}} .
\end{aligned}$$

(a) 10 points. Andy receives an endowment of one (1) unit of tea, and zero (0) units of coffee. Bob receives an endowment of zero (0) units of tea and one (1) unit of coffee:

$$\begin{aligned}
\overline{T}_A &= 1;\ \overline{C}_A = 0; \\
\overline{T}_B &= 0;\ \overline{C}_B = 1.
\end{aligned}$$

Construct the general equilibrium (GE) demand curves for coffee for each of these individuals, and construct the market general equilibrium demand curve.

Answer: First, budget constraints:

$$\frac{P_T}{P_T}T_A + \frac{P_C}{P_T}C_A = \frac{P_T}{P_T}\overbrace{\overline{T}_A}^{1} + \frac{P_C}{P_T}\overbrace{\overline{C}_A}^{0};$$

$$\frac{P_T}{P_T}T_B + \frac{P_C}{P_T}C_B = \frac{P_T}{P_T}\overbrace{\overline{T}_B}^{0} + \frac{P_C}{P_T}\overbrace{\overline{C}_B}^{1}$$

Now, tangency conditions:

$$\frac{dT_A}{dC_A} = -\frac{1}{3}\frac{T_A}{C_A} = -p \Rightarrow$$
$$T_A = 3pC_A$$
$$\frac{dT_B}{dC_B} = -\frac{T_B}{C_B} = -p \Rightarrow$$
$$T_B = pC_B$$

Combine each individual's tangency condition with his or her budget constraint to get his or her demand curve:

$$\begin{aligned}
C_A^d &= \frac{1}{4p}R_A; \\
C_B^d &= \frac{1}{2p}R_B
\end{aligned}$$

where R_A and R_B are Andy and Bob's **real incomes measured in units of tea.** (For future reference, note that combining these demand curves with each individual's respective tangency condition yields the most-preferred choices of tea as functions of p and R_i) :

$$\begin{aligned} T_A^d &= \frac{3}{4}R_A; \\ T_B^d &= \frac{1}{2}R_B. \end{aligned}$$

Back to the problem at hand. For the specified endowments, their GE demand functions are

$$C_A^d = \frac{1}{4p}$$

$$C_B^d = \frac{1}{2}$$

Adding up:

$$C^d \equiv C_A^d + C_B^d = \frac{1}{2} + \frac{1}{4p}$$

Miller time!

(b) 10 points. For this two-person economy, calculate the equilibrium relative price of coffee and the equilibrium quantities of coffee and tea consumed by each individual.

Answer: Equilibium condition:

$$C_A^d + C_B^d = \overbrace{1}^{\overline{C}_A + \overline{C}_B}$$

Substituting for C_i^d, $i = A, B$, into the equilibrium condition and solving for p:

$$p_a = \frac{1}{2}.$$

The subscript a is a mnemonic for "autarky." Subbing this back into the demand equations gives

$$\widehat{C}_A = \frac{1}{2}; \widehat{C}_B = \frac{1}{2};$$

Using the tangency conditions,

$$\widehat{T}_A = \frac{3}{4}; \widehat{T}_B = \frac{1}{4}.$$

(c) 10 points. In the French POW camp, Alphonse has identical preferences to Andy, and Baptiste has identical preferences to Bob. What is different are the endowments each receives: Alphonse receives an endowment of zero (0) units of tea, and one (1) unit of coffee, while Baptiste receives an endowment of one (1) unit of tea and zero (0) units of coffee**:**

$$\begin{aligned} \overline{T}_A &= 0;\ \overline{C}_A = 1; \\ \overline{T}_B &= 1;\ \overline{C}_B = 0. \end{aligned}$$

That is, the endowments have been switched relative to Andy and Bob. Construct the equilibrium demand curves for coffee for Alphonse and Baptiste and construct the market equilibrium demand curve.

Answer:the tangency conditions remain the same:

$$\begin{aligned} T_A &= 3pC_A; \\ T_B &= pC_B. \end{aligned}$$

The budget constraints are now:

$$\begin{aligned} T_A + pC_A &= p; \\ T_B + pC_B &= 1. \end{aligned}$$

Upon substitution of the tangency conditions into the budget constraints, we have

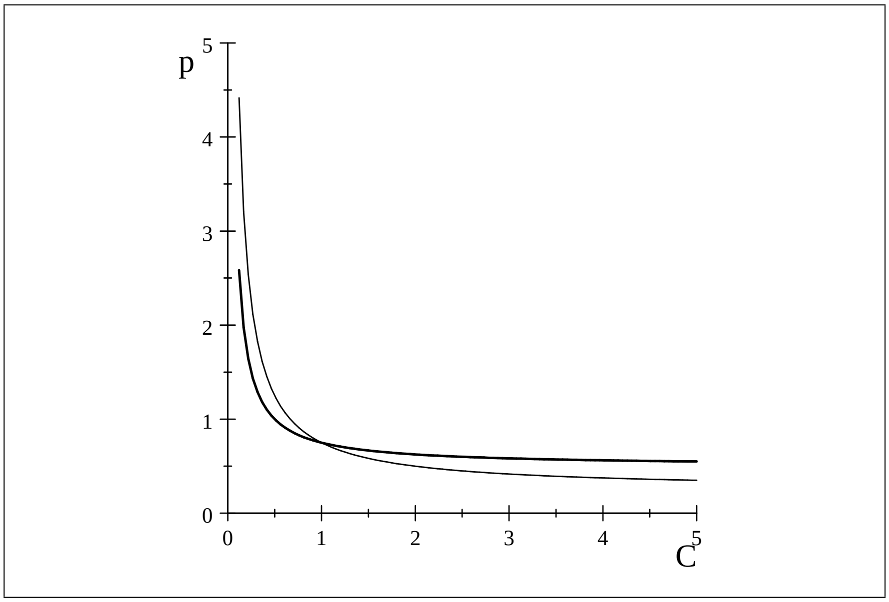
$$\begin{aligned} C_A^d &= \frac{1}{4}; \\ C_B^d &= \frac{1}{2p}. \end{aligned}$$

Aggregate or market demand:

$$C^d = \frac{1}{4} + \frac{1}{2p}$$

The two aggregate demand curves (one for the English and one for the French) differ, even though *aggregate* endowments are the same and preferences are the same in both scenarios. We depict the two demand curves below, where the thick line represents Andy and Bob's market demand curve:

$y = .25 + .5x^{-1}$



The point of this exercise: aggregate (market) demand functions depend on income distribution. This is a **caveat** for most of the models we use, in which we focus on model specifications that have demand functions invariant to income distribution. We focus on these models that don't have "income effects" because: (1) observation suggests it is not such a strong effect that it overturns qualitative predictions of our models; (2) we want to focus on other features without muddying the water with this complication, i.e., it has a pedagogical purpose.

(d) 5 points. Calculate the autarkic equilibrium relative price of coffee in the French camp.

Answer: Equating demand to supply:

$$\frac{1}{4} + \frac{1}{2p^*} = 1$$

This implies

$$p_a^* = \frac{2}{3}.$$

Now, substituting this value of p back into the individual demand functions yields

$$\begin{aligned} \widehat{C}_A^* &= \frac{1}{4}; \widehat{C}_B^* = \frac{3}{4} \\ \widehat{T}_A^* &= \frac{1}{2};\ \widehat{T}_B^* = \frac{1}{2}. \end{aligned}$$

(e) 5 points. Which of these individuals would be described as relatively coffee-loving?

Answer: Andy and Alphonse are tea-lovers: their indifference curves are "flatter" at any given point in the C-T plane; starting at teh same point in teh C-T plane, if they gave up as small as possible a unit of tea, tehy would need more coffee as compensation to get them back on teh same indiffernece curve

(f) Draw a schematic diagram that depicts this equilibrium and identifies a representation of the tariff revenue..

(g) Now imagine that the tariff revenue of $\frac{1}{6}$ unit of tea is distributed as a lump sum payment to Baptiste. Demonstrate (probably most easily done with the aid of a graph, but you are free to choose your own rhetorical devices) that Baptiste is better off in this situation of a tariff combined with a lump-sum redistribution of tariff revenues than he would have been in the autarkic situation.

6 Miscellany

1. Consider the following parametric example of a type of utility function:

$$U_i = T_i + \frac{\gamma_i}{\phi_i} (C_i)^{\phi_i}$$

where the parameters γ_i and ϕ_i obey the following restrictions:

$$\gamma_i > 0,\ \phi_i < 1,\ \phi_i \neq 0.$$

This can be rearranged as

$$T_i = U_i - \frac{\gamma_i}{\phi_i} (C_i)^{\phi_i}.$$

(a) 10 points. Find the marginal rate of substitution function associated with this utility function. (Hint: it is a function of C_i alone, not C_i *and* T_i, and can be found by taking a derivative; the MRS function is minus the slope of an indifference curve).

Answer:

$$-\frac{dT_i}{dC_i} = \gamma_i (C_i)^{(\phi_i - 1)}$$

(b) 10 points. What is the coffee demand function? Why do I not ask for the ordinary and the general equilibrium demand functions?

Answer: set the mrs equal to p and solve for C_i :

$$C_i^d = \left(\frac{p}{\gamma_i}\right)^{\left(\frac{1}{\phi_i - 1}\right)}$$

This is both an ordinary and a general equilibrium demand curve all at once. Nota bene: income has to affect the demand for something, which in this case is tea:

$$T_i^d = R_i^T - \left(\frac{p}{\gamma_i}\right)^{\left(\frac{\phi_i}{\phi_i - 1}\right)}$$

2. 10 points. Assume Andy's endowment is:

$$\begin{aligned}\overline{C}_A &= 3;\\ \overline{T}_A &= 25,\end{aligned}$$

i.e., the ordered pair $(3, 25)$, and his preferences are represented by

$$U_A = T_A + 10C_A - (C_A)^2 \; for \; C_A \leq 5;$$

$$U_A = T_A + 25 \; for \; C_A \geq 5.$$

Restrict attention to the range of values for prices such that $p > 0$. What is his demand curve for coffee? For tea?

Answer: For coffee, find $\frac{dT_A}{dC_A}$, set it to $-p$, and solve:

$$C_A^d = 5 - \frac{1}{2}p$$

For tea, we need to substitute the demand for coffee into the budget constraint:

$$\begin{aligned}T_A &= R_A - p\left(-\frac{1}{2}p + 5\right)\\ &= \overbrace{3p + 25}^{R_A} + \frac{1}{2} - 5p\\ &= -2p + 25.5\end{aligned}$$

3. Andy's preferences for coffee and tea are such that two of the members of his family of indifference curves, named I_A800 and I_A450, respectively, are described as:

$$\begin{aligned}I_A800 &: \quad T_A = \frac{800}{C_A};\\ I_A450 &: \quad T_A = \frac{450}{C_A}.\end{aligned}$$

(a) What are the functions that describe the slopes of each of these curves?

$$\begin{aligned}I_A800 &: \quad \frac{dT_A}{dC_A} = -\frac{800}{(C_A)^2};\\ I_A450 &: \quad \frac{dT_A}{dC_A} = -\frac{450}{(C_A)^2}.\end{aligned}$$

(b) Suppose $\overline{C}_A = 80$, $P_C = \$2.00/lb$, $P_T = \$4.00/lb$. Upon which of the above indifference curves would lie Andy's most-preferred feasible coffee-tea pair? By feasible we mean that he could afford it. Explain your reasoning.

Answer: I_A800. The most-preferred pair should be at that point on the BC at which the slope of an indifference curve is just equal to the slope of the BC, which is .5. Thus,

$$\begin{aligned} I_A800 &: \frac{dT_A}{dC_A} = -\frac{800}{(C_A)^2} = -\frac{1}{2}; \\ (C_A)^2 &= 1600; \\ C_A &= 40 \end{aligned}$$

If this is true, then the associated value of T_A on the I-curve is

$$T_A = \frac{800}{40} = 20.$$

This pair $(40, 20)$ is feasible (check that it satisfies the BC). For the other I-curve,

$$\begin{aligned} I_A450 &: \frac{dT_A}{dC_A} = -\frac{450}{(C_A)^2} = -\frac{1}{2}; \\ (C_A)^2 &= 900; \\ C_A &= 30. \end{aligned}$$

Hence, the associated value of T_A on this I-curve is

$$T_A = \frac{450}{30} = 15.$$

The pair $(30, 15)$ is less-preferred than $(40, 20)$ because at that point there is less of both goods compared to the other feasible point.

4. Suppose someone told you that two member's of Andy's family of indifference curves were described by the following two equations:

$$\begin{aligned} I_A800 &: T_A = \frac{800}{C_A}; \\ I_A20 &: T_A = 20 + \ln\left(\frac{C_A}{40}\right). \end{aligned}$$

If Andy's preferences satisfy the properties economist's assume in their model of the consumer, why would this person be wrong? Hint: look at the pair $(40, 20)$.

Answer: The point $(40, 20)$ lies on both I-curves. I-curves cannot cross, i.e., share a point.

5. Suppose Andy's preferences are such that the preceding two indifference curves I_A800 and I_A450 are in fact members of his indifference-curve family. This means that his preferences can be represented by the utility function:

$$U_A = C_A T_A$$

where U_A can take any value represented by the set of positive real numbers.

   (a) Re-write this equation with only T_A on the left-hand-side of the equality sign.

   A:

$$T_A = \frac{U_A}{C_A}.$$

   (b) Show that Andy's marginal rate of substitution function, i.e., the negative of the slope of an indifference curve, can be expressed as:

$$mrs = \frac{T_A}{C_A}.$$

   Hint: find $\frac{dT_A}{dC_A}$ as a function of U_A and C_A and then substitute in for U_A.

   A:

$$\begin{aligned} -\frac{dT_A}{dC_A} &= \frac{U_A}{(C_A)^2}; \\ -\frac{dT_A}{dC_A} &= \frac{T_A C_A}{(C_A)^2} = \frac{T_A}{C_A}. \end{aligned}$$

   You can also take the total derivative approach.

   (c) An individual general equilibrium demand curve answers the question: for any given set of prices, what is the quantity/unit of time chosen by an individual? Suppose that Andy produces $\overline{C}_A$ amount of coffee per year (and nothing else), derive Andy's demand curve for tea, and then coffee.

Answer: mrs=slope of BC:

$$\begin{aligned} \frac{T_A}{C_A} &= \frac{P_C}{P_T}; \\ C_A &= \frac{P_T}{P_C} \times T_A; \\ T_A &= C_A \times \frac{P_C}{P_T}. \end{aligned}$$

Sub this back into BC:

$$
\begin{aligned}
T_A &= \overline{C}_A \times \frac{P_C}{P_T} - \frac{P_C}{P_T}\left(\frac{P_T}{P_C} \times T_A\right); \\
2T_A &= \frac{\overline{C}_A}{\frac{P_T}{P_C}}; \\
T_A^d &= \frac{\overline{C}_A}{2\frac{P_T}{P_C}};
\end{aligned}
$$

$$
\begin{aligned}
C_A \times \frac{P_C}{P_T} &= \overline{C}_A \times \frac{P_C}{P_T} - C_A \times \frac{P_C}{P_T}; \\
C_A^d &= \frac{\overline{C}_A \times \frac{P_C}{P_T}}{2\overline{C}_A \times \frac{P_C}{P_T}} = \frac{1}{2}.
\end{aligned}
$$

7 Trade and arbitrage

1. Consider the following inverse excess supply and inverse excess demand functions for coffee:

$$
\begin{aligned}
p^* &= -2ED^* + 12; \\
p &= 2ES + 6.
\end{aligned}
$$

Arbitrageurs have the following cost function:

$$C(A) = \frac{c}{2}A^2; c > 0$$

with associated marginal cost function

$$MC(A) = cA.$$

(a) What are the autarkic prices?
Answer: Set ED, ES to zero:

$$p_a^* = 12, \ p_a = 6$$

(b) Assume $c = 2$. What are the equilibrium prices at home and abroad, and how much coffee is exported?
A: In equilibrium,

$$
\begin{aligned}
ED^* &= A; \ ES = A; \\
p^* - p &= cA (= 2A).
\end{aligned}
$$

So,

$$\begin{aligned}
\overbrace{-2ED^* + 12}^{p^*} - \overbrace{\{2ES + 6\}}^{p} &= cA; \\
-2 \times \overbrace{A}^{ED^*} + 12 - 2 \times \overbrace{A}^{ES} - 6 &= cA; \\
A(c+4) &= 6; \\
A &= \frac{6}{c+4} = 1; \\
p^* &= -2A + 12 = \frac{-12}{c+4} + 12 \\
&= 10; \\
p &= 2A + 6 = \frac{12}{c+4} + 6 \\
&= 8;
\end{aligned}$$

$$\begin{aligned}
A &= \frac{6}{c+4}; \\
p^* &= -2A + 12 = \frac{-12}{c+4} + 12; \\
p &= 2A + 6 = \frac{12}{c+4} + 6; \\
p^* - p &= cA = \frac{6c}{c+4}; \\
\pi_A &= (p^* - p)\,A - \frac{c}{2}A^2 = \left(\frac{6c}{c+4}\right)\left(\frac{6}{c+4}\right) - \frac{c}{2}\left(\frac{6}{c+4}\right)^2 \\
&= \left(\frac{6}{c+4}\right)^2\left(c - \frac{c}{2}\right) = \frac{c}{2}\left(\frac{6}{c+4}\right)^2
\end{aligned}$$

So, $p^* = 10,\ p = 8, A = 1$(one unit of coffee exported)

(c) Still assume $c = 2$. What are arbitrage profits?
Answer:

$$\pi_{Arb} = (p^* - p)\,A - \frac{c}{2}A^2 = 2 - 1 = 1$$

(d) Now assume the only think that changes is that now $c = \frac{2}{37}$. Recalculate the free-trade prices, the quantity exported (and imported) and the profits of the price-taking arbitrageur.
Answer:

$$\begin{aligned}
A &= \frac{6}{c+4} = \frac{6}{\frac{2+4(37)}{37}} = \frac{222}{150} = 1.48 \\
p^* &= -2A + 12 = 9.04; \\
p &= 2A + 6 = 8.96; \\
\pi_{arb} &= (p^* - p)\,A - \frac{c}{2}A^2 = (.08)\,(1.48) - \frac{1}{37}\,(1.48)^2 = .0592.
\end{aligned}$$

8 Instruments: Endowment Economy and tariffs

1. Consider a world with two countries, each of which has a perfectly competitive economy. Let the home country be the British POW camp, in which, for simplicity, there is only one individual (Andy) with an endowment of one (1) unit of coffee and zero (0) units of tea. Andy's preferences are such that his marginal rate of substitution function is

$$mrs_A = \frac{T_A}{C_A}.$$

Let p denote the relative price of coffee.

   (a) What is Andy's excess supply of coffee function?

   Answer: Step 1: Set $p = mrs$, then solve for T_A as a function of p and endowments:

$$p = \frac{T_A}{C_A};$$
$$T_A = p \times C_A.$$

   Substitute this expression for T_A into the budget constraint:

$$p \times C_A = p - pC_A.$$

   Solving for C_A yields:

$$C_A^d = \frac{1}{2}.$$

   Hence, because $C_A^S = \frac{1}{2}$,

$$ES = \frac{1}{2}.$$

   That is, excess supply is a constant.

   (b) In the foreign country, (the French POW camp), the lone POW (Baptiste) has preferences represented by the following utility function:

$$U_B = T_B + 10C_B - \frac{1}{2}(C_B)^2.$$

   Hence, his marginal rate of substitution function is

$$mrs_B = 10 - p^*.$$

   Baptiste has an endowment of eight (8) units of coffee ($\overline{C}_B = 8$). What is his excess demand function?

   Answer: From the tangency condition we have his demand function:

$$C_B^d = 10 - p^*.$$

   Hence, his excess demand function is

$$ED_B^* = 2 - p^*.$$

(c) Now assume zero transport costs. For this two-country world, what is the free trade equilibrium relative price of coffee and what is the quantity of coffee per unit of time consumed by Baptiste?

Answer:

$$2 - p^* = \frac{1}{2}$$

so

$$p^*_{FT} = \frac{3}{2}$$

Hence,

$$\widehat{C}_B = 8\frac{1}{2}.$$

(d) 15 points. Now suppose the French impose a specific tariff on coffee of one-third unit of tea per unit of coffee ($t^* = \frac{1}{3}$). What is the French relative price of coffee (denoted by p^*) after impostion of this tariff? What is the English relative price (denoted by p)? What is the tariff revenue collected by the French government?

A: With zero transport costs, $p^* = p + t^*$. Equilibrium requires ES = ED*, so

$$2 - p - t^* = \frac{1}{2}$$

or,

$$2 - p = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

This means

$$p_{FT} = \frac{7}{6}$$

Hence, (FYI)

$$p^*_{FT} = \frac{7}{6} + \frac{1}{3} = \frac{3}{2}.$$

Revenue?.

$$\overbrace{\frac{1}{3}}^{t^*} \times \frac{1}{2} = \frac{1}{6}.$$

Note that the inelastic excess supply function means that the effect of the tariff is to reduce the English price by the full amount of the tariff, leaving the French price the same as it was without the tariff. This is obviously a special case, but illustrates dramatically how imposition of a tariff can lower the world price.