Background on Turing Degrees and Jump Operators

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Basic Notation – Topological Spaces

Baire Space: $\mathbb{N}^{\mathbb{N}}$ with topology generated by subbasis, for $\sigma \in \mathbb{N}^*$,

$$N_\sigma := \{f \in \mathbb{N}^{\mathbb{N}} \mid \sigma \subset f\}$$

Cantor Space: $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ with subspace topology.

Proposition

1. $\mathbb{N}^{\mathbb{N}}$ *is homeomorphic to* $\mathbb{R} \setminus \mathbb{Q}$ *with the subspace topology.*
2. $2^{\mathbb{N}}$ *is compact (and is homeomorphic to any compact Hausdorff space without isolated points having a countable basis of clopen sets).*

Elements of $2^{\mathbb{N}}$ are freely identified with subsets of $\mathbb{N}$ via their characteristic functions.

Basic Notation – Partial Functions

Notation: $f :\subseteq A \to B$ is a function $f : \operatorname{dom} f \to B$ where $\operatorname{dom} f \subseteq A$.

Convergence: $f(x)\downarrow$ if $x \in \operatorname{dom} f$ (f *converges* or is *defined* at x)
$f(x)\uparrow$ otherwise (f *diverges* or is *undefined* at x).

Strong Equality: $f \simeq g$ if and only if f and g converge on same inputs and are equal when they converge.

Partial Recursive Functions

Definition (Partial Recursive)

Suppose $f :\subseteq \mathbb{N}^k \to \mathbb{N}$ is given.

$$f \text{ is } \textbf{partial recursive} \iff f \text{ is algorithmically computable}$$

where 'algorithm' is interpreted in your favorite programming language.

If e is the *Gödel number* of such an algorithm, write

$$\varphi_e^{(k)}(m_1, \ldots, m_k) \simeq f(m_1, \ldots, m_k)$$

We call e an **index** of f.

Equivalent Characterizations of Partial Recursiveness

Being a bit more precise:

Theorem

Suppose $f :\subseteq \mathbb{N} \to \mathbb{N}$ is given. The following are equivalent:

1. f *is* Turing machine computable.
2. f is register machine program computable.
3. f is μ-recursive.
4. f is λ-computable.

Church-Turing Thesis

A partial function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is computable by a digital computer (ignoring resource limitations) if and only if it is computable by any of the above equivalent definitions.

Partial Recursive Functionals

Definition (Partial Recursive)

Suppose $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}$ *is given.*

$$\Psi \text{ is } \textbf{partial recursive} \iff \Psi \text{ is algorithmically computable}$$

where 'algorithm' now includes oracle/black-box computations that make use of the function parameter.

If e is the *Gödel number* of such an algorithm, write

$$\varphi_e^{(k),f}(m_1, \ldots, m_k) \simeq \Psi(f, m_1, \ldots, m_k)$$

We call e an **index** of Ψ.

Some Basic Results

Theorem (Enumeration Theorem)

The partial functions $F(e, m_1, \ldots, m_k) \simeq \varphi_e^{(k)}(m_1, \ldots, m_k)$ *and* $\Psi(f, e, m_1, \ldots, m_k) \simeq \varphi_e^{(k),f}(m_1, \ldots, m_k)$ *are partial recursive.*

Theorem (Parametrization Theorem)

Suppose $F :\subseteq \mathbb{N}^{k+1} \to \mathbb{N}$ *is partial recursive. Then there exists a primitive recursive* $f : \mathbb{N} \to \mathbb{N}$ *such that*

$$F(e, m_1, \ldots, m_k) \simeq \varphi_{f(e)}^{(k)}(m_1, \ldots, m_k)$$

Turing Reducibility, Equivalence, and Degrees

Definition (Turing Reducibility)

Suppose $f, g \in \mathbb{N}^{\mathbb{N}}$*. The* **Turing reducibility** *preorder* $\leq_{\mathrm{T}}$ *is defined by*

$$f \leq_{\mathrm{T}} g \iff f \text{ is algorithmically computable using oracle } g$$
$$\iff f = \varphi_e^{(1),g} \text{ for some } e$$

f *and* g *are* **Turing equivalent**, $f \equiv_{\mathrm{T}} g$, *if and only if* $f \leq_{\mathrm{T}} g$ *and* $g \leq_{\mathrm{T}} f$.
$\equiv_{\mathrm{T}}$ *is an equivalence relation.*

Can similarly define *g-computability* ($g \in \mathbb{N}^{\mathbb{N}}$) for partial functions or predicates.

Definition (Turing Degree)

Suppose $f \in \mathbb{N}^{\mathbb{N}}$*. The* **Turing degree** *associated with* f *is*

$$\deg_{\mathrm{T}}(f) := \{g \in \mathbb{N}^{\mathbb{N}} \mid f \equiv_{\mathrm{T}} g\}$$

Basic Results

Proposition

Suppose $f, g \in \mathbb{N}^{\mathbb{N}}$ *and define* $(f \oplus g)(n) := \begin{cases} f(m) & \text{if } n = 2m \\ g(m) & \text{if } n = 2m+1 \end{cases}$. *Then*

$$\sup(\deg_{\mathrm{T}}(f), \deg_{\mathrm{T}}(g)) = \deg_{\mathrm{T}}(f) \vee \deg_{\mathrm{T}}(g) = \deg_{\mathrm{T}}(f \oplus g)$$

Proof.

Straight-forward. □

Proposition

Suppose $f \in \mathbb{N}^{\mathbb{N}}$. *Then there exists* $X \in 2^{\mathbb{N}}$ *such that* $f \equiv_{\mathrm{T}} X$.

Proof.

Let X be the characteristic function for the graph of f (under some suitable recursive pairing function of $\mathbb{N}$). □

Existence of Non-Recursive Functions

Proposition

There exists $f \in \mathbb{N}^{\mathbb{N}}$ which is non-recursive.

Proof.

The set of recursive 1-place functions is countable ($e \mapsto \varphi_e^{(1)}$ yields a surjection of $\mathbb{N}$ onto class of recursive functions), but $\mathbb{N}^{\mathbb{N}}$ is uncountable. □

Existence of Non-Recursive Functions – Halting Problem

A particular example:

Definition (Halting Problem)

$$0' := \{e \in \mathbb{N} \mid \varphi_e^{(1)}(e)\downarrow\}$$

Proposition

$0'$ *is non-recursive.*

Proof.

$$f(e) \simeq \begin{cases} 1 & \text{if } e \notin 0' \\ \text{undefined} & \text{if } e \in 0' \end{cases}$$

If $0'$ is recursive, so is f. Let e be an index for f. $e \in 0'$?

Case 1: $e \in 0'$. Then $f(e)\uparrow$ by definition, but $f(e)\downarrow$ by hypothesis.

Case 2: $e \notin 0'$. Then $f(e)\downarrow$ by definition, but $f(e)\uparrow$ by hypothesis.

□

Turing Jump

Relativizing the halting problem to an arbitrary oracle results in the following definition:

Definition (Turing Jump)

Suppose $f \in \mathbb{N}^{\mathbb{N}}$. The **Turing jump** *of f is defined by*

$$f' := \{e \in \mathbb{N} \mid \varphi_e^{(1),f}(e)\downarrow\}$$

Proposition

$f <_{\mathrm{T}} f'$

Proposition

If $f \leq_{\mathrm{T}} g$, then $f' \leq_{\mathrm{T}} g'$. Consequently, the Turing jump is well-defined on Turing degrees.

Iterated Turing Jumps

Can define iterated Turing jumps:

$$f^{(0)} := f$$
$$f^{(n+1)} := (f^{(n)})'$$

To extend past the finite ordinals, we set

$$f^{(\omega)} := \bigoplus_{n=0}^{\infty} f^{(n)} = \{\langle n, e\rangle \mid e \in f^{(n)}\}$$

In general, can define $f^{(\alpha)}$ for any *recursive ordinal*. (Requires some care to ensure well-definedness up to Turing equivalence.)

Theorems about the Turing Jump

Theorem (Friedberg's Jump Theorem/Jump Inversion Theorem)

If $0' \leq_T A$*, then there exists* B *such that* $A \equiv_T B' \equiv_T B \oplus 0'$.

Consequently, every $A \geq_T 0'$ is (Turing equivalent to) a Turing jump.

Theorem (Posner-Robinson Theorem)

Suppose $0 <_T Z \leq_T A$ *and* $0' \leq_T A$*. Then there exists* B *such that*

$$A \equiv_T B' \equiv_T B \oplus Z$$

Consequently, every non-recursive Z is, relative to some B, a Turing jump.

Relativized Arithmetical Hierarchy

Suppose $f \in \mathbb{N}^{\mathbb{N}}$.

Definition

$$\Sigma_0^{0,f} = \Pi_0^{0,f} = \Delta_0^{0,f} := \{R \mid R \subseteq \mathbb{N}^k \ (k \in \mathbb{N}) \text{ a } f\text{-recursive predicate}\}$$

$$\Sigma_{n+1}^{0,f} := \{S \mid S(\vec{m}) \equiv \exists n\, R(n, \vec{m}) \text{ for } R \in \Pi_n^{0,f}\}$$

$$\Pi_{n+1}^{0,f} := \{S \mid S(\vec{m}) \equiv \forall n\, R(n, \vec{m}) \text{ for } R \in \Sigma_n^{0,f}\}$$

$$\Delta_n^{0,f} := \Sigma_n^{0,f} \cap \Pi_n^{0,f}$$

Proposition

1. $\Sigma_n^{0,f}$ ($\Pi_n^{0,f}$) *is closed under conjunction, disjunction, and bounded quantification.*
2. $S \subseteq \mathbb{N}^k$ *is* $\Sigma_n^{0,f}$ *if and only if* S^c *is* $\Pi_n^{0,f}$.
3. *For each n,*

$$\Sigma_n^{0,f} \cup \Pi_n^{0,f} \subsetneq \Delta_{n+1}^{0,f} = \Sigma_{n+1}^{0,f} \cap \Pi_{n+1}^{0,f}$$

Recursively-Enumerable relative to an Oracle

Definition

$S \subseteq \mathbb{N}$ *is* **recursively enumerable relative to** $f \in \mathbb{N}^{\mathbb{N}}$ *if* S *is* $\Sigma_1^{0,f}$.

Proposition

Suppose $f \in \mathbb{N}^{\mathbb{N}}$ *and* $S \subseteq \mathbb{N}$. *The following are equivalent:*

1. S *is* $\Sigma_1^{0,f}$.
2. S *is the domain of some partial function partial recursive relative to* f.
3. $S = \varnothing$ *or* S *is the range of some* $g \leq_{\mathrm{T}} f$.

Proposition

Suppose $f \in \mathbb{N}^{\mathbb{N}}$ *and* $S \subseteq \mathbb{N}$. *Then* $S \leq_{\mathrm{T}} f$ *if and only if* S *is* $\Delta_1^{0,f}$.

$\Sigma_1^{0,f}$ and f'

An important interpretation of f' is as a uniform complete relativized r.e. set:

Theorem

f' is a complete $\Sigma_1^{0,f}$ set, i.e. if S is $\Sigma_1^{0,f}$, there exists a total recursive function g such that $g^{-1}[f'] = S$.

Iterated Turing jumps are closely related to higher levels of the relativized arithmetical hierarchy:

Theorem (Post's Theorem)

S is $\Sigma_1^{0,f^{(n)}}$ if and only if S is $\Sigma_{n+1}^{0,f}$.

Arithmetical Hierarchy of Subsets of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^k$

Definition

$$\Sigma_0^0 = \Pi_0^0 = \Delta_0^0 := \{R \mid R \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^k \ (k \in \mathbb{N}) \text{ a recursive predicate}\}$$

$$\Sigma_{n+1}^0 := \{S \mid S(f, \vec{m}) \equiv \exists n\, R(n, f, \vec{m}) \text{ for } R \in \Pi_n^0\}$$

$$\Pi_{n+1}^0 := \{S \mid S(f, \vec{m}) \equiv \forall n\, R(n, f, \vec{m}) \text{ for } R \in \Sigma_n^0\}$$

$$\Delta_n^0 := \Sigma_n^0 \cap \Pi_n^0$$

S is **arithmetical** *if $S \in \Sigma_n^0$ for some n.*

Proposition

1. *Σ_n^0 (Π_n^0) is closed under conjunction, disjunction, and bounded quantification.*
2. *$S \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^k$ is Σ_n^0 if and only if S^c is Π_n^0.*
3. *For each n,*

$$\Sigma_n^0 \cup \Pi_n^0 \subsetneq \Delta_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$$

Universal Predicates

Theorem

For each n and k, there exists a universal Σ^0_n (resp. Π^0_n) predicate $U \subseteq \mathbb{N}^{k+1}$, i.e. for each Σ^0_n (resp. Π^0_n) predicate $S \subseteq \mathbb{N}^k$ there exists $e \in \mathbb{N}$ such that

$$S(\vec{m}) \equiv U(e, \vec{m})$$

Proof (Sketch).

For $n = 1$, $U(e, \vec{m}) := \varphi^{(k)}_e(\vec{m}) \downarrow$. □

Corollary

For each n, k, there is an effective enumeration of the Σ^0_n (resp. Π^0_n) predicates $S \subseteq \mathbb{N}^k$.

Analogous results for predicates $\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k$.

Uniformization of Σ^0_1 Sets

Theorem

Suppose $S \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ is Σ^0_1. Then there exists $\hat{S} \subseteq S$ which is Σ^0_1 and for which if there exists n such that $S(f, n)$, then there exists a unique n such that $\hat{S}(f, n)$.

Proof.

Suppose $S(f, n) \equiv \exists m\, R(f, n, m)$ where R is recursive. Define

$$\hat{S}(f, n) \equiv \exists m(R(f, (m)_0, (m)_1) \wedge (n = (m)_0) \wedge \neg\exists(k < m)R(f, (k)_0, (k)_1))$$

which has the desired properties.
(Here $m \mapsto ((m)_0, (m)_1)$ is some recursive pairing function.) □

Π^0_1 Sets and Recursive Trees

Definition (Recursive Tree)

*A **recursive tree** is a recursive $T \subseteq \mathbb{N}^*$ closed under initial segments.*
*A **path** through T is an $f \in \mathbb{N}^\mathbb{N}$ such that $f \upharpoonright n \in T$ for all n.*

Proposition

$P \subseteq 2^\mathbb{N}$ (resp. $P \subseteq \mathbb{N}^\mathbb{N}$) is Π^0_1 if and only if P is the set of paths through a recursive subtree of 2^* (resp. $\mathbb{N}^*$).

Proof.

$$\begin{aligned} P &= \{X \in 2^\mathbb{N} \mid \varphi_e^{X,(1)}(e)\uparrow\} && \text{for some } e \\ &= \{X \in 2^\mathbb{N} \mid \forall n \varphi_e^{X\upharpoonright n,(1)}(e)\uparrow\} \\ &= \{X \in 2^\mathbb{N} \mid X \text{ a path through } T\} \end{aligned}$$

where $T = \{\sigma \in 2^* \mid \varphi_e^{\sigma,(1)}(e)\uparrow\}$ is recursive. Analogous for $\mathbb{N}^\mathbb{N}$ and $\mathbb{N}^*$. □

Properties of Π^0_1 Classes

Let $P_e = \{X \in 2^{\mathbb{N}} \mid \varphi_e^{(1),X}(e)\uparrow\}$.

Proposition

1. *For every $X \in 2^{\mathbb{N}}$, $X' = \{e \in \mathbb{N} \mid X \in P_e\}$.*
2. *If $Q \subseteq 2^{\mathbb{N}} \times \mathbb{N}^k$ is Π^0_1, there is $\alpha : \mathbb{N}^k \to \mathbb{N}$ primitive recursive such that for all $f \in 2^{\mathbb{N}}$ and $m_1, \ldots, m_k \in \mathbb{N}$,*

$$X \in P_{\alpha(m_1,\ldots,m_k)} \iff Q(X, m_1, \ldots, m_k)$$

Corollary

There is a primitive recursive function $v : \mathbb{N}^2 \to \mathbb{N}$ such that

$$P_{v(n,m)} = P_n \cap P_m$$

Basis Theorems for Π^0_1 Classes

Let P be a non-empty Π^0_1 subset of $2^{\mathbb{N}}$.

Theorem (Kleene Basis Theorem)

There exists $B \in P$ such that $B \leq_{\mathrm{T}} 0'$.

This can be substantially strengthened:

Theorem (Jump Inversion)

Suppose P is special, i.e. contains no recursive elements. Then for every $A \geq_{\mathrm{T}} 0'$, there exists $B \in P$ such that

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$$

In a different direction:

Theorem (Kreisel Basis Theorem)

If $0 <_{\mathrm{T}} Z$, then there exists $B \in P$ such that $Z \nleq_{\mathrm{T}} B$.

Analytical Hierarchy

Definition

$$\Sigma^1_0 = \Pi^1_0 = \Delta^1_0 := \{R \mid R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \ (k \in \mathbb{N}) \text{ an arithmetical predicate}\}$$

$$\Sigma^1_{n+1} := \{S \mid S(f, \underline{m}) \equiv \exists g\, R(f \oplus g, \underline{m}) \text{ for } R \in \Pi^1_n\}$$

$$\Pi^1_{n+1} := \{S \mid S(f, \underline{m}) \equiv \forall g\, R(f \oplus g, \underline{m}) \text{ for } R \in \Sigma^1_n\}$$

$$\Delta^1_n := \Sigma^1_n \cap \Pi^1_n$$

S is **analytical** *if $S \in \Sigma^1_n$ for some n.*

Analogous definition for subsets of $\mathbb{N}^k$, as well as a *relativized* analytical hierarchy.

As with the arithmetical hierarchy, there are complete Σ^1_n and Π^1_n sets for each n.

Hyperarithmetical Sets

Suppose $X \in 2^{\mathbb{N}}$.

Theorem (Kleene)

The following are equivalent:

1. $X \leq_{\mathrm{T}} 0^{(\alpha)}$ *for some recursive ordinal* α.
2. $X \subseteq \mathbb{N}$ *is* Δ^1_1.
3. X *is an element of any* ω*-model of ZFC.*

Definition

$X \in 2^{\mathbb{N}}$ *is* **hyperarithmetical** *if any of the equivalent conditions above hold. If* $f \in \mathbb{N}^{\mathbb{N}}$ *and* $f \equiv_{\mathrm{T}} X$*, then* f *is hyperarithmetical exactly when* X *is.*

$$\mathrm{HYP} := \{f \in \mathbb{N}^{\mathbb{N}} \mid f \text{ is hyperarithmetical}\}$$

Hyperarithmetical Reducibility

Relativizing the equivalent definitions of hyperarithmeticity yields the following:

Theorem

Suppose $X, Y \in 2^{\mathbb{N}}$. The following are equivalent:

1. $X \leq_{\mathrm{T}} Y^{(\alpha)}$ *for some ordinal* α *recursive in* Y.
2. $X \subseteq \mathbb{N}$ *is* $\Delta^{1,Y}_1$.
3. X *is an element of any* ω*-model of ZFC which contains* Y.

Definition

Suppose $X, Y \in 2^{\mathbb{N}}$. The **hyperarithmetical reducibility** *preorder* $\leq_{\mathrm{HYP}}$ *is defined by declaring* $X \leq_{\mathrm{HYP}} Y$ *if any of the equivalent conditions above hold.*
X *and* Y *are* **hyperarithmetically equivalent**, $X \equiv_{\mathrm{HYP}} Y$, *if and only if* $X \leq_{\mathrm{HYP}} Y$ *and* $Y \leq_{\mathrm{HYP}} X$. $\equiv_{\mathrm{HYP}}$ *is an equivalence relation.*

Existence of Non-Hyperarithmetical Functions – Kleene's $\mathcal{O}$

A cardinality argument shows that existence of non-hyperarithmetical functions. As a particular example:

Definition (Kleene's $\mathcal{O}$)

$\mathcal{O}$ is a fixed complete Π^1_1 subset of $\mathbb{N}$.

Theorem

$\mathcal{O}$ is not hyperarithmetical.

Proof.

Being a complete Π^1_1 set implies that it is not Δ^1_1. □

Σ^1_1 Sets as the Analogue for Π^0_1 Sets

One might expect that Σ^1_1 sets correspond to Σ^0_1 sets, but that is not the case.
Σ^0_1 sets instead correspond more closely to Π^1_1 sets.

Theorem (Π^1_1 Uniformization)

Suppose $S \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ is Π^1_1. Then there exists $\hat{S} \subseteq S$ which is Π^1_1 and for which if there exists n such that $S(f, \vec{m}, n)$, then there exists a unique n such that $\hat{S}(f, \vec{m}, n)$.

(Kondo's Theorem shows that Π^1_1 Uniformization holds for $S \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ (much harder).)
Σ^1_1 Uniformization fails badly!

Hyperjumps

Our characterizations of the Turing jump motivates the existence of a similar operation with analogous use of the analytical hierarchy (with the twist that $\Sigma_1^{0,X}$ corresponds to $\Pi_1^{1,X}$).

Definition

The **hyperjump** *of X is a fixed complete $\Pi_1^{1,X}$ set $\mathcal{O}^X$ (uniform in X).*

Proposition

If $X \leq_{\mathrm{HYP}} Y$, then $\mathcal{O}^X \leq_{\mathrm{T}} \mathcal{O}^Y$. Consequently, the hyperjump is well-defined on hyper degrees.

The Analogy between Arithmetical and Hyperarithmetical Theory

Arithmetical Theory	Hyperarithmetical Theory
Arithmetical Hierarchy	Analytical Hierarchy
Recursive = Δ_1^0	Hyperarithmetical = Δ_1^1
Turing reducibility	Hyperarithmetical reducibility
Π_1^0	Σ_1^1
Turing jump	Hyperjump
First-Order Logic	ω-logic

Properties of Σ^1_1 Classes

The analogy holds well in terms of the relation between the hyperjump and Σ^1_1 sets, as well as the structural properties of Σ^1_1 sets.

Proposition

1. *For every* $X \in 2^{\mathbb{N}}$,
$$\mathcal{O}^X \equiv_{\mathrm{T}} \{e \in \mathbb{N} \mid X \in P^*_e\}$$
2. *If* $K \subseteq 2^{\mathbb{N}} \times \mathbb{N}^k$ *is* Σ^1_1, *there is* $\alpha : \mathbb{N}^k \to \mathbb{N}$ *primitive recursive such that for all* $X \in 2^{\mathbb{N}}$ *and* $m_1, \ldots, m_k \in \mathbb{N}$,
$$X \in P^*_{\alpha(m_1,\ldots,m_k)} \iff K(X, m_1, \ldots, m_k)$$

Basis Theorems for Σ^1_1 Classes

Let K be a non-empty Σ^1_1 subset of $2^{\mathbb{N}}$.

Theorem (Gandy Basis Theorem)

There exists $B \in K$ such that $B <_{\mathrm{HYP}} \mathcal{O}$. Consequently, $\omega_1^B = \omega_1^{\mathrm{CK}}$.

Theorem (Hyperjump Inversion)

Suppose K is special, i.e. contains no hyperarithmetical elements. Then for every $A \geq_{\mathrm{T}} \mathcal{O}$, there exists $B \in K$ such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

Theorem (Kreisel Basis Theorem)

If Z is not hyperarithmetical, then there exists $B \in K$ such that $Z \nleq_{\mathrm{HYP}} B$.

Next time – Posner-Robinson for Turing Degrees of Hyperjumps

Theorem (Posner-Robinson for Turing Degrees of Hyperjumps)

Suppose $0 <_{\mathrm{HYP}} Z \leq_{\mathrm{T}} A$ *and* $\mathcal{O} \leq_{\mathrm{T}} A$*. Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z$$

Proof will involve Kumabe-Slaman Forcing and a stronger combined basis theorem for special Σ^1_1 sets.

Thank you!

Another Characterization of the Turing Jump

Suppose $B \in 2^{\mathbb{N}}$.
Consider the language of arithmetic with an additional unary predicate X, i.e. $\{0, S, +, \cdot, X\}$ where $\mathrm{ar}(0) = 0$, $\mathrm{ar}(S) = 1$, $\mathrm{ar}(+) = \mathrm{ar}(\cdot) = 2$, and $\mathrm{ar}(X) = 1$.
Define

$$\mathrm{PA} + (X = B) := \mathrm{PA} \cup \{X(\underline{n}) \mid n \in B\} \cup \{\neg X(\underline{n}) \mid n \notin B\}$$

Theorem

$$B' \equiv_{\mathrm{T}} \{\textit{Gödel numbers of theorems of } \mathrm{PA} + (X = B)\}$$

In particular, the halting problem is Turing equivalent to the problem of deciding whether a given sentence is a theorem of PA.

Hyperjumps - Alternative Characterization

Let Z_2 be the second-order theory of arithmetic.
A characterization of the hyperjump which is analogous to that of the Turing jump is in terms of theorems of $\mathsf{Z}_2 + (X = B)$. Say that φ is an ω-**theorem** of Φ if it true in every ω-model of Φ.

Theorem

$$\mathcal{O}^B \equiv_{\mathrm{T}} \{\textit{Gödel numbers of } \omega\textit{-theorems of } \mathsf{Z}_2 + (X = B)\}$$

Another characterization of $\mathcal{O}^B$ makes precise the notion of "notations for ordinals recursive in B".