The relationship of nilpotence to supernilpotence

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Overview of Talk

1. Basic Definitions

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2. Commutator Theory, Nilpotence, and Supernilpotence

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3. Higher Dimensional Congruence Relations

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6. Supernilpotence Need Not Imply Nilpotence

Basic Definitions

- Let $\{f_i\}_{i \in I}$ be a set of operation symbols and let $\sigma : I \to \omega$ be a function that assigns a finite arity to each function symbol. An **algebra** is a pair

$$\mathbb{A} = \langle A; \{f_i^{\mathbb{A}}\}_{i \in I} \rangle$$

where

1. A is a nonempty set called the **universe** of $\mathbb{A}$ and
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  1. $x(yz) \approx (xy)z$,
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  Varieties are exactly the HSP-closed classes (Birkhoff).

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- A **function clone** on a set A is a multi-sorted algebraic structure

$$\mathcal{C} = \langle C_1, C_2, \ldots, C_n, \ldots; \circ, \{\pi_i^n : n \geq 1 \text{ and } 0 \leq i < n\}\rangle$$

where

1. each $C_n \subseteq A^{A^n}$,
2. $\mathcal{C}$ contains all projection operations: $\pi_i^n(x_0, \ldots, x_{n-1}) = x_i$, and
3. $\mathcal{C}$ is closed under composition, e.g. for $f \in C_n$ and $g_0, \ldots, g_{n-1} \in C_m$

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- Let $\mathbb{A}$ be an algebra. The **clone of term operations** of $\mathbb{A}$ is denoted by $\mathrm{Clo}(\mathbb{A})$ and is defined to be the smallest function clone containing all of the basic operations of $\mathbb{A}$. (If $\mathbb{A}$ has null-ary operations we replace them by unary constant operations.)

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- Let $\mathbb{A}$ be an algebra. The **clone of polynomial operations** of $\mathbb{A}$ is denoted by $\mathrm{Pol}(\mathbb{A})$ and is the smallest function clone containing the basic operations of $\mathbb{A}$ and all constants.

Basic Definitions

- Let $\mathbb{A}$ be an algebra with universe A and $n \geq 1$ a natural number. A subset
$$R \subseteq A^n$$
is called an $\mathbb{A}$-**invariant relation** if it is closed under the basic operations of $\mathbb{A}$, equivalently, if R is a subalgebra of $\mathbb{A}^n$.

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- The invariant equivalence relations of an algebra $\mathbb{A}$ are called **congruences**. The collection of all congruences of an algebra forms an algebraic lattice under inclusion and is denoted by $\mathrm{Con}(\mathbb{A})$.

Commutator Theory

- The classical commutator for a universal algebra $\mathbb{A}$ is a binary operation
$$[\cdot,\cdot] : \mathsf{Con}(\mathbb{A})^2 \to \mathsf{Con}(\mathbb{A})$$
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- For example, an algebra $\mathbb{A}$ is said to be **abelian** if
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- The higher commutator is a higher arity operation that generalizes the binary commutator, e.g.
$$[\underbrace{\cdot,\ldots,\cdot}_{n}] : \mathsf{Con}(\mathbb{A})^n \to \mathsf{Con}(\mathbb{A})$$

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Definition

Let $\mathbb{A}$ be an algebra and let $\theta \in \mathrm{Con}(\mathbb{A})$. Set $[\theta]_0 = (\theta]^0 := \theta$ and

$$[\theta]_{i+1} := [[\theta]_i, [\theta]_i] \qquad \text{and} \qquad (\theta]^{i+1} = [(\theta]_i, \theta]_{TC}.$$

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3. If θ is such that $\underbrace{[\theta, \ldots, \theta]}_{n+1} = 0$, then θ is said to be ***n*-step supernilpotent**.

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  1. A finite Mal'cev algebra of finite type is supernilpotent if and only it is the product of prime power order nilpotent algebras. (Freese & McKenzie, Kearnes, Aichinger & Mudrinski)
  2. There is a polynomial time algorithm to solve the equation satisfiability problem for a finite supernilpotent Mal'cev algebra of finite type. (Kompatscher)

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- It follows from results of Wires that any supernilpotent algebra generating a modular variety is nilpotent.
- We can show any supernilpotent Taylor algebra is nilpotent. (A Taylor algebra is an algebra that satisfies some nontrivial idempotent Mal'cev condition.)
- Moore and M. have constructed a supernilpotent algebra that is not solvable and hence not nilpotent. Note, this algebra is necessarily infinite and not Taylor.

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$$\Big(t(\mathbf{a}_0, \mathbf{a}_1) \equiv_\delta t(\mathbf{a}_0, \mathbf{b}_1) \implies t(\mathbf{b}_0, \mathbf{a}_0) \equiv_\delta t(\mathbf{b}_0, \mathbf{b}_1) \Big)$$

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We write $C_{TC}(\alpha, \beta; \delta)$ whenever this is true.

- The term condition may be described as a condition that is quantified over a certain invariant relation of $\mathbb{A}$ which is called the algebra of (α, β)-matrices and is denoted $M(\alpha, \beta)$.

Matrices

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- We say that a relation R on a set A is 2-dimensional if $R \subseteq A^{2^2}$ (R is a set of squares whos vertices are labeled by elements of A.)

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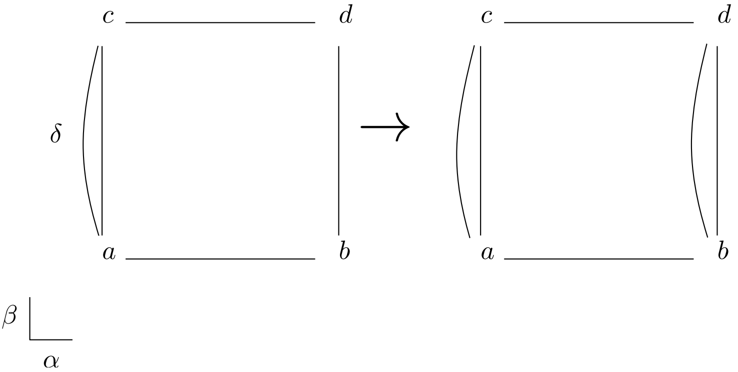
- $M(\alpha, \beta)$ is the subalgebra of $\mathbb{A}^{2^2}$ with generators

$$\left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x \equiv_\alpha y \right\} \bigcup \left\{ \begin{bmatrix} y & y \\ x & x \end{bmatrix} : x \equiv_\beta y \right\}$$

Matrices

For $\delta \in \mathrm{Con}(\mathbb{A})$ we have that α **centralizes** β **modulo** δ if the implication



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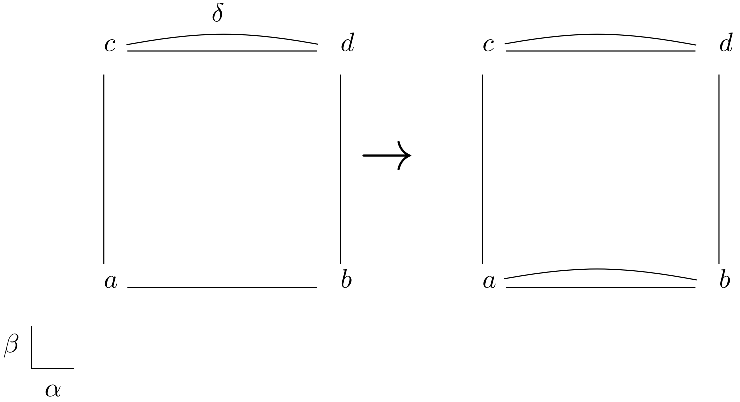
Similarly, we have that β **centralizes** α **modulo** δ if the implication



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Matrices

- The binary commutator is defined to be

$$[\alpha, \beta]_{TC} = \bigwedge\{\delta : C(\alpha, \beta; \delta)\}$$

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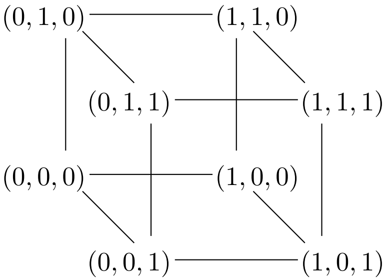

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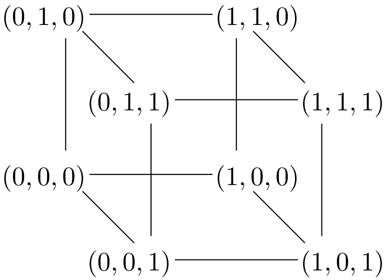

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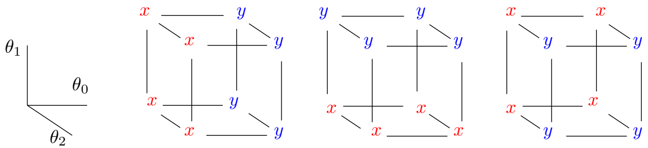
Matrices

- For congruences $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$, set $M(\theta_0, \theta_1, \theta_2) \leq \mathbb{A}^{2^3}$ to be the subalgebra generated by the following labeled cubes:



$M(\theta_0, \theta_1, \theta_2)$ is called the algebra of $(\theta_0, \theta_1, \theta_2)$-matrices.

Centrality

- For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:

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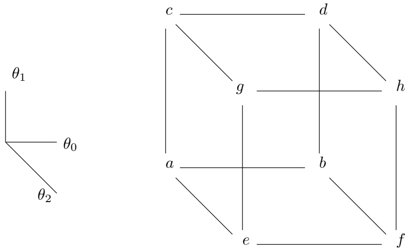

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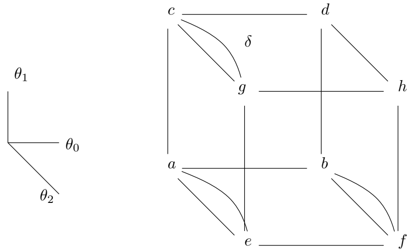

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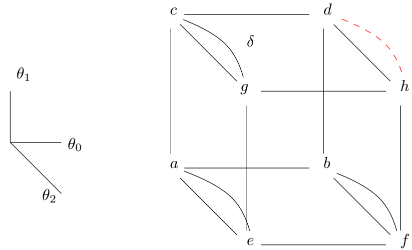

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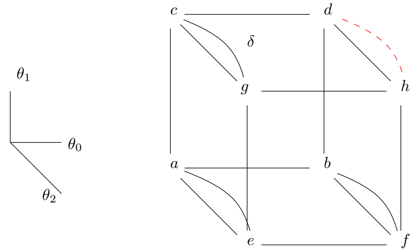
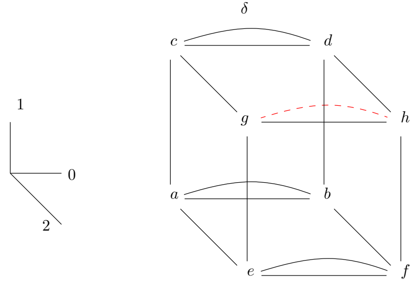


- This condition is abbreviated $C_{TC}(\theta_0, \theta_1, \theta_2; \delta)$.

Centrality

- Here is a picture of $C_{TC}(\theta_1, \theta_2, \theta_0; \delta)$:



Matrices

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- Higher centrality and the commutator for arity ≥ 4 are similarly defined.

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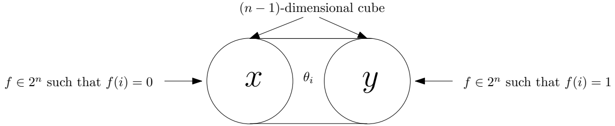
- **Observation:** The term condition definition of centrality involving n-many congruences $\theta_0, \ldots, \theta_{n-1}$ is a condition that is quantified over $(\theta_0, \ldots, \theta_{n-1})$-**matrices**, which are certain n-dimensional invariant relations

$$M(\theta_0, \ldots, \theta_{n-1}) \leq \mathbb{A}^{2^n}$$

that have generators of the form



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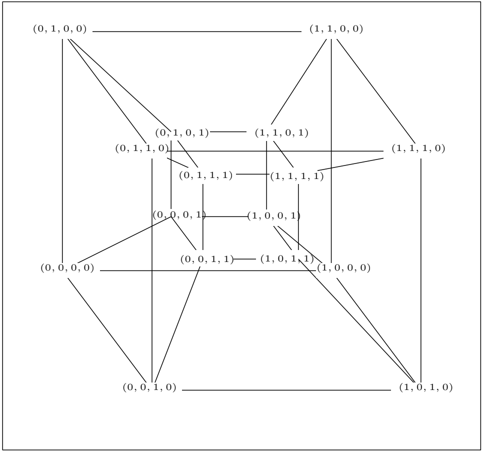

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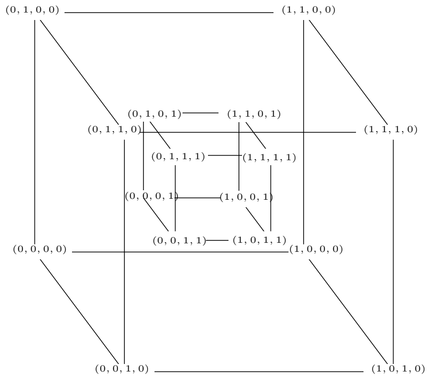

$(\mathbb{H}_n)_3^0$ and $(\mathbb{H}_n)_3^1$

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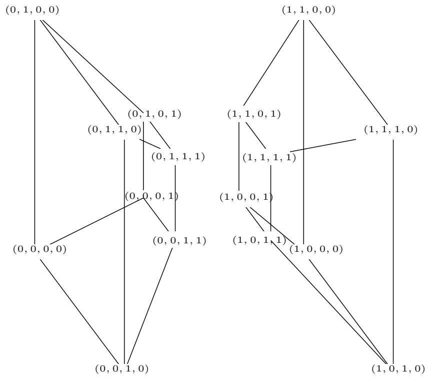

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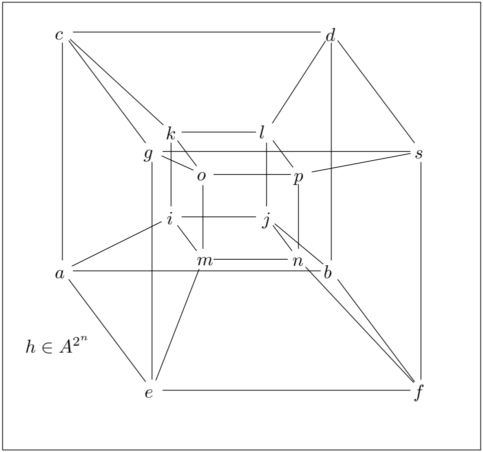

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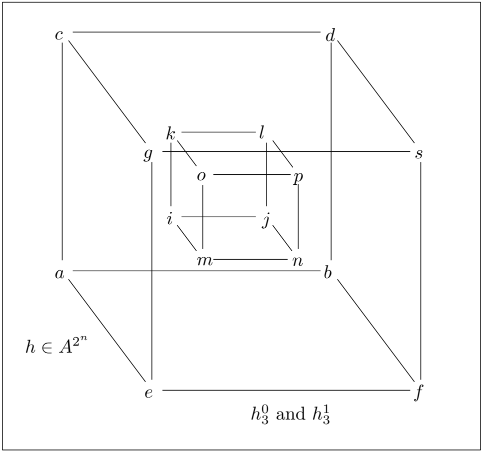

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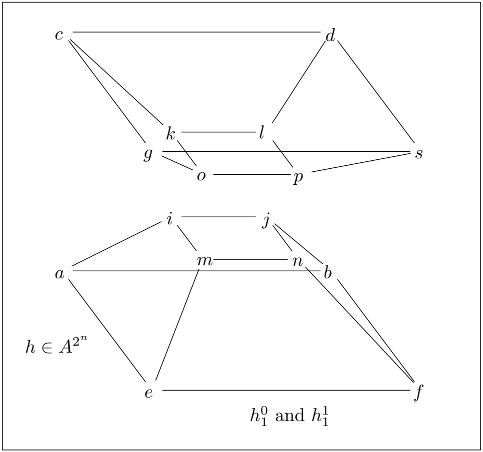

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- **Fact:** Suppose $\mathbb{A}$ is a member of a permutable variety, and take $(\theta_0, \ldots, \theta_{n-1}) \in \mathrm{Con}(\mathbb{A})^n$. Then,

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- This leads to a nice characterization of the commutator for permutable varieties.

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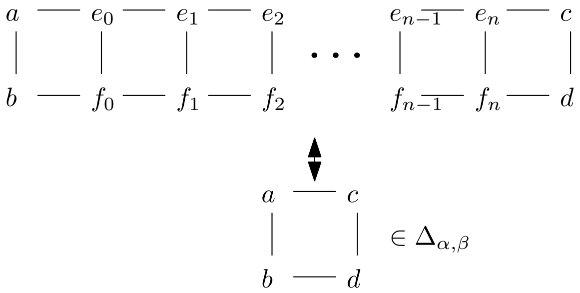

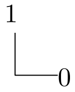

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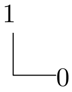
$$\begin{tikzcd}
a \arrow[r, dash] \arrow[d, dash] & e_0 \arrow[r, dash] \arrow[d, dash] & e_1 \arrow[r, dash] \arrow[d, dash] & e_2 \arrow[d, dash] & \cdots & e_{n-1} \arrow[r, dash] \arrow[d, dash] & e_n \arrow[r, dash] \arrow[d, dash] & c \arrow[d, dash] \\
b \arrow[r, dash] & f_0 \arrow[r, dash] & f_1 \arrow[r, dash] & f_2 & & f_{n-1} \arrow[r, dash] & f_n \arrow[r, dash] & d
\end{tikzcd}$$

$$\Updownarrow$$



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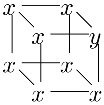
(2) 

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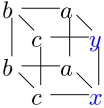
(3) 

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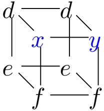
(4) 

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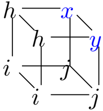
(5) 

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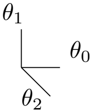

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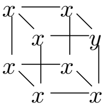
(2) 

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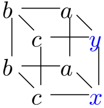
(3) 

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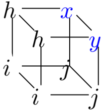
(5) 

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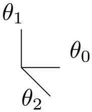
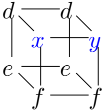


(4) 

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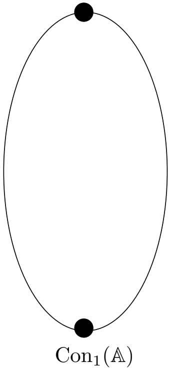
- There are n distinct embeddings from $\text{Con}_1(\mathbb{A})$ into $\text{Con}_n(\mathbb{A})$.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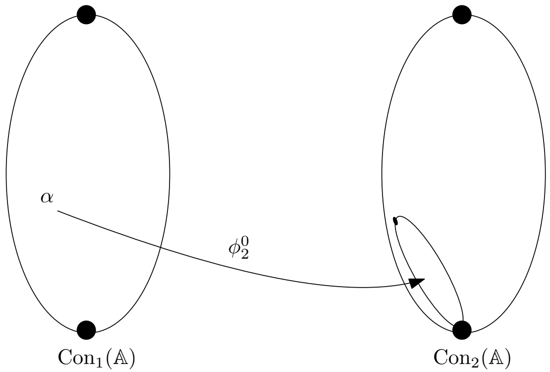
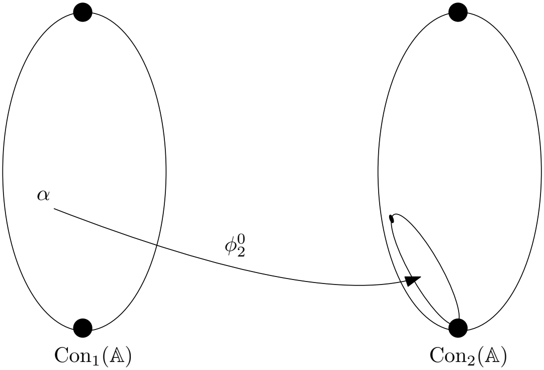


α
ϕ_2^0
$\mathrm{Con}_1(\mathbb{A})$
$\mathrm{Con}_2(\mathbb{A})$



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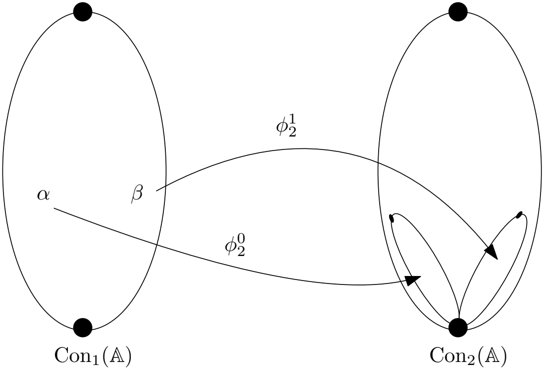

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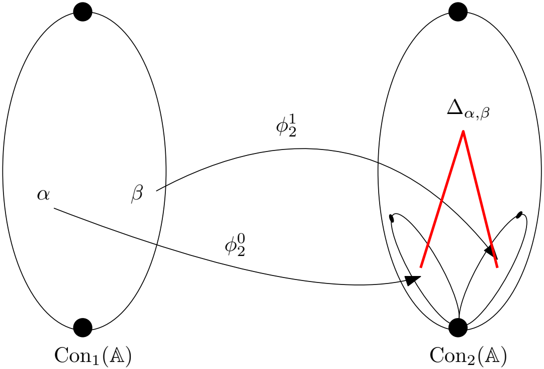
$\color{red}{\text{Define } \Delta_{\alpha,\beta} = \phi_2^0(\alpha) \vee \phi_2^1(\beta)}$



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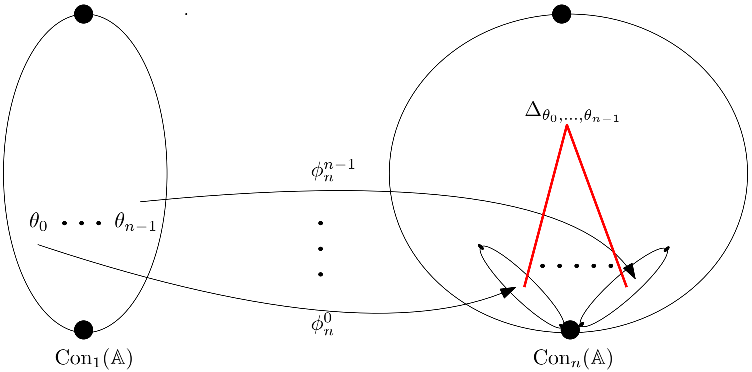
- Define $\phi_n^i : \mathrm{Con}_1(\mathbb{A}) \to \mathrm{Con}_n(\mathbb{A})$ by

$$\phi_n^i(\alpha) = \{\mathrm{Cube}_i(\langle x, y\rangle) : \langle x, y\rangle \in \alpha\}$$

Define $\Delta_{\theta_0,\ldots,\theta_{n-1}} = \bigvee_i \phi_n^i(\theta_i)$



Characterizing Joins

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- This generalizes to:

Theorem

Let $\mathbb{A}$ be an algebra and let $n \geq 1$. An n-dimensional equivalence relation θ is admissible if and only if θ is compatible with the n-ary polynomials of $\mathbb{A}$.

Proof Idea

Take
$$\begin{tikzcd}[arrows=dash]
{\color{red}a_0} \arrow[r] \arrow[d] & {\color{blue}b_0} \arrow[d] & {\color{red}a_1} \arrow[r] \arrow[d] & {\color{blue}b_1} \arrow[d] & {\color{red}a_2} \arrow[r] \arrow[d] & {\color{blue}b_2} \arrow[d] & {\color{red}a_3} \arrow[r] \arrow[d] & {\color{blue}b_3} \arrow[d] \\
{\color{orange}c_0} \arrow[r] & {\color{teal}d_0}\,, & {\color{orange}c_1} \arrow[r] & {\color{teal}d_1}\,, & {\color{orange}c_2} \arrow[r] & {\color{teal}d_2}\,, & {\color{orange}c_3} \arrow[r] & {\color{teal}d_3}
\end{tikzcd} \in \theta$$

Proof Idea

Take $\begin{matrix} a_0 & b_0 \\ c_0 & d_0 \end{matrix}, \begin{matrix} a_1 & b_1 \\ c_1 & d_1 \end{matrix}, \begin{matrix} a_2 & b_2 \\ c_2 & d_2 \end{matrix}, \begin{matrix} a_3 & b_3 \\ c_3 & d_3 \end{matrix} \in \theta$ Then,

$$\begin{matrix} a_0 & b_0 & b_0 & b_0 & b_0 \\ c_0 & d_0 & d_0 & d_0 & d_0 \\ c_0 & d_0 & d_0 & d_0 & d_0 \\ c_0 & d_0 & d_0 & d_0 & d_0 \\ c_0 & d_0 & d_0 & d_0 & d_0 \end{matrix} \qquad \begin{matrix} a_1 & a_1 & b_1 & b_1 & b_1 \\ a_1 & a_1 & b_1 & b_1 & b_1 \\ c_1 & c_1 & d_1 & d_1 & d_1 \\ c_1 & c_1 & d_1 & d_1 & d_1 \\ c_1 & c_1 & d_1 & d_1 & d_1 \end{matrix}$$

$$\begin{matrix} a_2 & a_2 & a_2 & b_2 & b_2 \\ a_2 & a_2 & a_2 & b_2 & b_2 \\ a_2 & a_2 & a_2 & b_2 & b_2 \\ c_2 & c_2 & c_2 & d_2 & d_2 \\ c_2 & c_2 & c_2 & d_2 & d_2 \end{matrix} \qquad \begin{matrix} a_3 & a_3 & a_3 & a_3 & b_3 \\ a_3 & a_3 & a_3 & a_3 & b_3 \\ a_3 & a_3 & a_3 & a_3 & b_3 \\ a_3 & a_3 & a_3 & a_3 & b_3 \\ c_3 & c_3 & c_3 & c_3 & d_3 \end{matrix} \quad \in \theta$$

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Compatibility with binary polynomials is sufficient to show compatibility with a 4-ary operation.

Characterizing Joins

- $\Delta_{\theta_0,\ldots,\theta_{n-1}} = \bigvee_i \phi_n^i(\theta_i)$ is therefore obtained by
    1. Closing $\bigcup \phi_n^i(\theta_i)$ under all n-ary polynomials and then

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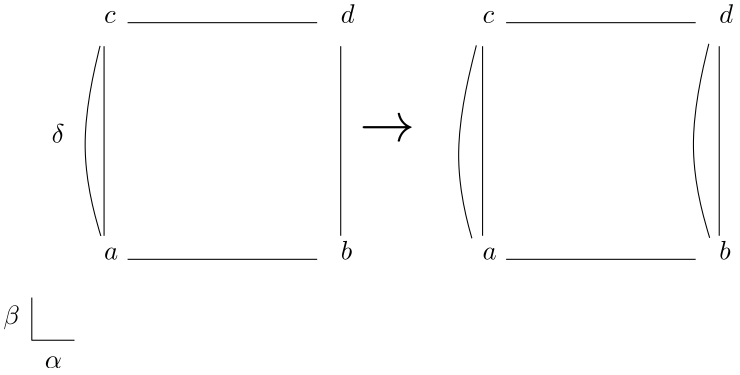
  1. Closing $\bigcup \phi_n^i(\theta_i)$ under all n-ary polynomials and then
  2. taking a sequence of transitive closures, cycling through all possible directions possibly ω-many times.
- Notice: $M(\theta_0, \ldots, \theta_{n-1}) \leq \Delta_{\theta_0,\ldots,\theta_{n-1}}$. We use this larger collection to define a stronger term condition.

Hypercentrality

For $\delta \in \mathrm{Con}(\mathbb{A})$ we have that α **hypercentralizes** β **modulo** δ if the implication



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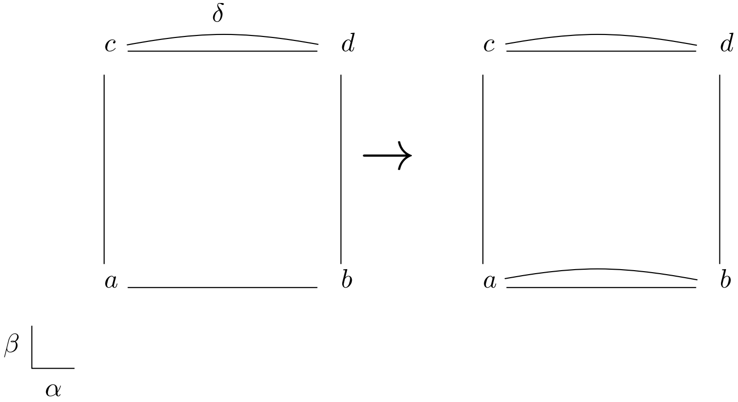

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- For congruences θ_0, θ_1 we set

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- Higher arity hypercentrality and the higher arity hypercommutator similarly defined.

Theorem (Binary Hyper Commutator)

Let $\mathbb{A}$ be an algebra. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, the following are equivalent:

1. $\langle x, y \rangle \in [\alpha, \beta]_H$
2. $\begin{bmatrix} x & y \\ x & x \end{bmatrix} \in \Delta_{\alpha,\beta}$
3. $\begin{bmatrix} a & y \\ a & x \end{bmatrix} \in \Delta_{\alpha,\beta}$ *for some* $a \in A$
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- A similar characterization of the higher arity hyper commutator also holds.

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  4. (2) and (3) imply that

     $$[[\theta, \dots, \theta]_{TC}, \theta, \dots, \theta]_{TC} = [[\theta, \dots, \theta]_H, \theta, \dots, \theta]_H$$

     $$\leq [\theta, \dots, \theta]_H = [\theta, \dots, \theta]_{TC}$$

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Define $A = O \cup R \cup G$ with G infinite, $O = \{o_i^j : i, j \in \omega\}$, and $R = \{r_i^j : i, j \in \omega\}$.

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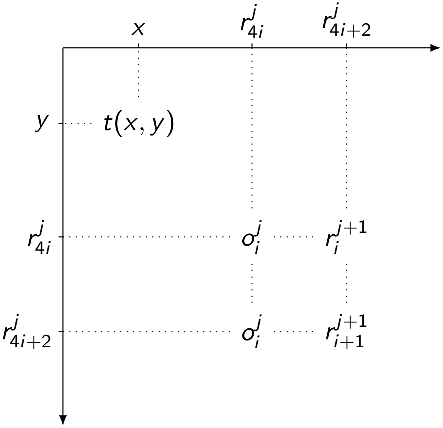

where t an injection into G otherwise.

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- $\mathbb{A}$ is not solvable and hence not nilpotent.
- $\mathbb{A}$ is 2-step supernilpotent. To prove this it suffices to show that

$$h = \begin{tikzcd}[row sep=small, column sep=small]
a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b \arrow[dr, dash] \arrow[dd, dash, dotted] & \\
& c \arrow[rr, dash] \arrow[dd, dash] & & e \arrow[dd, dash] \\
a \arrow[rr, dash, dotted] \arrow[dr, dash] & & b \arrow[dr, dash, dotted] & \\
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\end{tikzcd} \in M(1,1,1)$$

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  1. are not **solvable in dimension** n (no term in commutators up to arity n evaluated at 1 produces 0)
  2. but are n-step supernilpotent.
- Question: Let $[\mathcal{V}]$ be a chapter in the lattice of interpretability of types that does not lie above Olšák's variety. Is there a variety $\mathcal{W} \in [\mathcal{V}]$ with a supernilpotent algebra that is not nilpotent?