Join-Completions of Ordered Structures

Constantine Tsinakis

Vanderbilt University

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Abstract-Motivation

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H. Ono, Completions of Algebras and Completeness of Modal and Substructural Logics, Advances in Modal Logics 4 (2002), 1- 20.

H. Ono, Closure Operators and Complete Embeddings of Residuated Lattices, Studia Logica 74 (3) (2003), 427 - 440.

Featured Themes

(1) Abstract treatment of join-extensions

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(2) The role of nuclei and co-nuclei in logic

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(2) The role of nuclei and co-nuclei in logic

(3) Interaction of residuals with join extensions

(4) Applications to important classes of ordered structures

Join-Completions

Join-Extensions and Join-Completions

A poset $\mathbf{Q}$ is called be an extension of a poset $\mathbf{P}$ provided $P \subseteq Q$ and the order of $\mathbf{Q}$ restricts to that of $\mathbf{P}$. In case every element of $\mathbf{Q}$ is a join (in $\mathbf{Q}$) of elements of $\mathbf{P}$, we say that $\mathbf{Q}$ is a join-extension of $\mathbf{P}$ and that $\mathbf{P}$ is join-dense in $\mathbf{Q}$. We use the term join-completion for a join-extension that is a complete lattice. The concepts of a meet-extension and a meet-completion are defined dually.

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Join-completions, introduced by B. Banaschewski, are intimately related to representations of complete lattices studied systematically by J.R. Büchi.

B. Banaschewski, Hüllensysteme und Erweiterungen von Quasi-Ordunungen, Z. math. Logik Grundl. Math. 2 (1956), 117 - 130.

J. R. Büchi, Representations of complete lattices by sets, Portugaliae Math. 11 (1952), 151 - 167.

Preservation of Meets

If $\mathbf{Q}$ is a join-extension of $\mathbf{P}$, then the inclusion map $i : \mathbf{P} \to \mathbf{Q}$ preserves all existing meets.

Equivalently, if $X \subseteq P$ and $\bigwedge^{\mathbf{P}} X$ exists, then $\bigwedge^{\mathbf{Q}} X$ exists and $\bigwedge^{\mathbf{P}} X = \bigwedge^{\mathbf{Q}} X$

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Dually, If $\mathbf{Q}$ is a meet-extension of $\mathbf{P}$, then the inclusion map $i : \mathbf{P} \to \mathbf{Q}$ preserves all existing joins.

Lower and Upper Sets

A subset I of a poset $\mathbf{P}$ is said to be a lower set of $\mathbf{P}$ if whenever $y \in P$, $x \in I$, and $y \leq x$, then $y \in I$. Note that the empty set $\emptyset$ is a lower set.

A principal lower set is a lower set of the form

$$\downarrow a = \{x \in P \mid x \leq a\} \ (a \in P).$$

For $A \subseteq P$,

$$\downarrow A = \{x \in P \mid x \leq a, \ \text{for some a} \in \text{A}\}$$

denotes the smallest lower set containing A.

The set $\mathcal{L}(\mathbf{P})$ of lower sets of $\mathbf{P}$ ordered by set inclusion is a complete lattice; the join is the set-union, and the meet is the set-intersection.

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The upper sets of $\mathbf{P}$ are defined dually. Further, we write $\mathcal{U}(\mathbf{P})$ for the lattice of upper sets of $\mathbf{P}$,
$\uparrow a = \{x \in P \mid a \leq x\}$ for $a \in P$, and
$\uparrow A = \{x \in P \mid a \leq x, \ \text{for some a} \in \text{A}\}$ for $A \subseteq P$.

Canonical Representation of Join-Extensions

Each join-extension $\mathbf{Q}$ of a poset $\mathbf{P}$ can be identified with its canonical image $\dot{\mathbf{Q}}$:

$$\dot{\mathbf{Q}} = \{\downarrow x \cap P : x \in Q\}.$$

In particular, P can be identified with the poset $\dot{\mathbf{P}}$ of its principal lower sets:

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The largest join-extension of $\mathbf{P}$ is $\mathcal{L}(\mathbf{P})$. Thus, for any join-extension $\mathbf{Q}$ of $\mathbf{P}$, we have

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The smallest join-completion of $\mathbf{P}$, is the so called Dedekind-MacNeille completion $\mathcal{N}(\mathbf{P})$. Its canonical image consists of all lower sets that are intersections of principal lower sets. Lastly, we will have an occasion to consider the ideal completion $\mathcal{I}(\mathbf{P})$ of a join-semilattice $\mathbf{P}$.

Abstract Description of Join-Extensions

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E.g., $\mathcal{L}(\mathbf{P})$ can be described abstractly as a bi-algebraic distributive lattice whose poset of completely join-prime elements is isomorphic to $\mathbf{P}$. $\mathcal{N}(\mathbf{P})$, has an equally satisfying abstract description due to Banaschewski: it is the only join and meet-completion of P.

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$\mathcal{N}(\mathbf{P})$, has an equally satisfying abstract description due to Banaschewski: it is the only join and meet-completion of P.

The ideal completion $\mathcal{I}(\mathbf{P})$ of a join-semilattice (lattice) $\mathbf{P}$ can be described abstractly as an algebraic lattice whose join-semilattice (lattice) of compact elements is isomorphic to $\mathbf{P}$.

Abstract Description of Join-Extensions

Note that the inclusion map $i : \mathbf{P} \to \mathcal{N}(\mathbf{P})$ preserves all existing joins and meets. The Crawley completion $\mathcal{C}(\mathbf{P})$ – with canonical image consisting of all complete lower sets of $\mathbf{P}$, that is, lower sets that are closed with respect to any existing joins of their elements – is the largest join-completion with this property.

Closure Operators and Closure Retractions

Recall that a closure operator on a poset $\mathbf{P}$ is a map $\gamma : \mathbf{P} \to \mathbf{P}$ with the usual properties of being order-preserving ($x \leq y \Rightarrow \gamma(x) \leq \gamma(y)$), enlarging ($x \leq \gamma(x)$), and idempotent ($\gamma(\gamma(x)) = \gamma(x)$). It is completely determined by its image $\mathbf{P}_\gamma$ by virtue of the formula

$$\gamma(x) = \min\{c \in P_\gamma : x \leq c\}.$$

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An opposite direction also holds: Let us call a closure retraction of $\mathbf{P}$, a subposet $\mathbf{C}$ of $\mathbf{P}$ that satisfies:

$\min\{a \in C : x \leq a\}$ exists for all $x \in P$.

Closure Operators and Closure Retractions

Bijective Correspondence

- ⋆ If $\mathbf{C}$ is a closure retraction of $\mathbf{P}$, then $\gamma_{\mathbf{C}} : \mathbf{P} \to \mathbf{P}$ defined by $\gamma_{\mathbf{C}}(x) = \min\{c \in C : x \leq c\}$ is a closure operator.
- ⋆ Conversely, if $\gamma : \mathbf{P} \to \mathbf{P}$ is a closure operator, then $\mathbf{P}_\gamma = \gamma[\mathbf{P}]$ is a closure retraction of P.
- ⋆ Moreover, $\mathbf{P}_{\gamma_{\mathbf{C}}} = \mathbf{C}$ and $\gamma_{\mathbf{P}_\gamma} = \gamma$.

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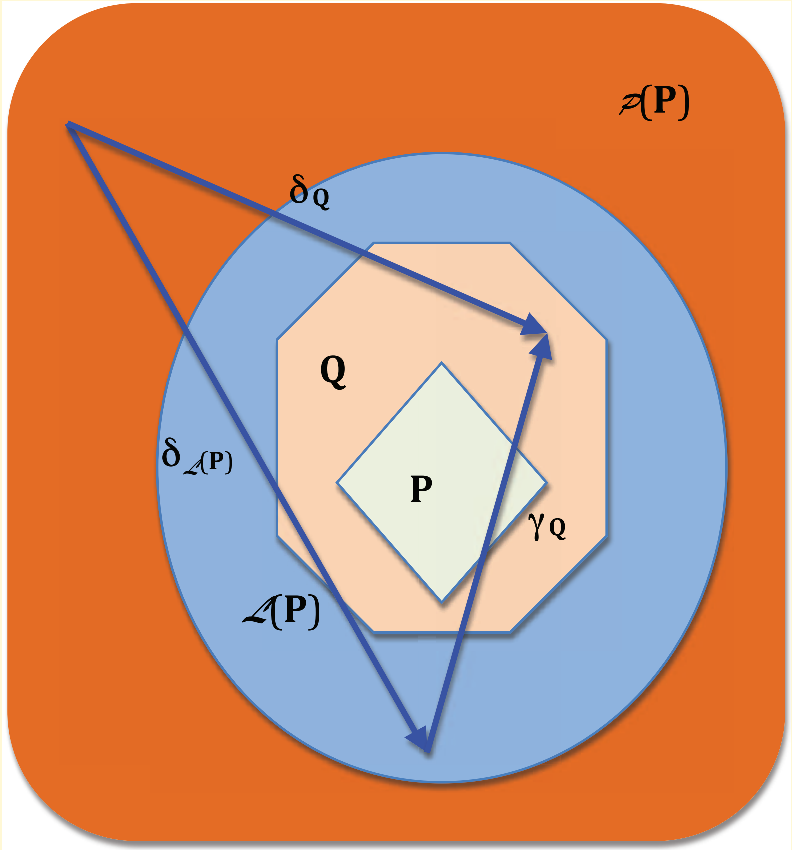
- ⋆ Moreover, $\mathbf{P}_{\gamma_{\mathbf{C}}} = \mathbf{C}$ and $\gamma_{\mathbf{P}_\gamma} = \gamma$.

If $\mathbf{Q}$ is a join-completion of $\mathbf{P}$, then $\mathbf{Q}$ is a closure retraction of $\mathcal{L}(\mathbf{P})$ and of $\wp(P)$. In what follows, we write $\gamma_{\mathbf{Q}}$ for the associated closure operator on $\mathcal{L}(\mathbf{P})$ and $\delta_{\mathbf{Q}}$ for the one on $\wp(P)$.

Back to Join-Completions



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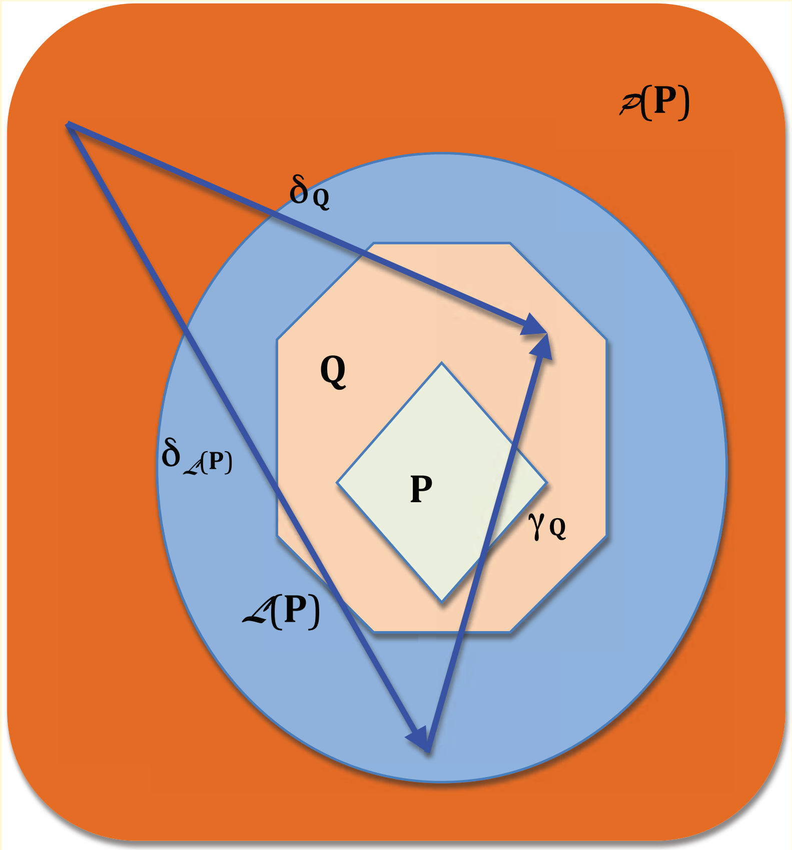

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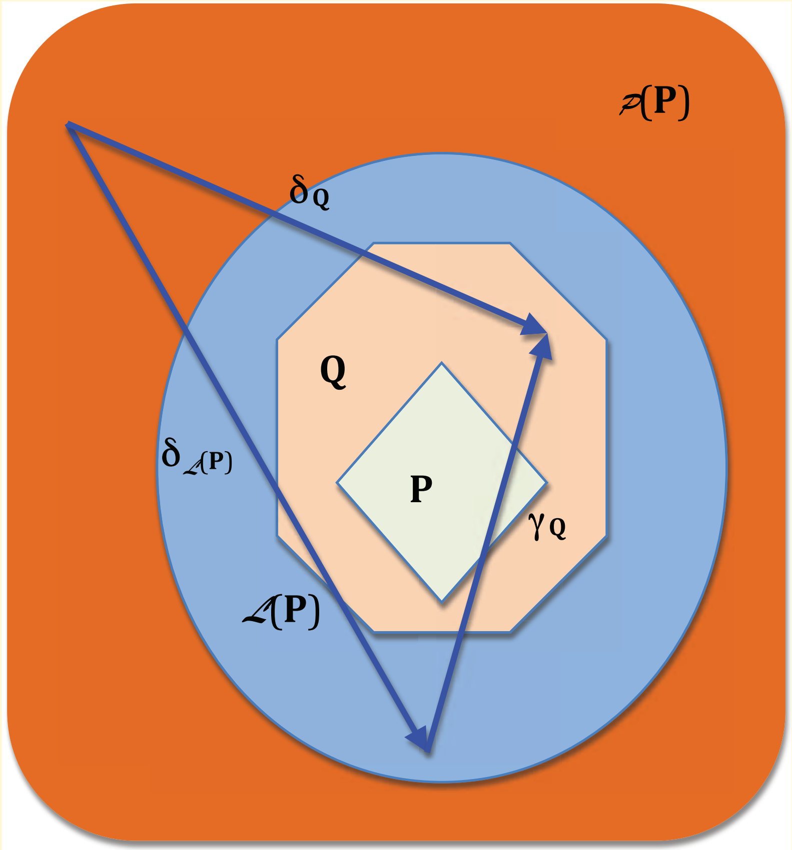

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There is a bijective correspondence between join-completions of $\mathbf{P}$ and closure operators δ on $\wp(P)$ with $P \subseteq \wp(P)_\delta \subseteq \mathcal{L}(P)$.

Join-Extensions of Ordered Structures

Residuals in Partially Ordered Monoids

Given a partially ordered monoid – pom for short – $\mathbf{P}$, the following question arises: Which join-completions of $\mathbf{P}$ are residuated lattices with respect to a, necessarily unique, multiplication that extends the multiplication of $\mathbf{P}$?

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Let $\mathbf{P} = \langle P, \cdot, \leq \rangle$ be a pom and let $x, y \in P$. We set:

$$x \backslash z = \max\{y \in P : xy \leq z\}, \text{ and}$$
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whenever these maxima exist.

$x \backslash z$ is read as "x under z"

z / x is read as "z over x"

Residuals in Partially Ordered Monoids

A residuated partially ordered monoid – residuated pom – $\mathbf{P}$ is one in which all quotients $x\backslash z$ and z/x exist. In particular, the binary operations $\backslash$ and $/$ are defined everywhere on P.

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Alternatively, a residuated pom $\mathbf{P}$ is one in which the binary operation $\cdot$ is residuated. This means that there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

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A residuated lattice is a residuated lattice-ordered monoid $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$

Residuals in Partially Ordered Monoids

Let $\mathbf{P}$ be a monoid. Then $\wp(\mathbf{P}) = \langle \wp(P), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$ is a residuated lattice where:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\},$$

$$X \backslash Y = \{z \mid X \cdot \{z\} \subseteq Y\}, \text{ and}$$

$$Y / X = \{z \mid \{z\} \cdot X \subseteq Y\}.$$

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Let $\mathbf{P}$ be a pom. Then $\mathcal{L}(\mathbf{P}) = \langle \mathcal{L}(P), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$ is a residuated lattice where:

$$X \cdot Y = \downarrow \{x \cdot y \mid x \in X, y \in Y\},$$

$$X \backslash Y = \{z \mid X \cdot \{z\} \subseteq Y\}, \text{ and}$$

$$Y/X = \{z \mid \{z\} \cdot X \subseteq Y\}.$$

Note: $(\downarrow x) \cdot (\downarrow y) = \downarrow (x \cdot y)$; hence $\dot{\mathbf{P}}$, is a submonoid of $\mathcal{L}(\mathbf{P})$

Nuclei and Subacts

A nucleus on a pom $\mathbf{P}$ is a closure operator γ on $\mathbf{P}$ such that $\gamma(a)\gamma(b) \leq \gamma(ab)$ (equivalently, $\gamma(\gamma(a)\gamma(b)) = \gamma(ab)$), for all $a, b \in P$.

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A closure retraction $\mathbf{C}$ of a residuated poset $\mathbf{P}$ is called a subact of $\mathbf{P}$ if $x/y,\ y\backslash x \in C$, for all $x \in C$ and $y \in P$.

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Let γ be a closure operator on a residuated pom $\mathbf{P}$, and let $\mathbf{P}_\gamma$ be the closure retraction associated with γ. Then γ is a nucleus iff $\mathbf{P}_\gamma$ is a subact of $\mathbf{P}$.

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A subact $\mathbf{P}_\gamma$ of a residuated pom $\mathbf{P}$ is a residuated pom. The product of two elements $x, y \in \mathbf{P}_\gamma$ is given by $x \circ_\gamma y = \gamma(x \cdot y)$, and the residuals are the restrictions on $\mathbf{P}_\gamma$ of the residuals of $\mathbf{P}$. In particular, if $\mathbf{P}$ is a residuated lattice, then so is $\mathbf{P}_\gamma$, with $x \vee_\gamma y = \gamma(x \vee y)$ and $x \wedge_\gamma y = x \wedge y$

Let $\mathbf{Q}$ be a join completion of a pom $\mathbf{P}$, let $\gamma_{\mathbf{Q}}$ be the associated closure operator on $\mathcal{L}(\mathbf{P})$, and let $\delta_{\mathbf{Q}}$ be the one on $\wp(\mathbf{P})$. TFAE:

(1) $\mathbf{Q}$ is a residuated lattice with respect to a multiplication extends the multiplication of $\mathbf{P}$.

(2) $a\backslash_{\mathcal{L}(\mathbf{P})} b \in Q$ and $b/_{\mathcal{L}(\mathbf{P})} a \in Q$, for all $a \in P$ and $b \in Q$.

(3) $\gamma_{\mathbf{Q}}$ is a nucleus on $\mathcal{L}(\mathbf{P})$ (equivalently, $\mathbf{Q}$ is a subact of $\mathcal{L}(\mathbf{P})$).

(4) $\delta_{\mathbf{Q}}$ is a nucleus on $\wp(\mathbf{P})$ (equivalently, $\mathbf{Q}$ is a subact of $\wp(\mathbf{P})$).

Furthermore, if the preceding conditions are satisfied, then the inclusion map $\mathbf{P} \hookrightarrow \mathbf{Q}$ preserves multiplication, all meets and all existing residuals.

Crucial Lemma

Let $\mathbf{P}$ be a pom and let $\mathbf{Q}$ be a join-completion of $\mathbf{P}$ that is a pom with respect to a multiplication that extends the multiplication of $\mathbf{P}$. Then for all $a, b \in P$, if $a \backslash_{\mathbf{P}} b$ exists, then $a \backslash_{\mathbf{Q}} b$ exists and

$$a \backslash_{\mathbf{P}} b = a \backslash_{\mathbf{Q}} b = a \backslash_{\mathcal{L}(\mathbf{P})} b = a \backslash_{\wp(\mathbf{P})} b.$$

Likewise for the other residual.

Applications I

The Dedekind-MacNeille Completion

If $\mathbf{P}$ be a residuated pom, then its Dedekind-MacNeille completion $\mathcal{N}(\mathbf{P})$ is a residuated lattice. Moreover, the inclusion map $\mathbf{P} \hookrightarrow \mathcal{N}(\mathbf{P})$ preserves products, residuals, and all existing meets and joins.

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All we need to prove is that if $a \in P$ and $b \in \mathcal{N}(P)$, then $a\backslash_{\mathcal{L}(\mathbf{P})} b \in \mathcal{N}(P)$ and $b/_{\mathcal{L}(\mathbf{P})} a \in \mathcal{N}(P)$.

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Let $a \in P$ and $b \in \mathcal{N}(P)$. Since $\mathbf{P}$ is meet-dense in $\mathcal{N}(\mathbf{P})$, there exists a family $(b_j \mid j \in J)$ of elements of P such that $b = \bigwedge_{j \in J} b_j$. We have:

$a\backslash_{\mathcal{L}(\mathbf{P})} b = a\backslash_{\mathcal{L}(\mathbf{P})} \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a\backslash_{\mathcal{L}(\mathbf{P})} b_j)$.

Thus, $a\backslash_{\mathcal{L}(\mathbf{P})} b \in \mathcal{N}(P)$, since $a\backslash_{\mathcal{L}(\mathbf{P})} b_j = a\backslash_{\mathbf{P}} b_j$.

Likewise, $b/_{\mathcal{L}(\mathbf{P})} a \in \mathcal{N}(P)$.

Join-Semilattice Monoids

A join-semilattice monoid, or simply js-monoid, is a pom that satisfies the equation

$$x(y \vee z)w \approx (xyw) \vee (xzw).$$

Note that if a js-monoid $\mathbf{P}$ has a least element 0, then $x0 = 0 = 0x$, for all $x \in P$.

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Think of the ideal completion $\mathcal{I}(\mathbf{P})$ of $\mathbf{P}$ as an algebraic lattice whose join-semilattice of compact elements is $\mathbf{P}$.

Join-Semilattice Monoids

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By Main Theorem, we need only need show that $a\backslash_{\mathcal{L}(\mathbf{P})} b \in \mathcal{I}(P)$ and $b/_{\mathcal{L}(\mathbf{P})} a \in \mathcal{I}(P)$, whenever $a \in P$ and $b \in \mathcal{I}(P)$. This is again simple:

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Let a and b as above. Define

$< a, b >= \{x \in P : a \cdot x \leq b\}$, $c = \bigvee^{\mathcal{I}(\mathbf{P})} < a, b >$.

We remark that the product $a \cdot x$ in the definition of the set $< a, b >$ takes place in $\mathbf{P}$ – or in $\mathcal{L}(\mathbf{P})$, since $\mathbf{P}$ is a submonoid of $\mathcal{L}(\mathbf{P})$. Is is almost immediate that $< a, b >= \downarrow c \cap P$, and so $c = a\backslash_{\mathcal{L}(\mathbf{P})} b$.

Applications II

Involutive Residuated Lattices

An element 0 of a residuated pom $\mathbf{P}$ is called cyclic if for all $x \in P, 0/x = x\backslash 0$; it is called dualizing if it satisfies $0/(x\backslash 0) = (0/x)\backslash 0 = x$, for all $x \in P$. If 0 is a cyclic element, we will write $x \to 0$ for $0/x = x\backslash 0$.

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An involutive residuated lattice is an algebra $Q = \langle Q, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ such that $Q' = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice and 0 is a cyclic dualizing element of Q'.

If in the preceding definition we replace the term 'lattice' by the term 'poset', we obtain the concept of an involutive residuated pom $Q = \langle Q, \leq, \cdot, \backslash, /, 1, 0\rangle$.

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The choice of the term 'involutive' reflects the fact that the map $x \mapsto x \to 0$ is an involution of the underlying order-structure. In fact, we have:

An involutive residuated lattice $\mathbf{Q}$ is term equivalent to an algebra $\bar{\mathbf{Q}} = \langle Q, \wedge, \vee, \cdot, 1, ' \rangle$ satisfying:

(1) $\langle Q, \cdot, 1 \rangle$ is a monoid;

(2) $\langle Q, \wedge, \vee, ' \rangle$ is an involutive lattice; and

(3) $xy \leq z \iff y \leq (z'x)' \iff x \leq (yz')'$, for all $x, y, z \in Q$.

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The Glivenko-Frink Theorem for Involutive RLs

A subact $\mathbf{C}$ of a residuated pom $\mathbf{P}$ is an involutive pom iff $\mathbf{C} = \mathbf{P}_{\gamma_0}$ for some cyclic element 0 of $\mathbf{P}$.

Recall that for a cyclic element 0 of a residuated pom $\mathbf{P}$, γ_0 denotes the nucleus $x \mapsto (x \to 0) \to 0$ on $\mathbf{P}$.

The Join-Completion of an Inv. Res. Pom

Let $\mathbf{P}$ be a residuated poset and let $\mathbf{Q}$ be a join-completion of $\mathbf{P}$ that is a residuated lattice with respect to a multiplication that extends the multiplication of $\mathbf{P}$. Then for every cyclic element 0 of $\mathbf{P}$, $\mathbf{Q}_{\gamma_0}$ is the Dedekind-MacNeille completion $\mathcal{N}(\mathbf{P}_{\gamma_0})$ of $\mathbf{P}_{\gamma_0}$.

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In particular:

(1) $\mathcal{L}(\mathbf{P})_{\gamma_0}$ is the Dedekind-MacNeille completion of $\mathbf{P}_{\gamma_0}$.

(2) The Dedekind-MacNeille completion $\mathcal{N}(\mathbf{P})$ of an involutive residuated pom $\mathbf{P}$ is an involutive residuated lattice, and the only join-completion of $\mathbf{P}$. Furthermore, the inclusion map $\mathbf{P} \hookrightarrow \mathcal{N}(\mathbf{P})$ preserves products, residuals, and all existing meets and joins.

A partially ordered group $\mathbf{G}$ is an involutive residuated pom, and hence its Dedekind-MacNeille completion $\mathcal{N}(\mathbf{G})$ is an involutive residuated lattice. This completion is of little interest, since the presence of least and greatest elements prevents $\mathcal{N}(\mathbf{G})$ from being a partially ordered group.

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Let $\mathcal{N}(\mathbf{G})^\sharp$ denote the involutive pom obtained from $\mathcal{N}(\mathbf{G})$ by removing its least and greatest elements. Note that $\mathcal{N}(\mathbf{G})^\sharp$ is conditionally complete, and it is a lattice precisely when $\mathbf{G}$ is directed.

ℓ-Groups

A partially ordered group $\mathbf{G}$ is an involutive residuated pom, and hence its Dedekind-MacNeille completion $\mathcal{N}(\mathbf{G})$ is an involutive residuated lattice. This completion is of little interest, since the presence of least and greatest elements prevents $\mathcal{N}(\mathbf{G})$ from being a partially ordered group.

Let $\mathcal{N}(\mathbf{G})^\sharp$ denote the involutive pom obtained from $\mathcal{N}(\mathbf{G})$ by removing its least and greatest elements. Note that $\mathcal{N}(\mathbf{G})^\sharp$ is conditionally complete, and it is a lattice precisely when $\mathbf{G}$ is directed.

It is well-known that $\mathcal{N}(\mathbf{G})^\sharp$ is a conditionally complete ℓ-group iff $\mathbf{G}$ is integrally closed (Krull, Lorenzen, Clifford, Everett and Ulam). [$\mathbf{G}$ is said to be integrally closed if whenever $x, y \in G$ such that $y \leq x^n$ $(n = 1, 2, \dots)$, then $x \geq 1$.]

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Suppose next that $\mathbf{G}$ is integrally closed. We observed that the map $x \mapsto x' = x \to 1$ is an involution of $\mathcal{N}(\mathbf{G})^\sharp$. Set $L_x = \downarrow x \cap G$, $U_x = \uparrow x \cap G$.

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We wish to show that each $a \in \mathcal{N}(G)^\sharp$ is invertible, or equivalently that $1 \leq aa'$, for all $a \in \mathcal{N}(G)^\sharp$. This reduces to $1 \leq x, \forall x \in U_{aa'}$. Fix a and x. We have $a = \bigwedge U_a$, and hence, $a' = \bigvee U_a{}^{-1}$. Now $aa' \leq x \Rightarrow (a \bigvee U_a^{-1} \leq x \Rightarrow au^{-1} \leq x, \forall u \in U_a \Rightarrow a \leq xu, \forall u \in U_a \Rightarrow a \leq \bigwedge\{xu \mid u \in U_a\} = x(\bigwedge U_a) = xa$ (since multiplication by an invertible element is an order-automorphism of $\mathcal{N}(\mathbf{G})^\sharp$). We have $a \leq xa$, and hence $a \leq x^n a$ (for $n = 1, 2, \ldots$). Fix $u \in U_a$ and $w \in L_a$. Then $w \leq x^n u$, and so $wu^{-1} \leq x^n$ (for $n = 1, 2, \ldots$). Since $\mathbf{G}$ is integrally closed, $x \geq 1$.