Chapter 10
Semi-linear Varieties of Lattice-Ordered Algebras

Antonio Ledda, Francesco Paoli and Constantine Tsinakis

Abstract We consider varieties of pointed lattice-ordered algebras satisfying a restricted distributivity condition and admitting a very weak implication. Examples of these varieties are ubiquitous in algebraic logic: integral *or* distributive residuated lattices; their $\{\cdot\}$-free subreducts; their expansions (hence, in particular, Boolean algebras with operators and modal algebras); and varieties arising from quantum logic. Given any such variety $\mathscr{V}$, we provide an explicit equational basis (relative to $\mathscr{V}$) for the semi-linear subvariety $\mathscr{W}$ of $\mathscr{V}$. In particular, we show that if $\mathscr{V}$ is finitely based, then so is $\mathscr{W}$. To attain this goal, we make extensive use of tools from the theory of *quasi-subtractive varieties*.

Keywords: Quasi-subtractive-varieties, Subtractive varieties, Semi-linear varieties, Residuated lattices, Open Filters

2010 *Mathematics Subject Classification:* 03G10, 03G27, 03C05

10.1 Introduction

A variety $\mathscr{V}$ of lattice-ordered algebras is said to be *semi-linear* in case it is generated by its totally ordered members (in more traditional algebraic parlance, the term 'representable' is often used in place of 'semi-linear'.) Due to the congruence

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distributivity of $\mathscr{V}$, $\mathscr{V}$ is semi-linear if and only if its subdirectly irreducible members are totally ordered ([11], Theorem 6.8, p. 165). Needless to say, semi-linearity is a welcome property insofar as it often makes a class of algebras very tractable for computation and proof purposes. Many well-understood varieties in algebraic logic are known to be semi-linear: examples include Abelian ℓ-groups and varieties arising from many-valued logic (such as MTL algebras and thus, in particular, BL algebras, MV algebras or Gödel algebras: [15]). Petr Hájek, besides giving fundamental contributions to the investigation of many such classes of algebras, has repeatedly underscored the central role played by semilinearity in fuzzy logic:

> Among the logics of residuated lattices, fuzzy logics [...] are distinguished by the property of semilinearity, i.e., completeness w.r.t. a class of linearly ordered residuated lattices. The main scope of mathematical fuzzy logic thus can be delimited as the study of *intuitionistic substructural semilinear logics* [3, p. 58]

On the other hand, one can easily find just as many important varieties that fail to be semi-linear. A prime example is given by the variety of (pointed) *residuated lattices* [20, 17, 25] and by several of its subvarieties, most notably ℓ-groups and Heyting algebras; we also mention orthomodular lattices [10] and interior algebras [4]. In these cases, it may be useful to be in a position to axiomatize the semi-linear subvariety[1] $\mathscr{W}$ of the variety $\mathscr{V}$ of our interest, relative to a given basis for $\mathscr{V}$ — and, in fact, elegant axiomatizations have been devised in many individual cases, for example residuated lattices [9], ℓ-groups [1], or Heyting algebras [31]. Yet, it is natural to ask the following question: given a variety $\mathscr{V}$ for which an equational basis is known, is it possible to provide a general criterion for axiomatizing its semi-linear subvariety, without having to proceed on a piecemeal fashion?

We address this problem from a fairly general standpoint. In fact, we consider varieties of pointed lattice-ordered algebras obeying a restricted distribution condition and admitting a binary implication term that satisfies a minimal set of reasonable properties. Examples of these varieties are ubiquitous in algebraic logic:

1. integral residuated lattices;
2. distributive residuated lattices;
3. the $\{\cdot\}$-free subreducts of the algebras under (1) or (2);
4. expansions of the algebras under (1) or (2) by any additional signature — hence, in particular, Boolean algebras with operators and modal algebras; and
5. some varieties arising from quantum logic, e.g. Chajda et al.'s *basic algebras* [12].

Given any such variety $\mathscr{V}$, we provide an explicit equational basis (relative to $\mathscr{V}$) for the semi-linear subvariety $\mathscr{W}$ of $\mathscr{V}$. In particular, we show that if $\mathscr{V}$ is finitely based, then so is $\mathscr{W}$. Our proof takes advantage of ideas developed in [9] for residuated lattices and in [13] for basic algebras, generalizing them to a more abstract

[1] From now on, when we speak of *the* semilinear subvariety of a given variety $\mathscr{V}$, we invariably mean its *largest* semilinear subvariety. This is the variety generated by all totally ordered members of $\mathscr{V}$, equivalently, all totally ordered subdirectly irreducible members of $\mathscr{V}$.

setting. This is in line with the approach taken by C. van Alten [29], who, using different techniques, provides a distinct axiomatization of the prelinear subquasivariety of a given quasivariety of lattice-ordered algebras.

To attain this goal, we put to good use some tools from the theory of *quasi-subtractive varieties* [21], a generalization of Gumm's and Ursini's subtractive varieties [19], introduced to account for some known isomorphism theorems between ideal and congruence lattices that are not corollaries of general theorems in the theory of subtractive varieties. The required machinery is briefly illustrated in § 10.2. The following section, § 10.3, is devoted to the introduction of the concept of an LI-algebra and to the proof of our main result. A final section discusses some special cases and applications of our criterion.

10.2 Preliminaries on quasi-subtractive varieties

All the results mentioned in this section are stated without a proof; all the relevant proofs can be found in [21].

A variety $\mathscr{V}$, of signature ν, such that there exists an essentially nullary term 1 that is equationally definable in $\mathscr{V}$ over ν, is 1-*subtractive* (or simply *subtractive* when no ambiguity is possible) if there is a binary term of signature ν, denoted by $\rightarrow$ and written in infix notation, such that $\mathscr{V}$ satisfies the following equations:

S1 $\quad x \rightarrow x \approx 1$
S2 $\quad 1 \rightarrow x \approx x$

$\mathscr{V}$ is called 1-*permutable* if for any algebra $\mathbf{A} \in \mathscr{V}$ and for any congruences θ, φ of $\mathbf{A}$, $[1^{\mathbf{A}}]_{\theta \circ \varphi} = [1^{\mathbf{A}}]_{\varphi \circ \theta}$, where$[1^{\mathbf{A}}]_{\theta \circ \varphi}$ and $[1^{\mathbf{A}}]_{\varphi \circ \theta}$ denote the equivalence classes of $1^{\mathbf{A}}$ relative to the congruences $\theta \circ \varphi$ and $\varphi \circ \theta$, respectively. In their paper [19], Gumm and Ursini essentially observe that a variety $\mathscr{V}$ with 1 is 1-permutable iff it is 1-subtractive.

In [21], the next generalization of the preceding concept was suggested:

Definition 10.1. *A variety $\mathscr{V}$, of signature ν, such that there exists a nullary term* 1 *and a unary term $\Box$ of the same signature, equationally definable in $\mathscr{V}$, is called* quasi-subtractive *with respect to* 1 *and* $\Box$ *iff there exists a binary term $\rightarrow$ (hereafter written in infix notation) of signature ν such that $\mathscr{V}$ satisfies the following equations:*

Q1 $\quad \Box x \rightarrow x \approx 1$
Q2 $\quad 1 \rightarrow x \approx \Box x$
Q3 $\quad \Box(x \rightarrow y) \approx x \rightarrow y$
Q4 $\quad \Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$

Observe that, given Q3, Q4 is equivalent to $(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$. Although the latter equation is simpler, Q4 is more reminiscent of the K axiom for modal algebras. On occasion, we will say that "$\rightarrow$ witnesses quasi-subtractivity with respect to 1 and $\Box$ for $\mathscr{V}$", possibly using some stylistical variants of this expression. Members of quasi-subtractive varieties will be called, by extension, quasi-subtractive as well.

In their article on assertionally equivalent quasivarieties [8], Blok and Raftery introduce a notion of *τ-class* that relativizes the usual notion of congruence class to a given *translation*, namely, to a finite set of equations in a single variable. If $\mathscr{V}$ is a variety of type ν, $\mathbf{A} \in \mathscr{V}$, $\theta \subseteq A^2$ and $\tau(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \leqslant n\}$ is a function from the formula algebra $\mathbf{Fm}$ of type ν to $\wp(\mathbf{Fm} \times \mathbf{Fm})$, the *$\tau$-class* of θ in $\mathbf{A}$ — in symbols $[\tau^{\mathbf{A}}]_\theta$ — is defined as

$$[\tau^{\mathbf{A}}]_\theta = \left\{ a \in \mathbf{A} : \delta_i^{\mathbf{A}}(a)\, \theta \varepsilon_i^{\mathbf{A}}(a) \text{ for every } i \leqslant n \right\}.$$

A variety $\mathscr{V}$ is said to be *τ-regular* if for any congruences θ, φ on any $\mathbf{A} \in \mathscr{V}$, $[\tau^{\mathbf{A}}]_\theta = [\tau^{\mathbf{A}}]_\varphi$ implies $\theta = \varphi$; if $\tau(x) = \{x \approx 1\}$, we get as a special case the standard notion of 1-regularity.

As shown in [7, Theorem 5.2], τ-regularity is a Mal'cev property: a variety $\mathscr{V}$ is τ-regular for $\tau(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \leqslant m\}$ iff there exist binary terms $p_1, ..., p_n$ such that:

$$\begin{gathered} \vDash_{\mathscr{V}} \tau(p_j(x,x)), \text{ for } j \leqslant n; \text{ and} \\ \{\tau(p_j(x,y)) : j \leqslant n\} \vDash_{\mathscr{V}} x \approx y, \end{gathered} \tag{10.1}$$

where $\{t_i \approx s_i : i \leqslant n\} \vDash_{\mathscr{V}} t \approx s$ means that for all $\mathbf{A} \in \mathscr{V}$ and for all $a \in A$, if $t_i^{\mathbf{A}}(a) = s_i^{\mathbf{A}}(a)$ for every $i \leqslant n$, then $t^{\mathbf{A}}(a) = s^{\mathbf{A}}(a)$. In case $m = 1$ (and rewriting δ_1 as δ and ε_1 as ε), we are also guaranteed [2, Theorem 5.2] that there exist $(2n+2)$-ary terms $t_1, ..., t_k$ such that $\mathscr{V}$ satisfies the identities

$$\begin{aligned} & x \approx t_1\left(x, y, \delta\left(\overrightarrow{p(x,y)}\right), \varepsilon\left(\overrightarrow{p(x,y)}\right)\right) \\ & t_j\left(x, y, \varepsilon\left(\overrightarrow{p(x,y)}\right), \delta\left(\overrightarrow{p(x,y)}\right)\right) \approx t_{j+1}\left(x, y, \delta\left(\overrightarrow{p(x,y)}\right), \varepsilon\left(\overrightarrow{p(x,y)}\right)\right), \\ & (1 \leqslant j < k) \\ & t_k\left(x, y, \varepsilon\left(\overrightarrow{p(x,y)}\right), \delta\left(\overrightarrow{p(x,y)}\right)\right) \approx y, \end{aligned} \tag{10.2}$$

where $\delta\left(\overrightarrow{p(x,y)}\right)$ is an abbreviation for the sequence $\delta(p_1(x,y)), ..., \delta(p_n(x,y))$, and similarly for $\varepsilon\left(\overrightarrow{p(x,y)}\right)$. A third equivalent characterization of τ-regularity is as follows: $\mathscr{V}$ is τ-regular in case its *τ-assertional logic*, whose consequence relation $\vdash_{\mathscr{V}}$ is defined by

$$\Gamma \vdash_{\mathscr{V}} t \text{ iff } \{\tau(s) : s \in \Gamma\} \vDash_{\mathscr{V}} \tau(t),$$

is strongly and finitely algebraizable with $\mathscr{V}$ as equivalent variety semantics.

Blok and Raftery also consider a property of τ-permutability appropriately generalizing the notion of 1-permutability to varieties which need not be pointed: a variety $\mathscr{V}$ is *τ-permutable* iff for any congruences θ, φ on any $\mathbf{A} \in \mathscr{V}$, $[\tau^{\mathbf{A}}]_{\theta \circ \varphi} = [\tau^{\mathbf{A}}]_{\varphi \circ \theta}$.

Every quasi-subtractive variety is $\{\Box x \approx 1\}$-permutable, while the converse statement need not hold [21]. For the sake of brevity, the notation "$\{\Box x \approx 1\}$" will be streamlined to "$(\Box x, 1)$" in every relevant context.

Every 1-subtractive variety with witness term $\rightarrow$ is automatically quasi-subtractive with witness terms $\rightarrow, 1$, and the identity term as box. The next table lists some other examples of quasi-subtractive varieties. Observe that some of these varieties are indeed subtractive but can be viewed as *properly* quasi-subtractive with a different choice of witness terms.

Variety	Ref.	$x \rightarrow y$	$\Box x$	1-Subtr.?
Residuated lattices	[17]	$(x\backslash y) \wedge 1$	$x \wedge 1$	Yes
Quasi-MV algebras	[23]	$x' \oplus y$	$x \oplus 0$	No
Var. with a comm. TD term	[6]	$p(x, p(x,y,x), 1)$	$p(x, 1, 1)$	
Pseudointerior algebras	[6]	$x \rightarrow y$	x°	No
Interior algebras	[4]	$\Box(\neg x \vee y)$	$\Box x$	Yes
Integral k-potent res. lattices	[17]	$(x\backslash y)^k$	x^k	Yes

The next concept of open filter is as central for the investigation of quasi-subtractive varieties as the Gumm-Ursini concept of ideal is for the investigation of subtractive varieties:

Definition 10.2. *Let $\mathscr{V}$ be a variety whose signature ν is as in Definition 10.1. A $\mathscr{V}$-*open filter term *in the variables $\overrightarrow{x}$ is an $n+m$-ary term $p(\overrightarrow{x}, \overrightarrow{y})$ of signature ν such that:*

$$\{\Box x_i \approx 1 : i \leqslant n\} \vDash_{\mathscr{V}} \Box p(\overrightarrow{x}, \overrightarrow{y}) \approx 1.$$

The wording "$\mathscr{V}$-open filter term" will be simplified to "open filter term" whenever this replacement is unambiguous. The same applies to "$\mathscr{V}$-open filter" below.

Definition 10.3. *Let $\mathscr{V}$ be as in Definition 10.2. A $\mathscr{V}$-*open filter *of $\mathbf{A} \in \mathscr{V}$ is a subset $F \subseteq A$ with the following properties:*

i) *F is closed with respect to all $\mathscr{V}$-open filter terms p: whenever $a_1, \ldots, a_n \in F, b_1, \ldots, b_m \in A$, $p\left(\overrightarrow{a}, \overrightarrow{b}\right) \in F$;*
ii) *for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.*

Observe that 1 is a member of any open filter since the constant term 1 is an open filter term.

In the theory of subtractive varieties, ideal generation can be nicely described. A similar result holds for open filters. If $\mathbf{A}$ is any algebra in a variety $\mathscr{V}$ of the appropriate signature, and we define for $X \subseteq A$:

$$\uparrow X = X \cup \{a : \Box a \in X\};$$
$$\Gamma X = \left\{p^{\mathbf{A}}\left(\overrightarrow{a}, \overrightarrow{b}\right) : \overrightarrow{a} \in X, \overrightarrow{b} \in A, p \text{ an open filter term}\right\},$$

we get:

Lemma 10.1. *Let $\mathcal{V}$ be a quasi-subtractive variety, $\mathbf{A} \in \mathcal{V}$ and $X \subseteq A$. The $\mathcal{V}$-open filter $[X)$ generated by X is precisely $\uparrow \Gamma X$.*

Among its consequences, the preceding theorem yields a characterization of joins of open filters and the following interesting property:

Lemma 10.2. *Let $\mathcal{V}$ be a quasi-subtractive variety. Then the lattice of open filters of any $\mathbf{A} \in \mathcal{V}$ is modular.*

If $\mathcal{V}$ is a 1-subtractive variety, the ideals of any $\mathbf{A} \in \mathcal{V}$ coincide with the deductive filters[2] on $\mathbf{A}$ of the 1-assertional logic of $\mathcal{V}$ [27]; if, moreover, $\mathcal{V}$ is 1-regular, the congruence lattice of any $\mathbf{A} \in \mathcal{V}$ is isomorphic to the lattice of such deductive filters and, therefore, to its ideal lattice [16]. What happens, instead, if the variety at issue is quasi-subtractive with respect to $\Box$ and 1 and $(\Box x, 1)$-regular? The next result is an analogue of Ursini's result for subtractive varieties:

Lemma 10.3. *If $\mathcal{V}$ is a quasi-subtractive variety and $\mathbf{A} \in \mathcal{V}$, then the $\mathcal{V}$-open filters of $\mathbf{A}$ coincide with the deductive filters on $\mathbf{A}$ of the $(\Box x, 1)$-assertional logic of $\mathcal{V}$.*

With this, we are halfway through our task. For the remaining half, we make a note of a result essentially due to Blok and Pigozzi [5], although they focus on the more general scenario of an arbitrary translation τ:

Theorem 10.1. *If $\mathcal{V}$ is $(\Box x, 1)$-regular, then the congruence lattice of any $\mathbf{A} \in \mathcal{V}$ is isomorphic to the lattice of deductive filters on $\mathbf{A}$ of the $(\Box x, 1)$-assertional logic of $\mathcal{V}$.*

By Lemma 10.3 and Theorem 10.1, we get:

Corollary 10.1. *If $\mathcal{V}$ is quasi-subtractive and $(\Box x, 1)$-regular, then in any $\mathbf{A} \in \mathcal{V}$ there is a lattice isomorphism between the congruence lattice of $\mathbf{A}$ and the lattice of $\mathcal{V}$-open filters on $\mathbf{A}$.*

Besides generalizing the correspondence theorem for ideal determined varieties, Corollary 10.1 subsumes many lattice isomorphism results that do not follow from the theorem itself, to be found e.g. in the theories of residuated lattices[3], of pseudointerior algebras, or of quasi-MV algebras.

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[2] If $F \subseteq A$ and $\mathbf{A}$ has the same signature as $\mathbf{Fm}$, F is said to be a deductive filter on $\mathbf{A}$ of the logic $(\mathbf{Fm}, \vdash)$ just in case F is closed with respect to all the $\vdash$-entailments: if $\Gamma \vdash t$ and $s^{\mathbf{A}}(\overrightarrow{a}) \in F$ for all $s \in \Gamma$, then $t^{\mathbf{A}}(\overrightarrow{a}) \in F$.

[3] The variety of residuated lattices is actually 1-ideal determined and, in fact, in every residuated lattice the lattice of congruences is isomorphic to the lattice of ideals in the sense of Gumm-Ursini, which in turn coincide with convex normal subalgebras of such. There is a further isomorphism theorem, however (namely, between congruences and *deductive filters* in the sense of [17]), which does not instantiate the correspondence theorem for ideal determined varieties, but follows from Corollary 10.1.

10.3 Axiomatizing the semi-linear subvariety

As a first step towards our goal, we need an umbrella heading that encompasses the varieties of lattice-ordered algebras of our interest. Therefore, we introduce the concept of *LI-algebra*, a label whose 'L' should be suggestive of 'lattice' and whose 'I' should remind of 'implication'.

Definition 10.4. *An* LI-algebra *is an algebra A that has a term reduct* $(A, \wedge, \vee, \rightsquigarrow, 1)$ *of signature* $(2,2,2,0)$ *such that:*

- $(A, \wedge, \vee, 1)$ *is a pointed lattice satisfying:*

$$\text{(D): } (x \vee y) \wedge 1 \approx (x \wedge 1) \vee (y \wedge 1)$$

- *The following conditions concerning* $\rightsquigarrow$ *are satisfied:*

(A1) $x \rightsquigarrow y \approx 1$ iff $x \leqslant y$
(A2) $1 \rightsquigarrow x \approx x \wedge 1$
(A3) $x \vee y \rightsquigarrow z \approx (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$
(A4) $z \rightsquigarrow x \wedge y \approx (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$

We assume that lattice operations bind more strongly that $\rightsquigarrow$. Let us now exemplify the preceding definition.

Example 10.1. *(Residuated lattices).* Recall that a *residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \wedge, \vee, \backslash, /, 1)$ such that (i) $(A, \cdot, 1)$ is a monoid, (ii) $(A, \wedge, \vee)$ is a lattice, and (iii) for all $x, y, z \in A, xy \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z$. A *pointed residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \wedge, \vee, \backslash, /, 1, 0)$ such that $(A, \cdot, \wedge, \vee, \backslash, /, 1)$ is a residuated lattice and 0 is a nullary operation. Residuated lattices and hence pointed residuated lattices form finitely based equational classes of algebras [9].

Not all residuated lattices can be viewed as instances of LI-algebras, because they fail, in general, to satisfy (D). However, all distributive (pointed) residuated lattices and all integral (pointed) residuated lattices are LI-algebras with $x \rightsquigarrow y = x\backslash y \wedge 1$. Therefore the class of LI-algebras includes, in particular: ℓ-groups; MTL algebras (thus, BL algebras, MV algebras and product algebras); Heyting algebras; and Sugihara algebras.

Example 10.2. *(Subreducts of residuated lattices).* Observe that nothing in Definition 10.4 hinges on the presence of a monoidal operation whose residual is $\rightsquigarrow$. Consequently, this definition equally applies to all the $(\wedge, \vee, \backslash, /, 1)$-subreducts of the residuated lattices in Example 10.1 (see [30] for a detailed study of these and other subreducts in the commutative case).

Example 10.3. *(Expansions of residuated lattices).* The property of being an LI-algebra is obviously preserved upon arbitrary expansions of the signature. As a result, any expansion of any residuated lattice in Example 10.1 continues to be a LI-algebra. In particular, Boolean algebras with operators and modal algebras make instances of our concept.

Example 10.4. *(Basic algebras).* Basic algebras were introduced in [12] as algebras arising from lattices with sectionally antitone involutions. The theory of basic algebras presents connections with the theories of MV algebras (which can be viewed as associative basic algebras), orthomodular lattices, and lattice-ordered effect algebras. Basic algebras are LI-algebras with $x \rightsquigarrow y = \neg x \oplus y$.

Throughout the rest of this paper, $\mathscr{V}$ will refer to a generic variety of LI-algebras. In the next lemmas, we list some arithmetical properties of $\mathscr{V}$.

Lemma 10.4. *Let* $\mathbf{A} \in \mathscr{V}$*, and let* $a, b \in A$*. The following equalities hold:*

(i) $a \rightsquigarrow b = (a \rightsquigarrow b) \wedge 1$
(ii) $a \wedge 1 \rightsquigarrow b = a \wedge 1 \rightsquigarrow b \wedge 1$
(iii) $a \leqslant 1$ *implies* $a \rightsquigarrow b = a \rightsquigarrow b \wedge 1$
(iv) $(a \rightsquigarrow b) \wedge 1 \leqslant a \wedge 1 \rightsquigarrow b \wedge 1$

Proof. (i) By (A1), (A3) and absorption, $(a \rightsquigarrow b) \wedge 1 = (a \rightsquigarrow b) \wedge (a \wedge b \rightsquigarrow b) = a \vee (a \wedge b) \rightsquigarrow b = a \rightsquigarrow b$.

(ii) By (A1), (A4) and (i), $a \wedge 1 \rightsquigarrow b \wedge 1 = (a \wedge 1 \rightsquigarrow b) \wedge (a \wedge 1 \rightsquigarrow 1) = (a \wedge 1 \rightsquigarrow b) \wedge 1 = a \wedge 1 \rightsquigarrow b$.

(iii) From (ii); (iv) By absorption and (A3),

$$a \rightsquigarrow b = a \vee (a \wedge 1) \rightsquigarrow b = (a \rightsquigarrow b) \wedge (a \wedge 1 \rightsquigarrow b),$$

whence by (i) and (ii)

$$(a \rightsquigarrow b) \wedge 1 = a \rightsquigarrow b \leqslant a \wedge 1 \rightsquigarrow b = a \wedge 1 \rightsquigarrow b \wedge 1.$$

□

Lemma 10.5. $\mathscr{V}$ *satisfies the quasiequation* $x \vee y \approx 1 \Longrightarrow x \rightsquigarrow y \approx y$*.*

Proof. Let $a, b \in \mathbf{A} \in \mathscr{V}$. Then $a \rightsquigarrow b = (a \rightsquigarrow b) \wedge 1 = (a \rightsquigarrow b) \wedge (b \rightsquigarrow b) = a \vee b \rightsquigarrow b = 1 \rightsquigarrow b = b \wedge 1 = b$, for $b \leqslant a \vee b = 1$. □

The crucial observation that paves the way for an application of the results in § 10.2 is the fact that $\mathscr{V}$ is quasi-subtractive and $(\Box x, 1)$-regular:

Lemma 10.6. $\mathscr{V}$ *is quasi-subtractive with respect to* 1 *and* $\Box x = x \wedge 1$*, as witnessed by* $x \to y = x \rightsquigarrow y$*; moreover,* $\mathscr{V}$ *is* $(\Box x, 1)$*-regular with respect to the same constant* 1 *and the same unary term* $\Box x$*.*

Proof. To show that $\mathscr{V}$ is quasi-subtractive, we need to check one by one the four conditions under Definition 10.1. However, (Q1) follows from (A1); (Q2) is exactly (A2); (Q3) amounts to Lemma 10.4.(i); finally, (Q4) follows from Lemma 10.4.(iv) and (A1).

Now, let us consider Equation (10.1) with $n = 2, m = 1, \delta_1(x) = x \wedge 1, \varepsilon_1(x) = 1, p_1(x,y) = x \rightsquigarrow y$ and $p_2(x,y) = y \rightsquigarrow x$. It is easy to check that this choice of witness terms vouches for the $(\Box x, 1)$-regularity of $\mathscr{V}$, given (A1) and Lemma 10.4.(i). □

Corollary 10.2. *If* $\mathbf{A} \in \mathscr{V}$*, the congruence lattice of* **A** *is isomorphic to the lattice of open filters of* **A**.

Proof. By Lemma 10.6 and Corollary 10.1. ☐

In the following, we consider the equation

$$\text{(S1)} \ \left(t\left(z \rightsquigarrow x_1,...,z \rightsquigarrow x_n, \overrightarrow{y}\right) \wedge 1\right) \vee (x_1 \rightsquigarrow z) \vee ... \vee (x_n \rightsquigarrow z) \approx 1$$

which is actually a family of equations, one for each $\mathscr{V}$-open filter term $t(\overrightarrow{x}, \overrightarrow{y})$ in the variables $\overrightarrow{x}$. Unwieldy as it may seem, (S1) can however be broken down into a conjunction of two more manageable conditions (cf. [29]).

Lemma 10.7. *(S1) is equivalent to the conjunction of*

$$\textit{(Prel)} \ (x \rightsquigarrow y) \vee (y \rightsquigarrow x) \approx 1$$

and

$$\textit{(Q)} \ x_1 \vee z \geqslant 1 \& ... \& x_n \vee z \geqslant 1 \Longrightarrow t(\overrightarrow{x}, \overrightarrow{y}) \vee z \geqslant 1.$$

Proof. From left to right, observe that (Prel) is a special case of (S1), because x is an open filter term and $x \rightsquigarrow y = (x \rightsquigarrow y) \wedge 1$. Moreover, let $a_i \vee c \geqslant 1$ for all $i \leqslant n$, whence $(a_i \vee c) \wedge 1 = 1$ and, by (D), $(a_i \wedge 1) \vee (c \wedge 1) = 1$. Applying Lemma 10.4.(ii) and Lemma 10.5, $c \wedge 1 \rightsquigarrow a_i = c \wedge 1 \rightsquigarrow a_i \wedge 1 = a_i \wedge 1$. Since open filters are generated by their open elements (Lemma 10.1), we are allowed to pick $a_i \leqslant 1$ (recall that $\Box a = a \wedge 1$), whence (S1) and Lemma 10.5 again give

$$\begin{aligned}
1 &= \left(t\left(c \wedge 1 \rightsquigarrow a_1,...,c \wedge 1 \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c \wedge 1) \vee ... \vee (a_n \rightsquigarrow c \wedge 1) \\
&= \left(t\left(a_1 \wedge 1,...,a_n \wedge 1, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c \wedge 1) \vee ... \vee (a_n \rightsquigarrow c \wedge 1) \\
&= \left(t\left(a_1,...,a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (c \wedge 1) \\
&= \left(t\left(a_1,...,a_n, \overrightarrow{b}\right) \vee c\right) \wedge 1
\end{aligned}$$

Conversely, replacing in (Q) the variables x_i by $z \rightsquigarrow x_i$, and the variable z by $(x_1 \rightsquigarrow z) \vee ... \vee (x_n \rightsquigarrow z)$, its consequent is exactly (S1) by Lemma 10.4.(i) and (D); its antecedent, however, follows from (Prel) for the same reasons. ☐

We can now state and prove the main result of this paper:

Theorem 10.2. *The semi-linear subvariety* $\mathscr{W}$ *of* $\mathscr{V}$ *is axiomatized by (S1).*

Proof. We first show that every totally ordered algebra in $\mathscr{V}$ satisfies (S1). We distinguish two cases. If there is an i such that $a_i \leqslant c$, then $a_i \rightsquigarrow c = 1$, whence our result follows since, by absorption and Lemma 10.4.(i),

$$\begin{aligned}
&\left(t\left(c \rightsquigarrow a_1,...,c \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1\right) \vee (a_1 \rightsquigarrow c) \vee ... \vee 1 \vee ... \vee (a_n \rightsquigarrow c) \\
&= (a_1 \rightsquigarrow c) \vee ... \vee 1 \vee ... \vee (a_n \rightsquigarrow c) \\
&= ((a_1 \rightsquigarrow c) \wedge 1) \vee ... \vee 1 \vee ... \vee ((a_n \rightsquigarrow c) \wedge 1) = 1
\end{aligned}$$

On the other hand, if for all i, $c \leqslant a_i$, then, since t is a $\mathscr{V}$-open filter term in $\overrightarrow{x}$,

$$t\left(c \rightsquigarrow a_1, \ldots, c \rightsquigarrow a_n, \overrightarrow{b}\right) \wedge 1 = t\left(\overrightarrow{1}, \overrightarrow{b}\right) \wedge 1 = 1,$$

whence our result, again, follows along similar lines.

It remains to prove that every subdirectly irreducible algebra in the subvariety axiomatized by (S1) is totally ordered. Let therefore $\mathbf{A}$ be such an algebra, and let $a, b \in A$ be such that $a \nleq b$ and $b \nleq a$, that is, $a \rightsquigarrow b \neq 1$ and $b \rightsquigarrow a \neq 1$. In particular, by Lemma 10.4.(i), $a \rightsquigarrow b, b \rightsquigarrow a < 1$. Owing to Lemma 10.7, $\mathbf{A}$ satisfies (Prel), whereby $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$. For $a \leqslant 1$ in A, let

$$a^{\perp} = \{b : a \vee b \geqslant 1\}$$

We now show that $\{a \rightsquigarrow b\}^{\perp\perp}$ and $\{b \rightsquigarrow a\}^{\perp\perp}$ are open filters that intersect to the smallest open filter $\uparrow 1$, and strictly include it. Sets of the form $a^{\perp}$ are open filters: by Lemma 10.7 they are closed with respect to open filter terms, while it is easy to check, using (D), that if $b \wedge 1 \in a^{\perp}$, then also $b \in a^{\perp}$. Consequently, so is $B^{\perp}$, for any nonempty B, because $B^{\perp} = \bigcap \left\{b^{\perp} : b \in B\right\}$. Also $\{a \rightsquigarrow b\}^{\perp\perp}$ and $\{b \rightsquigarrow a\}^{\perp\perp}$ are nonzero, because they respectively contain the elements $a \rightsquigarrow b$ and $b \rightsquigarrow a$, both outside the positive cone. Finally, let $c \in \{a \rightsquigarrow b\}^{\perp\perp}$ and $c \in \{b \rightsquigarrow a\}^{\perp\perp}$. Then for every y, if $y \vee (a \rightsquigarrow b) \geqslant 1$, it follows that $c \vee y \geqslant 1$. In particular, for $y = b \rightsquigarrow a$, we obtain that $c \vee (b \rightsquigarrow a) \geqslant 1$, and similarly, $c \vee (a \rightsquigarrow b) \geqslant 1$. Letting now $y = c$, we obtain $c = c \vee c \geqslant 1$. By Corollary 10.2, then, $\mathbf{A}$ has no monolith, a contradiction. □

Although Theorem 10.2 is not sufficient to ensure that $\mathscr{W}$ is finitely based in case $\mathscr{V}$ is, we can take advantage of the following result, proved in [24], that implies the existence of a *finite* axiomatization of $\mathscr{W}$ relative to $\mathscr{V}$, at least if $\mathscr{V}$ has a finite signature. Recall from Section 10.2 that $(\Box x, 1)$-regularity is a Mal'cev property, witnessed by terms $p_1, \ldots, p_n$. It also implies the existence of terms $t_1, \ldots, t_k$ abiding by the conditions specified in, respectively, Equation (10.1) and Equation (10.2).

Theorem 10.3. *Let $\mathscr{K}$ be a variety of signature ν that is quasi-subtractive with respect to $\Box$ and* 1*, and $(\Box x, 1)$-regular. Moreover, let the former property be witnessed by the term $x \rightarrow y$ and the latter be witnessed by $p_1(x,y), \ldots, p_n(x,y)$. Let $t_1, \ldots, t_k$ be as in Equation* (10.2)*. Suppose, finally, that $\mathbf{A} \in \mathscr{K}$ and that $F = \uparrow F \subseteq A$ contains* 1 *and is closed with respect to the terms $\Box x, \Box p_1(1, \Box x), \ldots, \Box p_n(1, \Box x)$. Then F is closed with respect to* all *the open filters terms (and so is an open filter) iff it closed with respect to the following terms:*

- $(\Box x \rightarrow (\Box y \rightarrow \Box z)) \rightarrow z$;
- *for any $j, l \in \{0, \ldots, n\}$, $i \in \{1, \ldots, k\}$, and any m-ary $f \in \nu$:*

$$\Box p_l\left(\Box p_j\left(f(\overrightarrow{x}), f\left(\alpha_1^i, \ldots, \alpha_m^i\right)\right), \Box p_j\left(f(\overrightarrow{x}), f\left(\beta_1^i, \ldots, \beta_m^i\right)\right)\right), \text{ and}$$
$$\Box p_l\left(\Box p_j\left(\Box f(\overrightarrow{x}), \Box f\left(\alpha_1^i, \ldots, \alpha_m^i\right)\right), \Box p_j\left(\Box f(\overrightarrow{x}), \Box f\left(\beta_1^i, \ldots, \beta_m^i\right)\right)\right),$$

where

$$\alpha_k^i = t_i(x_k, y_k, \Box y_1^k, ..., \Box y_n^k, \overrightarrow{\Box p(x_k, y_k)});$$
$$\beta_k^i = t_{i+1}(x_k, y_k, , \overrightarrow{\Box p(x_k, y_k)}, \Box y_1^k, ..., \Box y_n^k).$$

Corollary 10.3. *If $\mathscr{V}$ is finitely based, so is its semi-linear subvariety $\mathscr{W}$.*

Proof. By means of Theorem 10.2, we have exhibited a possibly infinite equational basis for $\mathscr{W}$ relative to $\mathscr{V}$, namely, the family of identities (S1). Now, $\mathscr{V}$ is quasi-subtractive with respect to $x \wedge 1$ and 1, as well as $(x \wedge 1, 1)$-regular. Its open filters contain a whenever they contain $a \wedge 1$, contain 1, and are closed with respect to the terms

$$\begin{aligned} x \wedge 1 = \Box x = \Box p_1(1, \Box x) &= 1 \rightsquigarrow \Box x; \\ \Box p_2(1, \Box x) &= \Box(\Box x \rightsquigarrow 1) \\ &= (x \wedge 1 \rightsquigarrow 1) \wedge 1 \\ &= 1. \end{aligned}$$

Therefore Theorem 10.3 applies, and we can streamline this basis to a finite one. □

10.4 Specializations and applications

We conclude this paper by pointing to the reader's attention some special cases and applications of the results in the preceding section.

If $\mathscr{V}$ is such that, for every $\mathbf{A} \in \mathscr{V}$, the pointed lattice term reduct $(A, \wedge, \vee, 1)$ has 1 as its top element, the situation drastically simplifies. In fact, while (D) is clearly redundant in this case, (A1) and (A2) imply that $\mathscr{V}$ is 1-ideal determined and its open filters coincide with its ideals in the sense of Gumm and Ursini.

To demonstrate the strength and applicability of Theorem 10.2, we will first identify $\mathscr{V}$ with the variety of residuated lattices satisfying (D), and $\mathscr{W}$ with its semilinear subvariety, deriving the characterization of $\mathscr{W}$ in [9][4] as a consequence of this theorem. To do so, we will use a known finite basis of open filter terms in order to streamline (S2) to a finite equational basis for $\mathscr{W}$ relative to $\mathscr{V}$. Subsequently, we prove that the basis obtained in this way can be reduced to the one in [9]. The same strategy will then be applied to the $(\wedge, \vee, \backslash, /, 1)$-subreducts of residuated lattices satisfying (D). The former application yields an alternative proof of a well-known result, whereas the last one is, to the best of our knowledge, new.

[4] It should be noted that a more delicate analysis in [9] demonstrates that (D) can be omitted from the hypothesis of the theorem. Such refinements of special instances of a general result are to be expected.

Theorem 10.4. *Let $\mathscr{V}$ be a variety of residuated lattices that satisfies (D). Then its semi-linear subvariety $\mathscr{W}$ is axiomatized by the single equation*

$$\textit{(S2)}\ \lambda_z\left((x \vee y) \backslash x\right) \vee \rho_w\left((x \vee y) \backslash y\right) \approx 1,$$

where $\lambda_y(x) = y \backslash xy \wedge 1$, $\rho_y(x) = yx/y \wedge 1$.

Proof. Since $\mathscr{V}$ is quasi-subtractive with respect to 1 and $x \wedge 1$, by Theorem 10.3 its open filters coincide with the deductive filters of its $(x \wedge 1, 1)$-assertional logic, namely, the extension of the substructural logic **FL** by the axiom

$$((\alpha \vee \beta) \wedge 1) \backslash ((\alpha \wedge 1) \vee (\beta \wedge 1)).$$

By results in [18, § 4.2], these "deductive filters" (in the sense of abstract algebraic logic) coincide with upsets of convex normal subalgebras, which are likewise called *deductive filters* by residuated lattice practicioners. It follows from the same results that, in order to ensure that a $\mathscr{V}$-open filter is closed under all open filter terms, it suffices to check that it is closed under the following three: xy (in the variables x, y), $\lambda_y(x), \rho_y(x)$ (in the variable x). By Theorem 10.2, therefore, an equational basis for $\mathscr{W}$ is given by:

$$\text{(S3)}\ 1 \leqslant (z \backslash x \wedge 1)(z \backslash y \wedge 1) \vee x \backslash z \vee y \backslash z$$
$$\text{(S4)}\ 1 \leqslant \lambda_z(y \backslash x \wedge 1) \vee x \backslash y$$
$$\text{(S5)}\ 1 \leqslant \rho_z(y \backslash x \wedge 1) \vee x \backslash y$$

What remains to be proved, then, is that (S3)-(S5) are jointly equivalent to (S2).[5] To begin with, observe that (S3) follows from (S4) or (S5) by letting $z = 1$. Note, next, that (S2) is equivalent to the quasi-identity

$$x \vee y \approx 1 \Rightarrow \lambda_z(x) \vee \rho_w(y) \approx 1$$

(see [9, Lemma 6.5]). This fact also implies that, in the presence of (D), (S2) is equivalent to

$$\text{(S2')}\ \lambda_z(x \backslash y \wedge 1) \vee \rho_w(y \backslash x \wedge 1) \approx 1.$$

Now (S4) and (S5) can be rewritten as

$$\text{(S4')}\ \lambda_z(x \backslash y \wedge 1) \vee (y \backslash x \wedge 1) \approx 1,$$

and

$$\text{(S5')}\ \rho_z(x \backslash y \wedge 1) \vee (y \backslash x \wedge 1) \approx 1.$$

It is now clear that (S2') is equivalent to the conjunction of (S4') and (S5'). □

[5] Compare [17, p. 426].

We now turn to the $(\wedge, \vee, \backslash, /, 1)$-subreducts of residuated lattices satisfying (D). We observed in Example 10.2 that these algebras form a class $\mathscr{K}$ of LI-algebras. As a consequence, $\mathscr{V} = V(\mathscr{K})$ is quasi-subtractive and $(\Box x, 1)$-regular, with the same witness term as for residuated lattices. It should be noted that the equations below involve all the operation symbols. A purely implicational characterization of the variety of semilinear integral residuated lattices, relative to the variety of integral residuated lattices, was conjectured in [28] and proven in [22]. Theorem 10.5 below presents a multiplication-free characterization of the semilinear subvariety of the variety $\mathscr{V}$ of residuated lattices.

Our first goal will be that of giving a manageable description of $\mathscr{V}$-open filters.

Lemma 10.8. *Let* $\mathbf{A} \in \mathscr{V}$*, and* $F \subseteq A$*. Then* F *is a* $\mathscr{V}$*-open filter of* $\mathbf{A}$ *iff it is upward closed, and it is closed under all interpretations of the following* $\mathscr{V}$*-open filter terms (in the variables* x, y*):* $x \wedge 1$*,* $(x\backslash z)\backslash z$*,* $z/(z/x)$*, and*

$$t(x,y,z) := (\Box x \rightsquigarrow (\Box y \rightsquigarrow \Box z)) \rightsquigarrow z.$$

Proof. By [18, Lemma 4.7], a subset F of the universe of a residuated lattice is a deductive filter (hence an open filter) in case it is upward closed, it is closed under modus ponens (if a, $a\backslash b \in F$, then $b \in F$), and it is closed under all interpretations of the open filter terms $x \wedge 1$, $(x\backslash z)\backslash z$, $z/(z/x)$. Observe that $F \subseteq A$ is a $\mathscr{V}$-open filter iff it obeys the same conditions. In fact, open filters are upward closures of congruence classes of 1, and the monoidal operation does not occur in the previous conditions. However, by Lemma 20 in [21], closure under all interpretations of the term $t(x,y,z)$ suffices to guarantee modus ponens in any quasi-subtractive algebra. ◻

Observe that the upward closure condition in Lemma 10.8 is equivalent to the provision that if $a \wedge 1 \in F$, then $a \in F$. Therefore, by Lemma 10.1, closure under all interpretations of the open filter terms in Lemma 10.8 guarantees closure under all interpretations of *any* open filter term.

Theorem 10.5. *The semilinear subvariety of* $\mathscr{V}$ *is axiomatized, relative to* $\mathscr{V}$*, by the equations:*

(S6) $1 \leqslant ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash v \wedge 1))\backslash v \vee x\backslash z \vee y\backslash z$;
(S7) $1 \leqslant z\backslash x \vee x\backslash z$;
(S8) $1 \leqslant ((z\backslash x \wedge 1)\backslash y)\backslash y \vee x\backslash z$;
(S9) $1 \leqslant y/(y/(z\backslash x \wedge 1)) \vee x\backslash z$.

Proof. By Lemma 10.8 and Theorem 10.2, the semilinear subvariety of $\mathscr{V}$ is axiomatized, relative to $\mathscr{V}$, by the equations (S7), (S8), (S9) and

(S6') $1 \leqslant ((z\backslash x \wedge 1)\backslash((z\backslash y \wedge 1)\backslash(v \wedge 1) \wedge 1) \wedge 1)\backslash v \vee x\backslash z \vee y\backslash z$

However,

$$
\begin{aligned}
&((z\backslash x\wedge 1)\backslash((z\backslash y\wedge 1)\backslash(v\wedge 1)\wedge 1)\wedge 1)\backslash v\vee x\backslash z\vee y\backslash z\\
=\ &((z\backslash x\wedge 1)\backslash((z\backslash y\wedge 1)\backslash v\wedge(z\backslash y\wedge 1)\backslash 1\wedge 1))\backslash v\vee x\backslash z\vee y\backslash z\\
=\ &((z\backslash x\wedge 1)\backslash((z\backslash y\wedge 1)\backslash v\wedge 1))\backslash v\vee x\backslash z\vee y\backslash z.
\end{aligned}
$$

□

Acknowledgements: We would like to express our appreciation to the unknown referees of this paper for reading it with care and for offering perceptive suggestions. After submitting this paper, we became aware of the paper [14] whose results provide a logical counterpart of some algebraic results contained in the present article. We warmly thank Petr Cintula for the pointer. The first author acknowledges the support of the Italian Ministry of Scientic Research within the FIRB project “Structures and dynamics of knowledge and cognition”, Cagliari: F21J12000140001.

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