

Lattice-theoretic properties of algebras of logic

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Abstract

In the theory of lattice-ordered groups, there are interesting examples of properties — such as projectability — that are defined in terms of the overall structure of the lattice-ordered group, but are entirely determined by the underlying lattice structure. In this paper, we explore the extent to which projectability is a lattice-theoretic property for more general classes of algebras of logic. For a class of integral residuated lattices that includes Heyting algebras and semilinear residuated lattices, we prove that a member of such is projectable iff the order dual of each subinterval $[a, 1]$ is a Stone lattice. We also show that an integral GMV algebra is projectable iff it can be endowed with a positive Gödel implication. In particular, a Ψ MV or an MV algebra is projectable iff it can be endowed with a Gödel implication. Moreover, those projectable involutive residuated lattices that admit a Gödel implication are investigated as a variety in the expanded signature. We establish that this variety is generated by its totally ordered members and is a discriminator variety.

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1. Introduction

Two issues arise quite naturally in the study of lattice-ordered algebras, in particular of algebras of logic:

1. To what extent are the properties of such algebras determined by the structure of the underlying lattices?
2. Can one achieve valuable insights into the behavior of a given class of lattice-ordered algebras by expanding its signature to include a term realizing an operation that is everywhere definable in the class? How can this strategy simplify its investigation with the tools of universal algebra?

Although these questions are in some measure independent, they are not as unrelated as they may appear. Hereafter, we will address them jointly, meaning to present a case-study as an example of a much more general situation that invites further inquiry.

1) As regards the first question, one can, for example, single out properties of a lattice-ordered algebra that are preserved under isomorphisms of its lattice reduct, or identify interesting cases where membership in some class can be described by first order lattice-theoretical conditions. This line of investigation has been pursued in the theories of MV algebras [?, Chapter 6] or of lattice-ordered groups (henceforth, ℓ -groups); results exemplifying this approach in the latter environment are the theorems by Conrad and Darnel to the effect that free Abelian ℓ -groups have a unique multiplication, or that an ℓ -group that is isomorphic as a lattice to a hyper-archimedean ℓ -group is semi-linear (representable) [?, Theorem 2.9].

A class of ℓ -groups that is known to be characterized purely in terms of its order structure is the class of *projectable* ℓ -groups — namely, ℓ -groups in which every principal polar is a cardinal summand (see e.g. [?]). This is a radical class of ℓ -groups that includes conditionally σ -complete ℓ -groups [?, p. 230] and vector lattices with the principal projection property. Considering: a) the significance of such examples in functional analysis and in other parts of mathematics (think of the Riesz decomposition theorem for order-complete vector lattices [?]); b) the historical relevance attached to a problem suggested by Birkhoff [?, Problem 117], [?], who challenged his readers to characterize projectable ℓ -groups; c) the fact that every representable ℓ -group can be embedded into a member of this class [?], projectable

ℓ -groups are definitely worth studying. One of the present authors has established that an ℓ -group is projectable iff every interval $[a, b]$ is a Stone lattice; as a consequence, projectability is preserved under lattice isomorphisms ([?]; [?]; [?]; [?]). Also, the negative cone of an ℓ -group is projectable iff its lattice reduct can be endowed with a *positive Gödel implication*. This interesting property immediately reverberates in an even more attractive form upon MV algebras, the relative pseudo-complement being in fact — in the bounded case — a Gödel implication.

2) It is not uncommon to find examples of order-theoretically relevant classes of algebras that fail to be varieties in their original signature, but they become so once their signature is expanded by a term realizing some definable operation. When this happens, these classes evidently become more tractable from the perspective of universal algebra and more easily investigated. A case in point is represented by *generalized Boolean algebras* (GBAs), which — unlike Boolean algebras proper — may fail to have a bottom element. GBAs, albeit not a variety in the language $(\wedge, \vee, 1)$, become equationally definable if a binary implication operation is added to their signature; more precisely, they can be viewed as a variety of residuated lattices [?]. Another instance is provided by the equivalent quasivariety semantics of *product Łukasiewicz logic*, obtained from infinite-valued Łukasiewicz logic by adding a set of axioms for the product T-norm [?]; this class is not a variety and, in particular, fails to coincide with the variety of *product MV algebras*, generated by the standard MV algebra over the $[0, 1]$ real interval endowed with the natural product [?]. However, it becomes a variety if expanded with a globalization¹ operator δ (and suitable additional equations) whose values on subdirectly irreducible algebras \mathbf{A} are given by $\delta(a) = 0$ if $a < 1$ and $\delta(a) = 1$ otherwise, for all $a \in A$ [?]. For projectable MV algebras, we have an analogous situation. Once their signature is expanded with a Gödel implication as suggested by the above-mentioned result, one obtains not only a variety, but a *discriminator* one at that. This observation has in fact been made by Cattaneo et al., see [?] and [?], even if their motivation for studying these hybrid structures, which they call *Heyting-Wajsberg algebras*, was of a different kind.

¹The globalization operator is sometimes also called *Baaz Delta*, after some pioneering work done on the topic by Matthias Baaz [?].

Against the backdrop of this preliminary discussion, the aim of this paper is twofold:

- We explore whether the lattice-theoretical characterization of projectable ℓ -groups and their negative cones in [?] carries over to arbitrary integral, distributive residuated lattices. We prove that:

Theorem A (see Theorem ??) *An integral, distributive residuated lattice satisfying*

$$x \vee y \approx 1 \implies xy \approx x \wedge y.$$

is projectable iff the order dual of each interval $[a, 1]$ is a Stone lattice.

In general, having a positive Gödel implication is a stronger condition than being projectable, although it is equivalent in some especially well-behaved cases:

Theorem B (see Corollary ??) *An integral GMV algebra is projectable iff it can be endowed with a positive Gödel implication.*

Theorem C (see Corollary ??) *A ΨMV (in particular, MV) algebra is projectable iff it can be endowed with a Gödel implication.*

- We investigate these particular projectable residuated lattices in the involutive case and in the expanded signature containing a Gödel implication, although we do not require, contrary to the suggestion by Cattaneo and his colleagues, that the latter be necessarily an MV algebra reduct. The additional involutivity assumption is rendered necessary by the need to effectively cope with the problem of filter generation without further expanding our language. For these algebras, hereby called *Gödel residuated lattices*, we prove:

Theorem D (see Theorem ??) *The variety of Gödel residuated lattices is generated by its totally ordered members.*

Theorem E (see Theorem ??) *The variety of Gödel residuated lattices is a discriminator variety.*

It is important to be mentioned at this point that, although it is not assumed that the residuated lattice reducts of Gödel residuated lattices are semi-linear, these algebras are in fact semi-linear as a consequence of Theorem D. In other words, a non-semi-linear residuated lattice cannot be endowed with a Gödel implication or a positive Gödel implication.

Our work presents connections with other approaches taken in the literature. Besides the already mentioned product MV algebras, it is known that many other varieties of logic — including MV algebras and, more generally, BL algebras — can be made into discriminator varieties if enriched with a globalization operator as defined above, and additional equations involving this operator. The same goal can be attained if one replaces δ by a Gödel negation, which analogously maps any element in a subdirectly irreducible algebra to the Boolean subalgebra with universe $\{0, 1\}$. Indeed, it is implicit in [?] and explicitly proved in [?] that MV algebras with globalization are term equivalent to Heyting-Wajsberg algebras. Subsequently, Chajda and Vychodil [?] showed that divisibility of the residuated lattice plays no role in getting a discriminator variety, since the same property obtains for every bounded commutative integral residuated lattice with globalization. Another related research stream is a series of papers by Cignoli and Torrens [? ? ?] who study integral residuated lattices with Boolean retracts², and by Cignoli and Esteva, who investigate integral residuated lattices with a Stonean negation and an additional involutive negation [?].

2. Preliminaries

The present section reviews some basic notions on pseudo-complemented lattices, Gödel algebras and (pointed) residuated lattices only to such an extent as is necessary to make this paper reasonably self-contained. For additional information on Gödel algebras and Gödel logic the reader is referred to [?] or [?], while [?], [?] or [?] can be profitably consulted as regards residuated lattices.

2.1. Pseudo-complemented lattices

A bounded lattice $\mathbf{L} = (L, \wedge, \vee, 1, 0)$ is said to be *pseudo-complemented* if for all $a \in L$, $\max\{x : a \wedge x = 0\}$ exists. This element is denoted by $\neg a$ and referred to as the *pseudo-complement* of a . Pseudo-complemented lattices need not be distributive, but we will henceforth assume that distributivity

²The homomorphisms that determine these retracts have, in fact, some properties in common with globalization operations. Over time, these authors proceeded at an increasing level of generality, carrying out their investigation first in the context of BL algebras, then of MTL algebras, and finally of arbitrary integral, bounded and commutative residuated lattices.

holds for the lattices under consideration. The map $\neg : L \rightarrow L$ is a self-adjoint order-reversing map, while the map sending a to its double pseudo-complement $\neg\neg a$ is a meet-preserving closure operator on L . By a classic result due to Glivenko, the image of this closure operator is a Boolean algebra $\mathbf{B}_{\mathbf{L}}$ with least element 0 and largest element 1. The complement of a in $\mathbf{B}_{\mathbf{L}}$ is precisely $\neg a$, whereas, for any pair of elements a, b of $\mathbf{B}_{\mathbf{L}}$ – also referred to as closed elements of L ,

$$a \vee^{\mathbf{B}_{\mathbf{L}}} b = \neg(\neg a \wedge^{\mathbf{L}} \neg b).$$

On the other hand, any existing meets in $\mathbf{B}_{\mathbf{L}}$ coincide with those in \mathbf{L} .

A pseudo-complemented lattice \mathbf{L} is called a *Stone lattice* if for all $a \in L$, $\neg a \vee \neg\neg a = 1$. It can be easily seen that \mathbf{L} is a Stone lattice if and only if $\mathbf{B}_{\mathbf{L}}$ is a sublattice of \mathbf{L} . Thus, in this case $\mathbf{B}_{\mathbf{L}}$ coincides with the Boolean algebra of complemented elements of \mathbf{L} .

2.2. Gödel algebras

A *relatively pseudo-complemented lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 1)$ of signature $(2, 2, 2, 0)$ such that (i) $(A, \wedge, \vee, 1)$ is a distributive lattice with top element 1; and (ii) for all $a, b, c \in A$, $a \wedge b \leq c$ iff $b \leq a \rightarrow c$. Thus, given $a, b \in A$, $a \rightarrow b$ is the *relative pseudo-complement* of a with respect to b , namely, the greatest element $x \in A$ such that $a \wedge x \leq b$. A *Heyting algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 1, 0)$ of signature $(2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \rightarrow, 1)$ is a relatively pseudo-complemented lattice and 0 is a bottom element with respect to the lattice ordering of \mathbf{A} . Heyting algebras form a variety, because the aforementioned biconditional can be replaced by the equations

$$\begin{aligned} x \rightarrow x &\approx 1; \\ x \wedge (x \rightarrow y) &\approx x \wedge y; \\ x \wedge (y \rightarrow x) &\approx x; \\ x \rightarrow (y \wedge z) &\approx (x \rightarrow y) \wedge (x \rightarrow z). \end{aligned}$$

Observe that the $(\wedge, \vee, 1, 0)$ reduct of a Heyting algebra is, in particular, a pseudo-complemented distributive lattice, with $\neg a = a \rightarrow 0$.

A (*positive*) *Gödel algebra* is a Heyting algebra (relatively pseudo-complemented lattice) satisfying the equation $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$. It is important to recall that the lattice reducts of positive Gödel algebras coincide with *relative Stone lattices*, i.e. lattices with a top element such that every interval $[b, 1]$ is a Stone lattice. The next Lemma provides some arithmetical properties of

Gödel algebras that will prove useful in the following. Observe that item (i) is nothing but the second equation in the previous display.

Lemma 1. *Any Gödel algebra satisfies the following equations:*

- (i) $x \wedge (x \rightarrow y) \approx x \wedge y$;
- (ii) $\neg x \vee (\neg x \rightarrow y) \approx 1$;
- (iii) $\neg x \vee \neg\neg x \approx 1$ and $\neg x \wedge \neg\neg x \approx 0$.

Gödel algebras play a prominent role in algebraic logic because they are the equivalent variety semantics of *Gödel logic* (also known as *Dummett's logic*, or *Dummett's LC*), which is both an intermediate logic (i.e. an extension of intuitionistic logic) and a fuzzy logic. As an intermediate logic, it stands out for its being sound and complete with respect to linearly ordered Kripke models, and as such it received considerable attention. **LC** has been widely investigated also within the community of mathematical fuzzy logic — it was observed early on that the variety of Gödel algebras is generated by the algebra $([0, 1], \wedge, \vee, \rightarrow, 1, 0)$, where \wedge and \vee are the minimum T-norm and the maximum T-conorm respectively, while \rightarrow , the residual³ of \wedge , behaves as follows for all $a, b \in [0, 1]$:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

Remark 2. *Observe that every bounded chain admits a unique Gödel implication, given by the above case-splitting definition. In particular, in every linearly ordered Gödel algebra $a \rightarrow b$ is 1 if $a \leq b$, and is b if $a > b$.*

2.3. Residuated lattices

A binary operation \cdot on a partially ordered set (A, \leq) is said to be *residuated* provided there exist binary operations \backslash and $/$ on A such that for all $a, b, c \in A$,

$$\text{(Res)} \quad a \cdot b \leq c \text{ iff } a \leq c/b \text{ iff } b \leq a \backslash c.$$

We refer to the operations \backslash and $/$ as the *left residual* and *right residual* of \cdot , respectively. When no danger of confusion is impending, we write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, \cdot is

³For a definition of residual, see below, § ??.

performed first, followed by \backslash and $/$, and finally by \wedge and \vee . The residuals may be viewed as generalized division operations, with x/y being read as “ x over y ” and $y\backslash x$ as “ y under x ”. In either case, x is considered the *numerator* and y is the *denominator*. They can also be viewed as generalized implication operators, with x/y being read as “ x if y ” and $y\backslash x$ as “if y then x ”. In either case, x is considered the *consequent* and y is the *antecedent*. We tend to favor \backslash in calculations, but any statement about residuated structures has a “mirror image” obtained by reading terms backwards (i.e., replacing xy by yx and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where \cdot is a monoid operation with unit element 1 and the partial order \leq is a lattice order. In this case, we add the monoid unit and the lattice operations to the signature and refer to the resulting structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ as a *residuated lattice*. A *pointed residuated lattice* (also called **FL**-algebra) is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ such that the reduct $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice; in other words, we impose no additional restrictions on the second constant 0. The class of residuated lattices will be denoted by \mathcal{RL} and that of pointed residuated lattices by \mathcal{PRL} . We adopt the convention that when a class is denoted by a string of calligraphic letters, then the members of that class will be referred to by the corresponding string of Roman letters. Thus an RL is a residuated lattice, and a PRL is a pointed residuated lattice.

Proposition 3. *\mathcal{RL} and \mathcal{PRL} are finitely based varieties in their respective signatures, for the residuation conditions (Res) can be replaced by the following equations (and their mirror images):*

- (i) $y \leq x \backslash (xy \vee z)$
- (ii) $x(y \vee z) \approx xy \vee xz$
- (iii) $y(y \backslash x) \leq x$

Given an RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ or a PRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, an element $a \in A$ is said to be *integral* if $1/a = 1 = a \backslash 1$, and \mathbf{A} itself is said to be *integral* if every member of A is integral. A pointed integral RL is bounded as a lattice, with 1 as a top and 0 as a bottom element, exactly when it satisfies the equation $0 \backslash x \approx 1$. Given a PRL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, an element $a \in A$ is said to be *dualizing* if $0/(a \backslash 0) = a = (0/a) \backslash 0$, and \mathbf{A} itself is said to be *involutive* if every member of A is dualizing. Moreover,

given $a \in A$, we define the polynomials $\rho_a(x) = ax/a \wedge 1$ (*right conjugation by a*) and $\lambda_a(x) = a \setminus xa \wedge 1$ (*left conjugation by a*). We use the term *semi-linear* for a variety that is generated by its linearly ordered members. This is equivalent, see [?] or [?], to the variety satisfying the equation

$$\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx 1.$$

Several varieties of crucial importance in algebra and logic are term equivalent to varieties of (pointed) residuated lattices. Lattice-ordered groups, for one, can be identified as those RLs that satisfy the equation $x(x \setminus 1) \approx 1$. Modulo term equivalence, pseudo-MV algebras (abbreviated, Ψ MV algebras), are axiomatized with respect to bounded integral PRLs by the equations $x/(y \setminus x) \approx x \vee y \approx (x/y) \setminus x$, whence MV algebras are obtained by adding the commutativity equation $xy \approx yx$. If one rereads carefully the definition of Heyting algebra given in § ??, it is not at all surprising that Heyting algebras make instances of PRLs; more precisely, they are those bounded integral PRLs satisfying the identity $xy \approx x \wedge y$, whence Gödel algebras can be identified as bounded integral PRLs for which product equals meet and the semi-linearity equation (whose form, in this commutative setting, can be duly simplified to the more familiar *prelinearity law* $x \setminus y \vee y \setminus x \approx 1$), is satisfied. If one adds on top the involutivity equation, one obtains an equational basis for Boolean algebras in the signature of PRLs. Both (the 0-free reducts of) MV algebras and negative cones of ℓ -groups are examples of integral RLs satisfying the equations $x/(y \setminus x) \approx x \vee y \approx (x/y) \setminus x$. Such integral RLs are called *integral GMV algebras*.

Throughout the rest of this paper, in the interests of brevity, by an RL we will mean an integral and distributive residuated lattice, and by a PRL we will mean an integral, distributive and bounded pointed residuated lattice. Every exception to this policy, for example in § ??, will be explicitly noted. We will not recap the arithmetical properties of (P)RLs; see [?] or [?] for extensive lists.

2.3.1. Filters in residuated lattices

Let \mathbf{A} be a (P)RL. A *multiplicative filter* F of \mathbf{A} is a filter of its lattice reduct that is closed under multiplication. We say that F is *normal* provided that for all $b \in F$ and $a \in A$, $\rho_a(b)$ and $\lambda_a(b)$ are in F . If $X \subseteq A$, we denote, respectively, by $L[X]$, $M[X]$, and $N[X]$ the lattice filter, multiplicative filter, and normal multiplicative filter in \mathbf{A} generated by X ; braces will be dropped

if $X = \{a\}$ is a singleton. $F(\mathbf{A})$, $MF(\mathbf{A})$, $NF(\mathbf{A})$ will respectively refer to the algebraic closure families of lattice filters, multiplicative filters and normal multiplicative filters (hereafter shortened to *normal filters*) of \mathbf{A} . With a mild abuse of notation, the same labels will sometimes be employed for the universes of such lattices. We set:

$$\begin{aligned} F \vee^L G &= L(F \cup G); \\ F \vee^M G &= M(F \cup G); \text{ and} \\ F \vee^N G &= N(F \cup G). \end{aligned}$$

If F is a normal filter of \mathbf{A} , then $\theta_F = \{(a, b) \in A^2 : a \setminus b \wedge b \setminus a \in F\}$ is a congruence on \mathbf{A} . Conversely, given a congruence θ on \mathbf{A} , the equivalence class $F_\theta = 1/\theta$ of 1 is a normal filter. Moreover, we have the following result from [?]:

Lemma 4. *The lattice $NF(\mathbf{A})$ of normal filters of \mathbf{A} is isomorphic to its congruence lattice $Con(\mathbf{A})$. The isomorphism is implemented by the mutually inverse maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$.*

Upon writing \widehat{X} for the submonoid of the corresponding reduct of \mathbf{A} generated by $X \subseteq A$, we also recall that:

Lemma 5. $M[X] = \left\{ a : y \leq a, \text{ for some } y \in \widehat{X} \right\}$.

An *iterated conjugation map* is a composition $\gamma = \gamma_1 \circ \dots \circ \gamma_n$, where each γ_i is a right conjugation or a left conjugation by an element $a_i \in A$. We denote by Γ the set of all iterated conjugation maps on \mathbf{A} . If $X \subseteq A$, we write $\Gamma[X]$ for the set $\{\gamma(a) : a \in X, \gamma \in \Gamma\}$, and, as above, we denote by $\widehat{\Gamma[X]}$ the submonoid of \mathbf{A} generated by $\Gamma[X]$. With this notation at hand, we recall the following result from [?]:

Lemma 6. $N[X] = \left\{ a : b \leq a, \text{ for some } b \in \widehat{\Gamma[X]} \right\}$.

Lemma 7. $F(\mathbf{A})$, $MF(\mathbf{A})$, and $NF(\mathbf{A})$ are algebraic distributive lattices, and hence relatively pseudo-complemented.

Proof. All three are algebraic closure families, and hence algebraic lattices [?, Thm. I.5.5]. Our standing hypothesis is that \mathbf{A} is an integral distributive RL. Hence, by a standard lattice-theoretic result, the filter lattice $F(\mathbf{A})$ is distributive [?, Thm. II.9.3]. We note for future reference the compact – that is, finitely generated elements of $F(\mathbf{A})$ – are the principal filters $L[a] = \uparrow a = \{x : x \in A, a \leq x\}$. With regard to $NF(\mathbf{A})$, note next that $\text{Con}(\mathbf{A})$ is a distributive lattice, since \mathbf{A} has an underlying lattice reduct [?, Thm. II.9.15]. Hence, by Lemma ??, so is $NF(\mathbf{A})$. The proof of the distributivity of $MF(\mathbf{A})$ is slightly more complicated and proceeds as follows:

- (i) The compact multiplicative filters of $MF(\mathbf{A})$ are of the form $M[a], a \in A$. Indeed, it is easy to check that if X is a finite subset of A , then $M[X] = M[\bigwedge X]$.
- (ii) If $a, b \in A$, then $(a \vee b)^{mn} \leq a^m \vee b^n$.
- (iii) If $a, b \in A$, then $M[a] \cap M[b] = M[a \vee b]$. Indeed, the inclusion $M[a \vee b] \subseteq M[a] \cap M[b]$ is clear. To prove the reverse inclusion, let $x \in M[a] \cap M[b]$. In view of Lemma ??, there exist $m, n \in \mathbb{N}$ such that $a^m \leq x$ and $b^n \leq x$. It follows that $(a \vee b)^{mn} \leq a^m \vee b^n \leq x$. Thus, $x \in M[a \vee b]$.
- (iv) Items (i) and (iii) above imply that the join semi-lattice $K(MF(\mathbf{A}))$ of compact multiplicative filters consists of the principal multiplicative filters. Moreover, it is a sublattice of $MF(\mathbf{A})$.
- (v) Taking into account the distributivity of \mathbf{A} , (i) and (iii) above imply the distributivity of $K(MF(\mathbf{A}))$: Given $a, b, c \in A$, $M[a] \vee (M[b] \cap M[c]) = M[a] \vee M[b \vee c] = M[a \wedge (b \vee c)] = M[(a \wedge b) \vee (a \wedge c)] = M[a \wedge b] \cap M[a \wedge c] = (M[a] \vee M[b]) \cap (M[a] \vee M[c])$.
- (vi) Let \mathbf{L} be an algebraic distributive lattice whose compact elements form a sublattice $K(\mathbf{L})$ of \mathbf{L} . \mathbf{L} is a *frame*, that is, the following distributive law holds for all $S \cup \{a\} \subseteq L$:

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}.$$

Therefore, \mathbf{L} is relatively pseudo-complemented, the relative pseudo-complement of a with respect to b being given by

$$a \rightarrow b = \bigvee \{x \in L : a \wedge x \leq b\}.$$

This applies, in particular, to the lattices under consideration.

■

In view of the preceding lemma, $F(\mathbf{A})$, $MF(\mathbf{A})$, and $NF(\mathbf{A})$ are pseudo-complemented lattices. The pseudo-complements in the first two lattices can be described in terms of polars: Given $X \subseteq A$, the *polar* X^\perp of X is the set⁴

$$\{y \in A : x \vee y = 1 \text{ for every } x \in X\}.$$

Whenever $X = \{a\}$ is a singleton, we will shorten $\{a\}^\perp$ to a^\perp , consistently with the notation employed at the beginning of this section, and call the latter set a *principal polar*. We observe that:

Lemma 8. *For all $X \subseteq A$, $X^\perp \in MF(\mathbf{A})$.*

Proof. Since \mathbf{A} is distributive, $X^\perp \in F(\mathbf{A})$. If $a, b \in X^\perp$ and $x \in X$, then $1 = (a \vee x)(b \vee x) = ab \vee ax \vee xb \vee x^2 \leq ab \vee x$. Therefore, X^\perp is closed under multiplication. ■

This yields the following result:

Corollary 9. *For all $X \subseteq A$, X^\perp is the pseudo-complement of $L[X]$ in $F(\mathbf{A})$, and the pseudo-complement of $M[X]$ in $MF(\mathbf{A})$.*

On the other hand, given an arbitrary $X \subseteq A$, X^\perp need not be a *normal* filter of \mathbf{A} ; if it is — in case, for example, \mathbf{A} is semi-linear (see below) — then it is the pseudo-complement of $M(X)$ in $NF(\mathbf{A})$. Otherwise, the pseudo-complement of $M(X)$ in $NF(\mathbf{A})$ is $N[X^\perp]$.

Lemma 10. $(L[a])^\perp = (M[a])^\perp = a^\perp$.

Proof. Use Lemmas ?? and ?? above. ■

3. Projectable residuated lattices

As recalled in the introduction, an ℓ -group \mathbf{A} is *projectable* whenever for all $a \in A$, $\mathbf{A} = a^\perp \boxplus a^{\perp\perp}$,⁵ where in the present context $a^\perp = \{b \in A : |a| \wedge |b| = 1\}$ and $|a| = a \vee a^{-1}$. Projectable ℓ -groups can be characterized purely in terms

⁴This is sometimes called the *co-annihilator* of X : see e.g., [?] or [?].

⁵Here $a^\perp \boxplus a^{\perp\perp}$ does not denote simply the direct sum of a^\perp and $a^{\perp\perp}$, but their *cardinal product*: the order in $a^\perp \boxplus a^{\perp\perp}$ is the Cartesian product of the orders in the summands.

of their lattice structure, because, as proved in [?] and [?], they coincide with ℓ -groups in which all closed intervals form a Stone lattice, which is equivalent to all these intervals admitting a Gödel implication. As this result highlights, projectability is a property of ℓ -groups that is entirely determined by their order structure. To get further insight into this, recall indeed that, given an ℓ -group \mathbf{A} : 1) principal polars are convex ℓ -subgroups of \mathbf{A} ; 2) projectability is equivalent to the property that for all $a \in A$, $\mathbf{A} = a^\perp \vee a^{\perp\perp}$ in the lattice of convex ℓ -subgroups of \mathbf{A} ; 3) the lattice of convex ℓ -subgroups of \mathbf{A} is isomorphic to the lattice of convex submonoids of its negative cone \mathbf{A}^- , and in particular $a^\perp \cap A^- = \{b \in A : a \vee b = 1\}$ for $a \in A^-$; the crucial observation here is that 4) for all $a \in A^-$, $\mathbf{A} = a^\perp \vee a^{\perp\perp}$, where the join is taken in the lattice of filters of the negative cone of \mathbf{A} ; this makes clear that projectability is an order-theoretic property. (See [?] for a lengthier discussion of these aspects.)

As already observed, negative cones of ℓ -groups make instances of RLs, with $a \setminus b = a^{-1}b \wedge 1$ and $b/a = ba^{-1} \wedge 1$. If we view them in such a guise, the notation a^\perp used in Section ?? is fully consistent with the different use of the same symbol mentioned in the preceding paragraph. Moreover, negative cones satisfy the quasi-equation

$$x \vee y \approx 1 \implies xy \approx x \wedge y. \quad (1)$$

It is therefore of interest to ascertain whether the above lattice-theoretic characterization of projectable ℓ -groups and their negative cones extends to the class \mathcal{A} of RLs satisfying that quasi-equation, which also includes all Heyting algebras and semi-linear RLs. By confining ourselves to integral residuated lattices we do not lose much generality, since congruences of any residuated lattice are determined by their negative cones.

Definition 11. \mathbf{A} is called *projectable* if it can be written as an internal direct sum $\mathbf{A} = a^\perp \boxplus a^{\perp\perp}$, for all $a \in A$.

The next lemma shows that projectability for the members of the envisaged class is a lattice-theoretic property, in the sense that it can be “captured” by the filter lattice of the underlying lattice-structure.

Lemma 12. *If \mathbf{A} satisfies quasi-equation (1) and is projectable, then*

$$\mathbf{A} = a^\perp \vee^L a^{\perp\perp} = a^\perp \vee^M a^{\perp\perp},$$

for all $a \in A$.

Proof. Suppose \mathbf{A} is projectable. Invoking the quasi-equation

$$x \vee y \approx 1 \implies xy \approx x \wedge y,$$

for all $b \in A$ there exist unique $b_1 \in a^\perp$ and $b_2 \in a^{\perp\perp}$ such that $b = b_1 b_2 = b_1 \wedge b_2$. In particular, $\mathbf{A} = a^\perp \vee^L a^{\perp\perp} = a^\perp \vee^M a^{\perp\perp}$. ■

Lemma 13. *If \mathbf{A} satisfies quasi-equation (1) and is projectable, then $a^\perp \in \text{NF}(\mathbf{A})$.*

Proof. Observe first that, if $b_1 \in a^\perp$ and $b_2 \in a^{\perp\perp}$, then $\lambda_{b_2}(b_1)$ and $\rho_{b_2}(b_1) \in a^\perp$. In fact, as $b_1 \vee b_2 = 1$, $b_1 b_2 = b_1 \wedge b_2 = b_2 b_1$ and so $b_1 \leq \lambda_{b_2}(b_1)$, whence the first part of our claim follows. Similarly, $\rho_{b_2}(b_1) \in a^\perp$. Now, let $b = b_1 b_2$, with $b_1 \in a^\perp$ and $b_2 \in a^{\perp\perp}$. We have:

$$\begin{aligned} \lambda_b(x) &= b_1 b_2 \backslash x b_1 b_2 \\ &= b_2 \backslash (b_1 \backslash x b_1 b_2) \\ &\geq b_2 \backslash (b_1 \backslash x b_1) b_2 \\ &= \lambda_{b_2}(\lambda_{b_1}(x)) \end{aligned}$$

whence $\lambda_b(x) \in a^\perp$ by our previous claim. Right conjugates are handled analogously. ■

The next important result, which aptly generalizes a corresponding result for ℓ -groups, is an immediate consequence of the previous lemma:

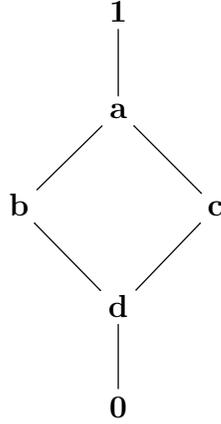
Theorem 14. *If \mathbf{A} satisfies quasi-equation (1), prelinearity, and is projectable, then it is semi-linear.*

Proof. By Lemma ??, under the projectability assumption, principal polars are closed under conjugates. This means that \mathbf{A} satisfies the quasi-equation

$$x \vee y \approx 1 \implies \lambda_w(x) \vee \rho_z(y) \approx 1.$$

By results in [?], every prelinear RL satisfying the preceding quasi-equation is semi-linear. ■

Example 15. We take note of the fact that the prelinearity assumption is essential in this result. The following subdirectly irreducible, non-totally ordered Heyting algebra is a Stone lattice, whence, a fortiori, it is projectable. Observe that $b \setminus c \vee c \setminus b = a < 1$.



We will now momentarily transfer the context of our discussion to a purely lattice-theoretic framework. We already recalled that an algebraic distributive lattice \mathbf{L} whose compact elements form a sublattice $K(\mathbf{L})$ is relatively pseudo-complemented, with

$$a \rightarrow b = \bigvee \{x \in L : a \wedge x \leq b\}.$$

As a matter of fact, \mathbf{L} has a bottom element and so it can be expanded to a Heyting algebra. The proof of the next lemma, which appears in [? , Chapter IX, Section 2, Theorem 8], is briefly sketched here for the reader's convenience.

Lemma 16. *If \mathbf{L} is an algebraic distributive lattice, every interval $[b, a]$ in \mathbf{L} , with $b \leq a$, is pseudo-complemented and, for all $c \in [b, a]$, the pseudo-complement and the double pseudo-complement of c are respectively given by:*

$$\begin{aligned} \neg c &= (c \rightarrow b) \wedge a; \\ \neg \neg c &= ((c \rightarrow b) \rightarrow b) \wedge a. \end{aligned}$$

Proof. Given c in $[b, a]$, observe that $b \leq \neg c \leq a$, whence $c \wedge \neg c \geq b$. On the other hand, $c \wedge \neg c = c \wedge (c \rightarrow b) \wedge a \leq b \wedge a = b$. So $c \wedge \neg c = b$. Let now $b \leq d \leq a$ and $c \wedge d = b$. In particular, then, $c \wedge d \leq b$, which amounts to $d \leq c \rightarrow b$. It follows that $d = d \wedge a \leq (c \rightarrow b) \wedge a$.

Further,

$$\begin{aligned} \neg\neg c &= \neg((c \rightarrow b) \wedge a) \\ &= (((c \rightarrow b) \wedge a) \rightarrow b) \wedge a \\ &= a \wedge ((c \rightarrow b) \rightarrow b). \end{aligned}$$

■

In particular, if $a = 1$, $\uparrow b$ is itself an algebraic distributive lattice and $K(\uparrow b) = \{b \vee c : c \in K(L)\}$.

Definition 17. \mathbf{L} is called *compactly Stonean* if it satisfies $\neg c \vee \neg\neg c = 1$, for all $c \in K(L)$. It is called *hyper-archimedean* if for all $c \in K(L)$, $c \vee \neg c = 1$ (i.e., if every compact element is complemented).

It is immediate that hyper-archimedean lattices, which arise naturally as congruence lattices of hyper-archimedean RLs (see [?] for a proof in the ℓ -group case), are compactly Stonean. Actually, more is true:

Proposition 18. *If \mathbf{L} is a hyper-archimedean distributive algebraic lattice and $b \in L$, then $\uparrow b$ is compactly Stonean.*

Proof. Let $c \in K(L)$ and $b \in L$. We first show that $c \rightarrow b = \neg c \vee b$. Since \mathbf{L} is distributive, $c \wedge (\neg c \vee b) \leq b$. Conversely, let $d \leq c \rightarrow b$, whence $c \wedge d \leq b$. Then $(c \wedge d) \vee \neg c \leq b \vee \neg c$, whence by the hyper-archimedean property

$$d \leq d \vee \neg c = (c \vee \neg c) \wedge (d \vee \neg c) \leq b \vee \neg c$$

(observe that we did not use the fact that c is compact, but only that it is complemented). Therefore,

$$\begin{aligned} (c \rightarrow b) \rightarrow b &= (\neg c \vee b) \rightarrow b \\ &= \neg c \rightarrow b \\ &= \neg\neg c \vee b \\ &= c \vee b \end{aligned}$$

Summing up, $(c \rightarrow b) \vee ((c \rightarrow b) \rightarrow b) = \neg c \vee b \vee c \vee b = 1$ and therefore $\uparrow b$ is compactly Stonean. ■

Moreover, there are examples of compactly Stonean lattices that are not hyper-archimedean, e.g. Example ???. Observe that a compactly Stonean lattice need not be a Stone lattice. In view of Lemma ??, $\uparrow b$ is compactly Stonean iff, for all $c \in K(L)$, $(c \rightarrow b) \vee ((c \rightarrow b) \rightarrow b) = 1$.

Proposition 19. *Let \mathbf{L} be an algebraic distributive lattice whose compact elements form a sublattice $K(\mathbf{L})$ of \mathbf{L} . The conditions below are equivalent:*

- (1) *for all $b \in L$, $\uparrow b$ is compactly Stonean;*
- (2) *for all $b \in L$ and for all $c \in K(L)$, $(c \rightarrow b) \vee ((c \rightarrow b) \rightarrow b) = 1$*

and imply the mutually equivalent conditions

- (3) *for all $c, b \in K(L)$, $(c \rightarrow b) \vee ((c \rightarrow b) \rightarrow b) = 1$*
- (4) *for all $a, b \in K(L)$, with $b \leq a$, $[b, a] \cap K(L)$ is a Stone lattice.*

Proof. We have already established the equivalence of (1) and (2), while clearly (2) implies (3). All we need to prove, therefore, is the equivalence of (3) and (4). In one direction, let $a, b \in K(L)$, with $b \leq a$. Notice that $[b, a] \cap K(L)$ is the lattice of compact elements of $[b, a]$. For $c \in [b, a] \cap K(L)$, (3) implies

$$\begin{aligned} ((c \rightarrow b) \wedge a) \vee (((c \rightarrow b) \rightarrow b) \wedge a) &= a; \\ ((c \rightarrow b) \wedge a) \wedge (((c \rightarrow b) \rightarrow b) \wedge a) &= b. \end{aligned}$$

We note that in an algebraic distributive lattice whose compact elements form a sublattice, if the join and meet of two elements are compact, then the elements themselves are compact (see, for example, [?], p. 71). So $[b, a] \cap K(L)$ is a Stone lattice by Lemma ??. Conversely, if (4) is satisfied, let $c, b \in K(L)$. Then, by assumption, for all $a \in K(L)$, $a \geq b \vee c$, we have that $((c \rightarrow b) \wedge a) \vee (((c \rightarrow b) \rightarrow b) \wedge a) = a$, i.e. $a \leq (c \rightarrow b) \vee ((c \rightarrow b) \rightarrow b)$. This proves the desired conclusion. ■

We now put to good use the preceding results by applying them to the lattice $F(\mathbf{A})$ of lattice filters of \mathbf{A} .

Theorem 20. *Suppose that an RL \mathbf{A} satisfies the quasi-equation (1):*

$$x \vee y \approx 1 \implies xy \approx x \wedge y,$$

Then \mathbf{A} is projectable iff the order dual of each interval $[a, 1]$, for $a \in A$, is a Stone lattice.

Proof. By Lemma ??, if \mathbf{A} is projectable then the lattice $F(\mathbf{A})$ of the lattice filters of \mathbf{A} is compactly Stonean, since principal filters correspond to compact elements of such a lattice. Therefore, by Proposition ?? each interval $[\{1\}, \uparrow a]$ in the sublattice of principal lattice filters of \mathbf{A} is a Stone lattice. The desired conclusion now follows in the light of the order reversing isomorphism between the lattice reduct of \mathbf{A} and the sublattice of principal filters in $F(\mathbf{A})$. ■

In particular, if \mathbf{A} is bounded or if it is an integral GMV algebra (we remark that this class satisfies Quasi-equation ?? by [? , Thm. 3.12], and it contains MV algebras and negative cones of ℓ -groups alike), we get something more:

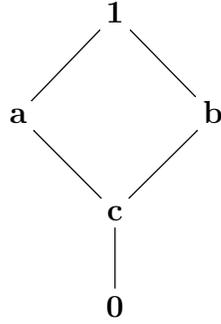
Corollary 21. *If \mathbf{A} is an integral GMV algebra, then \mathbf{A} is projectable iff its lattice reduct can be expanded to a positive Gödel algebra; if \mathbf{A} is a bounded integral GMV algebra (that is, a Ψ MV algebra), then \mathbf{A} is projectable iff its bounded lattice reduct can be expanded to a Gödel algebra.*

Proof. Since every interval $[a, 1]$ of \mathbf{A} is an involutive lattice under the mapping that sends x to $x \setminus a$, the order dual of an interval $[a, 1]$ is a Stone lattice iff $[a, 1]$ itself is a Stone lattice. This means that the lattice reduct of \mathbf{A} is a relative Stone lattice [? , Theorem 8.13], and this in turn implies that it can be expanded to a positive Gödel algebra. The second claim is straightforward from the first one. ■

Example ?? shows that the left-to-right direction of the equivalence in the first claim of Corollary ?? may fail for RLs that are not integral GMV algebras.

At this point, it is relevant to mention Mureşan’s work [?], [?], in which she defines and studies the so called *co-Stone residuated lattices*. They are defined as those bounded RLs (in the sense of this paper) in which each principal polar is the principal lattice filter of a complemented element of the underlying lattice reduct. These RLs are clearly projectable in our sense, but the converse is not true in general, as the next example shows:

Example 22. *Consider the integral residuated lattice \mathbf{A} , whose lattice structure and multiplication are specified below.*



·	0	c	a	b	1
0	0	0	0	0	0
c	0	0	0	c	c
a	0	c	a	c	a
b	0	0	0	b	b
1	0	c	a	b	1

Note that \mathbf{A} is projectable, but not co-Stone in Mureşan’s sense.

This example can also be used to demonstrate the importance of the quasi-equation (??). When an RL \mathbf{A} satisfies (1), the projectability of \mathbf{A} implies that the filter lattice of \mathbf{A} is compactly Stonean. In the absence of (??), this conclusion is not guaranteed. In the preceding example, note that $\uparrow b \vee^M \uparrow c = A$, but $\uparrow b \vee^L \uparrow c = \uparrow a$.

4. Gödel residuated lattices

Let us now take stock of the situation so far. Abstracting away from well-trodden topics in the literature on individual classes of residuated lattices, such as ℓ -groups or MV algebras, we singled out two interesting properties of (P)RLs, namely, that of being projectable and that of admitting a positive Gödel implication (or, in the bounded case, a Gödel implication). Both properties are purely lattice-theoretical, and they are mutually equivalent for integral GMV algebras. Projectable integral GMV algebras, or even projectable MV algebras, do not form a variety. However, Theorem ?? strongly suggests adding the definable Gödel implication to the signature in order to obtain an equational characterization thereof and to avail ourselves of the

powerful universal algebraic methods that such a move renders viable. The most natural way to do so is to consider the variety in the expanded signature that is axiomatized by, say, the MV algebra axioms plus the Gödel algebra axioms. Obvious as it is, this strategy does not in principle guarantee that the resulting variety is generated by its linearly ordered members, because further axioms might be needed that govern the interplay between the MV component and the Gödel component. We will prove that this is not the case, and we will do so under the general hypothesis that, instead of an MV algebra, we have an arbitrary involutive PRL⁶. In this exploratory paper, the additional involutivity assumption is made since the problem of describing congruence filters is, as we will presently see, considerably simplified. The present section provides the foundations of the structure theory of this variety, as well as a proof that it is a discriminator variety.

4.1. Definition and basic properties

Definition 23. We use the term *Gödel residuated lattice* for an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \rightarrow, 1, 0)$ of signature $(2, 2, 2, 2, 2, 2, 0, 0)$ such that:

- the reduct $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is an involutive PRL; and
- the reduct $(A, \wedge, \vee, \rightarrow, 1, 0)$ is a Gödel algebra.

The variety of Gödel residuated lattices will be denoted by \mathcal{GRL} .

Some pieces of notation will prove extremely convenient in what follows. We introduce three negations by putting $\neg x = x \rightarrow 0$, $\sim x = x \backslash 0$ and $-x = 0/x$; the symbol $\delta(x)$ will be used as shorthand for $\neg \sim x$.

Next, we list a representative example of the class of algebras we have just defined.

⁶Theorem ?? would also suggest considering a wider class, namely that of involutive PRLs with an additional Stonean negation. Although most of the structure theory that follows would carry over to this more general case, we did not investigate these algebras, for two main reasons: 1) unlike Gödel residuated lattices, hereafter defined, these algebras fail to be generalizations of Heyting-Wajsberg algebras; 2) the semi-linearity of their PRL reducts has to be explicitly postulated, whereas it follows from the axioms of Gödel residuated lattices. Involutive PRLs with an additional Stonean negation, on the other hand, are very similar to the symmetric Stonean residuated lattices of [?]: both have an involutive negation \sim and a Stonean negation \neg , but for Cignoli and Esteva $a \backslash 0 = \neg a$, while in the present case $a \backslash 0 = \sim a$.

Example 24. The algebra $\mathbf{S} = ([0, 1], \wedge, \vee, \cdot, \backslash, /, \rightarrow, 1, 0)$ over the real closed unit interval, where:

- the operation \cdot is the Łukasiewicz T-norm and $\backslash = /$ its residual;
- $([0, 1], \wedge, \vee, \rightarrow, 1, 0)$ is the standard Gödel algebra over $[0, 1]$ (cf. § ??),

is a GRL (actually, a commutative one). This algebra generates the variety of Heyting-Wajsberg algebras [?], mentioned in the Introduction.

Our first goal is describing how the PRL reduct and the Gödel algebra reduct of a GRL interact with each other. The next two lemmas provide some basic information concerning this mutual interplay.

Lemma 25. *In every $\mathbf{A} \in \mathcal{GRL}$, for all $a, b, c \in A$:*

- | | |
|--|--|
| <p>(i) $a \rightarrow b \leq a \backslash b, b/a;$</p> <p>(ii) $\delta(a) \leq a;$</p> <p>(iii) $\delta(a \vee b) = \delta(a) \vee \delta(b);$</p> <p>(iv) $\delta(a \wedge b) = \delta(a) \wedge \delta(b);$</p> <p>(v) $\delta(\neg a) = \neg a;$</p> | <p>(vi) $\delta(\delta(a)) = \delta(a);$</p> <p>(vii) $\delta(a) \vee \neg \delta(a) = 1;$</p> <p>(viii) $\delta(a) \delta(a) = \delta(a);$</p> <p>(ix) <i>if $a \leq b$, then $\delta(a) \leq \delta(b)$;</i></p> <p>(x) $\delta(ab) = \delta(a) \delta(b) = \delta(a) \wedge \delta(b).$</p> |
|--|--|

Proof.

(i) We have that $a \cdot (a \rightarrow b) \leq a \wedge (a \rightarrow b) \leq b$, so by residuation $a \rightarrow b \leq a \backslash b$. $a \rightarrow b \leq b/a$ is established similarly.

(ii) By (i), $\neg \sim a \leq \neg \sim a = a$.

(iii)-(iv) By De Morgan laws.

(v) By Lemma ??.(iii), $\neg a \vee \neg \neg a = 1$ and $\neg a \wedge \neg \neg a = 0$, whence, applying items (iii)-(iv), $\delta(\neg a) \vee \delta(\neg \neg a) = 1$ and $\delta(\neg a) \wedge \delta(\neg \neg a) = 0$. Taking into account item (ii), $\delta(\neg a) \vee \neg \neg a = 1$ and $\delta(\neg a) \wedge \neg \neg a = 0$, whence $\delta(\neg a) = \neg \neg \neg a = \neg a$.

(vi) By item (v), $\neg \sim \neg \sim a = \neg \sim a$.

(vii) By the preceding item, δ fixes every element of the Boolean algebra whose universe is $\{\neg a : a \in A\}$, whence our conclusion.

(viii) By item (vii), $\delta(a) \vee \delta(\neg a) = 1$. Multiplying both sides by $\delta(a)$, $\delta(a) \delta(a) \vee \delta(a) \delta(\neg a) = \delta(a)$. However, $\delta(a) \delta(\neg a) \leq \delta(a) \wedge \delta(\neg a) = 0$, whence our conclusion follows.

(ix) If $a \leq b$, then $a \wedge b = a$, whence $\delta(a \wedge b) = \delta(a) \wedge \delta(b) = \delta(a)$.

(x) By items (iv) and (viii), $\delta(a) \wedge \delta(b) = (\delta(a) \wedge \delta(b))^2 \leq \delta(a) \delta(b)$, while the reverse inequality holds in every PRL. In view of this and of the

preceding item, therefore, it will suffice to show that $\delta(a) \wedge \delta(b) \leq \delta(ab)$. In fact, $\delta(a) \wedge \delta(b) = \delta(\delta(a) \wedge \delta(b)) = \delta(\delta(a)\delta(b)) \leq \delta(ab)$. ■

Recall, see for example [?, p. 179], that a *conucleus* on an RL or PRL \mathbf{A} is an interior operator η such that for all $a, b \in A$, $\eta(a)\eta(b) \leq \eta(ab)$ and $\eta(1)\eta(a) = \eta(a) = \eta(a)\eta(1)$. In view of items (ii), (vi), (ix) and (x) in Lemma ??, we observe that δ is a conucleus on the PRL reduct of any GRL.

Lemma 26. *In every $\mathbf{A} \in \mathcal{GRL}$, the following conditions and their mirror images hold, for all $a, b, c \in A$:*

- (i) $a \setminus (a \rightarrow b) = a \setminus b$;
- (ii) $\neg a \setminus b = \neg a \rightarrow b$;
- (iii) $a \setminus b = \sim(-b \cdot a)$;
- (iv) $\delta(b/a) = \neg(a \cdot \sim b)$;
- (v) $\delta(b/a) \leq a \rightarrow b$;
- (vi) $\delta(b/a) = \delta(a \rightarrow b)$.
- (vii) $\delta(a)b = \delta(a) \wedge b = b\delta(a)$.

Proof.

(i) By Lemma ??.(i), $a \wedge (a \rightarrow b) = a \wedge b$. Dividing by a , $a \setminus (a \wedge (a \rightarrow b)) = a \setminus (a \wedge b)$. Thus,

$$a \setminus (a \rightarrow b) = a \setminus a \wedge a \setminus (a \rightarrow b) = a \setminus a \wedge a \setminus b = a \setminus b.$$

(ii) The inequality $\neg a \rightarrow b \leq \neg a \setminus b$ holds by Lemma ??.(i). Regarding the converse,

$$\begin{aligned} \neg a \setminus b &= \neg a \setminus (\neg a \rightarrow b) && \text{item (i)} \\ &= 1 \cdot (\neg a \setminus (\neg a \rightarrow b)) \\ &= (\neg a \vee (\neg a \rightarrow b)) (\neg a \setminus (\neg a \rightarrow b)) && \text{Lemma ??.(ii)} \\ &\leq \neg a \rightarrow b \end{aligned}$$

(iii) Taking into account that $(0/b) \setminus 0 = b$, we have: $\sim(-b \cdot a) = (-b \cdot a) \setminus 0 = ((0/b) \cdot a) \setminus 0 = a \setminus [(0/b) \setminus 0] = a \setminus b$.

(iv) By the mirror image of (iii), $b/a = -(a \cdot \sim b)$. So, $\sim(b/a) = \sim-(a \cdot \sim b) = a \cdot \sim b$. It follows that $\delta(b/a) = \neg \sim(b/a) = \neg(a \cdot \sim b)$.

(v) Note first that $a \leq \sim(-b \cdot -(a \cdot \sim b))$. Indeed, by using (iii) and its mirror image, we get: $a \leq (b/a) \setminus b = \sim(-b \cdot (b/a)) = \sim(-b \cdot -(a \cdot \sim b))$. Further, it is easy to see that $\sim(-b \cdot -(a \cdot \sim b)) \leq \sim(-b \cdot \neg(a \cdot \sim b))$. Indeed, by (i), $\neg(a \cdot \sim b) \leq -(a \cdot \sim b)$, whence multiplying by $-b$, $-b \cdot \neg(a \cdot \sim b) \leq -b \cdot -(a \cdot \sim b)$. By applying \sim to both sides, we obtain the required inequality. Hence, by applying transitivity, (ii), (iii) and (iv) above, we have $a \leq \sim$

$(-b \cdot \neg(a \cdot \sim b)) = \neg(a \cdot \sim b) \setminus b = \neg(a \cdot \sim b) \rightarrow b = \delta(b/a) \rightarrow b$. In conclusion, $a \leq \delta(b/a) \rightarrow b$, and $\delta(b/a) \leq a \rightarrow b$.

(vi) Condition (v) and the monotonicity and idempotency of δ imply that $\delta(b/a) \leq \delta(a \rightarrow b)$. The same properties of δ and (i) imply the reverse inequality.

(vii) The only nontrivial claim is $\delta(a) \wedge b \leq \delta(a) b$. We must therefore show that for all c , $\delta(a) b \leq c$ implies that $\delta(a) \wedge b \leq c$. However, if $\delta(a) b \leq c$, then $b \leq \delta(a) \setminus c = \delta(a) \rightarrow c$ by item (ii), whence $\delta(a) \wedge b \leq c$. ■

4.2. Structure theory

The main aim of this subsection is proving that \mathcal{GRL} is, as expected, generated by its linearly ordered members. In particular, it will follow from this result that axioms HW7 and HW8 in [?, p. 348] are not needed to prove the standard completeness of Heyting-Wajsberg algebras. To that effect, we will show that the congruences of any $\mathbf{A} \in \mathcal{GRL}$ are in bijective correspondence with a special subclass of the normal filters of its PRL reduct.

Definition 27. *Let $\mathbf{A} \in \mathcal{GRL}$. $F \subseteq A$ is a congruence filter of \mathbf{A} iff it is a multiplicative filter of the PRL reduct of \mathbf{A} closed under δ , that is, $\delta[F] \subseteq F$.*

Lemma 28. *Let $\mathbf{A} \in \mathcal{GRL}$, and let $F \subseteq A$ be a congruence filter of \mathbf{A} . Then, for all $a, b \in A$:*

- (i) $a, a \rightarrow b \in F$ imply $b \in F$;
- (ii) $\delta(a) \in F$ iff $a \in F$;
- (iii) $a \rightarrow b \in F$ iff $a \setminus b \in F$;
- (iv) F is normal.

Proof.

(i) Suppose $a, a \rightarrow b \in F$. Since $a \rightarrow b \leq a \setminus b$, it follows that $a \setminus b \in F$. But then, $a \cdot (a \setminus b) \leq b$ yields $b \in F$.

(ii) One direction follows from the definition of congruence filter, the other from Lemma ??.(ii) and the fact that F is a lattice filter.

(iii) By Lemma ??.(vi) and item (ii), $a \rightarrow b \in F$ iff $\delta(a \rightarrow b) = \delta(a \setminus b) \in F$ iff $a \setminus b \in F$.

(iv) It will suffice to show that, for every $a \in A$ and for every $b \in F$, $\lambda_a(b) = a \setminus ba \in F$. In fact, $a\delta(b) \leq \delta(b)a \leq ba$ by Lemma ??.(ii) and Lemma ??.(vii), whence $\delta(b) \leq a \setminus ba$; since $\delta(b) \in F$, this is enough for our purposes. ■

Lemma 29. *Let $\mathbf{A} \in \mathcal{GR}\mathcal{L}$. The congruence filter $\langle X \rangle$ generated by $X \subseteq A$ is*

$$\{b: \exists a_1, \dots, a_n \in X \text{ s.t. } \delta(a_1) \dots \delta(a_n) \leq b\}.$$

Proof. To begin with, observe that, by virtue of Lemma ??.(x),

$$\langle X \rangle = \{b: \exists a_1 \dots a_n \in X \text{ s.t. } \delta(a_1) \wedge \dots \wedge \delta(a_n) \leq b\}.$$

We will freely use this fact without further mention in the sequel. Clearly, $\langle X \rangle$ is a multiplicative order filter and, by Lemma ??.(iv)-(ix), it is closed with respect to δ . Thus $\langle X \rangle$ is a congruence filter and it contains X by Lemma ??.(ii). Now, suppose that F is a filter, $X \subseteq F$ and $b \in \langle X \rangle$, meaning $\delta(a_1) \wedge \dots \wedge \delta(a_n) \leq b$ for some $a_1, \dots, a_n \in X$. Then, for all $i \leq n$, $\delta(a_i) \in F$ and thus $b \in F$. ■

The next observation is crucial for our purposes. As it is easy to prove, its proof is left to the reader.

Lemma 30. *Let $\mathbf{A} \in \mathcal{GR}\mathcal{L}$. For $a, b \in A$, (i) $\langle a \rangle \cap \langle b \rangle = \langle a \vee b \rangle$; (ii) $\langle a \rangle \vee \langle b \rangle = \langle a \wedge b \rangle$.*

We now proceed to establish the correspondence between congruence filters and congruences.

Theorem 31. *There is a lattice isomorphism between the lattice $\text{Fil}(\mathbf{A})$ of congruence filters of any $\mathbf{A} \in \mathcal{GR}\mathcal{L}$ and its lattice of congruences $\text{Con}(\mathbf{A})$.*

Proof. If F is a congruence filter of \mathbf{A} , define $\theta_F = \{(a, b) \in A^2 : a \setminus b \wedge b \setminus a \in F\}$. We show that the maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta = 1/\theta$ are mutually inverse and induce such an isomorphism.

(i) θ_F is a congruence on \mathbf{A} . In light of Lemmas ?? and ??, θ_F is a congruence on its PRL reduct; what we must show is that it preserves \rightarrow . Suppose $(a, b) \in \theta_F$, i.e. $a \setminus b \wedge b \setminus a \in F$, whence by definition of congruence filter $a \setminus b, b \setminus a \in F$. By virtue of Lemma ??.(iii), $a \rightarrow b, b \rightarrow a \in F$. Observe that

$$(a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c))) = 1 \in F,$$

whence by Lemma ??.(i), $(b \rightarrow c) \rightarrow (a \rightarrow c) \in F$. Since $(b \rightarrow c) \rightarrow (a \rightarrow c) \leq (b \rightarrow c) \setminus (a \rightarrow c)$, also $(b \rightarrow c) \setminus (a \rightarrow c) \in F$. Similarly, $(a \rightarrow c) \setminus (b \rightarrow c) \in F$, whereby

$$(b \rightarrow c) \setminus (a \rightarrow c) \wedge (a \rightarrow c) \setminus (b \rightarrow c) \in F.$$

- (ii) F_θ is a filter of \mathbf{A} . Clearly, it is a normal filter of the PRL reduct, and it is easy to see that it is closed with respect to δ .
- (iii) $\theta_{1/\theta} = \theta$. We have that

$$\theta_{1/\theta} = \{(a, b) \in A^2 : (a \setminus b \wedge b \setminus a) \theta 1\}.$$

If $a\theta b$, then $a \setminus b, b \setminus a \theta 1$, whence $a \setminus b \wedge b \setminus a \theta 1$. On the other hand, if $(a \setminus b \wedge b \setminus a) \theta 1$, by absorption $a \setminus b \theta 1 \theta b \setminus a$. Thus in \mathbf{A}/θ , $a/\theta = b/\theta$, i.e. $a\theta b$.

- (iv) $F_{\theta_F} = F$. Immediate. ■

Theorem 32. *\mathcal{GRL} is a semi-linear variety.*

Proof. It will suffice to prove that every subdirectly irreducible member of \mathcal{GRL} is totally ordered. To this end, let \mathbf{A} be a subdirectly irreducible member of \mathcal{GRL} , and let $a, b \in A$. We have that $(a \rightarrow b) \vee (b \rightarrow a) = 1$, whence $\langle a \rightarrow b \rangle \cap \langle b \rightarrow a \rangle = \{1\}$ by Lemma ???. Combining Theorem ??? with the assumption that \mathbf{A} is subdirectly irreducible, we obtain that either $\langle a \rightarrow b \rangle = \{1\}$ or $\langle b \rightarrow a \rangle = \{1\}$, that is, either $a \leq b$ or $b \leq a$. ■

Corollary 33. *If \mathbf{A} is a subdirectly irreducible member of \mathcal{GRL} , then $\delta[A] = \{0, 1\}$.*

Proof. By Lemma ???.(vii), $\delta(a) \vee \neg\delta(a) = 1$, whereby, by Theorem ???, either $\delta(a) = 1$ or $\neg\delta(a) = 1$, and thus $\delta(a) = 0$. ■

More generally, $\delta[\mathbf{A}]$ is easily checked to be a Boolean algebra. Furthermore,

Lemma 34. *Let \mathbf{A} be a member of \mathcal{GRL} . Then there is a bijective correspondence between the congruence filters of \mathbf{A} and the filters of the Boolean algebra $\delta[\mathbf{A}]$.*

Proof. For a congruence filter F of \mathbf{A} , let $g(F) = F \cap \delta[A]$. It is clear that $g(F)$ is a Boolean filter of $\delta[\mathbf{A}]$. Now, suppose that $F \neq G$ are congruence filters of \mathbf{A} , whence without loss of generality there is $a \in F$ such that $a \notin G$. By Lemma ???.(ii), $\delta(a) \in F$ and $\delta(a) \notin G$, whence $g(F) = F \cap \delta[A] \neq G \cap \delta[A] = g(G)$.

It remains to be shown that every Boolean filter of $\delta[\mathbf{A}]$ is of the form $F \cap \delta[A]$, for some congruence filter F of \mathbf{A} . Thus, let H be a Boolean filter of $\delta[\mathbf{A}]$, and consider $\delta^{-1}[H]$.

- (i) $\delta^{-1}[H]$ is upward closed. In fact, if $\delta(a) \in H$ and $a \leq b$, by Lemma ??.(ix) $\delta(a) \leq \delta(b)$, whence $\delta(b) \in H$.
- (ii) $\delta^{-1}[H]$ is closed with respect to products. In fact, if $\delta(a), \delta(b) \in H$, then by Lemma ??.(x),

$$\delta(a \cdot b) = \delta(a) \cdot \delta(b) = \delta(a) \wedge \delta(b) \in H.$$

- (iii) $\delta^{-1}[H]$ is closed with respect to conjugates. In fact, $ba \leq ba$, whence $b \leq ba/a$. Therefore, using Lemma ??.(vi),

$$\delta(b) \leq \delta(ba/a) = \delta(a \rightarrow ba) = \delta(a \setminus ba),$$

which means that $\lambda_a(b) \in \delta^{-1}[H]$ whenever $b \in \delta^{-1}[H]$. Right conjugates are dealt with similarly.

- (iv) $\delta^{-1}[H]$ is closed with respect to δ , because $\delta(a) = \delta(\delta(a))$. It follows that $\delta^{-1}[H]$ is actually a congruence filter, and clearly $\delta(\delta^{-1}[H]) = H$. ■

The variety \mathcal{HW} of Heyting-Wajsberg algebras is clearly term equivalent to a subvariety of \mathcal{GRL} [?, p. 348], whose members are those GRLs having an MV algebra for a PRL reduct. It was shown in [?] that \mathcal{HW} is generated by the standard algebra \mathbf{S} over the $[0, 1]$ interval from Example ??. However, the proof originally given in that paper is strongly derivative in that it relies on the term equivalence between Heyting-Wajsberg algebras and MV algebras with globalization, known to be standard complete from Hájek's book [?]. A much more elegant proof of such a result was subsequently offered by König, mimicking Chang's original standard completeness proof for MV algebras [?]. Putting to good use the results in this paper, however, we can further simplify the extant proofs. In fact:

Theorem 35. *\mathcal{HW} is standard complete.*

Proof. Let \mathbf{S} be as in Example ??, and let \mathbf{A} be a subdirectly irreducible Heyting-Wajsberg algebra. By Theorem ??, we can assume that \mathbf{A} is a chain. Then the MV reduct \mathbf{A}_{MV} of \mathbf{A} is in $SP_u(\mathbf{S}_{MV})$, where \mathbf{S}_{MV} stands for the MV reduct of \mathbf{S} [?, Section 9.5]. Since every chain (in particular, every MV subchain of a chain in $P_u(\mathbf{S}_{MV})$) admits a *unique* Gödel implication (see Remark ??), \mathbf{A} is in $SP_u(\mathbf{S})$, whence our claim follows. ■

4.3. Congruence filters and open filters

Subtractive varieties were introduced by Gumm and Ursini [?] in an attempt to isolate the common relevant properties of varieties with a “good” theory of ideals. A variety \mathcal{V} with (at least) a constant 1 in its signature is *1-subtractive* (or simply *subtractive* when no ambiguity is possible) if there is a binary term, denoted by \rightarrow and written in infix notation, such that \mathcal{V} satisfies the equations $x \rightarrow x \approx 1$ and $1 \rightarrow x \approx x$. Gumm and Ursini observe that a variety \mathcal{V} with 1 is 1-subtractive just in case it is *1-permutable* — i.e. for any algebra $\mathbf{A} \in \mathcal{V}$ and for any congruences θ, φ of \mathbf{A} , $1^{\mathbf{A}}/\theta \circ \varphi = 1^{\mathbf{A}}/\varphi \circ \theta$. For these varieties one can establish a satisfactory theory of ideals (more precisely, a manageable concept of ideal generation); in particular, 1-ideal determined varieties (that is, 1-subtractive and 1-regular varieties) are especially well-behaved because in any of their members the lattice of congruences is isomorphic to the lattice of such “abstract” ideals [? ?].

As a matter of fact, however, several results along the same lines abound in the literature that are not corollaries of the above-mentioned result and do not fit into this framework. A case in point is the variety \mathcal{RL} of (not necessarily integral, bounded or distributive) residuated lattices. This variety is 1-ideal determined and, in fact, it is well-known that in every RL the lattice of congruences is isomorphic to the lattice of Gumm-Ursini ideals, which in turn coincide with convex normal subalgebras of such. The lattice of congruences of an RL, however, is also isomorphic to the lattice of its *deductive filters* (in the sense of [?]), and this correspondence does not arise as a special case of the general theorems we referred to earlier.

With an eye to widening the scope of these results, the following generalization of subtractive varieties was introduced in [?]:

Definition 36. *A variety \mathcal{V} whose signature ν includes a nullary term 1 and a unary term \square is called quasi-subtractive with respect to 1 and \square iff there is a binary term \rightsquigarrow of signature ν such that \mathcal{V} satisfies the equations*

$$\text{Q1 } \square x \rightsquigarrow x \approx 1$$

$$\text{Q2 } 1 \rightsquigarrow x \approx \square x$$

$$\text{Q3 } \square(x \rightsquigarrow y) \approx x \rightsquigarrow y$$

$$\text{Q4 } \square(x \rightsquigarrow y) \rightsquigarrow (\square x \rightsquigarrow \square y) \approx 1$$

Subtractive varieties are, in particular, quasi-subtractive if we let our \square above be the identity. But it is possible for a variety to be subtractive and, at

the same time, properly quasi-subtractive with a *different* choice of witness terms: for example, RLs are quasi-subtractive with $x \rightsquigarrow y = (x \setminus y) \wedge 1$ and $\Box x = x \wedge 1$.

The role played by ideals in the theory of Gumm and Ursini is taken up here by *open filters*, hereafter defined:

Definition 37. Let \mathcal{V} be a variety whose signature is as above. A \mathcal{V} -open filter term in the variables \vec{x} is an $n + m$ -ary term $p(\vec{x}, \vec{y})$ of signature ν such that

$$\{\Box x_i \approx 1 : i \leq n\} \vDash_{\mathcal{V}} \Box p(\vec{x}, \vec{y}) \approx 1.$$

A \mathcal{V} -open filter of $\mathbf{A} \in \mathcal{V}$ is a subset $F \subseteq A$ that is closed with respect to all \mathcal{V} -open filter terms p (whenever $a_1, \dots, a_n \in F, b_1, \dots, b_m \in A, p(\vec{a}, \vec{b}) \in F$) and such that for every $a \in A, a \in F$ iff $\Box a \in F$.

For example, \mathcal{RL} -open filters of RLs coincide with deductive filters of such. It can be proved more generally that, if \mathcal{V} is quasi-subtractive and $(\Box x, 1)$ -regular (a stronger property than 1-regularity), then in any $\mathbf{A} \in \mathcal{V}$ there is a lattice isomorphism between the lattice of congruences on \mathbf{A} and the lattice of \mathcal{V} -open filters on \mathbf{A} .

We have that:

Lemma 38. Any GRL is quasi-subtractive with $\Box x = \delta(x)$, and $x \rightsquigarrow y = \delta(x \rightarrow y)$.

Proof. Let $a, b \in A$. Q1) $\Box a \rightsquigarrow a = \delta(\delta(a) \rightarrow a) = \delta(1) = 1$. Q2) $1 \rightsquigarrow a = \delta(1 \rightarrow a) = \delta(a) = \Box a$. Q3) $\Box(a \rightsquigarrow b) = \delta(\delta(a \rightarrow b)) = \delta(a \rightarrow b) = a \rightsquigarrow b$. Q4) In view of Q3, this is a consequence of the fact that $\delta(a \rightarrow b) \leq \delta(a) \rightarrow \delta(b)$. ■

Theorem 39. In any GRL, the set of \mathcal{GRL} -open filters coincides with the set of congruence filters.

Proof. Let \mathbf{A} be a GRL, $F \subseteq A$ a congruence filter, $a_1, \dots, a_n \in F$, and $\vec{b} \in A$. By Theorem ??, $a_1 \theta_F 1, \dots, a_n \theta_F 1$. Therefore, $\delta(a_1) \theta_F 1, \dots, \delta(a_n) \theta_F 1$. Let $p(\vec{x}, \vec{y})$ be an open filter term in \vec{x} . Applying the definition of open filter to \mathbf{A}/θ_F , $\delta(p(a_1, \dots, a_n, \vec{b})) \theta_F 1$. But $\delta(p(a_1, \dots, a_n, \vec{b})) \leq p(a_1, \dots, a_n, \vec{b})$, whence $p(a_1, \dots, a_n, \vec{b}) \theta_F 1$, and then $p(a_1, \dots, a_n, \vec{b}) \in F$. So, every congruence filter is a \mathcal{GRL} -open filter. The converse follows from the fact that every quasi-subtractive variety has normal open filters [?, Theorem 23] and Theorem ??. ■

4.4. \mathcal{GRL} is a discriminator variety

A *discriminator algebra* [?] is an algebra \mathbf{A} on which the ternary discriminator operation

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a, & \text{otherwise} \end{cases}$$

is realized by a ternary term $t(x, y, z)$ of the signature of \mathbf{A} ; a *discriminator variety* is a variety whose subdirectly irreducible members are discriminator algebras. In their influential textbook on universal algebra [?, p. 186], Burris and Sankappanavar call discriminator varieties “the most successful generalization of Boolean algebras to date”. This remark of Burris and Sankappanavar can be justified in at least two ways. On the one hand, many important classes of algebras arising in algebra and logic form discriminator varieties, including the varieties of Boolean algebras, monadic algebras, n -dimensional cylindric algebras, Post algebras, and n -valued MV algebras. On the other hand, discriminator varieties turn out to be very easy and tractable to work with in general. In particular, algebras in discriminator varieties admit an extremely useful Boolean product representation, which typically gives a deep insight into the algebraic and logical properties of the class in question. Furthermore, they are congruence permutable with equationally definable principal congruences (hence, in particular, arithmetical) and semisimple.

Recall from our introduction that a globalization operation on a PRL \mathbf{A} is an operation δ such that, for all $a \in A$, $\delta(a) = 0$ if $a < 1$ and $\delta(a) = 1$ otherwise. Chajda and Vychodil [?] showed that every commutative RL with globalization is a discriminator algebra. A slight variant of their proof, which essentially constitutes the final part of Theorem ?? below, guarantees that the same also holds in the noncommutative case. The rest of the argument needed to establish that \mathcal{GRL} is a discriminator variety amounts to checking that our δ , as defined immediately after Definition ??, is actually a globalization operation on any subdirectly irreducible GRL.

Theorem 40. *\mathcal{GRL} is a discriminator variety.*

Proof. Our proof proceeds through a number of claims. Let \mathbf{A} be a subdirectly irreducible member of \mathcal{GRL} . Theorem ?? will be used throughout this proof without any special notice. We first claim that, for all $a \in A$, $\delta(a) = 0$

if $a < 1$, $\delta(a) = 1$ otherwise. The latter half of our claim being obvious, simply observe that if $a < 1$, then $\sim a > 0$ and thus $\neg \sim a = 0$.

We next define

$$x \leftrightarrow_{\delta} y = \delta(x \rightarrow y) \wedge \delta(y \rightarrow x)$$

and claim that, for all $a, b \in A$, $a \leftrightarrow_{\delta} b = 1$ if $a = b$, $a \leftrightarrow_{\delta} b = 0$ otherwise. Again, one half of our claim (this time the former half) is trivial. Suppose then that $a \neq b$, whence w.l.g. $a \not\leq b$ and thus $a > b$. By definition of Gödel implication, then, $a \rightarrow b = b$; on the other hand, necessarily $b < 1$. Summing up, $\delta(a \rightarrow b) = \delta(b) = 0$, whence $a \leftrightarrow_{\delta} b = 0$.

Having established these claims, it is easy to check that

$$t(x, y, z) = (x \leftrightarrow_{\delta} y) \setminus z \wedge ((x \leftrightarrow_{\delta} y) \vee x)$$

realizes the ternary discriminator on \mathbf{A} . ■

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