SSP $\stackrel{?}{=}$ RC.

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Reminder: Sauer-Shelah-Perles lemma

$\mathcal{F}$

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$Sh(\mathcal{F})$

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Let us fix a base set X and a family $\mathcal{F}$. A set $Y \subseteq X$ is **shattered** by $\mathcal{F}$ iff $\mathcal{F}|_Y = 2^Y$. Stated otherwise:

$$\forall Z \subseteq Y \quad \exists X \in \mathcal{F} \text{ s.t. } Z = Y \cap X.$$

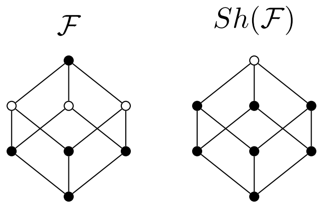
Lemma (Sauer-Shelah-Perles)

Every family $\mathcal{F}$ shatters at least as many elements as it has.

Alternatively, we can say that F is a subset of a boolean lattice B_n, and an element $y \in B_n$ is **shattered** by F if

$$\forall z \leq y \quad \exists X \in F \text{ s.t. } z = y \wedge x.$$

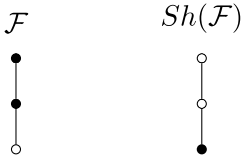
Lattices, satisfying SSP.



So, for original SSP lemma in the background we always have a boolean lattice B_n, which regulates how shattering is defined.

We can change B_N to arbitrary finite lattice to arbitrary lattice L, and say that $F \subseteq L$ **shatters** an element $y \in L$, iff

$$\forall z \leq y \;\; \exists x \in F \;\; \text{s.t.} \;\; z = y \wedge x.$$



We say that L satisfies Sauer-Shelah-Perles lemma (is SSP), if for any $F \subseteq L$ holds: $|F| \leq |Sh(F)|$.

Thus, all B_n are SSP, but, for example, a chain of length at least two is not.

Reminder: incidence algebras.

For a finite lattice L (locally finite pose P), an **incidence algebra** of L is a set of functions $\{f : I \to \mathbb{Z}\}$, where $I = \{(x, y) \in L^2 \mid x \le y\}$ with an associative **convolution**:

$$f * g(x, y) = \sum_{x \le z \le y} f(x, z) g(z, y).$$

Several special elements in incidence algebra are:

- $\delta(x, y) = \begin{cases} 1, x = y, \\ 0, x < y; \end{cases}$ δ is a unit of the algebra;
- $\zeta \equiv 1$;
- μ - unique left and right inverse of ζ, i.e. $\mu * \zeta = \delta$, $\zeta * \mu = \delta$;
- $\mu(x, y) = \begin{cases} 1 & x = y; \\ -\sum_{x \le z < y} \mu(x, z) & \textit{otherwise}. \end{cases}$

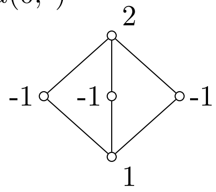
Sufficient condition for SSP.

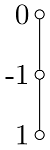
So, which finite lattices are SSP? There is one nice sufficient condition from *László Babai, Péter Frankl. Linear algebra methods in combinatorics.*

Theorem ((S) Babai, Frankl)

If a lattice L has a non-vanishing Möbius function μ, then it is SSP.

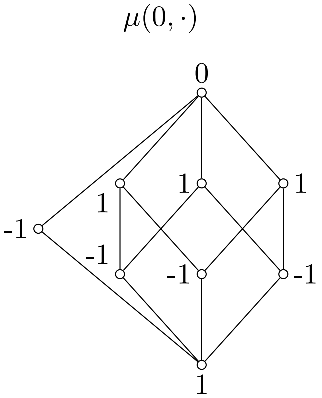
$\mu(0, \cdot)$





- As we see, for M_3 sufficient condition (S) holds, so M_3 is SSP. Same argument shows that M_n is SSP for all $n \geq 2$, including $M_2 = B_2$.
- For chains of length at least two, (S) does not hold. Although this condition is not necessary, such chains are not SSP.

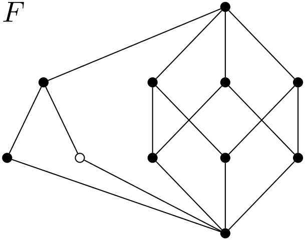
Some examples: (S) is not necessary



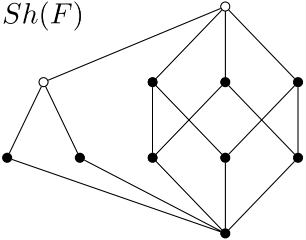
- For a lattice on the picture, μ vanishes on the pair $(0,1)$, however the corresponding lattice is SSP. We do not have a good criterion to easily see this, however this can be checked directly;
- This example can be generalized by adjoining an element in the similar way to an SSP lattice with $\mu(0,1) = -1$.

Very simple necessary condition

F



$Sh(F)$



Lemma

If L is SSP then it does not have a three-element chain as a subinterval.

Proof: If $x < y < z$ is such a subinterval, then $F = (z] - \{x\}$ can shatter only elements in $(z] - \{x, y\}$.

A lattice is **relatively complemented** if every interval is complemented. We refer to Anders Björner, *On complements in lattices of finite length*, 1981, where it is proved that L is RC iff it has no 3-element interval.

Corollary (N)

$SSP \Rightarrow RC$.

SSP $\stackrel{?}{=}$ RC.

The conjecture is stated in *Stijn Cambie, Bogdan Chornomaz, Zeev Dvir, Yuval Filmus, Shay Moran. A Sauer-Shelah-Perles lemma for lattices.*

Conjecture

$SSP = RC$.

- RC is obviously closed under direct products. Moreover, in Dilworth, *The Structure of Relatively Complemented Lattices*, 1950, it is proven that every RC lattice is a direct product of simple RC lattices. SSP class is also closed under direct products (the proof is easy).
- As SSP is closed under direct product, and as we have an example of SSP lattice with vanishing μ, we can construct an SSP lattice where μ will vanish almost everywhere;
- RC is also trivially closed under taking duals. We do not know whether it holds for RC.

Some notation.

When trying to approach SSP=RC conjecture we will use the following notation:

- Typically we deal with a finite RC lattice L, equipped with a partial relative complementation function $c(x, y, z)$ - complement of y in $[x, z]$, for $x \leq y \leq z$; $C(x, y, z)$ is a set of all complements of y in $[x, z]$. Thus, $c(x, y, z) \in C(x, y, z)$.
- We consider $F \subseteq L$. $\mathrm{Str}(F)$ is the set of shattered elements of F.
- The set of non-shattered elements of F is denoted by S, $S = L - \mathrm{Str}(F)$. It is an order-filter with minimal elements $x_1, \ldots, x_n$, forming an antichain.
- Each x_i is non-shattered *through* some y_i. Sometimes, instead of starting with F, we start with such x_i's and y_i's, calling it *system*.
- We denote $S_i = [x_i)$ and $C_i = \{u \mid u \wedge x_i = y + i\}$. Then $S = \bigcup_i S_i$, and we denote $C = \bigcup_i C_i$. Thus, $F \subseteq L - C$, and generally we can assume that $F = L - C$.

Some positive results for small n (n=1).

Note that to prove SSP=RC it is enough to show that for an arbitrary system $\mathcal{E} = x_i, y_i$ it holds: $|S_\mathcal{E}| \leq |C_\mathcal{E}|$, as that would imply:

$$|F| \leq |L - C_\mathcal{E}| \leq |L - S_\mathcal{E}| = |\,\mathrm{Str}(F)|.$$

It turns out that when an antichain x_i has one or two elements, we can prove that $|S_\mathcal{E}| \leq |C_\mathcal{E}|$. The case $n = 1$ is almost trivial.

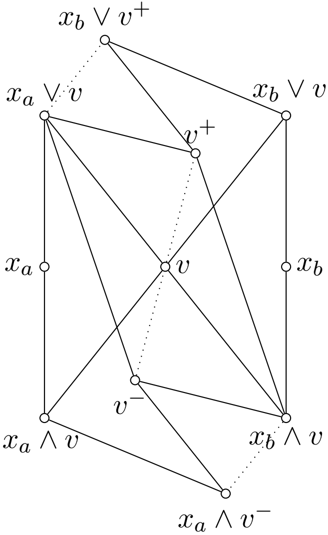
Lemma

For $y \leq x$, $|S_x| \leq |C_x|$.

Proof. We claim that the mapping $u \mapsto c(y, x, u)$ is injective and maps S_x to C_x. Indeed, $x \wedge c(y, x, u) = y$, so $c(y, x, u) \in C_x$. Also, $x \vee c(y, x, u) = u$ which proves injectivity.

Some positive results for small n (n=2).

We will use this simple but useful lemma



Lemma (*)

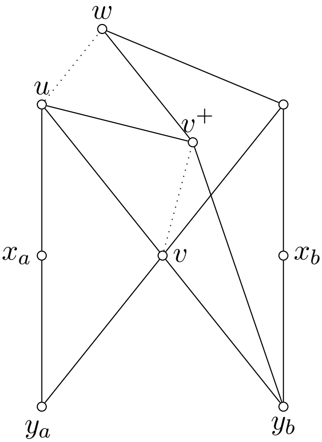
For arbitrary x_a, v, $x_b \in L$ there are elements v^- and v^+, $v^- \leq v \leq v^+$ such that

$$v^- \vee x_a = v \vee x_a = v^+ \vee x_a,$$
$$v^- \wedge x_b = v \wedge x_b = v^+ \wedge x_b,$$

and

$$v^+ \vee x_b \geq x_a;$$
$$v^- \wedge x_a \leq x_b.$$

Some positive results for small n (n=2).



Lemma

For a system $y_a \leq x_a, y_b \leq x_b$, it holds $|S| \leq |C|$.

Proof. Let $\alpha\colon S_a \to C_a$, $\beta\colon S_b - S_a \to C_b$ be:

$\beta(u) = c(y_b, x_b, u)$,

$$\alpha(u) = \begin{cases} v = c(y_a, x_a, u), & \text{if } v \notin \beta[S_b - S_a]; \\ v^+, & \text{otherwise .} \end{cases}$$

Then β is injective and maps into C_b. Also, for $u \in S_a$, $u = x_a \vee \alpha(u)$, hence α is injective, and it maps into $C = C_a \cup C_b$. Also, if $\alpha(u) = \beta(w)$ then $\alpha(u) = v^+$ for $v = c(y_a, x_a, u)$. But then $w = v^+ \vee x_b \geq a$, which is impossible as $w \in S_b - S_a$.

Graphs and types.

The case $n = 3$ is a stumbling stone for this approach. We still want to apply $(*)$ to this case and see what we can get.

Definition

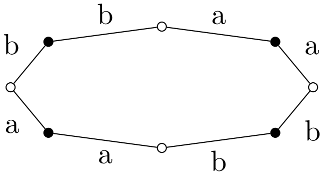
For a system $y_a \leq x_a$, $y_b \leq x_b$ and $y_c \leq x_c$, let Γ be a bipartite graph with:

- S and C are disjoint states of vertices of Γ. We say that vertices of S are "black" and of C are "white";
- edges of Γ are colored with colors a, b and c;
- there is an (undirected) a-edge from $u \in C$ to $v \in S$ if $u \vee x_a = c$.

- Every white vertex has exactly one a, one b and one c-edge;
- if $u \in L$ is in $S \cap C$, then it will correspond to two vertices in Γ, one black and one white;
- a black vertex u has an outgoing a-edge (in Γ) iff $u \in S_a$ (in L);
- edges of different colors can be parallel;
- this definition can be easily formulated for other n's.

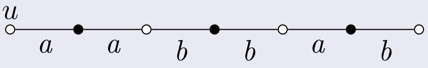
Graphs and types ($n = 2$ example).

Can this be a "proper" graph?



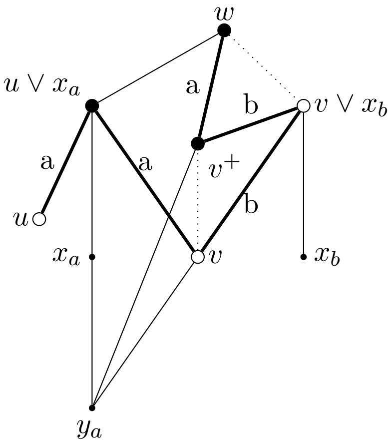
Lemma (Extension property)

For any $u \in C$ we should be able to map the following path into Γ, starting with u.



and similarly for a and b swapped.

Proof. By application of $(*)$ (with a and b swapped), as on the picture below



Proofs with graphs (n=2)

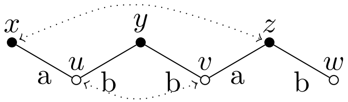
Let Γ be a bipartite graph with parts S and C and with edges colored in a, b, such that:

- every vertex from C has exactly one outgoing a-edge and one b-edge;
- every vertex in S has at least one outgoing edge;
- extension property is satisfied.

Lemma

For every Γ as above, it holds $|S| \le |C|$.

Proof. Suppose not, and let us take a minimal (in $|C|$) counterexample. As $|S| > |C|$, there is a black vertex x of degree 1, w.l.g. the outgoing edge is a going to u.



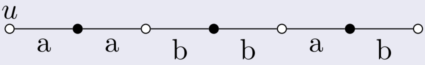
Then on the picture we have $y \ne x$, $z \ne x$ and hence $u \ne v$. Then Γ' obtained by "contracting" x with z and u with v has $|S| > |C|$ and has smaller $|C|$.

Proofs with graphs (n=3)

The "useful" property in this approach is extension property. How it should be adapted for $n = 3$?

Definition (E1: extension property for $n = 3$, naive)

For any $u \in C$ we should be able to map the following path into Γ, starting with u.



and similarly for any of the six pairs of letters from a, b, c.

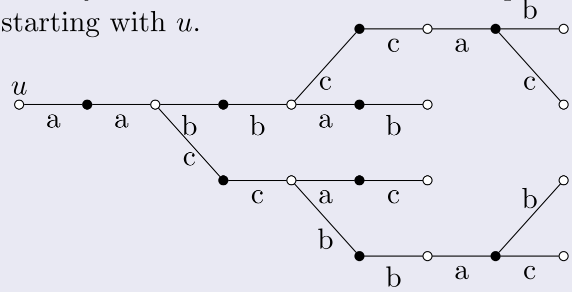
But it should be more than that.

Proofs with graphs (n=3)

The “useful” property in this approach is extension property. How it should be adapted for $n = 3$?

Definition (E2: extension property for $n = 3$)

For any $u \in C$ we should be able to map the following tree into Γ, starting with u.



and similarly for any of the six pairs of letters from a, b, c.

And even more than that.

Proofs with graphs (n=3)

For a white vertex u, its *type*, $T(u)$, is a graph like that:

$$\begin{tikzcd} & b & \\ a \arrow[ur] & & c \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ A \arrow[rr, leftrightarrow] & & C \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

- there are three vertices (n, in general), named after x's: a, b and c;
- vertex letters are either small or capitalized, independently of each other;
- edges are oriented and transitively closed (we don't draw loops);
- for a type T, we write $A \in T$ ($A \notin T$) if a is capitalized (small) in T, and $a \to b \in T$ if T has an edge from a to b; also, $A \to b \in T$ if $A \in T$ and $a \to b \in T$.

In Γ, constructed from L, and a white vertex $u \in \Gamma$, we put:

- $A \in T(u)$ iff $u \in C_a$;
- $a \to b \in T(u)$ iff $u \vee x_a \geq x_b$.

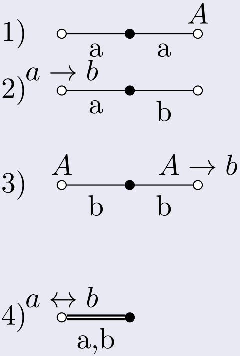
Proofs with graphs (n=3)

Now we can reformulate extension properties in terms of types:

Definition (E3: extension property for $n = 3$, with types)

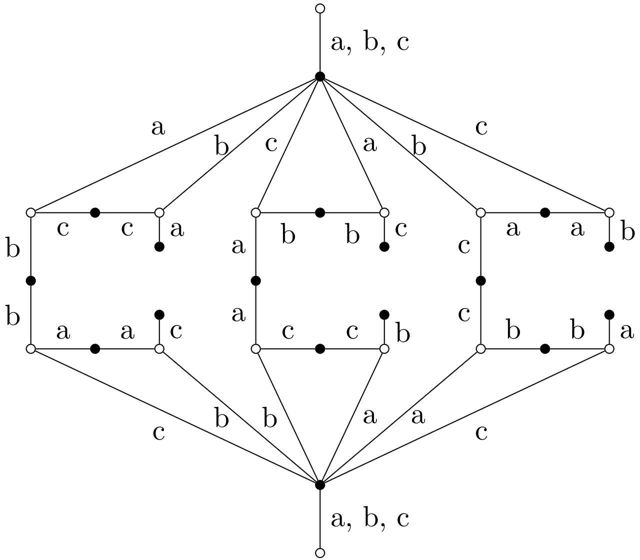
For any white vertex u:

1. there is an a-a-path from u to v such that $A \in T(v)$;
2. $a \to b \in T(u)$ iff there is an a-b-path from u;
3. if $A \in T(u)$ then there is a b-b-path from u to u^+ such that $A \to b \in T(u^+)$, and $E(T(u^+)) \supseteq E(T(u))$, that is, $T(u^+)$ has all edges that $T(u)$ has;
4. if $a \leftrightarrow b \in T(u)$ then a and b-edge from u go to the same black vertex.



and similarly for any of the six pairs of letters from a, b, c.

Counterexample by Stijn Cambie

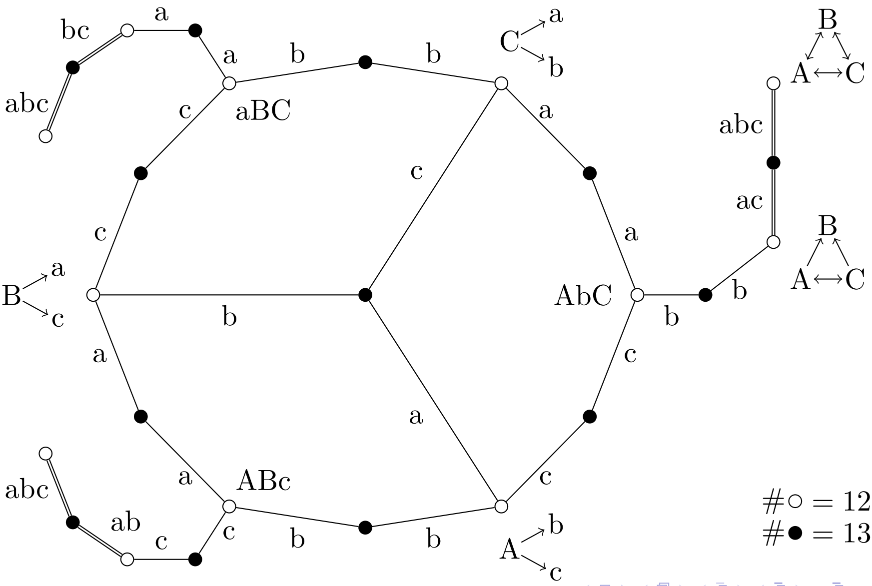


All in all,
$E3 \Rightarrow E2 \Rightarrow E1$.

This graph has 14 black elements, 17 white elements.

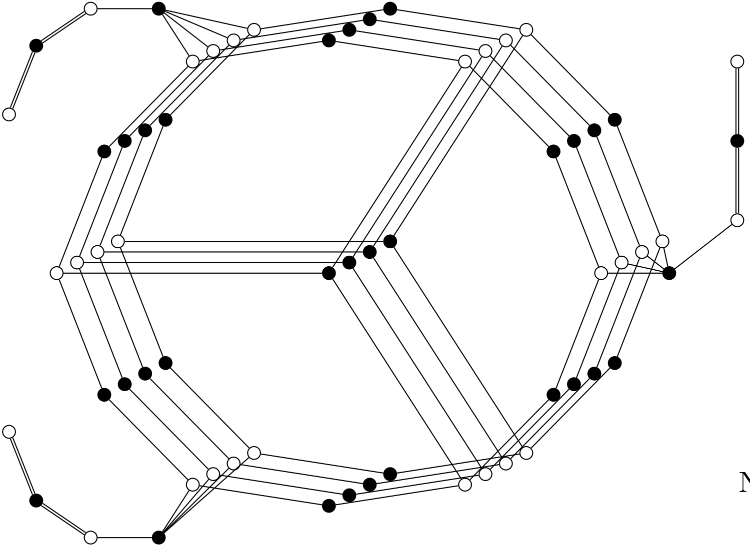
It satisfies $E2$ but not $E3$.

Counterexample to E3



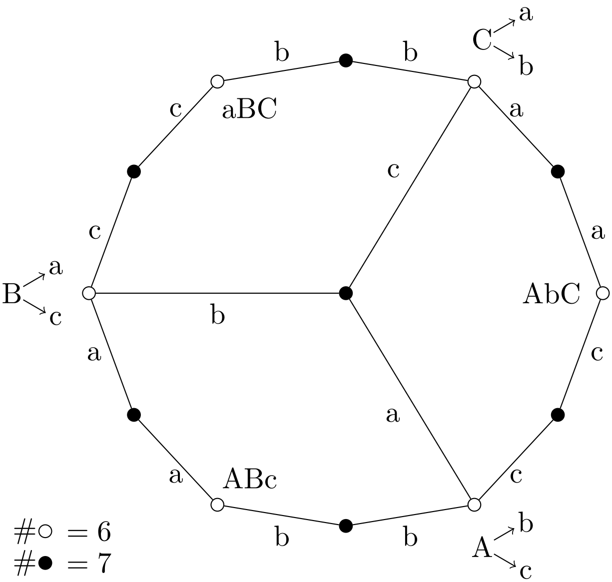
$\#\circ = 12$
$\#\bullet = 13$

Do we need those tails?



Not really.

Counterexample to E3



$\#\circ = 6$
$\#\bullet = 7$

We modify the definition of Γ: every white vertex has *at most* one a, b and c-edge;

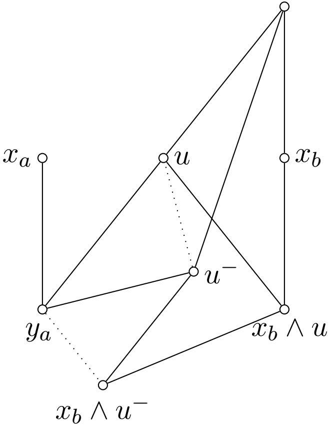
Extension properties also got modified:

1) For any white vertex u, *if there is an a-edge from u*, then there is an a-a-path from u to v such that $A \in T(v)$.

and so on...

Extension properties strengthening (conditions on y's)

So, E3 is stronger than E2, which is stronger than E1. Can we make those properties even stronger? Sure.

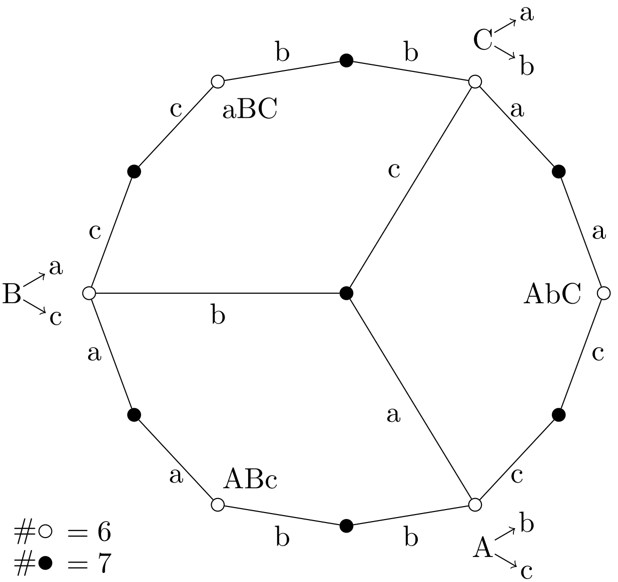


We now can make use of v^- from Lemma $(*)$ to get another extension property.

Definition (E3$^+$)

Unless $y_b \leq y_a$, for a white vertex u from Γ, if $A \in T(u)$ and there is a b-edge from u, then there is a b-b-path from u to u^- such that $A \in T(u^-)$, $B \notin T(u^-)$, and $E(T(u^-)) \subseteq E(T(u))$.

Extension properties strengthening



$\#\circ = 6$
$\#\bullet = 7$

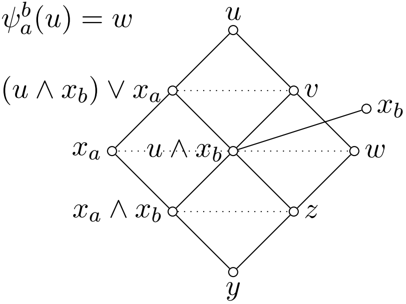
Looking at white vertices ABc, AbC and aBC, we see that E3$^+$ is not satisfied, so we can infer that (in L)

$$y_a = y_b = y_c,$$

and we call it y.

Extension properties strengthening (if all y's are equal)

$\psi_a^b(u) = w$



Lemma

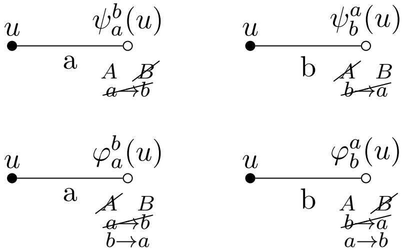
If $y_a = y_b$ then $|S_a \Delta S_b| \leq |C_a \Delta C_b|$.

$Y_a^b = \{u \in S_a - S_b \mid u \wedge x_b > x_a \wedge x_b\}$;

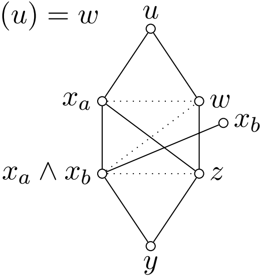
$X_a^b = \{u \in S_a - S_b \mid u \wedge x_b = x_a \wedge x_b\}$.

And there are $\psi_a^b \colon Y_a^b \to C_a - C_b$.

$\varphi_a^b \colon X_a^b \to C_b - C_a$, such that:

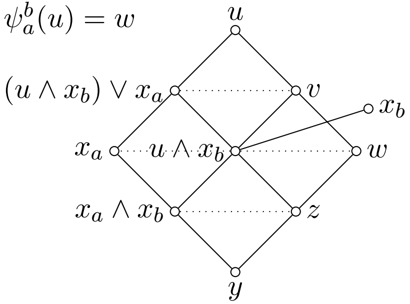


$\varphi_a^b(u) = w$

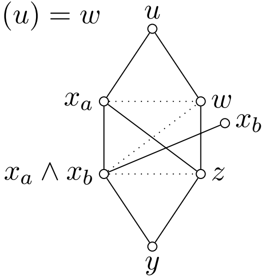


Just a side remark

$\psi_a^b(u) = w$



$\varphi_a^b(u) = w$



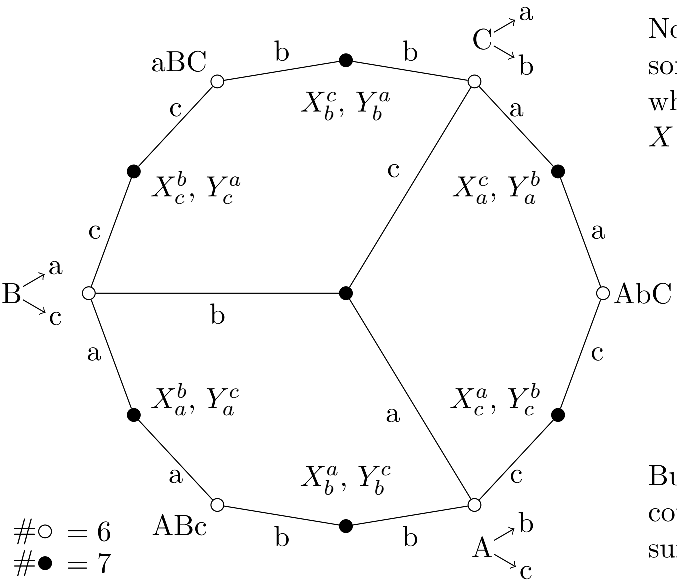
And, just in case:

$$\psi_a^b(u) = c\Big(c\big(y, x_a \wedge x_b, u \wedge x_b\big),\ u \wedge x_b,\ c\big(u \wedge x_b, (u \wedge x_b) \vee x_a, u\big)\Big)$$

$$\varphi_a^b(u) = c\Big(c\big(y, x_a \wedge x_b, x_a\big), x_a, u\Big)$$

Those are "injective". But is there a way to characterize injective polynomials in this language in general? God knows.

Extension properties strengthening



$\#\circ = 6$
$\#\bullet = 7$

Now we can say, for some black vertices, whether they lie in X's or in Y's.

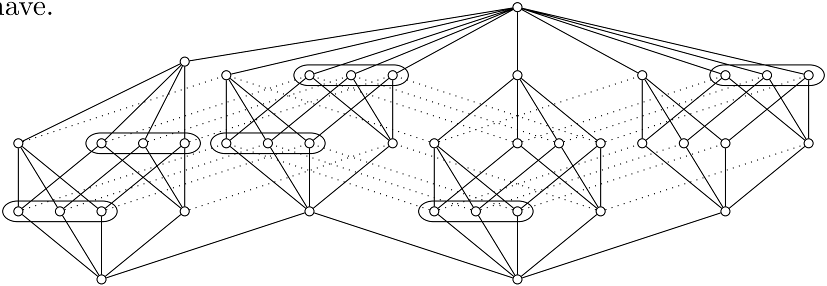
But all in all, the counterexample survives.

Generic elements (an example)

So, we wold like to try and construct a lattice, corresponding to this counterexample.

But it's doubtful that we can construct a lattice, which "literally" corresponds to this graph: in a lattice we should have a top element, x's and y's, their joins and meets, which are not on the graph;

Instead, we want the graph to describe how "generic elements" should behave.

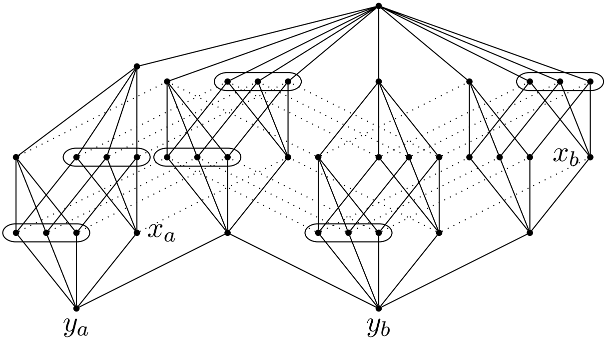


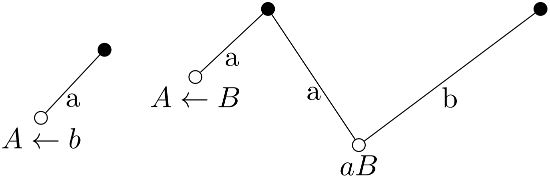
Generic elements

The encircled elements are "generic", we can have n of them in each ellipse, all of them are in $S \cup C$;

There are "generic" elements outside of $S \cup C$ and elements in $S \cup C$ which are not generic;

The example was used to illustrate that we can have $|S_a \Delta S_b| > |C_a \Delta C_b|$ in $n = 2$ case.



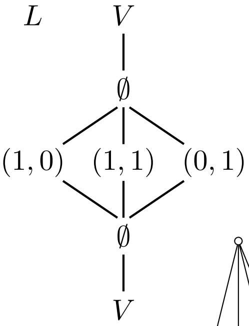


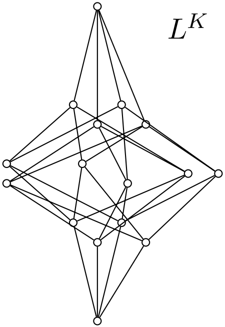
Pumping (a way to handle "genericity")

Definition

Let us take $N \geq 2$, a prime number; $D \geq 0$, an integer; $V = V(N, D)$, a D-dimensional vector space over $\mathbb{F}_N$; L, a finite lattice (not necessarily RC); $K\colon L \to \mathsf{Lin}\, V$. Then L^K is a poset, defined as:

- Elements of L^K are pairs (s, σ), where $s \in L$ and $\sigma \in V/K(s)$;
- The covering relation $\prec_{L^K}$ is defined by: $(p, \pi) \prec_{L^K} (q, \theta)$ iff $p \prec_L q$ and $\pi \cap \theta \neq \emptyset$;
- The partial order $\leq_{L^K}$ is a reflexive transitive closure of $\prec_{L^K}$.





RC lattices through pumping

Theorem

L^K is an RC lattice iff:

$(0^\dagger)$ *For e, z - top and bottom, $E = Z = V$.*

$(1^\dagger)$ *For all p, r, $P \wedge R \leq K_{p\wedge r}, K_{p\vee r}$.*

$(2^\dagger)$ *For all $p \leq q \leq r$, $Q \leq P \vee R$.*

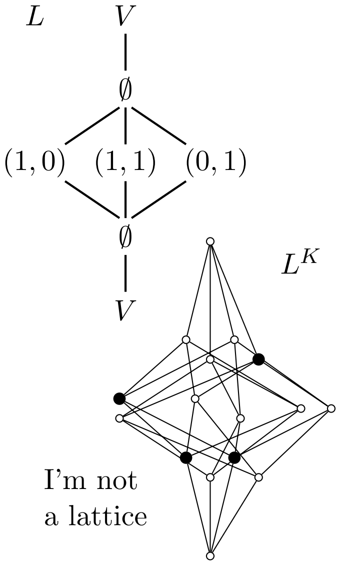
$(3^\dagger)$ *For any $p \leq r$ there is q, $p \leq q \leq r$, such that $Q = P \vee R$.*

$(4^\dagger)$ *For any 3-element interval $p \prec q \prec r$, $Q \lneq P \wedge R$.*

$(5^\dagger)$ *For $q, r \leq s, p$, it holds*

$$(S \vee R) \wedge (P \vee Q) \leq [S' \wedge Q'] \vee [P' \wedge R'],$$

where $S' = S \vee T, \ldots, R' = R \vee T$ and $T = (S\vee Q)\wedge(S\vee R)\wedge(P\vee Q)\wedge(P\vee R)$.



RC lattices through pumping (some notes)

Condition $(1^\dagger)$ is equivalent to: $\{q \in L \mid v \in Q\}$ is a sublattice of L, for all $v \in V$;

Condition $(2^\dagger)$ is equivalent to: $\{q \in L \mid Q \leq W\}$ is convex in L, for all $W \in \mathsf{Lin}\, V$ (thanks Ralph);

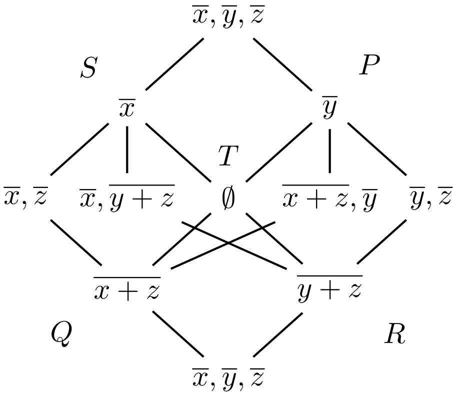
Element q from $(3^\dagger)$ has a complement s in $[p, r]$, and $S \leq P \wedge R$;

If L is RC, we don't need $(4^\dagger)$;

We need $(5^\dagger)$, but sometimes we don't;

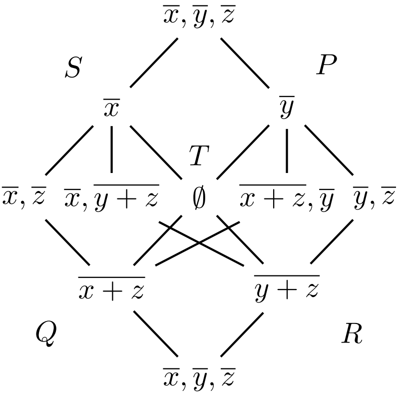
Conditions for L^K to be a lattice are hard to pinpoint.

I break condition $(5^\dagger)$



S.P.Q.R.


S·P·Q·R


$\overline{x}, \overline{y}, \overline{z}$
S
P
$\overline{x}$
$\overline{y}$
T
$\overline{x}, \overline{z}$
$\overline{x}, \overline{y+z}$
$\emptyset$
$\overline{x+z}, \overline{y}$
$\overline{y}, \overline{z}$
$\overline{x+z}$
$\overline{y+z}$
Q
R
$\overline{x}, \overline{y}, \overline{z}$

Pumping, simplified

Let us fix a basis S in V, $|S| = D$. Then linear closures of subsets of S form a distributive sublattice of $\mathsf{Lin}\, V$. Let $\Sigma\colon L \to 2^S$, $K_\Sigma\colon L \to \mathsf{Lin}\, V$, $K_\Sigma(x) = \mathsf{Lin}\,\Sigma(x)$, $L^\Sigma = L^{K_\Sigma}$

Theorem

L^Σ is an RC lattice iff:

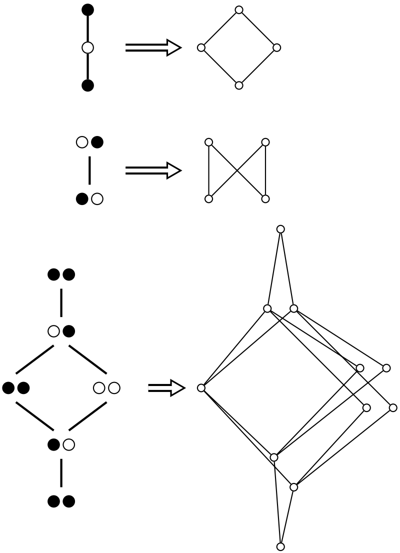
($0^\ddagger$) *For e, z, top and bot, $E = Z = V$*

($1^\ddagger$) *For all p, r, $P \cap R \leq \Sigma_{p\wedge r}, \Sigma_{p\vee r}$.*

($2^\ddagger$) *For all $p \leq q \leq r$, $Q \leq P \cup R$.*

($3^\ddagger$) *For any $p \leq r$ there is q, $p \leq q \leq r$, such that $Q = P \cup R$.*

($4^\dagger$) *For any 3-element interval $p \prec q \prec r$, $Q \subsetneq P \cap R$.*



PROFIT (maybe)

In context of pumping, we want to do the following:

- Take a lattice L with fixed x's and y's;
- Pump it;
- Make sure that S^K and C^K are "images" of S and C - for this we have to have $X \geq Y$ for each pair of x and y;
- Make sure we can take N "large enough" (we can);
- Let $S_{\max} \subseteq S$ and $C_{\max} \subseteq C$ are the elements, having biggest codimension in $S \cup C$. They will dominate S^K and C^K;
- Even if L is not RC, and $|S| \leq |C|$, we might construct K such that L^K is RC and $|S_{\max}| > |C_{\max}|$;
- Because elements in $S^K_{\max}$ and $C^K_{\max}$ dominate all others, we get $|S^K| > |C^K|$;
- PROFIT.

All in all, it's the codimension of K that we care about.

Which functions are expressible as codimensions?

Definition

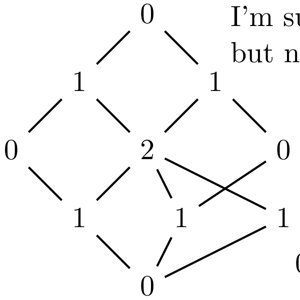
We say that $\phi\colon L \to \mathbb{Q}^+$ is realizable if $M \cdot \phi$ is a codimension function for some K, satisfying $(0^\dagger) - (5^\dagger)$, for some positive integer M.

Lemma

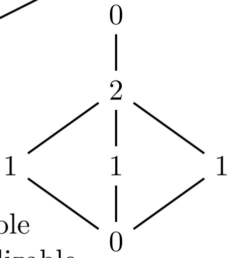
If ϕ is realizable then ϕ is submodular, that is

1. $\zeta(e) = \zeta(z) = 0$, *for e, z - top and bot;*
2. $\zeta(x) + \zeta(y) \geq \zeta(x \vee y) + \zeta(x \wedge y)$.

And we need only $(0^\dagger) - (2^\dagger)$ to prove it.



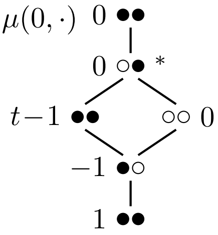
I'm submodular but not realizable



I'm K-realizable but not Σ-realizable

Wait, what about the Möbius function?

If we can construct L^K breaking SSP, then it breaks SSP for all N large enough; hence the Möbius function should be vanishing for all N large enough.



Lemma

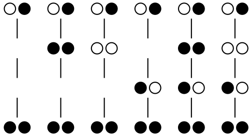
If L^K breaks SSP then the polynomial Möbius function $\mathcal{M}_{L,K}$ is vanishing, where $\mathcal{M}_{L,K}\colon \{(x,y) \mid x,y \in L, x \leq y\} \to \mathsf{Poly}(t)$ is defined as

$$\mathcal{M}_{L,K}(a,b) = \sum_C (-1)^{l(C)} \cdot t^{P_C},$$

where the sum is over all chains $C = k_0, k_1, \ldots, k_l$ from a to b, and P_C is

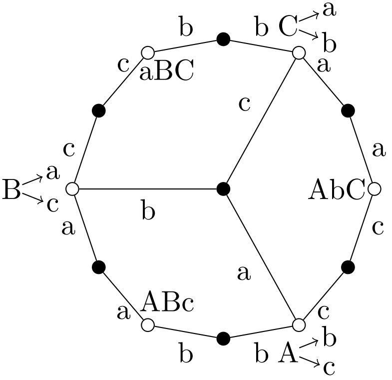
$$P_C = \sum_{i=0,\ldots,l} \dim K_i - \sum_{i=0,\ldots,l-1} \dim K_i \wedge K_{i+1} - \dim A \vee B$$

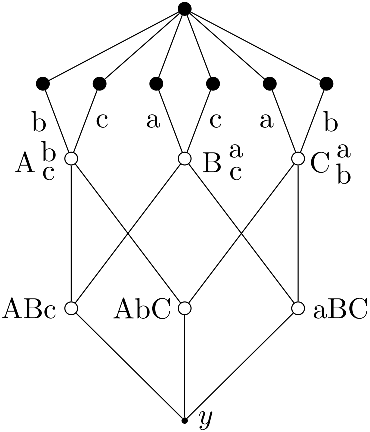
To compute $\mu(0,*)$:



-1 $+1$ $+t$ $+t$ $-t$ $-t$

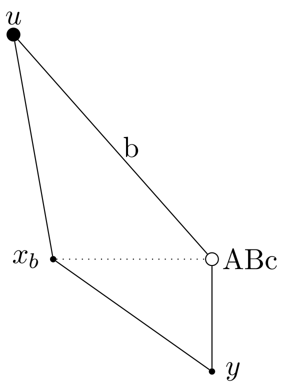
Trying to pump it up





That's how it ideally might look. "Fat" dots on the lattice are pumped, generic elements. All other elements should be either outside of $C \cup S$ or small. But it's not so simple.

Trying to pump it up (constraints on kernels)

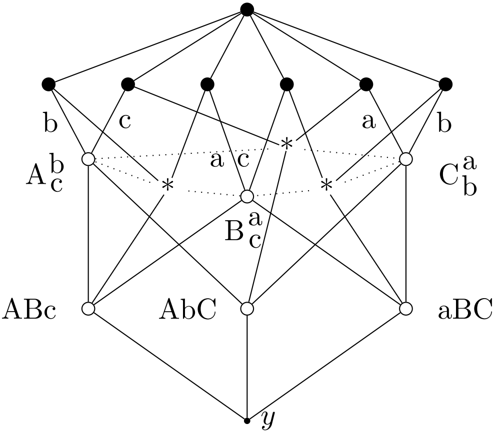


For pumping to preserve C and S, we should have $X_b \geq Y$, that is, x_b is "less generic" than y.

Element u should be "most generic" in $S \cup C$, thus $U \leq X_b$. Indeed, if $U > X_b$ then x_b itself is more generic than u; and if U is incomparable to X_b, then there is $v \in (x_b, u)$ such that $V \leq U \wedge X_b < U$. Then $v \in S_b$ and v is more generic than u.

Finally, $K_{ABc} = U$. As $U \leq X_b$, we can argue that x_b has a complement ABc in the interval $[y, u]$. Then $K_{ABc} \leq Y \vee U = Y$. Also, as $x_b \vee ABc = u$, $U \geq X_b \wedge K_{ABc} = K_{ABc}$. But if $U > K_{ABc}$, then u is less generic than ABc.

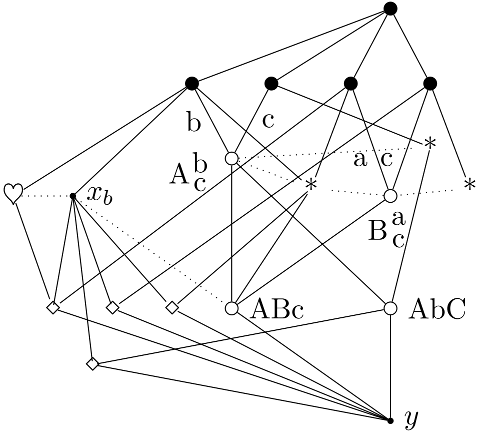
Trying to pump it up (some other elements)



Elements marked with $*$ are enforced by the construction of Lemma ($*$), dotted lines mark complementation.

They are sandwiched between fat black and fat white elements, so they cannot be "less generic" than those.

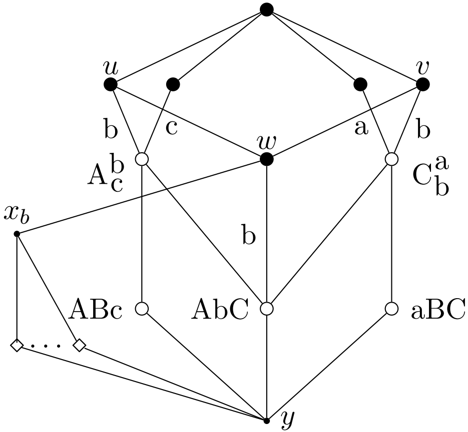
Hence we need to ensure they are not in $S \cup C$.



We need diamonds to ensure that elements which shouldn't be in C_b do not get there. They seem to might be made "small".

But then we hearts as complements of x_b in the intervals formed by diamonds and fat b's. Those should be "generic".

So now we have to ensure hearts do not get into C, and so on and so forth... But maybe we can tame this process.

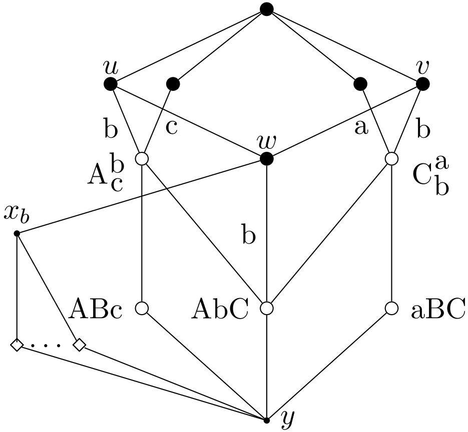


The element w in the center is the meet of u and v. It is generic as it is sandwiched between AbC and u, v, while $U = V = K_{AbC}$.

It is also in S_b and, because $u \in X_b^a \cap Y_b^c$ and $v \in X_b^c \cap Y_b^a$, u and v are incomparable, hence $w < u, v$.

All in all, w should be on the picture with graph, but it isn't.

Trying to pump it up (that which cannot be tamed)

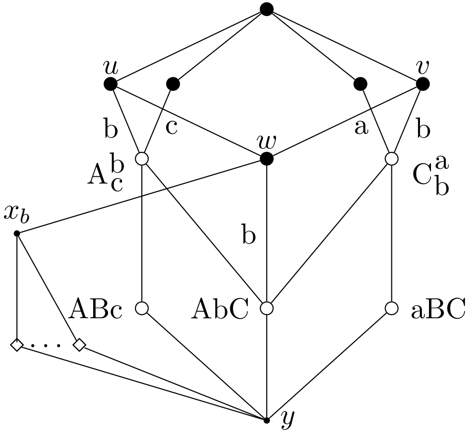


In a way, w correspond to the b-tail we'd cut from AbC. Yet we'd hoped to remove this tail from the picture by making it "not generic" - but we cannot.

Also, it might not look so bad - we got an additional element in S, thus S is now even bigger, right? But now we have to get B-element which joins to w, then apply Lemma (*), that is, everything which is required by extension properties.

All in all, w should be on the picture with graph, but it isn't.

That which cannot be tamed and what to do about it



TRY HARDER - just add those elements to the graph, construct new extension which takes them into account, yet beats SSP, and then lift it up to the lattice.

ALTERNATIVELY, there might be a "better pumping" out there, the one which would be able to kill that element.

BUT MAYBE, by studying this example, we can find another conditions which hold in our graphs - this would be helpful if we're trying to prove SSP=RC, instead of disproving it.

Thank you!

Pump a lattice $\rightarrow$??? $\rightarrow$ PROFIT