$\text{SSP} \stackrel{?}{=} \text{RC}.$

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Reminder: Sauer-Shelah-Perles lemma

$\mathcal{F}$

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Let us fix a base set X and a family $\mathcal{F}$. A set $Y \subseteq X$ is **shattered** by $\mathcal{F}$ iff $\mathcal{F}|_Y = 2^Y$.
Stated otherwise:

$$\forall Z \subseteq Y \quad \exists X \in \mathcal{F} \ \text{ s.t. } \ Z = Y \cap X.$$

$Sh(\mathcal{F})$

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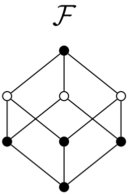
Lemma (Sauer-Shelah-Perles)

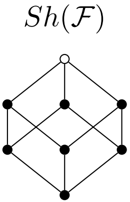
Every family $\mathcal{F}$ shatters at least as many elements as it has.

Alternatively, we can say that F is a subset of a boolean lattice B_n, and an element $y \in B_n$ is **shattered** by F if

$$\forall z \leq y \quad \exists X \in F \ \text{ s.t. } \ z = y \wedge x.$$

Lattices, satisfying SSP.

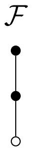


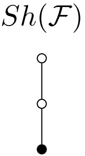


So, for original SSP lemma in the background we always have a boolean lattice B_n, which regulates how shattering is defined.

We can change B_N to arbitrary finite lattice to arbitrary lattice L, and say that $F \subseteq L$ **shatters** an element $y \in L$, iff

$$\forall z \leq y \;\; \exists x \in F \;\; \text{s.t.} \;\; z = y \wedge x.$$





We say that L satisfies Sauer-Shelah-Perles lemma (is SSP), if for any $F \subseteq L$ holds: $|F| \leq |Sh(F)|$.

Thus, all B_n are SSP, but, for example, a chain of length at least two is not.

SSP, an attempt at characterization.

The problem was stated in a preprint *Zeev Dvir, Yuval Filmus, Shay Moran. A Sauer-Shelah-Perles Lemma for Lattices. 2018*. They also gave the following sufficient condition:

Theorem ((S) Dvir, Filmus, Moran)

If a lattice L has a non-vanishing Möbius function μ, then it is SSP.

On the other hand the following necessary condition hold:

Lemma (N)

For a lattice L, define $\varphi, \psi \colon L \to \mathbb{Z}$ as

$$\varphi(x) = \big|[x)\big| = \zeta^2(x, 1);$$

$$\psi(x) = \sum_{z \le x} \mu(0, z)\varphi(z).$$

Then L is SSP implies $\varphi \le \psi$.

Reminder: incidence algebras.

For a finite lattice L (locally finite pose P), an **incidence algebra** of L is a set of functions $\{f : I \to \mathbb{Z}\}$, where $I = \{(x, y) \in L^2 \mid x \leq y\}$ with an associative **convolution**:

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

Several special elements in incidence algebra are:

- $\delta(x, y) = \begin{cases} 1, x = y, \\ 0, x < y; \end{cases}$ δ is a unit of the algebra;
- $\zeta \equiv 1$;
- μ - unique left and right inverse of ζ, i.e. $\mu * \zeta = \delta$, $\zeta * \mu = \delta$;
- $\mu(x, y) = \begin{cases} 1 & x = y; \\ -\sum_{x \leq z < y} \mu(x, z) & otherwise. \end{cases}$

Reminder: Möbius inversion formula.

Given a lattice L, for a pair of functions $f, g\colon L \to \mathbb{R}$ it holds:

$$f(x) \equiv \sum_{y \geq x} g(y) \quad \Leftrightarrow \quad g(x) \equiv \sum_{y \geq x} \mu(x,y) f(y);$$
$$f(y) \equiv \sum_{x \leq y} g(x) \quad \Leftrightarrow \quad g(y) \equiv \sum_{x \leq y} \mu(x,y) f(x).$$

One of the applications of Möbius inversion is inclusion-exclusion principle is the inclusion-exclusion formula:

$$\left| \bigcup_{i \in X} A_i \right| = \sum_{\emptyset \subsetneq Y \subseteq X} (-1)^{|Y|+1} \left| \bigcap_{j \in Y} A_j \right|.$$

This essentially comes from the fact that in a boolean lattice 2^S, for $X \subseteq Y \subseteq S$, $\mu(X,Y) = (-1)^{|Y|-|X|}$.

Proof of (S), setup

- For a given lattice L with nonvanishing μ let us consider an $|F|$-dimensional vector space V_F of functions $F \to \mathbb{R}$. We want to find a spanning set for V_F of size $|Sh(F)|$;
- For $X \subseteq L$ we denote by $\chi_X \colon L \to \mathbb{R}$ a characteristic function of X. A function χ_X^F is a restriction of χ_X to F;
- Trivially, a family $\chi_{[y)}^F$ for $y \in L$ spans V_F. We want to show that if $z \notin Sh(F)$, then $\chi_{[z)}^F$ is a linear combination of $\chi_{[w)}^F$, for $w < z$.

Proof of (S), magic

So, let $z \notin Sh(F)$, x_0 s.t. $x_0 \neq z \wedge p$, for all $p \in F$. Take arbitrary $p \in F$. We have:

$$\chi_{(p\wedge z]}(x) = \sum_{x \leq y} \chi_{p\wedge z}(y), \text{ for all } x$$

$$\Leftrightarrow \text{ [Möbius inversion]}$$

$$\chi_{p\wedge z}(x) = \sum_{x \leq y} \mu(x,y)\chi_{(p\wedge z]}(y) = \sum_{x \leq y} \mu(x,y)\chi_{[y)}(p \wedge z)$$

$$= \sum_{x \leq y \leq z} \mu(x,y)\chi_{[y)}(p).$$

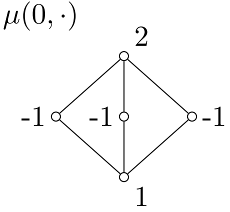
Take $x := x_0$, then:

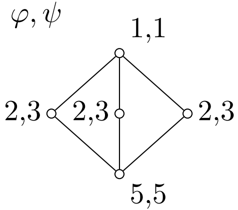
$$0 \equiv \chi_{p\wedge z}(x_0) = \sum_{x_0 \leq y \leq z} \mu(x_0,y)\chi_{[y)}(p), \text{ for all } p \in F.$$

$$0 \equiv \sum_{x_0 \leq y < z} \mu(x_0,y)\chi^F_{[y)} + \mu(x_0,z)\chi^F_{[z)}.$$

And we are done.

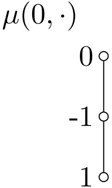
Some examples: M_3



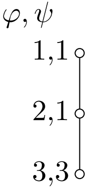


- As we see, for M_3 sufficient condition (S) holds, so M_3 is SSP. As a sanity check, we can see that $\varphi \leq \psi$, which means that necessary condition (N) also holds;
- The picture shows only $\mu(0, \cdot)$, while (S) states that μ has to be nonvanishing globally, that is, on all pairs. However, this is equivalent to $\mu(0, \cdot)$ to be nonvanishing on all principal filters of L, which are simple in this case.
- Same argument shows that M_n is SSP for all $n \geq 2$, including $M_2 = B_2$.

Some examples: chains



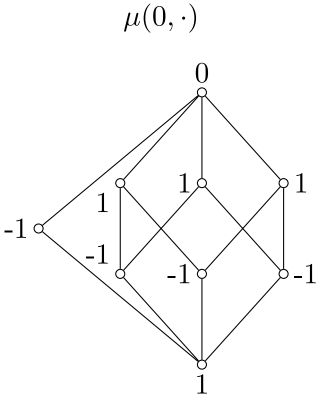
- For a chain C_2 of length 2 there is an element at which $\varphi > \psi$, so (N) does not hold and C_2 is not SSP. Yet again, this implies that μ is vanishing;
- For longer chains, we can apply (N) directly, however a strengthened version of necessarily condition holds, which can then be applied directly:



Lemma (N+)

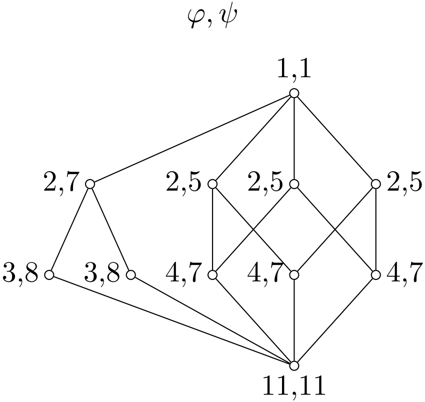
If L is SSP then (N) holds in every principal ideal.

Some examples: (S) is not necessary



- For a lattice on the picture, μ is vanishes on the pair $(0, 1)$, however the corresponding lattice is SSP. We do not have a good criterion to easily see this, however this can be checked directly;
- This example can be generalized by adjoining an element in the similar way to an SSP lattice with $\mu(0, 1) = -1$.

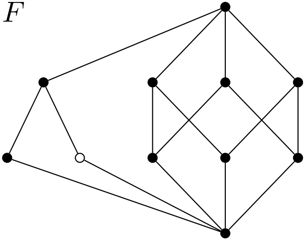
Some examples: (N+) is not sufficient



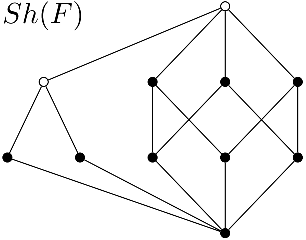
- For this lattice (N) holds. As all proper ideals here are boolean, and hence SSP, (N) holds for them as well. Thus, (N+) holds.
- Here we actually do have a good criterion to see that it is not SSP.

Very simple necessary condition

F



$Sh(F)$



Lemma

If L is SSP then it does not have a three-element chain as a subinterval.

Proof: If $x < y < z$ is such a subinterval, then $F = (z] - \{x\}$ can shatter only elements in $(z] - \{x, y\}$.

A lattice is **relatively complemented** if every interval is complemented. We refer to Anders Björner, *On complements in lattices of finite length*, 1981, where it is proved that L is RC iff it has no 3-element interval.

Corollary (N2)

$SSP \Rightarrow RC$.

Some final consideration

- We do not have an example, showing that RC is not strong enough to capture entire SSP. Hence, the conjecture:

Conjecture

$SSP = RC;$

- RC is obviously closed under direct products. Moreover, in Dilworth, *The Structure of Relatively Complemented Lattices*, 1950, it is proven that every RC lattice is a direct product of simple RC lattices. SSP are also happened to be closed under direct products. The proof is easy, but not absolutely trivial;
- As SSP is closed under direct product, and as we have an example of SSP lattice with vanishing μ, we can construct an SSP lattice where μ will vanish almost everywhere;
- RC is also trivially closed under taking duals. We do not know whether it holds for RC.