# RECOGNIZABILITY IN RESIDUATED LATTICES

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This work is dedicated to Hiroakira Ono
in recognition of his many contributions to mathematical logic.

Abstract. The objective of this work is to show that modules over a fixed residuated lattice $\mathbf{R}$ – that is, partially ordered sets acted upon by $\mathbf{R}$ – provide a suitable algebraic framework for extending the concept of a recognizable language as defined by Kleene. More specifically, we introduce the notion of a recognizable element of $\mathbf{R}$ by a finite module and provide a characterization of such an element in the spirit of Myhill's characterization of recognizable languages. Further, we investigate the structure of the set of recognizable elements of $\mathbf{R}$, and also provide sufficient conditions for a recognizable element to be recognized by a Boolean module.

## 1. Introduction.

The notions of a language, a finite state device, and a grammar, which are fundamental in Computer Science, have proved to be very closely related. If we try to determine which words belong to a language over an alphabet $\Sigma$, it might be possible to do it in some of the following ways. If there is a finite mechanical device, which is usually formalized as a finite state automaton, discerning the words that belong to the language from those that do not, we say that the language is *recognizable*. If there is a regular expression, which is an expression recursively defined in a specific way describing the language, we say that the language is *regular*. This concept was introduced by Kleene [18] during his investigations on the electronic models of the nervous systems. More specifically, the set of regular languages over an alphabet $\Sigma$ is the smallest set that contains the full language $\Sigma^*$ and the singletons $\{w\}$, for every word $w \in \Sigma^*$, and is closed under finite intersections and unions, complementation, complex multiplication of subsets of $\Sigma^*$, and the 'closure operation' $(\ )^*$. Kleene proved that the regular languages are exactly the recognizable languages. Another way of describing a language is by using some grammar, which roughly speaking is a mechanical way of obtaining all the words of the language by using a set of some specific rules for rewriting words. Further results from Chomsky and Miller [8] show the link between finite automata and grammars, namely the languages recognized by finite state automata are the same as the languages given by grammars of type 3, also called *regular* grammars.

Myhill [22] (see also [23]) proved an intrinsic characterization of regular languages in terms of the finiteness index of a certain syntactic equivalence relation between words. The main result of the article can be seen precisely as a generalization of this result, and it is proven in Section 4.

Several generalizations of recognizability have appeared in the literature. In particular, recognizable subsets of monoids, as well as recognizable elements of algebras of arbitrary type, have been extensively studied (see, for example, [10, 14, 26]). In fact, [14] relates the notion of recognition with duality theory and also makes substantial use of residuation.

Another well studied generalization of recognizable sets is recognizable series (see [26, Ch. III] or [3]).

In Section 2 we start revising the notions of an automaton and a language recognized by an automaton. We note that a language $L$ on an alphabet $\Sigma$ is recognized by a finite state automaton if and only if there is a surjective monoid homomorphism $\varphi : \Sigma^* \to \mathbf{M}$ onto some finite monoid $\mathbf{M}$ containing some set $T \subseteq M$ such that $L = \varphi^{-1}(T)$, which is the content of Theorem 2.5. We notice that the direct image map $\overline{\varphi} : \mathcal{P}(\Sigma^*) \to \mathcal{P}(M)$ is residuated, with residual $\varphi^{-1}$. And furthermore, $\mathcal{P}(\Sigma^*)$ has a natural structure of residuated lattice, the action of $\Sigma^*$ on $\mathbf{M}$ induced by $\varphi$ can be extended to an action of $\mathcal{P}(\Sigma^*)$ on $\mathcal{P}(M)$, that is, to a $\mathcal{P}(\Sigma^*)$-module, and the fact that $L$ is recognizable can be then expressed in terms of the residuation.

This motivates the notion of a recognizable element of a residuated lattice, which we investigate in Section 4, but first in Section 3 we develop the basics of the theory of modules over residuated lattices. Previous researchers have explored the concept of a module over a quantale, which essentially is an action of a quantale on a complete lattice. Such structures provide a suitable algebraic framework for extending the concept of a recognizable language (see [20]), but also for the study of some fundamental aspects of Algebraic Logic (see [13]). Here we consider the possibility of extending these ideas by letting the scalars come from an arbitrary residuated lattice and replacing the complete lattice by any partially ordered set. We notice that given a residuated lattice $\mathbf{R} = \langle R, \wedge, \vee, \cdot, e, \backslash, / \rangle$, it acts over itself by left multiplication, giving rise to an $\mathbf{R}$-module that we denote by $\mathbb{R} = \langle R, \cdot \rangle$. We define for every element $a$ of a residuated lattice $\mathbf{R}$ a special closure operator $\gamma_a$ on $\mathbb{R}$, and describe its basic properties in Proposition 3.22. This closure operator turns out to be crucial in deciding whether the element $a$ is recognizable.

In Section 4, as we mentioned before, we introduce the notion of a recognizable element of a residuated lattice. We then prove that this is the correct abstraction of the notion of a recognizable language to the context of residuated lattices, by showing that a language over an alphabet $\Sigma$ is recognizable by a finite state automaton if and only if it is recognizable as an element of the residuated lattice $\mathcal{P}(\Sigma^*)$, see Proposition 4.4. Next we prove Theorem 4.6, which is the main result of the section and of the article. It is a characterization of the recognizable elements of residuated lattices in the following terms:

**Theorem.** *Let $\mathbf{R}$ be a residuated lattice and $a \in R$. The following are equivalent:*

*(i) The element $a$ is recognizable.*
*(ii) There exists a structural closure operator[1] $\gamma$ on $\mathbb{R}$ with finite image such that $\gamma(a) = a$.*
*(iii) The image $\{x\backslash a : x \in R\}$ of the closure operator $\gamma_a$ is finite.*
*(iv) The set $\{a/x : x \in R\}$ is finite.*

Finally we devote Section 5 to two interesting problems. Our results shed light on them and could lead to their eventual resolution. We look for a Kleene's-like characterization of the recognizable elements of a residuated lattice. According to Kleene's Theorem, regular languages are exactly the languages recognized by a finite state automata. Therefore, in order to provide an appropriate generalization of this result for an arbitrary residuated lattice $\mathbf{R}$, we have to study the structure of the set of recognizable elements inside $\mathbf{R}$. We find that, whenever $\mathbf{R}$ has a top element, and only in this case, the set of recognizable elements is nonempty, contains the top element, and it is closed under (finite) meets and residuation. We also find that in the case $\mathbf{R} = \mathcal{P}(\mathbf{M})$, for some monoid $\mathbf{M}$, it is also closed

[1] See Definition 3.15.

under complementation and (finite) unions, although it may not contain all the singletons in general.

The second and last problem that we study in this section is the following: we notice that every recognizable language is recognized by a module whose poset reduct is indeed a Boolean algebra. This is not the general case for recognizable elements of residuated lattices. We provide conditions under which we can assure that a particular element can be recognized by a *Boolean module*, that is to say, a module whose poset reduct is a Boolean algebra.

We assume the reader has a basic knowledge of Universal Algebra and Lattice Theory. Recommended introductions to these can be found in [7] and [9].

## 2. Background and motivation

One of the goals of this paper is to define the concept of a *recognizable element* in an arbitrary residuated lattice. Since we are borrowing this term from the area of logic and computation, it will be helpful to give here a brief overview of the notion of a *recognizable language*, which also will provide us with the motivating example for our work. The reader is directed to [15] and [16] for a more detailed treatment of the subject.

Let $\Sigma$ be any set. We shall refer to $\Sigma$ as an *alphabet* and the elements of $\Sigma$ as *letters*. We use the symbol $\Sigma^*$ to denote the collection of all finite sequences (including the empty sequence) of letters from $\Sigma$. The members of $\Sigma^*$ are called *words* and we normally write a word as a string of adjacent letters. Thus, if $a, b$, and $c$ are letters, the word $\langle abbca \rangle$ will more often be written *abbca*. The empty sequence, also called the *empty word*, is denoted by $\varepsilon$. Clearly, $\Sigma^*$ forms a monoid – in fact the free monoid over $\Sigma$ – under the operation of concatenation with $\varepsilon$ in the role of the identity (for words $w, v \in \Sigma^*$, we write $wv$ for the concatenation of $w$ followed by $v$).

As with natural human languages, we often consider most random strings of letters to be gibberish and only a selected subset of $\Sigma^*$ will form a *language*. In examples coming from the realm of mathematics, languages are usually constituted by the so-called well-formed expressions. For instance, this is the case of the language of the formulas of classical logic, that are normally taken to be the well-formed expressions over the language $\Sigma = V \cup \{\wedge, \vee, \to, \neg, \top, \bot\} \cup \{\,),(\,\}$, where $V$ is a set of variables. But, there are also other examples that appeal for a more general definition of a language. For instance, we could identify the set $\mathbb{Z}[X]$ of polynomials in one variable and integer coefficients with a language over the set of symbols $\Sigma = \mathbb{Z}$, using the uniqueness of the expression of a polynomial as a sum of monomials. Indeed, under this identification, $\mathbb{Z}[X] = \mathbb{Z}^*$.

**Definition 2.1.** A *language* over the alphabet $\Sigma$ is a subset $L \subseteq \Sigma^*$. The *full language* over $\Sigma$ is $\Sigma^*$.

Now, this definition of a language might seem too general, as we have the intuition that languages are usually generated following some mechanical rules. Suppose that $L$ is a language over some alphabet $\Sigma$. One wonders if it is possible to decide, in finitely many steps of some automated process, whether a given word $w \in \Sigma^*$ is, or is not, a member of $L$. For example, such a task is carried out by compilers when parsing code written in some programming language. In our context, this leads to the idea of a *finite state automaton*.

**Definition 2.2.** A *finite state automaton* is a quintuple $\langle S, \Sigma, \star, i, F \rangle$ consisting of the following five components:

- a finite set $S$, called the set of *states*,

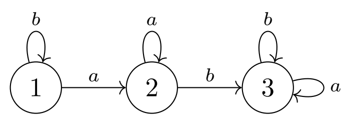
- a finite[2] set $\Sigma$, the *alphabet*,
- an *action* of $\Sigma^*$ on $S$, that is, a binary map $\star : \Sigma^* \times S \to S$ satisfying the properties:
  - (associativity) for all $w, v \in \Sigma^*$ and $s \in S$, $w \star (v \star s) = (wv) \star s$, and
  - (identity) for all $s \in S$, $\varepsilon \star s = s$.
- a state $i \in S$, called the *initial state*, and
- a set $F \subseteq S$, called the set of *final states*.

If one thinks of $S$ as actual states of some mechanical device and $\Sigma^*$ as potential instructions supplied to that device, then we think of the action as carrying out the instructions by transforming the device from one state into another; we are attempting to capture the internal workings of a computer in an algebraic setting. As an example, consider the finite state automaton $\langle S, \Sigma, \star, 1, \{3\}\rangle$ where $S = \{1, 2, 3\}$, $\Sigma = \{a, b\}$, and $\star : \Sigma^* \times S \to S$ is the map implicitly defined by the table:

| $\star$ | 1 | 2 | 3 |
|---|---|---|---|
| $a$ | 2 | 2 | 3 |
| $b$ | 1 | 3 | 3 |

It is clear that any function $\Sigma \times S \to S$ extends uniquely to an action of $\Sigma^*$ over $S$. One can depict this action as shown in Diagram (1), and thus this diagram can be used to represent the automaton, just by marking the initial state and set of final state in some way. If $w = aabb$ and $v = baab$, for example, then $w \star 1 = 2$ and $v \star 1 = 3$.

(1)




**Definition 2.3.** Given a finite state automaton $\langle S, \Sigma, \star, i, F\rangle$, we say that a language $L \subseteq \Sigma^*$ is *recognized* by this automaton if for any word $w \in \Sigma^*$, $w \in L$ if and only if $w \star i \in F$. A language $L$ is *recognizable* if there exists some finite state automaton that recognizes $L$.

*Example* 2.4. Letting $L = \{wbav : w, v \in \Sigma^*\}$, one can easily see that $L$ is recognized by the automaton of our previous example, and therefore, $L$ is recognizable.

It is well known that there is a bijective correspondence between actions of a monoid $\mathbf{M}$ on a set $S$ and monoid homomorphisms from $\mathbf{M}$ to the monoid $\mathrm{End}(S)$ of endomaps of $S$. Thus, we obtain the following characterization of recognizable languages, which can be found in [24].

**Theorem 2.5.** *A language $L$ is recognizable if and only if there exist a finite monoid $\mathbf{M}$, a (surjective) monoid homomorphism $\varphi : \Sigma^* \to \mathbf{M}$, and a subset $T \subseteq M$ such that $L = \varphi^{-1}(T)$.*

*Proof.* If $L$ is recognizable by $\langle S, \Sigma, \star, i, F\rangle$, then consider the monoid $\mathbf{M} \subseteq \mathrm{End}(S)$ with $M = \{\lambda_w : w \in \Sigma^*\}$, where $\lambda_w(s) = w \star s$, for every $s \in S$. Since $S$ is finite, hence $\mathbf{M}$ is finite and the map $\lambda : \Sigma^* \to \mathbf{M}$ determined by $\lambda : w \mapsto \lambda_w$ is a surjective monoid homomorphism. Now consider the set $T = \lambda[L] = \{\lambda_w : w \in L\}$, and let see that $L = \lambda^{-1}(T)$. It is clear that $L \subseteq \lambda^{-1}(T)$. In order to see the other inclusion, suppose that $v \in \lambda^{-1}(T)$. Hence, there exists $w \in L$ such that $\lambda_v = \lambda_w$, whence we have $v \star i = \lambda_v(i) = \lambda_w(i) = w \star i \in T$, and therefore $v \in L$.

[2] Usually, the alphabet is taken to be finite, because of the applications of automata to Computer Science, but none of the results that we mention here depend on the finiteness of the alphabet.

For the other implication, suppose that there is a monoidal homomorphism $\varphi : \Sigma^* \to \mathbf{M}$, and a subset $T \subseteq M$ such that $L = \varphi^{-1}(T)$. Hence, we can define an action $\star$ of $\Sigma^*$ on $M$ by $w \star x = \varphi(w) \cdot x$. Thus, if $M$ is finite, then $\langle M, \Sigma, \star, e_{\mathbf{M}}, T\rangle$ is a finite state automaton, and moreover

$$w \in L \;\Leftrightarrow\; w \in \varphi^{-1}(T) \;\Leftrightarrow\; \varphi(w) \in T \;\Leftrightarrow\; \varphi(w) \cdot e_{\mathbf{M}} \in T \;\Leftrightarrow\; w \star e_{\mathbf{M}} \in T.$$

Therefore $L$ is recognizable by $\langle M, \Sigma, \star, e_{\mathbf{M}}, T\rangle$. □

The previous discussion implies, in particular, that $\overline{\varphi}(L) = \overline{\varphi}(\varphi^{-1}(T))$ and $L = \varphi^{-1}(T) = \varphi^{-1}(\overline{\varphi}(L))$, where $\overline{\varphi}$ and $\varphi^{-1}$ are the direct and inverse images maps between the lattices $\mathcal{P}(\Sigma^*)$ and $\mathcal{P}(M)$. Notice that $\overline{\varphi}$ is a residuated map and $\varphi^{-1}$ is its residual. Pursuing this line of thought leads to the following observations:

- We can extend the action $\star$ of $\Sigma^*$ over $S$ to an action of $\mathcal{P}(\Sigma^*)$ on $\mathcal{P}(S)$ by:
$$A * X = \{w \star s : w \in A,\ s \in X\}.$$
- This extended action $* : \mathcal{P}(\Sigma^*) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is residuated in both coordinates.
- $L = \varphi^{-1}(T)$ translates in terms of the residuals of $*$.

This indicates that we can capture the concept of recognizability of languages in terms of residuation. We explain in more details these concepts in the next section.

## 3. Residuation, residuated lattices, and modules over residuated lattices

Residuation is a ubiquitous mathematical property, intimately related to Galois connections and to adjoint functors.

**Definition 3.1.** Given two partially ordered sets $\mathbf{P}$ and $\mathbf{Q}$, a map $f : P \to Q$ is *residuated* if there exists another map $g : Q \to P$, called the *residual* of $f$, so that for all $p \in P$ and $q \in Q$,

$$f(p) \leqslant q \;\Leftrightarrow\; p \leqslant g(q).$$

*Remark* 3.2. If $f : \mathbf{P} \to \mathbf{Q}$ is a residuated map with residual $g$, then among the many well-understood properties of $f$ and $g$ there are the following:

(i) Both $f$ and $g$ are order-preserving.
(ii) $f$ preserves arbitrary existing joins and $g$ preserves arbitrary existing meets.
(iii) $\gamma = g \circ f$ is a *closure operator* on $\mathbf{P}$: $\gamma$ is order-preserving, extensive ($x \leqslant \gamma(x)$, for all $x \in P$), and idempotent. Its associated *closure system* is $P_\gamma = \{g(x) : x \in Q\}$, which inherits a partial ordering from $\mathbf{P}$.
(iv) $\delta = f \circ g$ is an *interior operator* on $\mathbf{Q}$: $\delta$ is order-preserving, contractive ($\delta(x) \leqslant x$, for all $x \in Q$), and idempotent. Its associated *interior system* is $Q_\delta = \{f(x) : x \in Q\}$, which inherits a partial ordering from $\mathbf{Q}$.
(v) $f \circ g \circ f = f$ and $g \circ f \circ g = g$.
(vi) The corresponding restrictions of $f$ and $g$ determine an order-isomorphism and its inverse between $\mathbf{P}_\gamma$ and $\mathbf{Q}_\delta$.

**Definition 3.3.** A *residuated lattice* is a structure $\mathbf{R} = \langle R, \wedge, \vee, \cdot, \backslash, /, e\rangle$ comprising monoidal and lattice structures over the same underlying set $R$, and such that the product $\cdot$ is residuated in both coordinates with residuals $\backslash$ and $/$. This means that for every $a, b, c \in R$,

$$a \cdot b \leqslant c \;\Leftrightarrow\; b \leqslant a\backslash c \;\Leftrightarrow\; a \leqslant c/b.$$

General references in the theory of residuated lattices are [12], [21], [4], or [17]. The following properties can be readily proven:

**Proposition 3.4.** *If* $\mathbf{R}$ *is a residuated lattice, then the following hold for every* $a, b, c \in R$*:*

*(i)* $(a/b) \cdot b \leqslant a$ *and* $b \cdot (b\backslash a) \leqslant a$.
*(ii)* $e \leqslant a/a$ *and* $e \leqslant a\backslash a$.
*(iii)* $a/e = a = e\backslash a$.
*(iv)* $(a/b)/c = a/(cb)$ *and* $c\backslash(b\backslash a) = (bc)\backslash a$.
*(v) The operation* $\cdot$ *is order-preserving in both coordinates, while* $/$ *and* $\backslash$ *are order-preserving in their numerators and order-reversing in their denominators.* □

*Example* 3.5. One of the prototypical examples of a residuated lattice is a frame. A *frame* is a complete lattice $\mathbf{F}$ in which the meet operation $\wedge$ distributes over arbitrary joins. Therefore, $\wedge$ is residuated in both coordinates, and since it is commutative, left and right residuals coincide, and are often denoted by $\to$. This operation is determined by $b \to c = \bigvee\{x \in F : b \wedge x \leqslant c\}$. Thus, in a frame,

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

The induced residuated lattice is $\mathbf{F} = \langle F, \wedge, \vee, \wedge, \to, \leftarrow, \top\rangle$. But, we will in practice omit the operation $\leftarrow$ because, as we noted, both residuals coincide. We can do that whenever the product is commutative.

*Example* 3.6. Another standard example of a residuated lattice is the following: Given any monoid $\mathbf{M}$, we can define a residuated lattice $\mathcal{P}(\mathbf{M})$ as follows: $\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e_{\mathbf{M}}\}\rangle$ is the residuated lattice with the so-called *complex multiplication*, that is, for every $A, B \in \mathcal{P}(M)$, $A \cdot B = \{a \cdot b : a \in A, b \in B\}$. This product is residuated in both coordinates, and the residuals are determined by:

$$A\backslash C = \{b \in M : ab \in C, \text{ for all } a \in A\} \quad \text{and} \quad C/A = \{b \in M : ba \in C, \text{ for all } a \in A\}.$$

Those two examples are particular cases of *quantales*, which can be just seen as complete residuated lattices. Actually, residuals do not form part of the structure of the quantale, and therefore they are not residuated lattices, strictly speaking. But, since the residuals are uniquely determined by the order and the product, the identification of quantales with complete lattices is innocuous. The main difference is not on the structure itself, but on the morphisms, which in the case of quantales are residuated maps respecting the monoidal structure, but not necessarily the residuals. Quantales have arisen as partially ordered models of linear logic, which turned out to be a certain class of quantales. The precise connection between quantales and linear logic was made in [27]. See [25] for a detailed introduction of the theory of quantales. Abramsky and Vickers used in [1] (see also [2]) the notion of module over a quantale to investigate a variety of process semantics in a uniform algebraic framework. In their work, processes are certain modules over a given quantal of actions. We extend the notion of module over a quantale, allowing scalars to come from an arbitrary residuated lattice. Moreover, we do not require the module to be a complete algebra, but just a partially ordered set.

**Definition 3.7.** A *module* over a residuated lattice $\mathbf{R}$, or just an $\mathbf{R}$-module, is a pair $\mathbb{P} = \langle \mathbf{P}, *\rangle$ consisting of a partially ordered set $\mathbf{P} = \langle P, \leqslant\rangle$ and a map $* : \mathbf{R} \times \mathbf{P} \to \mathbf{P}$ satisfying the following three properties:

(i) $e * x = x$, for all $x \in P$,
(ii) $a * (b * x) = (a \cdot b) * x$, for all $a, b \in R$ and $x \in P$,
(iii) $*$ is residuated in both coordinates. That is, there exist two maps $\backslash_* : R \times P \to P$ and $/_* : P \times P \to R$ such that, for every $a \in R$ and $x, y \in P$,

$$a * x \leqslant y \iff x \leqslant a\backslash_* y \iff a \leqslant y/_* x.$$

*Example* 3.8. Given a monoid $\mathbf{M}$, we saw in Example 3.6 how to construct the residuated lattice $\mathcal{P}(\mathbf{M})$. Now, given a monoid action $\star : M \times S \to S$ of $\mathbf{M}$ on the a set $S$, we can define a $\mathcal{P}(\mathbf{M})$-module $\langle \mathcal{P}(S), * \rangle$ as follows: the set $\mathcal{P}(S)$ is ordered by inclusion, and the product $* : \mathcal{P}(M) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is given by: $A * X = \{a \star x : a \in A, x \in X\}$. It is easy to see that, indeed, this is a $\mathcal{P}(\mathbf{M})$-module, and that the residuals are given by $A\backslash_* Y = \{x \in S : a \star x \in Y, \text{ for all } a \in A\}$ and $Y/_* X = \{a \in M : a \star x \in Y, \text{ for all } x \in X\}$.

*Remark* 3.9. Notice that if $\mathbb{P} = \langle \mathbf{P}, * \rangle$ is an $\mathbf{R}$-module, then it follows immediately from the definition that $*$ is order-preserving in both coordinates and $\backslash_*$ and $/_*$ are order-preserving in their numerators and order-reversing in their denominators. Moreover, $*$, preserves existing arbitrary joins in both coordinates, and $\backslash_*$ and $/_*$ preserve existing arbitrary meets in their numerators and transform existing arbitrary joins in the denominators into meets.

**Definition 3.10.** Let $\mathbb{P}$ and $\mathbb{Q}$ be $\mathbf{R}$-modules. An $\mathbf{R}$-*morphism* $\varphi : \mathbb{P} \to \mathbb{Q}$ from $\mathbb{P}$ to $\mathbb{Q}$ is a residuated map $\varphi : \mathbf{P} \to \mathbf{Q}$ that preserves scalars; that is, for all $a \in R$ and $x \in P$, $\varphi(a *_{\mathbb{P}} x) = a *_{\mathbb{Q}} \varphi(x)$. An $\mathbf{R}$-module $\mathbb{Q}$ is a *submodule* of $\mathbb{P}$ if $Q \subseteq P$, and the inclusion map is an $\mathbf{R}$-morphism $i : \mathbb{Q} \to \mathbb{P}$. An *isomorphism* is a bijective $\mathbf{R}$-morphism between two $\mathbf{R}$-modules, in which case its inverse is also an $\mathbf{R}$-morphism.

*Remark* 3.11. Note that, given an $\mathbf{R}$-module $\mathbb{P}$, a submodule is just an $\mathbf{R}$-module $\mathbb{Q}$ such that $Q \subseteq P$ and the product of $\mathbb{Q}$ is the restriction of the product of $\mathbb{P}$. Therefore, submodules of an $\mathbf{R}$-module are determined by their underlying sets.

In general, $\mathbf{R}$-morphisms do not respect the residuals, but isomorphisms do.

**Proposition 3.12.** *If $\varphi : \mathbb{P} \to \mathbb{Q}$ is an isomorphism of $\mathbf{R}$-modules and $a \in R$, and $x, y \in P$, then $x/_{\mathbb{P}} y = \varphi(x)/_{\mathbb{Q}} \varphi(y)$, and $\varphi(a\backslash_{\mathbb{P}} x) = a\backslash_{\mathbb{Q}} \varphi(x)$.*

*Proof.* First, notice that an isomorphism of $\mathbf{R}$-modules is in particular an isomorphism of posets. Thus, if $b \in R$ is an arbitrary element, then

$$b \leqslant x/_{\mathbb{P}} y \iff b *_{\mathbb{P}} y \leqslant x \iff b *_{\mathbb{Q}} \varphi(y) \leqslant \varphi(x) \iff b \leqslant \varphi(x)/_{\mathbb{Q}} \varphi(y).$$

Which proves that $x/_{\mathbb{P}} y = \varphi(x)/_{\mathbb{Q}} \varphi(y)$. The other equality can be proved in a similar fashion. ☐

**Definition 3.13.** Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, we define $\langle p \rangle_{\mathbb{P}}$ to be the submodule of $\mathbb{P}$ with universe $\{a * p : a \in R\}$. We say that $\mathbb{P}$ is *cyclic* if $\mathbb{P} = \langle p \rangle_{\mathbb{P}}$ for some $p \in P$. In this case, $p$ is called a *generator* of $\mathbb{P}$.

*Example* 3.14. It can be readily seen that any residuated lattice $\mathbf{R}$ has the structure of an $\mathbf{R}$-module $\mathbb{R} = \langle \mathbf{R}, \cdot \rangle$. (Sometimes we say that $\mathbf{R}$ is a module over itself.) Moreover, $\mathbb{R}$ is cyclic, since it is generated by $e$, that is, $\mathbb{R} = \langle e \rangle_{\mathbb{R}}$. Thus, if $\mathbf{F}$ is a frame, the corresponding $\mathbf{F}$-module is $\mathbb{F} = \langle \mathbf{F}, \wedge \rangle$, and it is cyclic generated by its top element $\top$.

**Definition 3.15.** A *structural closure operator* on an $\mathbf{R}$-module $\mathbb{P} = \langle \mathbf{P}, * \rangle$ is a closure operator $\gamma$ on $\mathbf{P}$ – that is, an extensive, order-preserving, and idempotent endomap of $\mathbf{P}$ – such that for every $a \in R$ and $x \in P$,

$$a * \gamma(x) \leqslant \gamma(a * x). \tag{Str.}$$

The property (Str.) is called *structurallity.* In what follows, we often omit the adjective 'structural' when we speak about closure operators on an $\mathbf{R}$-module. Further, we denote by $P_\gamma = \{\gamma(x) : x \in P\}$ the *closure system* associated to $\gamma$.

*Remark* 3.16. Closure operators on an $\mathbf{R}$-module $\mathbb{P}$ can be ordered point-wise: $\gamma \leqslant \delta$ if and only if for every $x \in P$, $\gamma(x) \leqslant \delta(x)$. It is not difficult to see that $\gamma \leqslant \delta$ if and only if $P_\delta \subseteq P_\gamma$.

*Example* 3.17. A *nucleus* on a residuated lattice $\mathbf{R}$ is a closure operator on $\mathbf{R}$ satisfying the inequality,

$$\gamma(a) \cdot \gamma(b) \leqslant \gamma(a \cdot b), \tag{2}$$

for all $a, b \in R$. Obviously, every nucleus on a residuated lattice $\mathbf{R}$ is a closure operator on $\mathbb{R}$, but the converse need not be true. We however have:

**Proposition 3.18.** *The nuclei on a frame are precisely the meet-preserving closure operators on it.*

As we mentioned in Remark 3.2, a natural example of a closure operator arises by composing residuated maps with their residuals. In the case of $\mathbf{R}$-modules we obtain structural closure operators in the following way: let $\mathbb{P}$ be an $\mathbf{R}$-module, and $p \in P$ an arbitrary element. The map $\gamma_p : R \to R$ defined by $\gamma_p(a) = (a * p)/_* p$ is a closure operator on $\mathbf{R} = \langle R, \leqslant\rangle$. Actually, we can prove that $\gamma_p$ is a closure operator on $\mathbb{R}$.

**Lemma 3.19.** *Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, the map $\gamma_p$ defined above is a closure operator on $\mathbb{R}$, and $R_{\gamma_p} = \{x/_* p : x \in P\}$.*

*Proof.* Note that for every $a, b \in R$,

$$(a \cdot \gamma_p(b)) * p = a * (\gamma_p(b) * p) = a * \big(((b * p)/_* p) * p\big) = a * (b * p) = (ab) * p,$$

and therefore $a \cdot \gamma_p(b) \leqslant ((ab) * p)/_* p = \gamma_p(ab)$, which shows the structurally of $\gamma_p$. The equality $R_{\gamma_p} = \{x/_* p : x \in P\}$ follows from Remark 3.2.(iii), as $\gamma_p$ is the composition of the residuated $\_ * p$ and its residual $\_/_* p$. □

Given a closure operator $\gamma$ on an $\mathbf{R}$-module $\mathbb{P}$, the set $P_\gamma = \{\gamma(x) : x \in P\}$ inherits a partial ordering from $\mathbf{P}$, and one can define a scalar product $*_\gamma : R \times P \to P$ by $a *_\gamma x = \gamma(a * x)$. Moreover, for every $a \in R$ and $x, y \in P_\gamma$, $a *_\gamma x \leqslant y$ if and only if $\gamma(a * x) \leqslant y$, which is equivalent to $a * x \leqslant y$, because $y \in P_\gamma$ and $\gamma$ is expansive and order-preserving. Therefore, $a *_\gamma x \leqslant y$ if and only if $a \leqslant y/_* x$ if and only if $x \leqslant a\backslash_* y$. That is to say, $*_\gamma$ is residuated in both coordinates with residuals the corresponding restrictions of $\backslash_*$ and $/_*$. We also have that $e *_\gamma x = \gamma(e * x) = \gamma(x) = x$, and that $a *_\gamma (b *_\gamma x) = \gamma(a * \gamma(b * x)) \leqslant \gamma(\gamma(a * (b * x))) = \gamma(a * (b * x)) \leqslant \gamma(a * \gamma(b * x)) = a *_\gamma (b *_\gamma x)$, which proves that $(ab) *_\gamma x = \gamma((ab) * x) = \gamma(a * (b * x)) = a *_\gamma (b *_\gamma x)$. Thus, $\mathbb{P}_\gamma = \langle \mathbf{P}_\gamma, *_\gamma\rangle$ is an $\mathbf{R}$-module as well.

It is not difficult to prove that actually every closure operator $\gamma$ on $\mathbb{R}$ is a closure operator of the form $\gamma_p$ for a certain $\mathbf{R}$-module $\mathbb{P}$ and a certain $p \in P$. Indeed, we only have to consider $\mathbb{P} = \mathbb{R}_\gamma$, as defined in the previous paragraph, and $p = \gamma(e)$.

**Proposition 3.20.** *Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, the module $\mathbb{R}_{\gamma_p}$ is isomorphic to $\langle p\rangle_{\mathbb{P}}$, and therefore cyclic.*

*Proof.* Since the map $\_ * p : \mathbf{R} \to \mathbf{P}$ is residuated with residual $\_/_* p : P \to R$, then in virtue of Remark 3.2, the restriction $\varphi : \mathbf{R}_{\gamma_p} \to \langle\{a * p : a \in R\}, \leqslant\rangle$ of $\_ * p$ is an isomorphism of partially ordered sets. All we have to prove is that $\varphi$ also respects the scalars. Given $a \in R$ and $x \in R_{\gamma_p}$, we have

$$\begin{aligned}\varphi(a \cdot_{\gamma_p} x) = \varphi(\gamma_p(a \cdot x)) = \varphi\big(((a \cdot x) * p)/_* p\big) = \big(((a \cdot x) * p)/_* p\big) * p = (a \cdot x) * p \\ = a * (x * p) = a * \varphi(x).\end{aligned}$$

□

We have noted that the multiplication of a residuated lattice $\mathbf{R}$ induces an $\mathbf{R}$-module structure $\mathbb{R}$ on $\langle R, \leqslant\rangle$. An $\mathbf{R}$-module structure can also be defined on the dual partial ordering $\mathbf{R}^\partial = \langle R, \leqslant^\partial\rangle$:

**Proposition 3.21.** *Let $\mathbf{R}$ be a residuated lattice and let $\cdot^{\mathrm{d}} : R \times R \to R$ be defined by $a \cdot^{\mathrm{d}} x = x/a$, for all $a, x \in R$. Then the structure $\mathbb{R}^{\mathrm{d}} = \langle \mathbf{R}^\partial, \cdot^{\mathrm{d}}\rangle$ is an $\mathbf{R}$-module.*

*Proof.* First note that for every $x \in R$, $e \cdot^{\mathrm{d}} x = x/e = x$, and that for every $a, b, x \in R$,

$$a \cdot^{\mathrm{d}} (b \cdot^{\mathrm{d}} x) = (x/b)/a = x/(ab) = (ab) \cdot^{\mathrm{d}} x.$$

Therefore, it only remains to prove that $\cdot^{\mathrm{d}} : \mathbf{R} \times \mathbf{R}^\partial \to \mathbf{R}^\partial$ is residuated in both coordinates. For every $a, x, y \in R$, we have

$$a \cdot^{\mathrm{d}} x \leqslant^\partial y \iff y \leqslant a \cdot^{\mathrm{d}} x \iff y \leqslant x/a \iff y \cdot a \leqslant x \iff a \leqslant y\backslash x.$$

Therefore, the maps $\backslash_{\mathrm{d}} : R \times R \to R$ and $/_{\mathrm{d}} : R \times R \to R$ determined by $a\backslash_{\mathrm{d}} y = y \cdot a$ and $y/_{\mathrm{d}} x = y\backslash x$ are the residuals of $\cdot^{\mathrm{d}}$, since

$$a \cdot^{\mathrm{d}} x \leqslant^\partial y \iff y \cdot a \leqslant x \iff x \leqslant^\partial y \cdot a \iff x \leqslant^\partial a\backslash_{\mathrm{d}} y$$

and

$$a \cdot^{\mathrm{d}} x \leqslant^\partial y \iff a \leqslant y\backslash x \iff a \leqslant y/_{\mathrm{d}} x.$$ □

Thus, given a residuated lattice $\mathbf{R}$ and a particular element $a \in R$, by using the $\mathbf{R}$-module $\mathbb{R}^{\mathrm{d}}$, we can define a closure operator $\gamma_a$ on $\mathbb{R}$ as follows:

$$\gamma_a(x) = (x \cdot^{\mathrm{d}} a)/_{\mathrm{d}} a = (a/x)\backslash a.$$

**Proposition 3.22.** *Given a residuated lattice $\mathbf{R}$ and an element $a \in R$, the closure operator $\gamma_a$ on $\mathbb{R}$ defined above has the following properties:*

*(i)* $R_{\gamma_a} = \{x\backslash a : x \in R\}$.
*(ii)* $\gamma_a(a) = a$.
*(iii) If $\gamma$ is a closure operator on $\mathbb{R}$, then $\gamma(a) = a$ if and only if $\gamma \leqslant \gamma_a$.*

*Proof.* (i) By Lemma 3.19, the closure system associated to $\gamma_a$ is $R_{\gamma_a} = \{x/_{\mathrm{d}} a : x \in R\} = \{x\backslash a : x \in R\}$.

(ii) By (i), $a = e\backslash a \in R_{\gamma_a}$, that is $\gamma_a(a) = a$.

(iii) In virtue of Remark 3.16, it is enough to show that if $\gamma$ is an operator on $\mathbb{R}$, then $\gamma(a) = a$ if and only if $R_{\gamma_a} \subseteq R_\gamma$. Suppose that $\gamma(a) = a$. By (i), every element in $R_{\gamma_a}$ is of the form $x\backslash a$ for some $x \in R$. Since by the structurallty of $\gamma$ we have that $x \cdot \gamma(x\backslash a) \leqslant \gamma(x \cdot (x\backslash a)) \leqslant \gamma(a) = a$, by the motonicity of $\gamma$ and the hypothesis, hence $\gamma(x\backslash a) \leqslant x\backslash a \leqslant \gamma(x\backslash a)$. That is, $x\backslash a \in R_\gamma$. For the other implication, simply notice that if $\gamma \leqslant \gamma_a$, then by (ii) and Remark 3.16, $a \in R_{\gamma_a} \subseteq R_\gamma$, and therefore $\gamma(a) = a$. □

## 4. Recognizable elements in residuated lattices

Going back to our discussion about recognizable languages, we recall that a language $L$ in the alphabet $\Sigma$ is recognizable if and only if there exists a finite state automaton $\langle S, \Sigma, \star, i, F\rangle$ such that for every $w \in \Sigma^*$, $w \in L$ if and only if $w * i \in F$. According to Example 3.8, we can extend the action $\star$ of $\Sigma^*$ on $S$ to obtain a $\mathcal{P}(\Sigma^*)$-module $\langle \mathcal{P}(S), *\rangle$. Now, $\{i\}$ and $F$ are in $\mathcal{P}(S)$. Further,

$$F/_* \{i\} = \{w \in \Sigma^* : w \star x \in F, \text{ for all } x \in \{i\}\} = \{w \in \Sigma^* : w \star i \in F\} = L.$$

Thus, the notion of recognizable language can be captured in terms of modules over residuated lattices. We have stablished the following proposition, which also suggests the definition of a recognizable element in a residuated lattice.

**Proposition 4.1.** *A language $L$ in the alphabet $\Sigma$ is recognizable by a finite state automaton $\langle S, \Sigma, \star, i, F\rangle$ if and only if $L = F/_{*}\{i\}$, where $/_{*}$ is the residual of the $\mathcal{P}(\Sigma^*)$-module $\langle \mathcal{P}(S), *\rangle$.* □

**Definition 4.2.** An element $a$ of a residuated lattice $\mathbf{R}$ is said to be *recognizable* provided there exists a finite $\mathbf{R}$-module $\mathbb{P}$ and elements $i, t \in P$ such that $a = t/_{*}i$. If the preceding conditions are satisfied, we also say that $a$ is *recognized* by $\mathbb{P}$, $i$ and $t$.

*Remark* 4.3. If an element $a$ of a residuated lattice $\mathbf{R}$ is recognized by $\mathbb{P}$, $i$ and $t$, then we can always assume that $\mathbb{P}$ is cyclic and generated by $i$. Indeed, consider the submodule $\langle i\rangle_{\mathbb{P}}$ of $\mathbb{P}$ and let $t' = a * i$. Since $a = t/_{*}i$, we have that $t' = a * i \leqslant t$, and hence $a \leqslant (a * i)/_{*}i = t'/_{*}i \leqslant t/_{*}i = a$. It follows that $a$ is recognized by $\langle i\rangle_{\mathbb{P}}$, $i$, and $t'$. This corresponds to trimming the automaton, disregarding the unreachable states.

We prove next that the definition of a recognizable element in a residuated lattice is the correct abstraction of the concept of a recognizable language, in the sense that the recognizable languages in an alphabet $\Sigma$ are exactly the recognizable elements of the residuated lattice $\mathcal{P}(\Sigma^*)$.

**Proposition 4.4.** *If $L$ is a language in the alphabet $\Sigma$, then $L$ is recognizable as a language if and only if it is recognizable as an element of $\mathcal{P}(\Sigma^*)$.*

*Proof.* ($\Rightarrow$) In virtue of Proposition 4.1, we have that if $L$ is recognizable by the finite state automaton $\langle S, \Sigma, \star, i, F\rangle$, then $L$ is an element of $\mathcal{P}(\Sigma^*)$ recognized by the module $\mathbb{P} = \langle \mathcal{P}(S), *\rangle$ and the elements $\{i\}$ and $F$, and since $S$ is finite, then so is $\mathbb{P}$.

($\Leftarrow$) If $L$ is recognizable as an element of $\mathcal{P}(\Sigma^*)$, then there is a finite $\mathcal{P}(\Sigma^*)$-module $\mathbb{P} = \langle \mathbf{P}, *\rangle$ and two elements $i, t \in P$ such that $L = t/_{*}i$. We can define the map $\star : \Sigma^* \times P \to P$ by $w \star x = \{w\} * x$, which can readily be proven to be an action of $\Sigma^*$ on $P$. Furthermore,

$$\begin{aligned} L = t/_{*}i &= \max\{A \in \mathcal{P}(\Sigma^*) : A * i \leqslant t\} = \{w \in \Sigma^* : \{w\} * i \leqslant t\} = \{w \in \Sigma^* : w \star i \leqslant t\} \\ &= \{w \in \Sigma^* : w \star i \in {\downarrow} t\}, \end{aligned}$$

where ${\downarrow} t = \{x \in P : x \leqslant t\}$. Hence $\langle P, \Sigma, \star, i, {\downarrow} t\rangle$ is a finite state automaton and $L$ is recognized by it. □

Recognizability of elements in a residuated lattice is a notion invariant up to $\mathbf{R}$-isomorphisms. This is an immediate consequence of Proposition 3.12.

**Corollary 4.5.** *If $\mathbb{P}$ and $\mathbb{Q}$ are two isomorphic $\mathbf{R}$-modules, then an element $a \in R$ is recognized by $\mathbb{P}$ if and only if it is also recognized by $\mathbb{Q}$.* □

The next theorem is an intrinsic characterization of recognizable elements of a residuated lattice. This result in conjunction with Proposition 4.4 indicate that, in order to determine whether a language in an alphabet $\Sigma$ is recognizable or not, we do not need to look for finite automata or homomorphisms from $\Sigma^*$ onto finite monoids, but instead we can do it by just analyzing the structure of $\mathcal{P}(\Sigma^*)$ as a residuated lattice. Note that, given a residuated lattice $\mathbf{R}$ and an element $a \in R$, we can consider the module $\mathbb{R}^{\mathrm{d}}$, and as we saw $a = (a \cdot^{\mathrm{d}} a)/_{\mathrm{d}}a$. Therefore, if $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$ is finite, then it recognizes $a$. Moreover, by Proposition 3.20, we know that there is an isomorphism between $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$ and $\mathbb{R}_{\gamma_a}$. Therefore, by Corollary 4.5, if $\mathbb{R}_{\gamma_a}$

is finite, then $a$ is recognizable. We prove that this is not only a sufficient condition, but indeed a characterization.

**Theorem 4.6.** *Let* $\mathbf{R}$ *be a residuated lattice and* $a \in R$. *The following are equivalent:*

*(i)* *The element* $a$ *is recognizable.*
*(ii)* *There exists a closure operator* $\gamma$ *on* $\mathbb{R}$ *such that* $\gamma(a) = a$ *and* $R_\gamma$ *is finite.*
*(iii)* *The* $\mathbf{R}$*-module* $\mathbb{R}_{\gamma_a}$ *is finite. That is, the set* $\{x\backslash a : x \in R\}$ *is finite.*
*(iv)* *The* $\mathbf{R}$*-module* $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$ *is finite. That is, the set* $\{a/x : x \in R\}$ *is finite.*

*Proof.* We will show that (i) and (ii) are equivalent, that (ii) and (iii) are equivalent, and that (iii) and (iv) are equivalent as well.

(i) $\Rightarrow$ (ii): Suppose that $a$ is recognized by $\mathbb{P}$, $i$, and $t$. Without loss of generality, we assume that $\mathbb{P}$ is cyclic and $i$ is a generator of $\mathbb{P}$. Consider the closure operator $\gamma_i$ on $\mathbb{R}$. By Proposition 3.20, $\mathbb{R}_{\gamma_i} \cong \langle i\rangle_{\mathbb{P}} = \mathbb{P}$, and therefore $R_{\gamma_i}$ is finite. Let $b \in R$ such that $b * i = t$. Hence, $\gamma_i(b) = (b * i)/_{*}\, i = t/_{*}\, i = a$, and thus $a \in R_{\gamma_i}$.

(ii) $\Rightarrow$ (i): Suppose that there is a closure operator $\gamma$ on $\mathbb{R}$ such that $\mathbb{R}_\gamma$ is finite and $\gamma(a) = a$. Hence, by Proposition 3.22, $\gamma \leqslant \gamma_a$, and therefore $\gamma(e) \leqslant \gamma_a(e) = (a/e)\backslash a = a\backslash a$. Thus, $a \cdot \gamma(e) \leqslant a$, and it follows that $a \leqslant a/\gamma(e) \leqslant a/e = a$, because $e \leqslant \gamma(e)$. Therefore, $a = a/\gamma(e)$, which proves that $a$ is recognized by $\mathbb{R}_\gamma$, $\gamma(e)$, and $a$. Note that $\mathbb{R}_\gamma$ is cyclic with generator $\gamma(e)$.

(ii) $\Leftrightarrow$ (iii): In virtue of Proposition 3.22, if $\gamma$ is a closure operator on $\mathbb{R}$ such that $\gamma(a) = a$, then $\gamma \leqslant \gamma_a$, and therefore $R_{\gamma_a} \subseteq R_\gamma$. If furthermore $R_\gamma$ is finite, then it follows that $R_{\gamma_a}$ is also finite. The other implication is obvious, since $\gamma_a(a) = a$, as we saw in Proposition 3.22.

(iii) $\Leftrightarrow$ (iv): In virtue of Proposition 3.20, there is an isomorphism between $\mathbb{R}_{\gamma_a}$ and $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$, and therefore a bijection between their universes. We saw in Proposition 3.22 that $R_{\gamma_a} = \{x\backslash a : x \in R\}$. Finally, notice that $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}} = \{x \cdot^{\mathrm{d}} a : x \in R\} = \{a/x : x \in R\}$. □

*Remark* 4.7. Modules over residuated lattices, as defined above, are sometimes called *left-modules*, since the action of the residuated lattice is on the left, and thus the notion of a recognizable element of a residuated lattice could more accurately be called *left-recognizable*. Analogously, one could define *right-modules*, in which the action of the residuated lattice is on the right, and thereby obtain a notion of a *right-recognizable* element. However, the equivalences of the previous theorem establish that the notions of a left-recognizable and right-recognizable element coincide.

Last theorem is a generalization of Myhill's Theorem [22]. Given a language $L$ over an alphabet $\Sigma$, we can define its *syntactic congruence*, which is actually the congruence on the monoid $\Sigma^*$, by

$$w_1 \approx_L w_2 \iff \text{for all } u, v \in \Sigma^*,\ (uw_1v \in L \iff uw_2v \in L).$$

This is the largest monoid congruence on $\Sigma^*$ that *saturates* $L$, that is, such that it does not relate words in $L$ with words outside $L$. We could also define the right-congruence on $\Sigma^*$ by

$$w_1 \sim_L w_2 \iff \text{for any } v \in \Sigma^*,\ (w_1v \in L \iff w_2v \in L).$$

Myhill's Theorem characterizes the recognizable languages as those for which both $\approx_L$ and $\sim_L$ are of finite index (see [19]).

**Theorem 4.8.** *For a language* $L$ *over an alphabet* $\Sigma$, *the following are equivalent:*

*(i)* $L$ *is recognizable.*

*(ii)* $\approx_L$ *is of finite index, that is, the quotient* $\Sigma^*/{\approx_L}$ *is finite.*
*(iii)* $\sim_L$ *is of finite index, that is, the quotient* $\Sigma^*/{\sim_L}$ *is finite.*

This can be readily proved to be a consequence of Theorem 4.6. We present now a few examples to illustrate the previous discussion.

*Example* 4.9. Let $\mathbf{R} = \langle \mathbb{N}\cup\{\infty\}, \wedge, \vee, \cdot, \backslash, /, 1\rangle$, where $\cdot$ is the usual multiplication in the set $R = \mathbb{N}\cup\{\infty\}$ of extended natural numbers, ordered as usual, and in which $\infty\cdot x = x\cdot\infty = \infty$ if $x \neq 0$, and $\infty\cdot 0 = 0\cdot\infty = 0$. One can verify that this is a residuated lattice and for any $a \in \mathbb{N}$, $x\backslash a \in \{0, 1, 2, \ldots, a\} \cup \{\infty\}$ and $x\backslash\infty = \infty$, for every $x \in R$. Therefore, every element of $\mathbf{R}$ is recognizable.

*Example* 4.10. Let $\mathbb{Z}$ be the residuated lattice of all integers under the usual addition. Then, no element of $\mathbb{Z}$ is recognizable. In fact, no element of any non-trivial $\ell$-group is recognizable. This is so because a non-trivial $\ell$-group is infinite, and for a fixed element $a$ of an $\ell$-group $\mathbf{G}$, $\{x\backslash a : x \in G\} = \{x^{-1}a : x \in G\} = G$.

*Example* 4.11. Consider[3] the residuated lattice $\mathcal{P}(\langle\mathbb{N}, +, 0\rangle) = \langle\mathcal{P}(\mathbb{N}), \cap, \cup, +, \backslash, /, \{0\}\rangle$, in which the monoidal operation is defined as follows: $A + B = \{a + b : a \in A, b \in B\}$ for every $A, B \in \mathcal{P}(\mathbb{N})$. All finite members of $\mathcal{P}(\langle\mathbb{N}, +, 0\rangle)$ are recognizable, and so are some of its infinite members.

Let $A \in \mathcal{P}(\mathbb{N})$ be finite and nonempty, and $X \in \mathcal{P}(\mathbb{N})$ an arbitrary element. It follows from the inequality $X + (X\backslash A) \subseteq A$, that if $X \neq \emptyset$, then $X\backslash A \subseteq [0, \max A]$, and $\emptyset\backslash A = \mathbb{N}$. Therefore, $\{X\backslash A : X \in \mathcal{P}(\mathbb{N})\} \subseteq \mathcal{P}([0, \max A]) \cup \{\mathbb{N}\}$. This inequality shows that $\{X\backslash A : X \in \mathcal{P}(\mathbb{N})\}$ is finite, and hence $A$ is recognizable. Furtheremore, for every $X \in \mathcal{P}(\mathbb{N})$, $X\backslash\emptyset = \emptyset$, and therefore $\emptyset$ is also recognizable.

If $E$ and $O$ are the sets of even and odd numbers, respectively, then $\{X\backslash E : X \in \mathcal{P}(\mathbb{N})\} = \{\emptyset, E, O, \mathbb{N}\}$. Therefore, $E$ is recognizable. Notice that this also proves that $\emptyset$, $O$, and $\mathbb{N}$ are recognizable. We will see in Example 5.4 that all cofinite sets, which are infinite, are also recognizable.

Finally, we give an example of an infinite set which is not recognizable. Let $T = \{n(n+1)/2 : n \in \mathbb{N}\}$ the set of triangular numbers. Then, for every $n \in \mathbb{N}$, $\{n\}\backslash T = \{t - n : t \in T, t \geqslant n\}$. Now, one can verify that $\{n\}\backslash T \neq \{m\}\backslash T$ for $n \neq m$ (just considering the two first elements of each one of these sets), and thus $T$ is not recognizable.

In the remainder of this section we investigate how recognizable elements are affected by certain special maps between residuated lattices. Given two residuated lattices, $\mathbf{R}'$ and $\mathbf{R}$, a residuated monoidal homomorphism $\varphi : \mathbf{R}' \to \mathbf{R}$, and an $\mathbf{R}$-module $\mathbb{P} = \langle\mathbf{P}, *\rangle$, one can define an $\mathbf{R}'$-module $\mathbb{P}' = \langle\mathbf{P}, *'\rangle$ in the following way: for every $a \in R'$, and every $x \in P$, let $a *' x = \varphi(a) * x$. It is easy to see that for all $x \in P$, and $a, b \in R'$ $a *' (b *' x) = (a \cdot' b) *' x$, and $e *' x = x$, since $\varphi$ preserves products and the neutral element. To show is that $*' : \mathbf{R}' \times \mathbf{P} \to \mathbf{P}$ is residuated in both coordinates, note that if $\varphi^+$ is the residual of $\varphi$, then for all $a \in R'$ and $x, y \in P$ we have:

$$a *' x \leqslant y \;\Leftrightarrow\; \varphi(a) * x \leqslant y \;\Leftrightarrow\; \varphi(a) \leqslant y/_* x \;\Leftrightarrow\; a \leqslant \varphi^+(y/_* x) \;\Leftrightarrow\; x \leqslant \varphi(a)\backslash_* y.$$

Thus, the residuals of $*'$ are given by $y/_{*'} x = \varphi^+(y/_* x)$ and $a\backslash_{*'} y = \varphi(a)\backslash_* y$.

**Definition 4.12.** Given two residuated lattices, $\mathbf{R}'$ and $\mathbf{R}$, an $\mathbf{R}$-module $\mathbb{P} = \langle\mathbf{P}, *\rangle$, and a residuated monoidal homomorphism $\varphi : \mathbf{R}' \to \mathbf{R}$, we say that the $\mathbf{R}'$-module $\mathbb{P}' = \langle\mathbf{P}, *'\rangle$ is obtained from $\mathbb{P}$ and $\varphi$ by *restriction of scalars*.

[3]This example was suggested by N. Galatos.

**Proposition 4.13.** *Let $\varphi : \mathbf{R}' \to \mathbf{R}$ be a residuated monoidal homomorphism, $\mathbb{P}$ an $\mathbf{R}$-module, $\mathbb{P}'$ the $\mathbf{R}'$-module be obtained by restriction of scalars from $\mathbb{P}$ and $\varphi$, and let $a' \in R'$, $a \in R$ be arbitrary elements. If $\mathbb{P}$ recognizes $a$, then $\mathbb{P}'$ recognizes $\varphi^+(a)$. Moreover, if $\varphi$ is injective and $\mathbb{P}$ recognizes $\varphi(a')$, then $\mathbb{P}'$ recognizes $a'$; and if $\varphi$ is surjective and $\mathbb{P}'$ recognizes $a'$, then $\mathbb{P}$ recognizes $\varphi(a')$.*

*Proof.* In order to prove the first part, notice that if $a \in R$ is recognized by $\mathbb{P}$, and $t, i \in P$, then $a = t/_{*} i$ and then $\varphi^+(a) = \varphi^+(t/_{*} i) = t/_{*'} i$. For the second part, we recall that if $\varphi$ is residuated and injective, then its residual $\varphi^+$ is a left inverse, that is, $\varphi^+\varphi(a') = a'$, for every $a' \in R'$. Thus, if $\varphi(a')$ is recognized by $\mathbb{P}$ and $t, i \in P$, then $\varphi(a') = t/_{*} i$, whence we obtain $a' = \varphi^+\varphi(a') = \varphi^+(t/_{*} i) = t/_{*'} i$. Analogously, if $\varphi$ is surjective, then $\varphi^+$ is a right inverse, and if $a'$ is recognized by $\mathbb{P}'$ and $t, i \in P$, then $a' = t/_{*'} i = \varphi^+(t/_{*} i)$, whence we obtain $\varphi(a') = \varphi\varphi^+(t/_{*} i) = t/_{*} i$. □

*Remark* 4.14. In view of the preceding proposition, recognizability is a notion invariant under isomorphisms of residuated lattices. That is, if $\varphi$ is an isomorphism between $\mathbf{R}'$ and $\mathbf{R}$, and $a' \in R$, then $a'$ is recognizable if and only if $\varphi(a')$ is recognizable.

## 5. Regular elements and Boolean-recognizability

We will devote this section to the study of two problems, providing some interesting results that might lead to the eventual resolution of the problems. The first one is finding a Kleene-like characterization of the recognizable elements of a residuated lattice, while the second seeks a characterization of those elements of a residuated lattice that are recognizable by Boolean cyclic modules.

A celebrated result due to Kleene establishes that recognizable languages coincide with regular languages. The set of *regular* languages on an alphabet $\Sigma$ is the smallest set containing the full language $\Sigma^*$, the singleton languages $\{w\}$, for every $w \in \Sigma^*$, and is closed under finite intersections and unions, complex multiplication,[4] complementation, and the operation $(\,)^*$.

As was noted earlier, recognizable subsets of monoids, that is, recognizable elements of residuated lattices arising from monoids, have received considerable attention in the literature (see [10, 26]). While Kleene's theorem does not hold in this setting, McKnight's theorem asserts that every recognizable subset of a finitely generated monoid is rational (the equivalent notion of a regular language); see [26, Proposition 2.5].

A similar characterization for recognizable elements in a residuated lattice would most likely require an appropriate abstraction of the corresponding terms: whereas intersection, union, and complex multiplication correspond to the meet, join, and multiplication operations of the residuated lattice, respectively, it is not obvious what the proper abstraction of the other operations should be.

As we have already observed that not every residuated lattice has recognizable elements. We will see that Kleene's characterization strongly depends on the fact that the residuated lattice $\mathcal{P}(\Sigma^*)$ is of the form $\mathcal{P}(\mathbf{M})$, and even on the monoidal properties of $\Sigma^*$. Nevertheless, we can prove that whenever a residuated lattice has recognizable elements, the set they form is closed under some operations.

**Proposition 5.1.** *A residuated lattice $\mathbf{R}$ has recognizable elements if and only if it has a top element $\top$, in which case the set of recognizable elements of $\mathbf{R}$ contains $\top$, and is closed under (finite) meets and residuation by arbitrary elements. In other words, given*

[4] See Example 3.6.

*two recognizable elements $a, b \in R$ and arbitrary $c \in R$, the elements $a \wedge b, c\backslash a$ and $a/c$ are recognizable.*

*Proof.* If a residuated lattice $\mathbf{R}$ does not have a top element, then it is easy to see that every closure operator $\gamma$ on $\mathbf{R}$ would have an infinite associated closed system $R_\gamma$, because of the expansiveness of $\gamma$, and therefore for every $a \in R$, $R_{\gamma_a}$ would be infinite, and hence no element of $\mathbf{R}$ would be recognizable. On the other hand, if $\mathbf{R}$ has a top element $\top$, then from $x \cdot \top \leqslant \top$, which is true for every $x \in R$, one could derive that $\top \leqslant x\backslash\top$, and therefore $R_{\gamma_\top} = \{\top\}$, which shows that $\top$ is recognizable.

Note that $x\backslash(a \wedge b) = x\backslash a \wedge x\backslash b$, and hence $R_{\gamma_{a\wedge b}} = \{x\backslash(a \wedge b) : x \in R\} \subseteq \{s \wedge t : s \in R_{\gamma_a},\ t \in R_{\gamma_b}\}$. Since $R_{\gamma_a}$ and $R_{\gamma_b}$ are finite by hypotheses, so is $R_{\gamma_{a\wedge b}}$, and therefore $a \wedge b$ is recognizable.

In order to prove that $c\backslash a$ is recognizable, just notice that $c\backslash a \in R_{\gamma_a}$, and therefore $\gamma_a(c\backslash a) = c\backslash a$ and $R_{\gamma_a}$ is finite, whence in virtue of Theorem 4.6, $c\backslash a$ is recognizable.

Finally, in order to prove that $a/c$ is recognizable, consider the inclusion $\langle a/c\rangle_{\mathbb{R}^{\mathrm{d}}} = \{(a/c)/x : x \in R\} = \{a/(xc) : x \in R\} \subseteq \{a/x : x \in R\} = \langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$, and since $\langle a\rangle_{\mathbb{R}^{\mathrm{d}}}$ is finite, so is $\langle a/c\rangle_{\mathbb{R}^{\mathrm{d}}}$, what shows that $a/c$ is recognizable. ☐

*Remark* 5.2. Proposition 5.1 gives another reason why no element of a non-trivial $\ell$-group is recognizable (see Example 4.10), since such an algebra is unbounded.

There is a very straightforward argument why the complement of a recognizable language $L$ over an alphabet $\Sigma$ is recognizable: if $L$ is recognized by $\langle S, \Sigma, \star\rangle$ with initial state $i$ and set of final states $F$, then $L$ is the set of all words $w \in \Sigma^*$ such that $w \star i \in F$, and therefore the complement $L'$ of $L$ is the set of all words $w \in \Sigma^*$ such that $w \star i \notin F$, that is to say, $L'$ is recognized by the same automaton $\langle S, \Sigma, \star\rangle$ with the same initial state $i$ and set of final states $F'$, the complement of $F$. Nevertheless, the modules $\langle \mathcal{P}(S), *\rangle_{\gamma_L}$ and $\langle \mathcal{P}(S), *\rangle_{\gamma_{L'}}$ might look very different. Actually, the property that the set of recognizable elements is closed under complementation is true for every residuated lattice that arises as in Example 3.6, see for example [10, 26]. For the sake of completeness, we provide a short proof of this result.

**Proposition 5.3.** *Let $\mathbf{M}$ be a monoid. Then, the set of recognizable elements of the residuated lattice $\mathcal{P}(\mathbf{M})$ is closed under complementation and under (finite) unions.*

*Proof.* First of all notice that, for every $A \in \mathcal{P}(M)$, $\emptyset\backslash A = M$, and for every $X \neq \emptyset$,

$$X\backslash A = \Big(\bigcup_{x\in X}\{x\}\Big)\backslash A = \bigcap_{x\in X}\big(\{x\}\backslash A\big).$$

Thus, it is clear that $\mathcal{P}(M)_{\gamma_A} = \{X\backslash A : X \in \mathcal{P}(M)\}$ is finite if and only if $\big\{\{x\}\backslash A : x \in M\big\}$ is finite. Now, for every $x \in M$, we have:

$$\{x\}\backslash A = \{y \in M : xy \in A\} = \{y \in M : xy \notin A\}' = \{y \in M : xy \in A'\}' = \big(\{x\}\backslash A'\big)'$$

Hence, $\{x\}\backslash A' = \big(\{x\}\backslash A\big)'$, whence by Theorem 4.6 it follows that if $A$ is recognizable, then so is $A'$. Finally, we only need to notice that the empty union is $\emptyset = M'$, which is therefore recognizable, and that if $A$ and $B$ are recognizable, then $A \cup B = (A' \cap B')'$ which is also recognizable in virtue of Proposition 5.1. ☐

*Example* 5.4. Let consider the residuated lattice $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ of Example 4.11. Since we proved that its finite elements are recognizable, its cofinite elements are also recognizable. As we mentioned before, the lattices $\mathcal{P}(\mathbb{N})_{\gamma_A}$ and $\mathcal{P}(\mathbb{N})_{\gamma_{A'}}$ might look very different. For instance, whereas (as we will see) the lattice of $\mathcal{P}(\mathbb{N})_{\gamma_{\{n\}}}$ is that of Diagram (4), the lattice of $\mathcal{P}(\mathbb{N})_{\gamma_{\{n\}'}}$ is isomorphic to the Boolean algebra $\mathcal{P}(\{0,1,\ldots,n\})$.

The fact that singletons $\{w\}$, for $w \in \Sigma^*$, are recognizable languages, not only depends (from our perspective) on the fact that the residuated lattice $\mathcal{P}(\Sigma^*)$ is of the form $\mathcal{P}(\mathbf{M})$, but also on the monoidal properties of $\Sigma^*$.

**Proposition 5.5.** *Let $\mathbf{M}$ be a monoid and $a \in M$ an arbitrary element, and let consider the residuated lattice $\mathcal{P}(\mathbf{M})$. If $a$ has a finite number of divisors then $\{a\}$ is recognizable. If $\mathbf{M}$ is cancellative and $\{a\}$ is recognizable, then $a$ has a finite number of divisors.*

*Proof.* As we showed in the proof of Proposition 5.3, for any $A \in \mathcal{P}(\mathbf{M})$, $A$ is recognizable if and only if $\big\{\{x\}\backslash A : x \in M\big\}$ is finite. Given $a, x \in M$, we have that

$$\{x\}\backslash\{a\} = \{y \in M : xy = a\}.$$

Therefore, this set is empty, except when $x$ is a divisor of $a$, whence it follows that if $a$ has a finite number of divisors, then it is recognizable. For the other implication, observe that, given two different divisors $x, x'$ of an element $a$, there exist two elements $y, y'$ such that $xy = a = x'y'$, and by the cancellativity of $\mathbf{M}$, we obtain $y \neq y'$. Therefore, the sets $\{x\}\backslash\{a\}$ and $\{x'\}\backslash\{a\}$ are different. Thus, if $a$ has an infinite number of divisors, $\{a\}$ is not recognizable. □

*Example* 5.6. If $\mathbf{M}$ is an infinite group, then no singleton of $\mathcal{P}(\mathbf{M})$ is recognizable. On the other hand, notice that 0 has an infinite number of divisors in $\langle\mathbb{N},\cdot,1\rangle$, but for every $x \in \mathbb{N}$, $\{x\}\backslash\{0\} = \{y \in \mathbb{N} : xy = 0\} = \{0\}$ if $x \neq 0$, and $\{0\}\backslash\{0\} = \mathbb{N}$. Therefore, $\{0\}$ is recognizable, and since all the positive natural numbers have only a finite number of divisors, it follows that every singleton of $\mathcal{P}(\langle\mathbb{N},\cdot,1\rangle)$ is recognizable.

*Remark* 5.7. As we mentioned in the introduction, there is another well studied generalization of recognizable sets, namely recognizable series (see [26, Ch. III] or [3]). One can replace $\mathcal{P}(\Sigma^*)$ by a semiring $K\langle\langle M\rangle\rangle$ of formal series over a monoid $M$ with coefficients in a semiring $K$ (see [26, Ch. III] or [3]). In fact, one has to assume that every element of $M$ has a finite number of divisors, which is related to the assumption in Proposition 5.5. Unlike recognizable elements in residuated lattices, recognizable series have Kleene's characterization due to Schützenberger (see [3]).

We next direct our attention to another problem that does not focus on the structure of the set of recognizable elements of a residuated lattice, but on the structure of the modules that recognize them. We notice that, by virtue of Theorem 2.5, given a recognizable language $L$, there exists a surjective homomorphism of monoids $\varphi : \Sigma^* \to \mathbf{M}$ and a set $T \subseteq M$ such that $\mathbf{M}$ is finite and $L = \varphi^{-1}(T)$. As we mentioned before, this map extends to a residuated map $\overline{\varphi} : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\mathbf{M})$, which is also a homomorphism of monoids and whose residual is $\varphi^{-1}$. We can consider the residuated lattice $\mathcal{P}(\mathbf{M})$ as a module $\mathbb{P}$ over itself, and since it is finite (because $\mathbf{M}$ is finite), each of its elements is recognizable. In particular $T$ is recognizable by $\mathbb{P}$ which implies, in light of Proposition 4.13, that $L = \varphi^{-1}(T)$ is recognized by $\mathbb{P}'$, the $\mathcal{P}(\Sigma^*)$-module obtained by $\mathbb{P}$ and $\overline{\varphi}$ by restriction of scalars. Note that since $\mathbb{P}$ is a cyclic $\mathcal{P}(\mathbf{M})$-module and $\overline{\varphi}$ is surjective, $\mathbb{P}'$ is also cyclic, and moreover the lattice reduct of $\mathbb{P}'$ is a Boolean algebra.

**Definition 5.8.** Given a residuated lattice $\mathbf{R}$, we say that it is *Boolean* if its lattice reduct is so. Also, given an $\mathbf{R}$-module $\mathbb{P}$, we say that $\mathbb{P}$ is *Boolean*[5] if its lattice reduct is so.

Thus, we have stablished the following result.

**Proposition 5.9.** *Every recognizable language on any alphabet is recognized by a Boolean cyclic module.* □

The questions that follow are completely natural: are all the recognizable elements of residuated lattices recognized by Boolean cyclic modules? If not, which elements are? And in particular, given an element $a$ of a residuated lattice $\mathbf{R}$, when is $\mathbf{R}_{\gamma_a}$ a Boolean module? As we will see, not every recognizable element in a residuated lattice can be recognized by a Boolean cyclic module (see Example 5.25). However, let us analyze a bit further the case of recognizable languages. Since $\overline{\varphi}$ is onto $\mathcal{P}(\mathbf{M})$, because $\varphi : \Sigma^* \to \mathbf{M}$ is surjective, we have that $\gamma = \varphi^{-1}\overline{\varphi}$ is a closure operator on $\mathcal{P}(\Sigma^*)$ such that $\mathcal{P}(\Sigma^*)_\gamma \cong \mathcal{P}(M)$, as lattices. We prove now that this closure operator satisfies the equation

$$\gamma(a \wedge \gamma(b)) = \gamma(a) \wedge \gamma(b), \tag{3}$$

which has interesting consequences, as we will see.

**Proposition 5.10.** *Given any map $\varphi : X \to Y$, and its extension to a residuated map $\overline{\varphi} : \mathcal{P}(X) \to \mathcal{P}(Y)$, for every $A, B \subseteq X$, we have*

$$\overline{\varphi}\big(\varphi^{-1}\overline{\varphi}(A) \cap B\big) = \overline{\varphi}(A) \cap \overline{\varphi}(B).$$

*Moreover, the closure operator $\gamma = \varphi^{-1}\overline{\varphi}$ satisfies equation* (3).

*Proof.* Since $\overline{\varphi} = \overline{\varphi}\varphi^{-1}\overline{\varphi}$, and $\overline{\varphi}(S \cap T) \subseteq \overline{\varphi}(S) \cap \overline{\varphi}(T)$, for any $S, T \subseteq X$, we get $\overline{\varphi}\big(\varphi^{-1}\overline{\varphi}(A) \cap B\big) \subseteq \overline{\varphi}(A) \cap \overline{\varphi}(B)$. Conversely, choose $y \in \overline{\varphi}(A) \cap \overline{\varphi}(B)$. Then there exist $a \in A$ and $b \in B$ such that $y = \varphi(a) = \varphi(b)$. Thus, $b \in \varphi^{-1}\overline{\varphi}(A)$, and therefore we get $y = \varphi(b) \in \overline{\varphi}\big(\varphi^{-1}\overline{\varphi}(A) \cap B\big)$. Hence, the equation holds. Finally, since $\varphi^{-1}$ is the residual of a residuated map, hence it preserves meets, and therefore we have

$$\varphi^{-1}\overline{\varphi}\big(\varphi^{-1}\overline{\varphi}(A) \cap B\big) = \varphi^{-1}\big(\overline{\varphi}(A) \cap \overline{\varphi}(B)\big) = \varphi^{-1}\overline{\varphi}(A) \cap \varphi^{-1}\overline{\varphi}(B),$$

that is, $\gamma$ satisfies equation (3). □

**Theorem 5.11.** *Let $\mathbf{B}$ be a Boolean algebra and $\gamma$ a closure operator on $\mathbf{B}$. If $\gamma$ satisfies equation* (3) *for all $a, b \in B$, then $\mathbf{B}_\gamma$ is also a Boolean algebra.*

*Proof.* Since $\mathbf{B}$ is a lattice, then $\mathbf{B}_\gamma$ also inherits a structure of lattice, where $x \wedge_\gamma y = x \wedge y$ and $x \vee_\gamma y = \gamma(x \vee y)$. We are going to show that $\mathbf{B}_\gamma$ is actually a bounded complemented distributive lattice.

(i) Bounds: Clearly, by the monotonicity of $\gamma$, $\gamma(\bot)$ and $\gamma(\top) = \top$ are the bottom and top elements of $\mathbf{B}_\gamma$.

(ii) Complements: For any element $x \in B_\gamma$, we will see that $\gamma(x')$ is its complement in $\mathbf{B}_\gamma$. Indeed, using equation (3):

$$x \wedge_\gamma \gamma(x') = \gamma(x) \wedge_\gamma \gamma(x') = \gamma(x) \wedge \gamma(x') = \gamma(\gamma(x) \wedge x') = \gamma(x \wedge x') = \gamma(\bot).$$

We also have:

$$x \vee_\gamma \gamma(x') = \gamma(x \vee \gamma(x')) \geqslant \gamma(x \vee x') = \gamma(\top) = \top.$$

[5]The reader may be familiar with the concept of a Boolean module as introduced by Brink in [6], but those are not exactly the same kind of structures, as Brink's Boolean modules are modules over relation algebras.

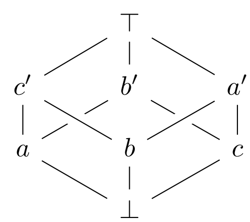
(iii) Distributivity: If $x, y, z \in \mathbf{B}_\gamma$, then using equation (3) we have:

$$\begin{aligned} x \wedge_\gamma (y \vee_\gamma z) &= x \wedge \gamma(y \vee z) = \gamma(x) \wedge \gamma(y \vee z) = \gamma(\gamma(x) \wedge (y \vee z)) = \gamma(x \wedge (y \vee z)) \\ &= \gamma((x \wedge y) \vee (x \wedge z)) = (x \wedge y) \vee_\gamma (x \wedge z) = (x \wedge_\gamma y) \vee_\gamma (x \wedge_\gamma z). \end{aligned}$$ □

*Remark* 5.12. Obviously, if $\mathbf{B}$ is a complete Boolean algebra, then $\mathbf{B}_\gamma$ is also a complete Boolean algebra.

The converse of Theorem 5.11 is not true. That is, equation (3) is a sufficient but not a necessary condition for $\mathbf{B}_\gamma$ to be a Boolean algebra. The following is an example of such a case.

*Example* 5.13. Let $\mathbf{B}$ be the Boolean algebra represented by the diagram:




The map $\gamma : B \to B$ defined by $\gamma(\bot) = \bot$, $\gamma(a) = a$, $\gamma(c) = c$, and $\gamma(x) = \top$, otherwise, is a closure operator and has a Boolean image, but $\gamma$ fails equation (3), because $\gamma(a' \wedge \gamma(a)) = \gamma(a' \wedge a) = \gamma(\bot) = \bot$, but $\gamma(a') \wedge \gamma(a) = \top \wedge a = a$.

It is easy to see that the Boolean image $\mathbf{B}_\gamma$ is not necessarily a Boolean subalgebra of $\mathbf{B}$, just because $\gamma(\bot)$ may not be $\bot$. But, the following proposition states that, under the assumption of equation (3) for a closure operator $\gamma$ on a Boolean algebra $\mathbf{B}$, $\gamma(\bot) = \bot$ is an equivalent condition to $\mathbf{B}_\gamma$ being a Boolean subalgebra of $\mathbf{B}$.

**Corollary 5.14.** *Let $\gamma$ be a closure operator on a Boolean algebra $\mathbf{B}$ satisfying equation* (3)*. Then, the following are equivalent.*

*(i)* $\mathbf{B}_\gamma$ *is a Boolean subalgebra of* $\mathbf{B}$*.*
*(ii)* $\gamma(\bot) = \bot$*.*
*(iii)* $\gamma(a) = \bot$ *if and only if* $a = \bot$*.*

*Proof.* (i) $\Leftrightarrow$ (ii): If $\mathbf{B}_\gamma$ is a Boolean subalgebra of $\mathbf{B}$, then clearly $\gamma(\bot) = \bot$. For the reverse direction, assume $\gamma(\bot) = \bot$. It is enough to show that $\mathbf{B}_\gamma$ is closed under complementation. If it is true, then $\mathbf{B}_\gamma$ is also closed under finite joins, because $x \vee y = (x' \wedge y')' \in B_\gamma$, for any $x, y \in B_\gamma$. Therefore, we need to show that for any $x \in B_\gamma$, $\gamma(x') = x'$. But, by the part (ii) of the proof of Theorem 5.11, and by our hypotheses, we know that $x \wedge \gamma(x') = x \wedge_\gamma \gamma(x') = \gamma(\bot) = \bot$, and therefore $\gamma(x') \leqslant x'$, because in a Boolean algebra $a \wedge b = \bot$ implies $b \leqslant a'$. It follows that $\gamma(x') = x'$, as we wanted to prove.

(ii) $\Leftrightarrow$ (iii): This equivalence is trivial. □

Equation (3) has an interesting application is the setting of frames. It is well known that given a nucleus $\gamma$ on a frame $\mathbf{F}$, the image $\mathbf{F}_\gamma$ is also a frame.[6] But, the following proposition states that equation (3), which is satisfied by every nucleus, is sufficient to prove this result.

[6]This is because if $\gamma$ is a nucleus on a frame $\mathbf{F}$, then $\gamma$ is a closure operator on $\mathbb{F}$, and therefore $\mathbb{F}_\gamma = \langle \mathbf{F}_\gamma, \wedge \rangle$ is an $\mathbf{F}$-module. In particular, $\wedge$ is residuated on both coordinates on $\mathbf{F}_\gamma$, and therefore it distributes with respect to arbitrary joins.

**Proposition 5.15.** *Let $\gamma$ be a closure operator on a frame* $\mathbf{F}$ *satisfying equation* (3)*. Then, the image* $\mathbf{F}_\gamma$ *is also a frame.*

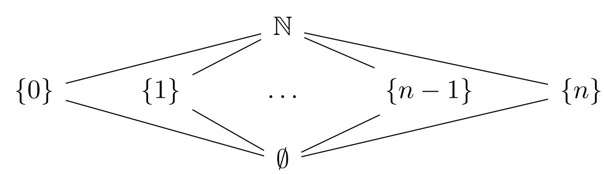
*Proof.* We know that the image $\mathbf{F}_\gamma$ is a complete lattice and the meets in $\mathbf{F}$ and in $\mathbf{F}_\gamma$ coincide. We need to show that for any $x \in F_\gamma$ and any family $\{x_i : i \in I\} \subseteq F_\gamma$, $x \wedge \bigvee_I^{\mathbf{F}_\gamma} x_i = \bigvee_I^{\mathbf{F}_\gamma}(x \wedge x_i)$. We have:

$$x \wedge \bigvee_I^{\mathbf{F}_\gamma} x_i = x \wedge \gamma\Big(\bigvee_I^{\mathbf{F}} x_i\Big) = \gamma(x) \wedge \gamma\Big(\bigvee_I^{\mathbf{F}} x_i\Big) = \gamma\Big(\gamma(x) \wedge \bigvee_I^{\mathbf{F}} x_i\Big) = \gamma\Big(x \wedge \bigvee_I^{\mathbf{F}} x_i\Big)$$
$$= \gamma\Big(\bigvee_I^{\mathbf{F}}(x \wedge x_i)\Big) = \bigvee_I^{\mathbf{F}_\gamma}(x \wedge x_i). \qquad \square$$

We next apply equation (3) to closure operators of the form $\gamma_a$ for a recognizable element $a$ of a Boolean residuated lattice. As the example below shows, such operators may not satisfy equation (3).

*Example* 5.16. Consider the residuated lattice $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ of Example 4.11. For any pair of numbers $n, m \in \mathbb{N}$, its easy to compute $\{n\}\backslash\{m\} = \{m\}/\{n\} = \{m - n\}$ if $n \leqslant m$, and otherwise $\{n\}\backslash\{m\} = \{m\}/\{n\} = \emptyset$. Therefore $\gamma_{\{n\}}(\{m\}) = (\{n\}/\{m\})\backslash\{n\} = \{n - m\}\backslash\{n\} = \{n - (n - m)\} = \{m\}$ if $m \leqslant n$, and $\gamma_{\{n\}}(\{m\}) = (\{n\}/\{m\})\backslash\{n\} = \emptyset\backslash\{n\} = \mathbb{N}$ otherwise. Moreover, its not difficult to see that if $A \subseteq \mathbb{N}$ contains more than one number, then $\gamma_{\{n\}}(A) = \mathbb{N}$. Therefore, the closure system associated to $\gamma_{\{n\}}$ is:

(4)




which evidently is a Boolean algebra only when $n = 1$. Hence, by Theorem 5.11, for every $n \neq 1$, the closure operator $\gamma_{\{n\}}$ on $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ fails equation (3).

Given a frame $\mathbf{F}$, the closure operators on $\mathbb{F}$ of the form $\gamma_a$, for some $a \in F$, are special. As we see in the following theorem (see also [11]), they produce Boolean algebras.

**Theorem 5.17.** *Given a frame* $\mathbf{F}$ *and an arbitrary element* $a \in F$*, the image* $\mathbf{F}_{\gamma_a}$ *is a Boolean algebra.*

*Proof.* Since $\gamma_a$ is a closure operator on $\mathbb{F}$, $\gamma_a$ is a nucleus on $\mathbf{F}$, by virtue of Proposition 3.18. It follows that $\gamma_a$ satisfies equation (3), and thus by Proposition 5.15, $\mathbf{F}_{\gamma_a}$ is also a frame. It remains to prove that $\mathbf{F}_{\gamma_a}$ is complemented. First, notice that for every $a \in F$, $\bot \to a = \top$, and since $\top$ is the neutral element of $\mathbf{F}$ as a residuated lattice, we also have that $\top \to a = a$ and $a \to a = \top$. Therefore the bottom and top elements of $\mathbf{F}_{\gamma_a}$ are $\gamma(\bot) = a$ and $\top$, respectively. We prove now that for every $x \in F_{\gamma_a}$, $x \to a$ is the complement of $x$ in $\mathbf{F}_{\gamma_a}$.

Let $x \in F_{\gamma_a}$ be an arbitrary element. Since $x \wedge (x \to a) \leqslant a$, and $a$ is the bottom element of $\mathbf{F}_{\gamma_a}$ and hence $a \leqslant x \wedge (x \to a)$, we have that $x \wedge (x \to a) = a$. Thus

$$x \vee_{\gamma_a} (x \to a) = \gamma_a(x \vee (x \to a)) = \big((x \vee (x \to a)) \to a\big) \to a = \big((x \to a) \wedge ((x \to a) \to a)\big) \to a$$
$$= \big((x \to a) \wedge \gamma_a(x)\big) \to a = \big((x \to a) \wedge x\big) \to a = a \to a = \top. \qquad \square$$

We have proved that every recognizable element $a$ of a frame $\mathbf{F}$, is recognized by a Boolean cyclic module, namely $\mathbb{F}_{\gamma_a}$. On the other hand, we have seen that it is not true in general that for any residuated lattice $\mathbf{R}$ and any element $a \in R$, $\mathbb{R}_{\gamma_a}$ is a Boolean module, even if $\mathbf{R}$ is Boolean. We close this paper by providing sufficient conditions for a recognizable element of a Boolean residuated lattice to be recognized by a Boolean cyclic module. We first introduce some additional concepts (see [25]).

**Definition 5.18.** Let $\mathbf{R}$ be a residuated lattice and let $a$ an arbitrary element of $R$.

- $a$ is *cyclic* if $a/x = x\backslash a$, for every $x \in R$.
- $a$ is *two-sided* if $a\backslash a = a/a = \top$.
- $a$ is *semiprime* if for every $x \in R$, $x^2 \leqslant a \;\Rightarrow\; x \leqslant a$.
- $a$ is *localic* if it is cyclic, two-sided, and semiprime.

We center our attention on localic elements because of the following theorem (see [5]).

**Theorem 5.19.** *Let $\mathbf{R}$ be a complete residuated lattice. Then, there exists a residuated map $f : \mathbf{R} \to \mathbf{B}$ of $\mathbf{R}$ onto a Boolean algebra $\mathbf{B}$ if and only if $\mathbf{R}$ contains a localic element.* □

The characterizations of the following lemma will be used extensively.

**Lemma 5.20.** *Let $\mathbf{R}$ be a residuated lattice, and $a \in R$ an arbitrary element.*

*(i) $a$ is cyclic if and only if for every $x, y \in R$, $xy \leqslant a \;\Leftrightarrow\; yx \leqslant a$.*
*(ii) $a$ is two-sided if and only if for every $x \in R$, $xa \leqslant a$ and $ax \leqslant a$.*
*(iii) if $a$ is cyclic and two-sided, then it is semiprime if and only if for every $x \in R$, $x\backslash a = x^2\backslash a$.*

*Proof.* (i) This is an immediate consequence of residuation.

(ii) For an element $a \in R$, $a\backslash a = \top$ is equivalent to say that for every $x \in R$, $x \leqslant a\backslash a$, or which is the same as for every $x \in R$, $ax \leqslant a$; and analogously for $a/a = \top$ and for every $x \in R$, $xa \leqslant a$.

(iii) If for every $x \in R$, $x\backslash a = x^2\backslash a$, then it is easy to see that it is semiprime, because if $x^2 \leqslant a$, then $e \leqslant x^2\backslash a = x\backslash a$, and therefore $x \leqslant a$. In order to see the other implication suppose that $a$ is cyclic, two-sided and semiprime. Thus, if $y \leqslant x^2\backslash a$, then $x^2 y \leqslant a$, and therefore $xyx \leqslant a$, by the cyclicity of $a$, which implies that $(yx)^2 = (yx)(yx) \leqslant y(xyx) \leqslant ya \leqslant a$, because $a$ is two-sided. And therefore, $yx \leqslant a$, because it is semiprime, whence we obtain $xy \leqslant a$, again by cyclicity, and hence $y \leqslant x\backslash a$. Now, if $y \leqslant x\backslash a$, then $xy \leqslant a$, and therefore $x^2 y = x(xy) \leqslant xa \leqslant a$, because $a$ is two-sided, and hence $y \leqslant x^2\backslash a$. Thus, we have proved that $y \leqslant x^2\backslash a$ if and only if $y \leqslant x\backslash a$, which implies that $x\backslash a = x^2\backslash a$. □

*Remark* 5.21. Notice that if $a$ is a cyclic element of a residuated lattice, then $\gamma_a$ is a nucleus of $\mathbf{R}$. First, since $\gamma_a$ is a structural closure operator on $\mathbb{R}$, hence for every $x, y \in R$, $x\gamma_a(y) \leqslant \gamma_a(xy)$. Now, for $x, y \in R$, using the cyclicity of $a$ we have

$$y(a/(xy))\gamma_a(x) = y((xy)\backslash a)\gamma_a(x) = y(y\backslash(x\backslash a))\gamma_a(x) \leqslant (x\backslash a)\gamma_a(x) = (a/x)\gamma_a(x) \leqslant a.$$

Thus, using again the cyclicity of $a$, we have $(a/(xy))\gamma_a(x)y \leqslant a$, whence it follows that $\gamma_a(x)y \leqslant (a/(xy))\backslash a = \gamma_a(xy)$. Now, using this inequality and the structurallity of $\gamma_a$ we obtain:

$$\gamma_a(x)\gamma_a(y) \leqslant \gamma_a(x\gamma_a(y)) \leqslant \gamma_a(\gamma_a(xy)) = \gamma_a(xy).$$

**Theorem 5.22.** *Let $\mathbf{R}$ be a complete residuated lattice. For every localic element $a \in R$, the module $\mathbb{R}_{\gamma_a}$ is Boolean.*

*Proof.* To begin with, we are going to show that $\cdot_{\gamma_a}$ and $\wedge$ coincide in $\mathbb{R}_{\gamma_a}$. Indeed $x, y \in R_{\gamma_a}$. We claim that $\gamma_a(x \cdot y) = x \wedge y$. Notice that since $a$ is two-sided, we have $(a/x)(xy) = ((a/x)x)y \leqslant ay \leqslant a$. Thus, $xy \leqslant (a/x)\backslash a = \gamma_a(x) = x$. Analogously, and using the cyclicity of $a$, we have $(xy)(a/y) = (xy)(y\backslash a) = x(y(y\backslash a)) \leqslant xa \leqslant a$, and again by the cyclicity of $a$, it follows that $(a/y)(xy) \leqslant a$. Thus, $xy \leqslant (a/y)\backslash a = \gamma_a(y) = y$. Hence, $xy \leqslant x \wedge y$, whence we have $\gamma_a(xy) \leqslant x \wedge y$.

In order to prove the other inequality, suppose that $t \leqslant x \wedge y$. Since $t \leqslant x$, hence $x\backslash a \leqslant t\backslash a$, and thus $t(x\backslash a) \leqslant a$, and by the cyclicity of $a$, $(x\backslash a)t \leqslant a$. Now, since $t \leqslant y$, hence $t((xy)\backslash a) = t(y\backslash(x\backslash a)) \leqslant y(y\backslash(x\backslash a)) \leqslant x\backslash a$. Therefore, we have

$$\begin{aligned}\big(((xy)\backslash a)t\big)^2 &= \big(((xy)\backslash a)t\big)\big(((xy)\backslash a)t\big) = ((xy)\backslash a)\big(\big(t((xy)\backslash a)\big)t\big) \leqslant ((xy)\backslash a)\big((x\backslash a)t\big)\\ &\leqslant ((xy)\backslash a)a \leqslant a.\end{aligned}$$

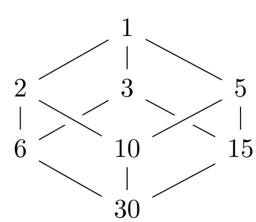
Since $a$ is semiprime, $((xy)\backslash a)t \leqslant a$, and hence by the cyclicity of $a$, $(a/(xy))t \leqslant a$, which implies $t \leqslant (a/(xy))\backslash a = \gamma_a(xy)$. Thus we have proved that, in particular, $x \wedge y \leqslant \gamma_a(xy)$, as we wanted.

Therefore we have that indeed $\wedge$ and $\cdot_{\gamma_a}$ coincide in $\mathbb{R}_{\gamma_a}$, and since $\cdot_{\gamma_a}$ is residuated in both coordinates, so is $\wedge$, which means that $\wedge$ distributes over arbitrary joins of $\mathbf{R}_{\gamma_a}$. Therefore $\mathbf{R}_{\gamma_a}$ is actually a frame that contains the element $a$. Now, since the residuals of $\cdot_{\gamma_a}$ are the restictions of the residuals of $\cdot$, the corresponding restriction of $\gamma_a$ is a closure operator on $\mathbb{R}_{\gamma_a}$, which is actually the identity. By virtue of Theorem 5.17, $\mathbf{R}_{\gamma_a}$ is a Boolean algebra, and this completes the proof. ☐

The following is an immediate consequence of the preceding theorem.

**Corollary 5.23.** *Let* $\mathbf{R}$ *be a complete residuated lattice. Every recognizable localic element is recognized by a Boolean cyclic module.* ☐

*Example* 5.24. Consider the residuated lattice of the natural numbers $\mathbf{R} = \langle \mathbb{N}, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ where the order is given by division as follows: $n \leqslant m \;\Leftrightarrow\; m \mid n$. This is a complete commutative residuated lattice, and therefore we can denote both residuals by $\to$. Notice also that the bottom and the top elements of this residuated lattice are $\bot = 0$ and $\top = 1$. It is easy to see that $m \vee n = (m, n)$ the greatest common divisor of $m$ and $n$, and $n \to m = \frac{m}{(m,n)}$, if $m$ or $n$ are different from 0, and $0 \to 0 = 1$. Therefore, $R_{\gamma_0} = \{0, 1\}$, and $R_{\gamma_m}$ is the set of the divisors of $m$, if $m \neq 0$. Thus, every element of this complete residuated lattice is recognizable. Moreover, since it is commutative, every element is cyclic, and given any $a, x \in \mathbb{N}$, $a \mid ax = xa$, which means that $ax \leqslant a$ and $xa \leqslant a$, and therefore $a$ is two-sided. Furthermore, every square-free natural number is semiprime, and hence localic. Thus, by Corollary 5.23, every square-free natural number is recognized by a Boolean module. For example, the number 30 is recognized by the Boolean module $\mathbb{R}_{\gamma_{30}}$, whose Hasse diagram is:




The converse of Corollary 5.23 is not true in general, that is, there are elements of complete residuated lattices that are recognized by Boolean cyclic modules, but are not localic. As we saw in Example 5.16, $\{1\}$ is recognized by a four-element Boolean cyclic module. Nevertheless, $\{1\}$ is not two-sided, since $\{1\}\backslash\{1\} = \{0\} \neq \mathbb{N}$, and therefore it is not localic.

We end this section proving that not every recognizable element can be recognized by a Boolean cyclic module. The question whether any element of a residuated lattice can be recognized by a Boolean module, not necessarily cyclic, is still open.

*Example* 5.25. Consider the residuated lattice $\mathbf{R} = \langle \mathbb{N} \cup \{\infty\}, \wedge, \vee, \cdot, 1\rangle$ of Example 4.9. As we saw, every element of this residuated lattice is recognizable. But the cyclic $\mathbf{R}$-modules, are all chains, because $\mathbf{R}$ itself is a chain and the cyclic $\mathbf{R}$-modules are always of the form $\mathbb{R}_\gamma$ for some closure operator $\gamma$ on $\mathbb{R}$. Given an element $n \in R$, and a closure operator $\gamma$ on $\mathbb{R}$ that fixes $n$, we have that $\gamma \leqslant \gamma_n$ by Proposition 3.22, and therefore $\mathbf{R}_{\gamma_n} \subseteq \mathbf{R}_\gamma$. Therefore, all we have to do is showing an element $n \in R$ such that $R_{\gamma_n}$ has more that two elements. For instance, it is easy to see that $R_{\gamma_5} = \{0, 1, 2, 5, \infty\}$, and therefore there is no Boolean cyclic module that recognizes 5.

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