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Hulls of Ordered Algebras: Projectability, Strong Projectability and Lateral Completeness

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Abstract

There has been compelling evidence during the past decade that lattice-ordered groups (ℓ-groups) play a far more significant role in the study of algebras of logic than it had been previously anticipated. Their key role has emerged on two fronts: First, a number of research articles have established that some of the most prominent classes of algebras of logic may be viewed as ℓ-groups with a modal operator. Second, and perhaps more importantly, recent research has demonstrated that the foundations of the Conrad Program for ℓ-groups can be profitably extended to a much wider class of algebras, namely the variety of e-cyclic residuated lattices – that is, residuated lattices that satisfy the identity $x\backslash e \approx e/x$. Here, the term *Conrad Program* refers to Paul Conrad's approach to the study of ℓ-groups that analyzes the structure of individual or classes of ℓ-groups by primarily focusing on their lattices of convex ℓ-subgroups.

The present article, building on the aforementioned works, studies existence and uniqueness of the laterally complete, projectable and strongly projectable hulls of of e-cyclic residuated lattices. While these hulls first

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made their appearance in the context of functional analysis, and in particular the theory of Riesz spaces, their introduction into the study of algebras of logic adds new tools and techniques in the area and opens up possibilities for a deep exploration of their logical counterparts.

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There has been compelling evidence during the past decade that lattice-ordered groups (ℓ-groups) are of fundamental importance in the study of algebras of logic[1] – and that their role is likely to become even more crucual in the future. For example, a key result [40] in the theory of MV algebras is the categorical equivalence between the category of MV algebras and the category of unital Abelian ℓ-groups. Likewise, the non-commutative generalization of this result in [26] establishes a categorical equivalence between the category of pseudo-MV algebras and the category of unital ℓ-groups. Further, the generalization of these two results in [38] shows that one can view GMV algebras as ℓ-groups with a suitable modal operator. Likewise, the work in [38] offers a new paradigm for the study of various classes of cancellative residuated lattices by viewing these structures as ℓ-groups with a suitable modal operator (a conucleus).

In a different direction, articles [14] and [35] have demonstrated that large parts of the Conrad Program can be profitably extended to the much wider class of *e-cyclic* residuated lattices, that is, those satisfying the identity $x\backslash e \approx e/x$. The term *Conrad Program* traditionally refers to Paul Conrad's approach to the study of ℓ-groups, which analyzes the structure of individual ℓ-groups, or classes of ℓ-groups, by means of an overriding inquiry into the lattice-theoretic properties of their lattices of convex ℓ-subgroups. Conrad's papers [16–20, 22] in the 1960s pioneered this approach and extensively vouched for its usefulness. A survey of the most important consequences of

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[1] We use the term *algebra of logic* to refer to residuated lattices – algebraic counterparts of propositional substructural logics – and their reducts. Substructural logics are non-classical logics that are weaker than classical logic, in the sense that they may lack one or more of the structural rules of contraction, weakening and exchange in their Genzen-style axiomatization. These logics encompass a large number of non-classical logics related to computer science (linear logic), linguistics (Lambek Calculus), philosophy (relevant logics), and many-valued reasoning.

this approach to ℓ-groups can be found in [3], while complete proofs for most of the surveyed results can be found in Conrad's "Blue Notes" [21], as well as in [4] and [24].

The present article studies existence and uniqueness of the laterally complete, projectable and strongly projectable hulls of e-cyclic semilinear residuated lattices. The study of these concepts has a rich history that can be traced back to the theory of Riesz spaces, also referred to in the literature as vector lattices. For example, lateral completions of Riesz spaces were considered in [46], where it is shown that an Archimedean Riesz space can be embedded in the laterally complete Riesz space of almost finite continuous functions on a Stone space. The main result of [41] states that any conditionally complete Riesz space has a unique extension in which every disjoint subset has a supremum (in modern terminology, a lateral completion). An elegant proof of this result was obtained in [44, 49], and an extension to arbitrary Riesz spaces was established in [2]. In another particularly relevant article [45], it is proved that any conditionally complete Riesz space is *strongly projectable* (in the terminology of Riesz spaces, it satisfies the *strong projection property*). In regards to this property, we also mention [48].

The transfer of the preceding ideas and results to the theory of ℓ-groups, and in particular their development in the non-Archimedean and non-commutative contexts, was by no means straightforward, since many of the original proofs made extensive use of scalar multiplication and required the previously mentioned representation of Archimedean Riesz spaces as Riesz spaces of almost finite continuous functions on a Stone space. Among the many noteworthy contributions in this topic, we mention [1, 6–8, 11, 12, 20, 22, 23, 36].

It may be worthwhile to add a few general comments regarding the necessity and importance of the extensions we consider in this article. Given two classes $\mathcal{L}, \mathcal{K}$ of algebras of the same signature, with $\mathcal{L} \subseteq \mathcal{K}$, let us say that $\mathcal{K}$ has a "sufficient supply" of algebras in $\mathcal{L}$, if each member of $\mathcal{K}$ can be embedded into a member of $\mathcal{L}$. For example, it is particularly desirable for a class of ordered algebras to have a sufficient supply of order-complete algebras. Indeed, not only such objects support computations involving arbitrary joins and meets, but they often possess special properties that the original algebras may lack. The correspondence between an algebra and its extension provides a vehicle for transferring properties back and forth, provided that the two algebras are not "too far apart" from each other. A typical desideratum in this respect would be that the latter be an essential extension of the former, but in this article we use the slightly stronger condition of density.[2]

There is usually little to gain, in this context, from lattice-theoretic completions, such as the Dedekind-MacNeille completion or the ideal completion. For example, it is shown in [9] that the only proper subvarieties of Heyting algebras that are closed under the Dedekind-MacNeille completion are the trivial subvariety and the variety of Boolean algebras. Even worse, the varieties of Abelian ℓ-groups and Riesz spaces are examples of ordered algebras that possess no non-trivial order-complete members. In fact, even restricted versions of completeness – such as conditional (bounded) completeness – impose severe restrictions on the structure of an ℓ-group (or Riesz space). Indeed, it is well known that such a structure admits a conditionally complete extension of the same type if and only if it is Archimedean [10, XIII, §2]. On the other hand, the Archimedean property is not necessary for the embedding of an Abelian ℓ-group into a laterally complete Abelian ℓ-group, but one may ask whether a "minimal" laterally complete extension an Archimedean ℓ-group is Archimedean.

Thus, in a search for interesting extensions in varieties of ordered algebras we should deflect our attention from order-complete ones and focus on ones that satisfy restricted forms of order completeness or share interesting properties with order-complete ones. This is precisely what we do in this article. Our work, which owes great debt to P. Conrad's articles [20] and [22], expands the Conrad Program to the vastly more general framework of e-cyclic residuated lattices. This variety encompasses most varieties of notable significance in algebraic logic, including ℓ-groups, MV algebras, pseudo-MV algebras, GMV algebras, semilinear GBL algebras, BL algebras, Heyting algebras, commutative residuated lattices, and integral residuated lattices. A byproduct of our work is the introduction of new tools and techniques into the study of algebras of logic.

A featured result of this work is the construction, for any given e-cyclic and semilinear residuated lattice, of an orthocomplete (strongly projectable and laterally complete) extension in which the original algebra is dense. More specifically, we have:

Theorem A (Theorem 49)**.** *Any algebra* $\mathbf{L}$ *in a variety* $\mathcal{V}$ *of* e*-cyclic semilinear residuated lattices is densely embeddable in a laterally complete member*

[2] See Definition 38.

of $\mathcal{V}$*.*

The strategy for establishing this result is the following: In Section 5 we study the partitions of the Boolean algebra of polars (introduced in Section 3) of an e-cyclic residuated lattice $\mathbf{L}$ – which we simply call *partitions of* $\mathbf{L}$, by mild abuse of terminology. We show that they form a directed poset and, in fact, a join-semilattice. The partitions of $\mathbf{L}$ are used to define a directed system of algebras in Section 7, whenever $\mathbf{L}$ is semilinear (see Section 4). We discuss in Section 6 a general method for obtaining the direct limit of a directed system of algebras. We make use of this description to construct the direct limit of the directed system of algebras induced by the directed poset of partitions of $\mathbf{L}$. We prove that this limit, denoted $\mathcal{O}(\mathbf{L})$, enjoys many interesting properties. In particular, $\mathbf{L}$ is densely embeddable in $\mathcal{O}(\mathbf{L})$ (see Definition 38 and Theorem 39), and furthermore it is laterally complete (see Definition 40 and Theorem 48). This immediately yields Theorem A.

In Section 8 we advance our study of $\mathcal{O}(\mathbf{L})$ by proving that $\mathcal{O}(\mathbf{L})$ is also strongly projectable (see Definition 51). In fact, we prove more:

Theorem B (Theorem 55)**.** *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice. Then* $\mathcal{O}(\mathbf{L})$ *is strongly projectable.*

Hence, Theorems 39, 48, and 55 have the following consequence:

Theorem C (Corollary 57)**.** *If* $\mathbf{L}$ *is any algebra in a variety* $\mathcal{V}$ *of e-cyclic semilinear residuated lattices, then* $\mathcal{O}(\mathbf{L})$ *is an orthocomplete dense extension of* $\mathbf{L}$ *that belongs to* $\mathcal{V}$*.*

We also introduce in this section the algebra $\mathcal{O}_{<\omega}(\mathbf{L})$, which is a subalgebra of $\mathcal{O}(\mathbf{L})$, but generally smaller. While $\mathcal{O}(\mathbf{L})$ is laterally complete, $\mathcal{O}_{<\omega}(\mathbf{L})$ may fail this property. Nonetheless, $\mathbf{L}$ is also densely embeddable in $\mathcal{O}_{<\omega}(\mathbf{L})$, which is strongly projectable (Theorem 58).

Lastly, in Section 9, we investigate the existence and uniqueness of minimal extensions of a GMV algebra, which are laterally complete, projectable, strongly projectable, or orthocomplete. We refer to these extensions as *hulls*. In this direction we prove the following results:

Theorem D (Theorem 65)**.** *Any algebra* $\mathbf{L}$ *in a variety* $\mathcal{V}$ *of semilinear GMV algebras has a unique, up to isomorphism, laterally complete hull that belongs to* $\mathcal{V}$*.*

In order to achieve this result, we explore the relationship between the lattices of convex subalgebras of an e-cyclic residuated lattice and a dense extension of it. This discussion concludes with Proposition 53, which asserts that the Boolean algebras of their polars are isomorphic. We also make use of the fact, proved in Section 2, that any GMV algebra satisfies the Riesz interpolation property (Proposition 15, see also Definition 14). Using similar techniques we also obtain:

Theorem E (Theorem 74)**.** *Any algebra $\mathbf{L}$ in a variety $\mathcal{V}$ of semilinear GMV algebras has a unique, up to isomorphism, projectable hull, strongly projectable hull, and orthocomplete hull that belongs to $\mathcal{V}$.*

Moreover, we generalize in the context of GMV algebras a well-known result for ℓ-groups, namely lateral completeness and projectability imply strong projectability, and therefore we obtain:

Theorem F (Proposition 70)**.** *If a GMV algebra is laterally complete and projectable, then it is orthocomplete.*

We end this paper proving that $\mathcal{O}_{<\omega}(\mathbf{L})$ is actually the unique, up to isomorphims, strongly projectable hull of $\mathbf{L}$:

Theorem G (Theorem 76)**.** *If $\mathbf{L}$ is an algebra in a variety $\mathcal{V}$ of semilinear GMV algebras, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is the strongly projectable hull of $\mathbf{L}$ in $\mathcal{V}$.*

1. Background

In this section we recall basic facts about residuated lattices. Varieties of residuated lattices provide algebraic semantics for substructural logics and encompass important classes of algebras, such as ℓ-groups. We refer the reader to [13, 27, 31, 37] for further details.

A *residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ satisfying:

(a) $(L, \cdot, e)$ is a monoid;

(b) $(L, \vee, \wedge)$ is a lattice; and

(c) $\backslash$ and $/$ are binary operations satisfying:

$$x \cdot y \leqslant z \quad \Leftrightarrow \quad y \leqslant x\backslash z \quad \Leftrightarrow \quad x \leqslant z/y.$$

We refer to the operation $\cdot$ as *multiplication* and the operations $\backslash$ and $/$ as the *left residual* and *right residual* of the multiplication, respectively. As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parentheses, the multiplication is performed first, followed by the residuals, and finally by the lattice operations. The class of residuated lattices is a finitely based variety (see e.g. [5, 13, 37]), which we denote throughout this paper by $\mathcal{RL}$.

The existence of residuals has the following basic consequences, which will be used throughout this paper without explicit reference.

Lemma 1. *Let* $\mathbf{L}$ *be a residuated lattice.*

(1) The multiplication preserves all existing joins in each argument. That is, if $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq L$*, then*

$$\bigvee_{x \in X,\, y \in Y} xy \text{ exists and } \Big(\bigvee X\Big)\Big(\bigvee Y\Big) = \bigvee_{x \in X,\, y \in Y} xy.$$

(2) The residuals preserve all existing meets in the numerator and convert existing joins to meets in the denominator, i.e., if $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq L$*, then for any* $z \in L$*,* $\bigwedge_{x \in X} x\backslash z$*,* $\bigwedge_{y \in Y} z\backslash y$ *exist and*

$$\Big(\bigvee X\Big)\backslash z = \bigwedge_{x \in X} x\backslash z, \qquad z\backslash\Big(\bigwedge Y\Big) = \bigwedge_{y \in Y} z\backslash y.$$

The right residual $/$ *satisfies the corresponding properties.*

(3) The following identities[3] *(and the corresponding identities for* $/$*) hold in* $\mathbf{L}$*:*

(a) $y(y\backslash x) \leqslant x$*;*

(b) $(x\backslash y)z \leqslant x\backslash yz$*;*

(c) $x\backslash y \leqslant zx\backslash zy$*;*

(d) $(x\backslash y)(y\backslash z) \leqslant x\backslash z$*;*

(e) $xy\backslash z = y\backslash(x\backslash z)$*;*

(f) $x\backslash(y/z) = (x\backslash y)/z$*;*

(g) $e\backslash x = x$*;*

(h) $e \leqslant x\backslash x$*;*

(i) $x(x\backslash x) = x$*; and*

(j) $(x\backslash x)^2 = x\backslash x$*.*

We will have the occasion to consider pointed residuated lattices. A *pointed residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e, 0)$ of signature

[3] Some of them are expressed as inequalities, but are clearly equivalent to identities.

$(2, 2, 2, 2, 2, 0, 0)$ such that $(L, \wedge, \vee, \cdot, \backslash, /, e)$ is a residuated lattice. In other words, a pointed residuated lattice is simply a residuated lattice with an extra constant 0. Pointed residuated lattices are also referred to in the literature as *FL-algebras*, as they provide algebraic semantics for the Full Lambek calculus, and its subvarieties correspond to substructural logics. Residuated lattices may be identified with pointed residuated lattices satisfying the identity $e \approx 0$. The statements of the general results will be expressed in terms of residuated lattices, rather than pointed residuated lattices. However, they hold for the latter as well, since their proofs use congruences (normal convex subalgebras) and convex subalgebras of the residuated lattice reducts of these algebras.

A subvariety of $\mathcal{RL}$ of particular interest is the variety $\mathcal{CRL}$ of commutative residuated lattices, which satisfies the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. We always think of this variety as a subvariety of $\mathcal{RL}$, but we abuse notation by suppressing the operations $\backslash$ and $/$ into one, often denoted by $\to$, in describing their members.

Given a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$, an element $a \in A$ is said to be *integral* if $e/a = e = a\backslash e$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral; this is equivalent to e being its top element. We denote by $\mathcal{IRL}$ the variety of all integral residuated lattices. If F is a non-empty subset of a residuated lattice $\mathbf{L}$, we write F^- for the set of *negative* elements of F, that is, $F^- = \{x \in F \mid x \leqslant e\}$. The *negative cone* of $\mathbf{L}$ is the integral residuated lattice $\mathbf{L}^-$ with domain L^-, monoid and lattice operations the restrictions to L^- of the corresponding operations in $\mathbf{L}$, and residuals $\backslash_-$ and $/_-$ defined by

$$x\backslash_- y = (x\backslash y) \wedge e \quad \text{and} \quad y/_- x = (y/x) \wedge e,$$

where $\backslash$ and $/$ denote the residuals in $\mathbf{L}$.

We call a residuated lattice *e-cyclic* if it satisfies the identity $e/x \approx x\backslash e$. Three important subvarieties of e-cyclic residuated lattices are the variety $\mathcal{CRL}$ of commutative residuated lattices, the variety $\mathcal{IRL}$ of integral residuated lattices, and the variety of ℓ-groups, which occupies a very special place among the varieties of residuated lattices. An element a of a residuated lattice $\mathbf{L}$ is said to be *invertible* provided $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The residuated lattices in which every element is invertible are precisely the ℓ-groups. Perhaps a word of caution is appropriate here. An

ℓ-group is usually defined in the literature as an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, {}^{-1}, e)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, {}^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations, see [4, 29]). The variety of ℓ-groups is term equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the equation $(e/x)x \approx e$; the term equivalence is given by $x^{-1} = e/x$, $x/y = xy^{-1}$, and $x\backslash y = x^{-1}y$. Throughout this paper, the members of this subvariety will be referred to as ℓ-groups.

The variety $\mathcal{GMV}$ of GMV algebras will be featured prominently in the discussion of Section 9 related to the existence of various hulls.

Definition 2. A *GMV algebra* is a residuated lattice that satisfies the equations

$$x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x, \tag{GMV}$$

or, equivalently, the equations

$$x/(y\backslash x \wedge e) \approx x \vee y \approx (x/y \wedge e)\backslash x. \tag{GMV'}$$

It is straightforward to verify that the variety $\mathcal{LG}$ of ℓ-groups is a subvariety of $\mathcal{GMV}$. Subvarieties of pointed GMV algebras include the variety of MV algebras (axiomatized by $0 \wedge x \approx 0$, $xy = yx$, and $(x\backslash y)\backslash y \approx x \vee y$), as well as that of pseudo-MV algebras (axiomatized by $0 \wedge x \approx 0$ and $(x\backslash y)\backslash y \approx x \vee y$). We refer the reader to [50] for an extensive discussion of these topics.

We note that the variety $\mathcal{IGMV}$ of integral GMV algebras is axiomatized, relative to $\mathcal{IRL}$, by the equations:

$$(y/x)\backslash y \approx x \vee y \approx y/(x\backslash y), \tag{IGMV}$$

while the variety $\mathcal{LG}^-$ of *negative cones of ℓ-groups* is axiomatized relative to the latter variety (see [5, Theorem 6.2]) by the equations:

$$x\backslash xy \approx y \approx yx/x. \tag{LG$^-$}$$

Extending the results in [26, 40], the main result of [28] establishes a categorical equivalence between the class of GMV algebras and the class of ℓ-groups endowed with a suitable unary operator. Further, it is shown that this equivalence restricts to one involving integral GMV algebras and negative cones of ℓ-groups endowed with a nucleus whose image generates the negative cone as a semigroup. For the purposes of this paper, we only need a few, but crucial, pieces from the chain of lemmas leading to the proof of these results. We start with a definition:

Definition 3. [32] A residuated lattice $\mathbf{L}$ is said to be the *inner direct product* of its subalgebras $\mathbf{B}$ and $\mathbf{C}$ – in symbols, $\mathbf{L} = \mathbf{B} \otimes \mathbf{C}$ – if $\mathbf{B} \vee \mathbf{C} = \mathbf{L}$, where the join is taken in the lattice of subalgebras of $\mathbf{L}$ and the map $(b, c) \mapsto bc$ is an isomorphism from $\mathbf{B} \times \mathbf{C}$ to $\mathbf{L}$. In other words: (i) every $a \in L$ can be written uniquely as a product bc, for some $b \in B$ and $c \in C$; (ii) each element in B commutes with every element in C; and (iii) $b_1c_1 \leqslant b_2c_2$ – with $b_1, b_2 \in B$ and $c_1, c_2 \in C$ – if and only if $b_1 \leqslant b_2$ and $c_1 \leqslant c_2$.

Theorem 4. [28, Theorem 5.2] *In a GMV algebra* $\mathbf{L}$ *the following hold:*

(1) The invertible elements of $\mathbf{L}$ *form a subalgebra* $\mathbf{G}(\mathbf{L})$ *of* $\mathbf{L}$*, which is an* ℓ*-group.*

(2) The integral elements of $\mathbf{L}$ *form a subalgebra* $\mathbf{I}(\mathbf{L})$ *of* $\mathbf{L}$*, which is an integral GMV algebra.*

(3) $\mathbf{L}$ *has an inner product decomposition* $\mathbf{L} = \mathbf{G}(\mathbf{L}) \otimes \mathbf{I}(\mathbf{L})$*.*[4]

In view of the preceding theorem, every element a of a GMV algebra $\mathbf{L}$ can be written uniquely as $a = g_a i_a$, where $g_a \in G(L)$ and $i_a \in I(L)$. We use the term *inner factorization* of a to refer to the expression $a = g_a i_a$.

We also state the following lemma for future reference. The statement of the lemma refers to the following two identities, the so called *left prelinearity law* LP and the *right prelinearity law* RP:

$$((x\backslash y) \wedge e) \vee ((y\backslash x) \wedge e) \approx e, \tag{LP}$$

$$((y/x) \wedge e) \vee ((x/y) \wedge e) \approx e. \tag{RP}$$

The importance of the prelinearity laws will emerge in Section 4. Note incidentally that the identities of Lemma 5 are easy consequences of Theorem 4, Theorem 6 below, and the fact that these identities hold in ℓ-groups.

Lemma 5. *[28, Lemmas 2.7 and 2.9] Any GMV algebra satisfies the following identities:*

(1) $x/x \approx e \approx x\backslash x$

[4] Theorem 4 was independently proved in the setting of DRℓ-monoids by T. Kovář in his unpublished thesis "A general theory of dually residuated lattice ordered monoids" (Palacký University, Olomouc, 1996).

(2) $x\backslash e \approx e/x$ *(that is,* $\mathbf{L}$ *is e-cyclic)*

(3) $xy\backslash e \approx (x\backslash e)(y\backslash e)$

(4) $x \wedge (e/x) \leqslant e$

(5) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$

(6) $x \approx (x \wedge e)(x \vee e)$

(7) $\mathbf{L}$ *satisfies both prelinearity laws* (LP) *and* (RP)*.*

We close this section by recalling that if $\mathbf{L}$ is a GMV algebra, then its integral part $\mathbf{I}(\mathbf{L})$ is of the form $\mathbf{H}^-_\gamma$ where $\mathbf{H}^-$ is the negative cone of an ℓ-group and γ is a *nucleus* on it. That is, γ is a closure operator satisfying $\gamma(x)\gamma(y) \leqslant \gamma(xy)$ for all $x, y \in H^-$. The domain of $\mathbf{H}^-_\gamma$ is the set $H^-_\gamma = \gamma(H^-)$ and the operations of $\mathbf{H}^-_\gamma$ are the restrictions to H^-_γ of the corresponding operations of $\mathbf{H}^-$, except that multiplication on H^-_γ is defined by $x \circ_\gamma y = \gamma(xy)$. Moreover, H^-_γ is a lattice filter in $\mathbf{H}^-$, and the ℓ-group $\mathbf{H}$ can be constructed in such a way that H^-_γ generates $\mathbf{H}^-$ as a monoid.

Summarizing, we have:

Theorem 6. [28, Theorem 3.12] *Given an integral GMV algebra* $\mathbf{L}$*, there exists a negative cone* $\mathbf{H}^-$ *of an* ℓ*-group* $\mathbf{H}$ *and a nucleus* γ *on* $\mathbf{H}^-$ *such that* $\mathbf{H}^-_\gamma = \mathbf{L}$*. Moreover,* H^-_γ *is a lattice filter of* H^- *that generates it as a semigroup.*

2. Convex subalgebras

In this section, we start with a brief review of relevant properties of the lattice of convex subalgebras of an e-cyclic residuated lattice (see definition below), and then proceed with the study of its special properties in the setting of GMV algebras. An extensive study of related topics can be found in [14].

A subset C of a poset $\mathbf{P} = (P, \leqslant)$ is *order-convex* (or simply *convex*) in $\mathbf{P}$ if for every $a, b, c \in P$, whenever $a, c \in C$ with $a \leqslant b \leqslant c$, then $b \in C$. For a residuated lattice $\mathbf{L}$, we write $\mathcal{C}(\mathbf{L})$ for the set of all convex subalgebras of $\mathbf{L}$, partially ordered by set-inclusion. In fact, refer to the discussion below, it can be shown that $\mathcal{C}(\mathbf{L})$ is an algebraic lattice (see Theorem 11).

For any $S \subseteq L$, we let $C[S]$ denote the smallest convex subalgebra of $\mathbf{L}$ containing S. As is customary, we call $C[S]$ the convex subalgebra *generated* by S and let $C[a] = C[\{a\}]$. We refer to $C[a]$ as the *principal* convex subalgebra of $\mathbf{L}$ generated by the element a. The principal convex subalgebras of $\mathcal{C}(\mathbf{L})$ are the compact members of $\mathcal{C}(\mathbf{L})$, since by Lemma 10.(3) below, every finitely generated convex subalgebra of $\mathbf{L}$ is principal.

An important concept in the theory of ℓ-groups is the notion of an absolute value. This idea can be fruitfully generalized in the context of residuated lattices [14, 42].

Definition 7.

(1) The *absolute value* of an element x in a residuated lattice $\mathbf{L}$ is the element

$$|x| = x \wedge (e/x) \wedge e.$$

(2) If $X \subseteq L$, we set $|X| = \{|x| \mid x \in X\}$.

The proof of the following lemma is routine:

Lemma 8. *Let $\mathbf{L}$ be a residuated lattice, $x \in L$, and $a \in L^{-}$. The following conditions hold:*

(1) $x \leqslant e$ if and only if $|x| = x$;

(2) $|x| \leqslant x \leqslant |x| \backslash e$;

(3) $|x| = e$ if and only if $x = e$;

(4) $a \leqslant x \leqslant a \backslash e$ if and only if $a \leqslant |x|$; and

(5) if $H \in \mathcal{C}(\mathbf{L})$, then $x \in H$ if and only if $|x| \in H$.

Recall that, in view of Theorem 4, every element a of a GMV algebra $\mathbf{L}$ has a unique inner factorization $a = g_a i_a$, where $g_a \in G(L)$ and $i_a \in I(L)$.

Lemma 9. *Let $\mathbf{L}$ be a GMV algebra and $a \in L$.*

(1) $|a| = a \wedge (e/a)$.

(2) If $a = g_a i_a$ is the inner factorization of a, then $|a| = |g_a| \cdot i_a$ is the inner factorization of $|a|$.

Proof. Condition (1) follows from Lemma 5.(4). Using Conditions (2) and (3) of the same lemma, we have $|a| = |g_a i_a| = (g_a i_a) \wedge (g_a i_a \backslash e) = (g_a i_a) \wedge (g_a \backslash e)(i_a \backslash e) = (g_a \wedge (g_a \backslash e))(i_a \wedge (i_a \backslash e)) = |g_a|(i_a \wedge e) = |g_a| \cdot i_a$. □

In what follows, given a subset S of a residuated lattice $\mathbf{L}$, we write $\langle S \rangle$ for the submonoid of $\mathbf{L}$ generated by S. Thus, $x \in \langle S \rangle$ if and only if there exist elements $s_1, \ldots, s_n \in S$ such that $x = s_1 \cdots s_n$.

The next lemma provides an intrinsic description of the convex subalgebra generated by a subset of an e-cyclic residuated lattice. We notice that the assumption of e-cyclicity is needed to prove that the sets defined by the righthand expressions of (1) and (2) are closed under the residual operations.

Lemma 10. *[14] Let $\mathbf{L}$ be an e-cyclic residuated lattice.*

(1) For $S \subseteq L$,

$$\begin{aligned} C[S] = C[|S|] &= \{x \in L \mid h \leqslant x \leqslant h\backslash e, \ \textit{for some } h \in \langle |S| \rangle\} \\ &= \{x \in L \mid h \leqslant |x|, \ \textit{for some } h \in \langle |S| \rangle\}. \end{aligned}$$

(2) For $a \in L$,

$$\begin{aligned} C[a] = C[|a|] &= \{x \in L \mid |a|^n \leqslant x \leqslant |a|^n\backslash e, \ \textit{for some } n \in \mathbb{N}\} \\ &= \{x \in L \mid |a|^n \leqslant |x|, \ \textit{for some } n \in \mathbb{N}\}. \end{aligned}$$

(3) For $a, b \in L$, $C[a] \cap C[b] = C[|a| \vee |b|]$ and $C[a] \vee C[b] = C[|a| \wedge |b|] = C[|a||b|]$.

(4) If H is a convex subalgebra of $\mathbf{L}$, then $H = C[H^-]$.

Lemma 10 yields the following results.

Theorem 11. *[14] If $\mathbf{L}$ is an e-cyclic residuated lattice, then:*

(1) $\mathcal{C}(\mathbf{L})$ is a distributive algebraic complete lattice.

(2) The poset $\mathcal{K}(\mathcal{C}(\mathbf{L}))$ of compact elements of $\mathcal{C}(\mathbf{L})$ – consisting of the principal convex subalgebras of $\mathbf{L}$ – is a sublattice of $\mathcal{C}(\mathbf{L})$.

The next lemma connects the lattice $\mathcal{C}(\mathbf{L})$ of convex subalgebras of a GMV algebra $\mathbf{L}$ with those of $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$.

Lemma 12. *Let $\mathbf{L}$ be a GMV algebra.*

1. *If* $H \in \mathcal{C}(\mathbf{L})$*, then* $H = (H \cap G(L)) \otimes (H \cap I(L))$*.*

2. *If* $H_1 \in \mathcal{C}(\mathbf{G}(\mathbf{L}))$ *and* $H_2 \in \mathcal{C}(\mathbf{I}(\mathbf{L}))$*, then* $H_1 \vee^{\mathcal{C}(\mathbf{L})} H_2 = H_1 \otimes H_2$*.*

Proof. (1) Clearly, if $\mathbf{H} \in \mathcal{C}(\mathbf{L})$ then $\mathbf{H}$ is a GMV algebra, and therefore $\mathbf{H} = \mathbf{G}(\mathbf{H}) \otimes \mathbf{I}(\mathbf{H})$, in view of Theorem 4.(3). The statements follows from the fact that $G(H) = H \cap G(L)$ and $I(H) = H \cap I(L)$.

(2) Let $H_1 \in \mathcal{C}(\mathbf{G}(\mathbf{L}))$, $H_2 \in \mathcal{C}(\mathbf{I}(\mathbf{L}))$ and $H = H_1 \vee^{\mathcal{C}(\mathbf{L})} H_2$. Consider an element $h \in H$, and let $h = g_h i_h$ be the inner decomposition of h. Lemma 9 implies that $|h| = |g_h| i_h$. Further, in view of Lemma 10, there exist $h_1 \in H_1^-$ and $h_2 \in H_2^-$ such that $h_1 h_2 \leqslant |h| = |g_h| i_h \leqslant e$. Thus $h_1 \leqslant |g_h|$ and $h_2 \leqslant i_h$, which implies that $|g_h|$ and hence g_h is in H_1 by Lemma 8.(5), and also i_h is in H_2. It follows that $h \in H = H_1 \otimes H_2$ and hence $H \subseteq H_1 \otimes H_2$. The other inclusion trivially holds. □

Corollary 13. *For any GMV algebra* $\mathbf{L}$*,* $\mathcal{C}(\mathbf{L}) \cong \mathcal{C}(\mathbf{G}(\mathbf{L})) \times \mathcal{C}(\mathbf{I}(\mathbf{L}))$*.*

We close this section with two results that will be useful for the considerations of Section 9. We note that a special case of Proposition 15 appears in [43].

Definition 14. A residuated lattice $\mathbf{L}$ is said to satisfy the *Riesz interpolation property* if, for all $b_1, \ldots, b_n, a \in L^-$ satisfying $b_1 \cdots b_n \leqslant a$, there exist $a_1, \ldots, a_n \in L^-$ such that $a = a_1 \cdots a_n$ and $b_i \leq a_i \leq e$, for $1 \leq i \leq n$.

Proposition 15. *Any GMV algebra satisfies the Riesz interpolation property.*

Proof. Let $\mathbf{L}$ be a GMV algebra. In view of Theorem 4, every element $a \in L$ has an inner factorization $a = g_a i_a$, with $g_a \in G(L)$ and $i_a \in I(L)$. Thus it will suffice to show that ℓ-groups and integral GMV algebras satisfy the Riesz interpolation property.

It is well known and easy to prove that ℓ-groups satisfy this property. For example, consider the case $n = 2$: if $b_1, b_2, a \leqslant e$ are elements of an ℓ-group $\mathbf{G}$ such that $b_1 b_2 \leqslant a$, then $a_1 a_2 = a$, where $a_1 = b_1 \vee a$ and $a_2 = (b_1 \vee a)^{-1} a$. An easy induction proves the property for arbitrary n.

Let now $\mathbf{M}$ be an integral GMV algebra. By Theorem 16, there exists a negative cone $\mathbf{H}^-$ of an ℓ-group $\mathbf{H}$ and a nucleus γ on $\mathbf{H}^-$ such that $\mathbf{H}^-_\gamma = \mathbf{M}$. Let $b_1, \ldots, b_n, a$ be elements of $\mathbf{M}$ such that $b_1 \cdot_\gamma \cdots \cdot_\gamma b_n \leqslant a$. Then $b_1 b_2 \cdots b_n \leqslant a$ in $\mathbf{H}^-$, and so there exist elements $a_1, \ldots, a_n \in H^-$

such that $a_1 \cdots a_n = a$ and $b_i \leq a_i \leq e$, for $1 \leq i \leq n$. It follows that $a = \gamma(a) = \gamma(a_1 \cdots a_n) = \gamma(a_1) \cdot_\gamma \cdots \cdot_\gamma \gamma(a_n)$ and $b_i \leqslant \gamma(a_i) \leq e$, for $1 \leq i \leq n$. $\square$

Proposition 16. *Let* $\mathbf{L}$ *be a GMV algebra and let* H, K *be convex subalgebras of* $\mathbf{L}$*. Then the join* $H \vee^{\mathcal{C}(\mathbf{L})} K$ *of* H *and* K *in* $\mathcal{C}(\mathbf{L})$ *is the submonoid* $\langle H \cup K \rangle$ *of* $\mathbf{L}$ *generated by* $H \cup K$*.*

Proof. Let us first observe that the theorem, or equivalently the inclusion $H \vee^{\mathcal{C}(\mathbf{L})} K \subseteq \langle H \cup K \rangle$, holds under the additional assumption that $\mathbf{L}$ is integral. Indeed, let $a \in H \vee^{\mathcal{C}(\mathbf{L})} K$. In view of Lemma 10, there exist elements $x_1, \ldots, x_n \in H$ and $y_1, \ldots, y_n \in K$ such that $x_1 y_1 \cdots x_n y_n \leqslant a$. Now $\mathbf{L}$ satisfies the Riesz interpolation property by Proposition 15. It follows that there exist elements $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ in L such that, $z_1 w_1 \cdots z_n w_n = a$, and moreover, for $1 \leqslant i \leqslant n$, $x_i \leqslant z_i$ and $y_i \leqslant w_i$. Since H, K are convex subalgebras of $\mathbf{L}$, the last two inequalities imply that each z_i is in H, and each w_i is in K. Hence, $H \vee^{\mathcal{C}(\mathbf{L})} K \subseteq \langle H \cup K \rangle$, as was to be shown.

Removing the assumption of integrality for $\mathbf{L}$, we have shown that the theorem holds for $\mathbf{I}(\mathbf{L})$. In view of Lemma 12, it will suffice to note that the result holds for $\mathbf{G}(\mathbf{L})$, and more precisely that $(H \cap G(L)) \vee^{\mathcal{C}(\mathbf{G}(\mathbf{L}))} (K \cap G(L)) \subseteq \langle (H \cap G(L)) \cup (K \cap G(L)) \rangle$. Let $a \in (H \cap G(L)) \vee^{\mathcal{C}(\mathbf{G}(\mathbf{L}))} (K \cap G(L))$. Then, invoking Lemma 10 and the case of integral GMV algebras, we have that $a \wedge e$ and $a^{-1} \wedge e$ are in $\langle (H \cap G(L)) \cup (K \cap G(L)) \rangle$. But then, so is $(a^{-1} \wedge e)^{-1}$ and $a = (a \wedge e)(a^{-1} \wedge e)^{-1}$. $\square$

Lemma 17. *[28, Lemma 2.10] Any GMV algebra satisfies the quasi-identity*

$$x \vee y = e \implies xy = x \wedge y. \tag{1}$$

Proof. In view of Theorem 4, it will suffice to prove that ℓ-groups and integral GMV algebras satisfy the quasi-identity. This is a well-known fact for ℓ-groups and follows directly from the ℓ-group identity $x(y \vee z)^{-1} y \approx x \wedge y$.

Suppose next that $\mathbf{L}$ in an integral GMV algebra. By Theorem 6, $\mathbf{L}$ is the image of a nucleus γ on the negative cone $\mathbf{H}^-$ of an ℓ-group $\mathbf{H}$. If $x \vee y = e$ in $\mathbf{L}$, then $x \vee y = e$ in $\mathbf{H}^-$, and so $xy = x \wedge y$ in $\mathbf{H}^-$. But then $x \circ_\gamma y = \gamma(xy) = \gamma(x \wedge y) = x \wedge y$. $\square$

Proposition 18. *Let* $\mathbf{L}$ *be a GMV algebra and let* H, K *be convex subalgebras of* $\mathbf{L}$*. If* $H \cap K = \{e\}$*, then* $H \vee^{\mathcal{C}(\mathbf{L})} K = H \otimes K$*.*

Proof. In view of Theorem 4 and Lemma 12, it will suffice to prove that the conclusion of the proposition holds for integral GMV algebras and ℓ-groups.

Let us first assume that $\mathbf{L}$ is an integral GMV algebra. In view of Proposition 16, $J = H \vee^{\mathcal{C}(\mathbf{L})} K$ is the submonoid $\langle H \cup K \rangle$ of $\mathbf{L}$ generated by $H \cup K$. Note that if $x \in K$ and $y \in H$ then $x, y \leqslant x \vee y \leqslant e$, and so by convexity $x \vee y \in H \cap K = \{e\}$, therefore $x \vee y = e$. It follows from Lemma 17 that every element of H commutes with every element of K and therefore, by quasi-identity (1), every element of J is of the form $x \wedge y$ with $x \in H$ and $y \in K$. We next show that if $x \wedge y \leqslant z \wedge w$, with $x, z \in H$ and $y, w \in K$, then $x \leqslant z$ and $y \leqslant w$. We have that $x \wedge y = (x \wedge y) \wedge (z \wedge w) = (x \wedge z) \wedge y$, and using the distributivity of $\langle L, \wedge, \vee \rangle$ (see Lemma 5), $x \vee y = e = (x \wedge z) \vee y$. Thus, $x \wedge z = x$, or $x \leqslant z$. Likewise, $y \leqslant w$. In particular, $x \wedge y = z \wedge w$, with $x, z \in H$ and $y, w \in K$, if and only if $x = z$ and $y = w$. We have shown that $J = H \otimes K$ whenever $\mathbf{L}$ is an integral GMV algebra.

The case for ℓ-groups is already known and can be distilled by results of Chapter 1 in [21]. Since these notes are not easily accessible, we provide a direct proof of this case. Suppose $\mathbf{L}$ is an ℓ-group. By the analysis of the previous paragraph, every element of H^- commutes with every element of K^-, and hence every element of $H^- \cup H^+$ commutes with every element of $K^- \cup K^+$. Since every element of $x \in L$ can be written as $x = (x \wedge e)(x^{-1} \wedge e)^{-1}$, it follows that every element of H commutes with every element of K. We have shown so far that $J = HK = \{xy \mid x \in H, y \in K\}$.

It remains to prove that this product is an inner direct decomposition of $\mathbf{J}$. Let $z = xy \in J$, with $x \in H$ and $y \in K$, and set $w = (x^{-1} \wedge e)(y^{-1} \wedge e) = (x^{-1} \wedge e) \wedge (y^{-1} \wedge e)$. We have $zw = xy(x^{-1} \wedge e)(y^{-1} \wedge e) = x(x^{-1} \wedge e)y(y^{-1} \wedge e) = (x \wedge e)(y \wedge e)$, and so $(z \wedge e)w = zw \wedge w = (x \wedge e)(y \wedge e)(x^{-1} \wedge e)(y^{-1} \wedge e) = |x||y|$, that is, $(z \wedge e)w = |x||y|$.

Suppose now that $z \leqslant e$. Then the last equality gives $zw = |x||y|$, and so $z = |x||y|w^{-1} = (x \wedge e)(y \wedge e)$. If $x \nleqslant e$, then $x \wedge e < x$, and so $z = (x \wedge e)(y \wedge e) < xy = z$, which is a contradiction. We have shown that $z = xy \leqslant e$ if and only if $x \leqslant e$ and $y \leqslant e$. In particular, $z = xy = e$ if and only if $x = e$ and $y = e$. This proves that $J = H \otimes K$ whenever $\mathbf{L}$ is an ℓ-group. ∎

As it is proved in [13] (see also [27]), for any residuated lattice $\mathbf{L}$, the congruences of $\mathbf{L}$ are completely determined[5] by its convex normal subalgebras,

[5] Actually, their corresponding lattices are isomorphic.

which we define in what follows.

Let $\mathbf{L}$ be a residuated lattice. Given an element $u \in L$, we define

$$\lambda_u(x) = (u\backslash xu) \wedge e \quad \text{and} \quad \rho_u(x) = (ux/u) \wedge e,$$

for all $x \in L$. We refer to λ_u and ρ_u as *left conjugation* and *right conjugation* by u. A set $X \subseteq L$ is said to be *normal* if it is closed under conjugates. In other words, for all $x \in X$ and $u \in L$, $\lambda_u(x), \rho_u(x) \in X$.

Lemma 19 ([13,31]; see also [50] or [27])**.** *Given a normal convex subalgebra $\mathbf{H}$ of $\mathbf{L}$, $\Theta_{\mathbf{H}} = \{\langle x, y\rangle \in L^2 \mid (x\backslash y) \wedge (y\backslash x) \wedge e \in H\}$ is a congruence relation of $\mathbf{L}$. Conversely, given a congruence relation Θ, the equivalence class $[e]_\Theta$ is a normal convex subalgebra. This correspondence establishes an isomorphism between the congruence lattice of $\mathbf{L}$ and the lattice of its normal convex subalgebras.*

In what follows, if $\mathbf{H}$ is a normal convex subalgebra of $\mathbf{L}$, we write $\mathbf{L}/H$ for the quotient algebra $\mathbf{L}/\Theta_{\mathbf{H}}$, and denote the equivalence class of an element $x \in L$ by $[x]_H$.

3. Polars

As we have mentioned in Section 2, the lattice $\mathcal{C}(\mathbf{L})$ of convex subalgebras of an e-cyclic residuated lattice $\mathbf{L}$ is an algebraic distributive lattice. In particular, it is relatively pseudo-complemented and satisfies the join-infinite distributive law. Thus, for all $X, Y \in \mathcal{C}(\mathbf{L})$, the relative pseudocomplement $X \to Y$ of X relative to Y is given by:

$$X \to Y = \max\{Z \in \mathcal{C}(\mathbf{L}) \mid X \cap Z \subseteq Y\}.$$

The next lemma provides an element-wise description of $X \to Y$ in terms of the absolute value, and in particular one for the pseudocomplement $X^\perp = X \to \{e\}$ of X.

Lemma 20. *If $\mathbf{L}$ is an e-cyclic residuated lattice, then $\mathcal{C}(\mathbf{L})$ is a relatively pseudo-complemented lattice. Specifically, given $X, Y \in \mathcal{C}(\mathbf{L})$,*

$$X \to Y = \{a \in L \mid |a| \vee |x| \in Y, \text{ for all } x \in X\}, \tag{2}$$

and in particular,

$$X^\perp = \{a \in L \mid |a| \vee |x| = e, \text{ for all } x \in X\}. \tag{3}$$

For any subset $X \subseteq L$, we define the set $X^\perp$ as in Equation (3). It can be easily seen that $X^\perp = C[X]^\perp$, so $X^\perp$ is always a convex subalgebra. We refer to $X^\perp$ as the *polar* of X; in case $X = \{x\}$, we write $x^\perp$ instead of $\{x\}^\perp$ (or $C[x]^\perp$) and refer to it as the *principal polar* of x. Furthermore, notice that for every $X \subseteq L$, $X^\perp = |X|^\perp$, by virtue of Lemma 10.(1).

We state the following lemma for future reference:

Lemma 21. *If* $\mathbf{L}$ *is an e-cyclic residuated lattice, then for every* $x, y \in L$,

$$(|x| \vee |y|)^{\perp\perp} = x^{\perp\perp} \cap y^{\perp\perp}.$$

Proof. By virtue of Lemma 10:

$$\begin{aligned}(|x| \vee |y|)^{\perp\perp} &= C[|x| \vee |y|]^{\perp\perp} = (C[x] \cap C[y])^{\perp\perp} = C[x]^{\perp\perp} \cap C[y]^{\perp\perp} \\ &= x^{\perp\perp} \cap y^{\perp\perp}.\end{aligned}$$ □

The map $^\perp : \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{L})$ is a self-adjoint inclusion-reversing map, while the map sending $H \in \mathcal{C}(\mathbf{L})$ to its double polar $H^{\perp\perp}$ is an intersection-preserving closure operator on $\mathcal{C}(\mathbf{L})$. Therefore, a set H is a polar if and only if $H = H^{\perp\perp}$. By Glivenko's classical result, the image of this operator is a (complete) Boolean algebra $\mathrm{Pol}(\mathbf{L})$ with least element $\{e\}$ and largest element L. The complement of H in $\mathrm{Pol}(\mathbf{L})$ is $H^\perp$ and for any family $\{H_i \mid i \in I\}$ in $\mathrm{Pol}(\mathbf{L})$

$$\bigvee\nolimits_{i \in I}^{\mathrm{Pol}(\mathbf{L})} H_i = \big(\bigvee\nolimits_{i \in I}^{\mathcal{C}(\mathbf{L})} H_i\big)^{\perp\perp} = \big(\bigcup\nolimits_{i \in I} H_i\big)^{\perp\perp}.$$

We refer to $\mathrm{Pol}(\mathbf{L})$ as the *algebra of polars* of $\mathbf{L}$. Thus, $\mathrm{Pol}(\mathbf{L})$ is a complete Boolean algebra whose top and bottom elements are L and $\{e\}$, respectively.

Lemma 22. *Let* $\mathbf{L}$ *be a GMV algebra,* $H_1 \in \mathcal{C}(\mathbf{G}(\mathbf{L}))$*,* $H_2 \in \mathcal{C}(\mathbf{I}(\mathbf{L}))$ *and* $H = H_1 \otimes H_2$*. Then*

$$H^{\perp_{\mathbf{L}}} = H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \otimes H_2^{\perp_{\mathbf{I}(\mathbf{L})}} = H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_2^{\perp_{\mathbf{I}(\mathbf{L})}}.$$

Proof. We prove the first equality. The second equality follows from Lemma 12.2. Let x be an arbitrary element of L with inner decomposition $x = g_x \cdot i_x$. By

Lemma 9, $|x| = |g_x| \cdot i_x$ and hence,

$$\begin{aligned}
x \in H^{\perp_{\mathbf{L}}} &\Leftrightarrow |x| \vee |a| = e, \text{ for every } a \in H = H_1 \otimes H_2 \\
&\Leftrightarrow (|g_x|i_x) \vee (|a_1|a_2) = e, \text{ for every } a_1 \in H_1, \text{ and } a_2 \in H_2 \\
&\Leftrightarrow (|g_x| \vee |a_1|)(i_x \vee a_2) = e, \text{ for every } a_1 \in H_1, \text{ and } a_2 \in H_2 \\
&\Leftrightarrow (|g_x| \vee |a_1|) = e = (i_x \vee a_2), \text{ for every } a_1 \in H_1, \text{ and } a_2 \in H_2 \\
&\Leftrightarrow g_x \in H_1^{\perp_{\mathbf{G}}} \text{ and } i_x \in H_2^{\perp_{\mathbf{I}}} \\
&\Leftrightarrow x \in H_1^{\perp_{\mathbf{G}}} \otimes H_2^{\perp_{\mathbf{I}}}
\end{aligned}$$

□

The next two results are of interest, since they connect the polars of $\mathbf{L}$ with those of $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$.

Proposition 23. *Let* $\mathbf{L}$ *be a GMV algebra and* $A \in \mathcal{C}(\mathbf{L})$*. Then* $A \in \mathrm{Pol}(\mathbf{L})$ *if and only if there exist* $B \in \mathrm{Pol}(\mathbf{G}(\mathbf{L}))$ *and* $C \in \mathrm{Pol}(\mathbf{I}(\mathbf{L}))$ *such that* $A = B \otimes C$.

Proof. In light of Lemma 12, there exist $A_1 \in \mathcal{C}(\mathbf{G}(\mathbf{L}))$ and $A_2 \in \mathcal{C}(\mathbf{I}(\mathbf{L}))$ such that $A = A_1 \otimes A_2$. Hence Lemma 22 yields

$$A^{\perp\perp_{\mathbf{L}}} = A_1^{\perp\perp_{\mathbf{G}(\mathbf{L})}} \otimes A_2^{\perp\perp_{\mathbf{I}(\mathbf{L})}},$$

which immediately implies the claim. □

Proposition 24. *Let* $\mathbf{L}$ *be a GMV algebra and let* $a \in L$ *have inner decomposition* $a = g_a i_a$*. Then*

$$a^{\perp_{\mathbf{L}}} = g_a^{\perp_{\mathbf{G}(\mathbf{L})}} \otimes i_a^{\perp_{\mathbf{I}(\mathbf{L})}}.$$

Proof. Notice that $C_{\mathbf{L}}[a] = C_{\mathbf{G}(\mathbf{L})}[g_a] \otimes C_{\mathbf{I}(\mathbf{L})}[i_a]$. The inclusion from left to right follows from the equality $a = g_a \cdot i_a$, while the reverse inclusion follows from the fact that $g_a, i_a \in C_{\mathbf{L}}[a]$. Thus,

$$\begin{aligned}
a^{\perp_{\mathbf{L}}} &= C_{\mathbf{L}}[a]^{\perp_{\mathbf{L}}} = (C_{\mathbf{G}(\mathbf{L})}[g_a] \otimes C_{\mathbf{I}(\mathbf{L})}[i_a])^{\perp_{\mathbf{L}}} = C_{\mathbf{G}(\mathbf{L})}[g_a]^{\perp_{\mathbf{G}(\mathbf{L})}} \otimes C_{\mathbf{I}(\mathbf{L})}[i_a]^{\perp_{\mathbf{I}(\mathbf{L})}} \\
&= g_a^{\perp_{\mathbf{G}(\mathbf{L})}} \otimes i_a^{\perp_{\mathbf{I}(\mathbf{L})}}.
\end{aligned}$$

□

4. Semilinearity

Some prominent varieties of residuated lattices and pointed residuated lattices – including Abelian ℓ-groups and MV algebras – are generated by their linearly ordered members. We refer to such varieties as *semilinear*,[6] and denote the variety of all semilinear residuated lattices by $\mathcal{S}em\mathcal{RL}$. Thus, a residuated lattice is semilinear if and only if it is a subdirect product of totally ordered residuated lattices.

It is well known (see [4]) that the class $\mathcal{R}ep\mathcal{LG}$ of representable ℓ-groups form a variety and, in fact, it can be axiomatized relative to $\mathcal{LG}$ by the equation:

$$(x^{-1}yx \vee y^{-1}) \wedge e \approx e.$$

An analogous result was shown in [13, 31]: the class $\mathcal{S}em\mathcal{RL}$ is a variety, and it can be axiomatized, relative to $\mathcal{RL}$, by either of the equations below:

$$\lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y) \approx e, \tag{SL1}$$

$$\lambda_u(x/(x \vee y)) \vee \rho_v(y/(x \vee y)) \approx e. \tag{SL2}$$

The next theorem generalizes the well-known results on semilinear ℓ-groups as well as all analogous results characterizing semilinear members of some classes of residuated lattices – see [25] for pseudo-MV algebras, [34] for pseudo-BL algebras, [33] for GBL algebras (DRℓ-monoids), and [47] for integral residuated lattices.

Theorem 25. *[14] For a variety $\mathcal{V}$ of residuated lattices, the following statements are equivalent:*

(1) $\mathcal{V}$ is semilinear.

(2) $\mathcal{V}$ satisfies either of the equations (SL1) *and* (SL2).

(3) $\mathcal{V}$ satisfies either of the prelinearity laws and the quasi-identity

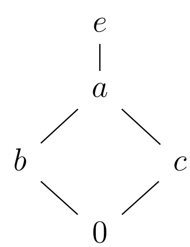
$$x \vee y \approx e \quad \Rightarrow \quad \lambda_u(x) \vee \rho_v(y) \approx e. \tag{4}$$

If in addition $\mathcal{V}$ is a variety of e-cyclic residuated lattices, the preceding conditions are equivalent to the condition:

[6] The more traditional, but less descriptive, name for these varieties is *representable*, specially for ℓ-groups, for which we will keep the name for traditional reasons.

(4) *$\mathcal{V}$ satisfies either of the prelinearity laws and for every $\mathbf{L} \in \mathcal{V}$, all (principal) polars in $\mathbf{L}$ are normal.*

It is well known, and easy to prove, that representable ℓ-groups satisfy both prelinearity laws. Thus, in view of the preceding result, a variety of ℓ-groups is semilinear if and only if all polars of every algebra in the variety are normal. Normality of polars alone is not sufficient to imply semilinearity in general. For example, the variety of Heyting algebras satisfies the normality condition on polars, since it is a variety of commutative pointed residuated lattices, but it is not semilinear. For example, the Heyting algebra below is subdirectly irreducible but not totally ordered.



5. Partitions

In this section, we introduce the notion of partition of a complete Boolean algebra and study the particular case of the Boolean algebras of polars of a residuated lattice. Recall that a *partition* of a set X is a nonempty set $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\emptyset \notin \mathcal{C}$, for every pair of different elements $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, and $\bigcup \mathcal{C} = X$. Our notion of partition generalizes this one to arbitrary complete Boolean algebras. Indeed, a partition of a set X is just a partition of the Boolean algebra $\mathcal{P}(X)$.

We say that two elements $a, b \neq \perp$ of a Boolean algebra are *disjoint* provided $a \wedge b = \perp$. A word of caution is in order. According to Definition 40, we say that two (negative) elements x, y of an e-cyclic residuated lattices are disjoint if $x \vee y = e$, which in the particular case of integral residuated lattices, is exactly the dual notion of the one that we have just defined. This should not lead to any confusion, as the Boolean algebras under consideration are the algebras of polars of residuated lattices. The two notions of disjointness will be clearly separated by the context, and Lemma 47 will elucidate the connection between these homonymous concepts.

Definition 26. Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, \bot, \top \rangle$ be a non-trivial complete Boolean algebra. A *partition* of $\mathbf{B}$ is a maximal set of disjoint elements of $B \setminus \{\bot\}$, that is to say, it is a set $\mathcal{C} \subseteq B$ such that

(1) $\bot \notin \mathcal{C}$,

(2) for every $c, d \in \mathcal{C}$, if $c \neq d$ then $c \wedge d = \bot$, and

(3) if $a \in B$ is such that $a \neq \bot$, then there exists $c \in \mathcal{C}$ such that $a \wedge c \neq \bot$.

The following result, which is an immediate consequence of the preceding definition, provides an alternative characterization of partitions:

Lemma 27. *A subset $\mathcal{C} \subseteq B$ is a partition of $\mathbf{B}$ if and only if it satisfies the following conditions:*

(1) $\bot \notin \mathcal{C}$,

(2) for every $c, d \in \mathcal{C}$, if $c \neq d$ then $c \wedge d = \bot$, and

(3) $\bigvee \mathcal{C} = \top$.

Further, any subset $\mathcal{C}$ of B that satisfies conditions (1) and (2) can be extended to a partition, for instance $\mathcal{C} \cup \{\neg(\bigvee \mathcal{C})\}$.

The set $\mathbb{D}$ of partitions of $\mathbf{B}$ can be ordered in the following manner: given two partitions $\mathcal{C}$ and $\mathcal{A}$, we say that $\mathcal{A}$ is a *refinement* of $\mathcal{C}$, and write $\mathcal{C} \preccurlyeq \mathcal{A}$, if for every $a \in \mathcal{A}$ there exists a (necessarily unique) $c \in \mathcal{C}$ such that $a \leqslant c$. It is easily checked that $\preccurlyeq$ is a partial order on $\mathbb{D}$. We in fact prove that $\langle \mathbb{D}, \preccurlyeq \rangle$ is a join semilattice, and hence any two partitions have a least common refinement. Indeed let $\mathcal{C}, \mathcal{D}$ be partitions. We claim that

$$\mathcal{A} = \bigl\{ c \wedge d \neq \bot \mid c \in \mathcal{C},\ d \in \mathcal{D} \bigr\} \tag{5}$$

is their join in $\langle \mathbb{D}, \preccurlyeq \rangle$. Let us first verify that $\mathcal{A}$ is actually a partition. Observe that $\bot \notin \mathcal{A}$ by definition. If $a = c \wedge d$ and $a' = c' \wedge d'$ are in $\mathcal{A}$, with $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, and $a \neq a'$, then $c \neq c'$ or $d \neq d'$, and in either case $a \wedge a' = (c \wedge c') \wedge (d \wedge d') = \bot$. And finally, if $a \in B$ is such that $a \neq \bot$, then there exists $c \in \mathcal{C}$ such that $a \wedge c \neq \bot$, by the maximality of $\mathcal{C}$, and therefore there exists $d \in \mathcal{D}$ such that $(a \wedge c) \wedge d \neq \bot$, by the maximality of $\mathcal{D}$. Thus, we have found $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $c \wedge d \neq \bot$, and therefore $c \wedge d \in \mathcal{A}$, and $a \wedge (c \wedge d) \neq \bot$, which proves the maximality of $\mathcal{A}$. Lastly, $\mathcal{A}$ is clearly a refinement of $\mathcal{C}$ and $\mathcal{D}$, and any other refinement of $\mathcal{C}$ and $\mathcal{D}$ must also be a refinement of $\mathcal{A}$.

Lemma 28. *Let* $\mathbf{B}$ *be a complete Boolean algebra and* $\mathcal{C}, \mathcal{A}$ *be partitions of* $\mathbf{B}$*, and let* $\mathcal{A}_c = \{a \in \mathcal{A} \mid a \leqslant c\}$*. Then the following are equivalent:*

1. $\mathcal{C} \preccurlyeq \mathcal{A}$*;*
2. *for every* $c \in \mathcal{C}$*,* $\mathcal{A}_c$ *is a partition of the Boolean algebra* $[\bot, c]$*;*
3. *for every* $c \in \mathcal{C}$*,* $c = \bigvee \mathcal{A}_c$*; and*
4. *for every* $c \in \mathcal{C}$*,* $\neg c = \bigwedge \{\neg a \mid a \in \mathcal{A}_c\}$*.*

Proof. $(1 \Rightarrow 2)$: Let $c \in \mathcal{C}$ and let $\mathcal{A}_c = \{a \in \mathcal{A} \mid a \leqslant c\}$. Obviously $\bot \notin \mathcal{A}_c$ and if $a, b \in \mathcal{A}_c$ and $a \neq b$, then $a \wedge b = \bot$, since $\mathcal{A}$ is a partition. Now, by Lemma 27 all we need to show is that $c = \bigvee \mathcal{A}_c$, which is true by virtue of the distributivity law, and the facts that $\mathcal{C}$ is a refinement of $\mathcal{A}$ and $\bigvee \mathcal{A} = \top$, by hypothesis.

$(2 \Rightarrow 1)$: Consider $b \in \mathcal{A}$. Then $b \neq \bot$, and therefore $b \wedge c \neq \bot$, for some $c \in \mathcal{C}$. Obviously $b \wedge c \in [\bot, c]$, and since $\mathcal{A}_c$ is a partition of $[\bot, c]$, there exists $a \in \mathcal{A}$ such that $a \leqslant c$ and $a \wedge b \wedge c \neq \bot$. Thus, $a \wedge b \neq \bot$, and since $\mathcal{A}$ is a partition, $b = a \leqslant c$. We have established that $\mathcal{C} \preccurlyeq \mathcal{A}$.

$(2 \Leftrightarrow 3)$: This equivalence is an immediate consequence of Lemma 27.

$(3 \Leftrightarrow 4)$: This equivalence follows from two facts: (i) complementation in a Boolean algebra is a dual order-automorphism; and (ii) arbitrary joins and meets in $[\bot, c]$ coincide with those in $\mathbf{B}$. □

Given an e-cyclic residuated lattice $\mathbf{L}$, we denote by $\mathbb{D}(\mathbf{L})$ the join-semilattice of partitions of the Boolean algebra $\mathrm{Pol}(\mathbf{L})$ of polars of $\mathbf{L}$.[7] Recall that if $\mathbf{L}$ is semilinear then each polar of $\mathbf{L}$ is a convex normal subalgebra, by Theorem 25. Thus the following corollary is an immediate consequence of Lemma 28.

Corollary 29. *Let* $\mathbf{L}$ *be an* e*-cyclic residuated lattice and* $\mathcal{C}, \mathcal{A}$ *be partitions of* $\mathrm{Pol}(\mathbf{L})$*, and let* $\mathcal{A}_C = \{A \in \mathcal{A} \mid A \subseteq C\}$*. Then the following are equivalent:*

1. $\mathcal{C} \preccurlyeq \mathcal{A}$*;*
2. *for every* $C \in \mathcal{C}$*,* $\mathcal{A}_C$ *is a partition of the Boolean algebra* $[\{e\}, C]$*;*
3. *for every* $C \in \mathcal{C}$*,* $C = \bigvee^{\mathrm{Pol}(\mathbf{L})} \mathcal{A}_C$*; and*

[7] See Section 3.

4. *for every* $C \in \mathcal{C}$, $C^\perp = \bigcap\{A^\perp \mid A \in \mathcal{A}_C\}$.

If moreover $\mathbf{L}$ *is semilinear, then the previous conditions are equivalent to*

5. *The homomorphism* $f : \mathbf{L}/C^\perp \to \prod\{\mathbf{L}/A^\perp \mid A \in \mathcal{A}_C\}$, *defined by* $f([a]_{C^\perp}) = ([a]_{A^\perp} \mid A \in \mathcal{A}_C)$ *provides a subdirect representation of* $\mathbf{L}/C^\perp$ *in terms of the algebras* $\{\mathbf{L}/A^\perp \mid A \in \mathcal{A}_C\}$.

Proof. The equivalence of the first four conditions follows from Lemma 28. Further, in view of Theorem 25, all polars of $\mathbf{L}$ are normal. Hence, 4 and 5 are equivalent. $\square$

Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice. As we mentioned already, in view of Theorem 25, all polars of $\mathbf{L}$ are normal, and hence for every $C^\perp \in \mathrm{Pol}(L)$ one can form the quotient algebra $\mathbf{L}/C^\perp$. For every partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$, we define the product $\mathbf{L}_{\mathcal{C}} = \prod_{C\in\mathcal{C}} \mathbf{L}/C^\perp$. We will see that if $\mathcal{C}$ and $\mathcal{A}$ are two partitions such that $\mathcal{C} \preccurlyeq \mathcal{A}$ then we can define an injective homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$. The family of homomorphisms of residuated lattices $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A}\}$ satisfies a compatibility property, namely, given three partitions $\mathcal{A} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{C}$, we have

$$\phi_{\mathcal{C}} = id_{\mathcal{C}} \quad \text{and} \quad \phi_{\mathcal{B}\mathcal{C}} \circ \phi_{\mathcal{A}\mathcal{B}} = \phi_{\mathcal{A}\mathcal{C}}.$$

Recall that $\mathbb{D}(\mathbf{L})$ is an join-semilattice, and in particular a directed set. Thus, we can form the direct limit of this family and obtain a residuate lattice $\mathcal{O}(\mathbf{L})$ that will contain all the algebras $\mathbf{L}_{\mathcal{C}}$ in a minimal way. Next section is devoted to the construction of the direct limit of any family of compatible homomorphisms and its basic properties.

6. Direct Limits

The direct limit of a directed family of algebras of the same signature is usually obtained as a suitable homomorphic image of the coproduct of this family. In this section, we describe an explicit construction of the direct limit of a family of algebras (see Theorem 31) that will facilitate the proofs of the results in Sections 7 and 8 on (lateral) completions of e-cyclic semilinear residuated lattices and GMV algebras, respectively. We note that the aforementioned construction is briefly mentioned in [15, (p. 114)] and [30, (Exercises 32 and 33, pp. 155-156)], and somewhat implicitly in [20]. In the sequel we consider exclusively direct limits of algebras and algebra homomorphisms.

Recall that a partially ordered set $(I, \leqslant)$ is said to be *directed* if for any $i, j \in I$ there is a $k \in I$ such that $i, j \leqslant k$. Let $\mathcal{K}$ be a category of algebras and algebra homomorphisms, $(I, \leqslant)$ a directed set, and $\{\mathbf{A}_i \mid i \in I\}$ a family of objects of $\mathcal{K}$. A family $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$ of homomorphisms in $\mathcal{K}$ is a *directed system* for $\{\mathbf{A}_i \mid i \in I\}$ if for every $i \in I$, $f_{ii} = id_{\mathbf{A}_i}$, and for $i \leqslant j \leqslant k$, $f_{jk} \circ f_{ij} = f_{ik}$. That is to say, for any i, j, k such that $i \leqslant j \leqslant k$, the diagram

$$\begin{tikzcd} \mathbf{A}_i \arrow[rr, "f_{ik}"] \arrow[dr, "f_{ij}"'] & & \mathbf{A}_k \\ & \mathbf{A}_j \arrow[ur, "f_{jk}"'] & \end{tikzcd}$$

commutes.[8] Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$ in $\mathcal{K}$, a family of homomorphisms $\{\phi_i : \mathbf{A}_i \to \mathbf{A} \mid i \in I\}$ is said to be *compatible* with the system provided the equation $\phi_j \circ f_{ij} = \phi_i$ holds, for all $i \in I$. Such a family is called a *direct limit* of the directed system if it is "minimal" among the families of homomorphisms compatible with it, in the sense that it satisfies the following universal property: for any family $\{\psi_i : \mathbf{A}_i \to \mathbf{B} \mid i \in I\}$ compatible with $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$, there exists a unique $\psi : \mathbf{A} \to \mathbf{B}$ rendering the following diagram commutative, for all $i \in I$:

$$\begin{tikzcd} \mathbf{A} \arrow[r, dashed, "\exists ! \psi"] & \mathbf{B} \\ \mathbf{A}_i \arrow[u, "\phi_i"] \arrow[ur, "\psi_i"'] & \end{tikzcd} \tag{6}$$

It can be seen that direct limits are unique up to isomorphism: whenever $\{\phi_i : \mathbf{A}_i \to \mathbf{A} \mid i \in I\}$ and $\{\psi_i : \mathbf{A}_i \to \mathbf{B} \mid i \in I\}$ are direct limits of the same system, then there exists a unique isomorphism $\psi : \mathbf{A} \to \mathbf{B}$ rendering commutative the diagram (6). Very often, the common target of the homomorphisms of the direct limit of a system is also called the *direct limit* of the system.

Intuitively, the elements of the direct limit are determined by "approximations", which are elements in the algebras of the system. Thanks to the compatibility of the morphisms of the system and directedness, these approximations can be chosen in algebras with arbitrarily large indices. As we will

[8] More formally, one can think of a directed system in a category $\mathcal{K}$ as a functor $F : \mathcal{I} \to \mathcal{K}$, where the directed set $\mathcal{I} = (I, \leqslant)$ is regarded as a category.

see, the elements of the limit can be represented by sequences of elements in the algebras such that, from one index on, they respect the compatibility law of the system. The behavior of the sequences for indices preceding this index is irrelevant. Thus, two sequences such that from one index on are identical should be considered the same element in the limit. Formally, let $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$ be a directed system in a class $\mathcal{K}$ of algebras of the same signature, and consider the set T of *threads* in $\prod_{i\in I} A_i$:

$$T = \big\{a \in \textstyle\prod_{i\in I} A_i \mid \exists k \forall j \geqslant k,\ a_j = f_{kj}(a_k)\big\}. \tag{7}$$

In the definition of T, and in the sequel, we write a_i instead of $a(i)$, for $a \in \prod_{i\in I} A_i$ and $i \in I$. We define the following binary relation $\sim$ on T, for all $a, b \in T$:

$$a \sim b \quad \Leftrightarrow \quad \exists k \forall j \geqslant k,\ a_j = b_j. \tag{8}$$

The following result can be easily proved.

Lemma 30. *The set T is the universe of a subalgebra $\mathbf{T}$ of $\prod_{i\in I} \mathbf{A}_i$, and moreover $\sim$ is a congruence of $\mathbf{T}$.*

Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$ and the set T of threads defined in (7), we call $i \in I$ a *witness* of $a \in T$, or just a *witness* for a, if for every $k \geqslant i$, $a_k = f_{ik}(a_i)$. By the very definition of T, every thread has a witness and the set of witnesses of a thread is closed upwards.

Now we fix an arbitrary element $u \in \prod_{i\in I} \mathbf{A}_i$, and define the map $\phi_i : A_i \to T$ as follows for all $a \in A_i$:

$$\phi_i(a)_j = \begin{cases} f_{ij}(a) & \text{if } i \leqslant j; \\ u_j & \text{otherwise.} \end{cases} \tag{9}$$

One can easily verify that for each $a \in T$, and each witness i for a, $a \sim \phi_i(a_i)$. The map ϕ_i defined in (9) induces a map $\overline{\phi}_i : A_i \to T/{\sim}$ defined, for all $a \in A_i$, by:

$$\overline{\phi}_i(a) = [\phi_i(a)]_\sim. \tag{10}$$

In what follows, we denote by $\mathbf{A}$ the quotient $\mathbf{T}/{\sim}$. The next result shows that $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{A} \mid i \in I\}$ is the direct limit of the directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$. We sketch its proof for the reader's convenience.

Theorem 31. *Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$, the family of homomorphisms $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{A} \mid i \in I\}$ in Equation* (10) *is the direct limit of $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$. That is, $\mathbf{A}$ has the universal property:*

$$\begin{tikzcd}
\mathbf{A}_i \arrow[rr, "f_{ij}"] \arrow[dr, bend right, "\overline{\phi}_i"] \arrow[ddr, bend right=50, "\psi_i"'] & & \mathbf{A}_j \arrow[dl, bend left, "\overline{\phi}_j"'] \arrow[ddl, bend left=50, "\psi_j"] \\
& \mathbf{A} \arrow[d, dashed, "\exists ! \psi"'] & \\
& \mathbf{B} &
\end{tikzcd}$$

for any family $\{\psi_i : \mathbf{A}_i \to \mathbf{B} \mid i \in I\}$ of homomorphisms compatible with $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$, there is a unique $\psi : \mathbf{A} \to \mathbf{B}$ such that, for every $i \in I$, $\psi \circ \overline{\phi}_i = \psi_i$.

Proof. We leave to the reader to verify that the system $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{T}/{\sim} \mid i \in I\}$ is indeed a family of homomorphisms compatible with the directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$

Suppose that $a, b \in T$ are such that $a \sim b$, and let i, j be witnesses for a, b, respectively, and k such that a and b agree from k on. Let us consider any $r \geqslant i, j, k$, which exists since I is a directed set. Thus, a and b agree on r and therefore

$$\psi_i(a_i) = \psi_r(f_{ir}(a_i)) = \psi_r(a_r) = \psi_r(b_r) = \psi_r(f_{jr}(b_j)) = \psi_j(b_j).$$

Therefore, we can define the map $\psi : \mathbf{A} \to \mathbf{B}$ in the following way: for every $[a]_\sim \in \mathbf{A}$,

$$\psi([a]_\sim) = \psi_i(a_i),$$

where i is any witness for a.

Let σ be an n-ary operation symbol in the signature and $a^1, \ldots, a^n \in T$ with common witness k. Then, it can be easily seen that k is also a witness for $\sigma^{\mathbf{T}}(a^1, \ldots, a^n)$, and hence:

$$\begin{aligned}
\psi(\sigma^{\mathbf{A}}([a^1]_\sim, \ldots, [a^n]_\sim)) &= \psi([\sigma^{\mathbf{T}}(a^1, \ldots, a^n)]_\sim) = \psi_k(\sigma^{\mathbf{T}}(a^1, \ldots, a^n)_k) \\
&= \psi_k(\sigma^{\mathbf{A}_k}(a^1_k, \ldots, a^n_k)) = \sigma^{\mathbf{B}}(\psi_k(a^1_k), \ldots, \psi_k(a^n_k)) \\
&= \sigma^{\mathbf{B}}(\psi([a^1]_\sim), \ldots, \psi([a^n]_\sim)).
\end{aligned}$$

That ψ renders the diagram commutative is a direct consequence of the fact that, for every $i \in I$, and every $a \in \mathbf{A}_i$, i is a witness for $\phi_i(a)$. As regards the uniqueness, note that if i is a witness of $a \in T$, then $a \sim \phi_i(a_i)$, and therefore if $\psi' : \mathbf{A} \to \mathbf{B}$ is a map rendering commutative the diagram, then

$$\psi'([a]_\sim) = \psi'([\phi_i(a_i)]_\sim) = \psi'(\overline{\phi}_i(a_i)) = \psi_i(a_i) = \psi([a]_\sim). \qquad \square$$

We introduce the concept of proxy that will facilitate the discussion in the sequel.

Definition 32. If $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j \mid i, j \in I,\ i \leqslant j\}$ is a directed system, $i \in I$, and $x \in \mathbf{A}_i$, we call x a *proxy* of $\overline{\phi}_i(x)$ at i.

Note that if $[a]_\sim \in \mathbf{A}$, and i is a witness for a, then a_i is a proxy of $[a]_\sim$ at i. Consequently, every element of the limit has a proxy at some index i, and the set of indices where a particular element has a proxy is closed upwards. Moreover, if $i \leqslant j$, $x \in \mathbf{A}_i$ and $y = f_{ij}(x)$, then x is a proxy of an element $s \in \mathbf{A}$ at i if and only if y is a proxy of s at j.

We note the following result for future reference:

Lemma 33. *If all the homomorphisms of a directed system of algebras are embeddings, then the homomorphisms of the direct limit are also embeddings.*

Under the assumptions of the preceding lemma, whenever an element of the direct limit $\mathbf{A}$ has a proxy in $i \in I$, this proxy is unique. Note also that, as a consequence of Theorem 31, the direct limit of a directed system is the quotient of a subalgebra of the product of the algebras of the system. Thus, varieties are closed under direct limits. In fact, it can be shown that the same result holds for quasivarieties.[9]

7. $\mathcal{O}(\mathbf{L})$ is laterally complete

We devote this section to the construction of a laterally complete[10] extension, $\mathcal{O}(\mathbf{L})$, of an arbitrary semilinear e-cyclic residuated lattice $\mathbf{L}$. The fundamental property connecting $\mathbf{L}$ and $\mathcal{O}(\mathbf{L})$ is the fact that $\mathbf{L}$ is a dense[11]

[9] Actually, a stronger result can be proved. Namely, given a set of quasi-equations Σ and a directed system of algebras indexed on I, if the set $F \subseteq I$ of indices of the algebras satisfying Σ is cofinal in I, that is, for every index $i \in I$ there is another index $j \in F$ such that $i \leqslant j$ and Σ valid in the algebra indexed by j, then the direct limit also satisfies Σ.

[10] Refer to Definition 40 below.

[11] Refer to Definition 38 below.

subalgebra of $\mathcal{O}(\mathbf{L})$. Using the direct limit construction of $\mathcal{O}(\mathbf{L})$, we obtain the main result of this section, Theorem 49, which asserts that any algebra in a variety $\mathcal{V}$ of semilinear e-cyclic residuated lattices can be densely embedded into a laterally complete algebra in $\mathcal{V}$. Most of the effort in proving these results goes into verifying that $\mathcal{O}(\mathbf{L})$ is laterally complete (Theorem 48). It is worth mentioning that it is a trivial matter to embed a semilinear residuated lattice $\mathbf{L}$ into a laterally complete one. Indeed, $\mathbf{L}$ can be embedded into a product of chains, which is clearly laterally complete. The dense embeddability of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$ guarantees that the latter is not "too large," in fact it is an essential extension of the former.

Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice. In view of Theorem 25, all polars of $\mathbf{L}$ are normal, and hence for every $C \in \mathrm{Pol}(L)$ one can form the quotient algebra $\mathbf{L}/C^{\perp}$. For every partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$, we define the product $\mathbf{L}_{\mathcal{C}} = \prod_{C\in\mathcal{C}} \mathbf{L}/C^{\perp}$.

If $\mathcal{C}$ and $\mathcal{A}$ are two partitions with $\mathcal{C} \preccurlyeq \mathcal{A}$, we define a homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$ as follows (see Diagram (11)): for every $A \in \mathcal{A}$, there is a unique $C \in \mathcal{C}$ such that $A \subseteq C$. Then, $C^{\perp} \subseteq A^{\perp}$, whence there exists a homomorphism $f_{CA} : \mathbf{L}/C^{\perp} \to \mathbf{L}/A^{\perp}$. Composing with the canonical projection $\pi_C : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}/C^{\perp}$, we obtain a homomorphism $f_{CA}\pi_C : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}/A^{\perp}$. Then, by the couniversal property of the product $\mathbf{L}_{\mathcal{A}}$, there exists a unique homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$ such that for all $A \in \mathcal{A}$, $\pi_A\phi_{\mathcal{C}\mathcal{A}} = f_{CA}\pi_C$, where $\pi_A : \mathbf{L}_{\mathcal{A}} \to \mathbf{L}/A^{\perp}$ is the canonical projection.

$$\begin{tikzcd}
\mathbf{L}_{\mathcal{C}} \arrow[r, dashed, "\phi_{\mathcal{C}\mathcal{A}}"] \arrow[d, two heads, "\pi_C"'] \arrow[dr, "f_{CA}\circ\pi_C"] & \mathbf{L}_{\mathcal{A}} \arrow[d, two heads, "\pi_A"] \\
\mathbf{L}/C^{\perp} \arrow[r, "f_{CA}"'] & \mathbf{L}/A^{\perp}
\end{tikzcd} \tag{11}$$

We can describe $\phi_{\mathcal{C}\mathcal{A}}$ as follows: every element $x \in \mathbf{L}_{\mathcal{C}}$ is of the form $x = ([x_C]_{C^{\perp}} \mid C \in \mathcal{C})$, with $x_C \in L$. Then, $\phi_{\mathcal{C}\mathcal{A}}(x) = ([y_A]_{A^{\perp}} \mid A \in \mathcal{A})$, where for every $A \in \mathcal{A}$, $y_A = x_C$, for the unique $C \in \mathcal{C}$ such that $A \subseteq C$. Recall that the ordered set $\mathbb{D}(\mathbf{L})$ of all partitions is a join-semilattice. Hence any two partitions have a least common refinement. Using the previous description one can easily show that $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A}\}$ is a directed system. We denote the direct limit of this system by $\mathcal{O}(\mathbf{L})$. Our objective in this section is to prove that $\mathcal{O}(\mathbf{L})$ is laterally complete and has $\mathbf{L}$ as a dense subalgebra (see Definitions 38 and 40 below).

Let us specialize the discussion of Section 6 to the system $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A}\}$. We start with the subalgebra $\mathbf{T}$ of $\prod_{\mathcal{C}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{C}}$ whose universe is the set of threads defined in Equation (7):

$$T = \left\{ l \in \prod_{\mathcal{B}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{B}} \mid \exists \mathcal{C}\, \forall \mathcal{A} \succcurlyeq \mathcal{C},\ l_{\mathcal{A}} = \phi_{\mathcal{C}\mathcal{A}}(l_{\mathcal{C}}) \right\}.$$

Then we obtain $\mathcal{O}(\mathbf{L})$ as the quotient of $\mathbf{T}$ by the congruence $\sim$ in Equation (8), which in this case reads as follows: $l \sim k$ if there exists a partition $\mathcal{C}$ such that for any refinement $\mathcal{A}$ of it, $l_{\mathcal{A}} = k_{\mathcal{A}}$. Therefore, the elements of $\mathcal{O}(\mathbf{L})$ will be the equivalence classes of the elements $l = (l_{\mathcal{C}} \mid \mathcal{C} \in \mathbb{D}(\mathbf{L})) \in T$. As we have noted in the previous section, for every partition $\mathcal{C}$, there exists a homomorphism $\overline{\phi}_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathcal{O}(\mathbf{L})$. More specifically, we first fix an element of $\prod_{\mathcal{B}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{B}}$, in this case we can conveniently choose the identity element e. We then define, for every $x \in \mathbf{L}_{\mathcal{C}}$, $\overline{\phi}_{\mathcal{C}}(x) = [\phi_{\mathcal{C}}(x)]_{\sim}$, where $\phi_{\mathcal{C}}(x) \in \prod_{\mathcal{C}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{C}}$ is such that

$$\phi_{\mathcal{C}}(x)_{\mathcal{A}} = \begin{cases} \phi_{\mathcal{C}\mathcal{A}}(x) & \text{if } \mathcal{C} \preccurlyeq \mathcal{A} \\ e_{\mathcal{C}} & \text{otherwise.} \end{cases} \tag{12}$$

Furthermore, since all the homomorphisms $\phi_{\mathcal{C}\mathcal{A}}$ are embeddings (see Lemma 37), the same is true for the homomorphisms $\overline{\phi}_{\mathcal{C}}$ by Lemma 33.

It is important, for any element $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C}) \in \mathbf{L}_{\mathcal{C}}$, to distinguish all polars C such that $[x_C]_{C^\perp} \neq [e]_{C^\perp}$. A criterion is provided by the next lemma, whose easy proof is left to the reader.

Lemma 34. *If $\mathbf{L}$ is an e-cyclic residuated lattice and $H \in \mathrm{Pol}(\mathbf{L})$ is normal, then the following statements are equivalent:*

(i) $[a]_H = [e]_H$,

(ii) $a \in H$,

(iii) $C[a] \subseteq H$,

(iv) $C[a] \cap H^\perp = \{e\}$,

(v) $a^{\perp\perp} \cap H^\perp = \{e\}$.

In what follows, some concepts relative to an element $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C}) \in \mathbf{L}_{\mathcal{C}}$ will be defined in terms of the representatives x_C of the equivalent classes $[x_C]_{C^\perp}$. The following lemma shows that these notions are independent of the choice of representatives.

Lemma 35. *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice,* $H \in \mathrm{Pol}(\mathbf{L})$ *be a normal convex subalgebra, and* $a, b \in L$*. If* $[a]_{H^\perp} = [b]_{H^\perp}$ *then* $a^{\perp\perp} \cap H = b^{\perp\perp} \cap H$*.*

Proof. Suppose that $[a]_{H^\perp} = [b]_{H^\perp}$. Then, $[|a|]_{H^\perp} = [|b|]_{H^\perp}$, and therefore by Lemma 19, $(|a|\backslash|b|) \wedge (|b|\backslash|a|) \wedge e \in H^\perp$. This means that there exists $c \in H^\perp$ such that $|a|c \leqslant b$ and $|b|c \leqslant a$, and hence, for any $d \in b^\perp$:

$$|c| = e \cdot |c| = (|d| \vee |b|)|c| = |d||c| \vee |b||c| \leqslant |d| \vee |b|c \leqslant |d| \vee |a|.$$

Therefore, for any $h \in H$, $e = |c| \vee |h| \leqslant |d| \vee |a| \vee |h|$, whence, $|h| \vee |d| \in a^\perp$. If moreover $h \in a^{\perp\perp}$, then $|h| \vee |d| \in a^{\perp\perp}$, and then $|h| \vee |d| \in a^{\perp\perp} \cap a^\perp = \{e\}$. Therefore $|h| \vee |d| = e$. Thus, for any $h \in a^{\perp\perp} \cap H$ and $d \in b^\perp$, $|h| \vee |d| = e$; whence $h \in b^{\perp\perp}$, as we wanted. □

The converse of Lemma 35 is not true in general. For example, consider the three element Gödel chain and let $H = \left\{0, \frac{1}{2}, 1\right\}$, $a = 0$ and $b = \frac{1}{2}$. It can be seen that $0^{\perp\perp} \cap H = \frac{1}{2}^{\perp\perp} \cap H$, but $[0]_{H^\perp} = [0]_{\{e\}} \neq \left[\frac{1}{2}\right]_{\{e\}} = \left[\frac{1}{2}\right]_{H^\perp}$.

Definition 36. Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, and let $\mathcal{O}(\mathbf{L})$ be the direct limit of $\{\phi_{\mathcal{CA}} : \mathbf{L}_\mathcal{C} \to \mathbf{L}_\mathcal{A} \mid \mathcal{C} \preccurlyeq \mathcal{A}\}$. Given a partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$ and an element $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C}) \in \mathbf{L}_\mathcal{C}$, we define the *support* of x at $\mathcal{C}$ to be the set

$$\mathrm{Supp}(x) = \{C \in \mathcal{C} \mid [x_C]_{C^\perp} \neq [e]_{C^\perp}\}.$$

It is clear from the definition that, for any $x \in \mathbf{L}_\mathcal{C}$, x is equal to the identity element $e_\mathcal{C}$ of $\mathbf{L}_\mathcal{C}$ if and only if $\mathrm{Supp}(x) = \emptyset$.

Lemma 37. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice and let* $\mathcal{C}, \mathcal{A}$ *two partitions such that* $\mathcal{C} \preccurlyeq \mathcal{A}$*. For every* $C \in \mathcal{C}$ *let* $\mathcal{A}_C = \{A \in \mathcal{A} \mid A \subseteq C\}$*. Then:*

1. *For all* $x \in \mathbf{L}_\mathcal{C}$*,* $C \in \mathrm{Supp}(x)$ *if and only if* $A \in \mathrm{Supp}(\phi_{\mathcal{CA}}(x))$*, for some* $A \in \mathcal{A}_C$*; and*

2. $\phi_{\mathcal{CA}}$ *is injective.*

Proof. Both (1) and (2) follow directly from Corollary 29.(5). Let $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C}) \in \mathbf{L}_\mathcal{C}$ and let $y = ([y_A]_{A^\perp} \mid A \in \mathcal{A}) = \phi_{\mathcal{CA}}(x) \in \mathbf{L}_\mathcal{A}$. As we noted above, if $C \in \mathcal{C}$ and $A \in \mathcal{A}_C$, then we can choose $y_A = x_C$. Thus, for any $C \in \mathcal{C}$, since $\mathbf{L}/C^\perp$ is a subdirect product of the algebras in $\{\mathbf{L}/A^\perp \mid A \in \mathcal{A}_C\}$, $[x_C]_{C^\perp} \neq [e]_{C^\perp}$ if and only if $[y_A]_{A^\perp} \neq [e]_{A^\perp}$ for some $A \in \mathcal{A}_C$. □

As noted in Lemma 37, $\phi_{\mathcal{C}\mathcal{A}}$ is injective whenever $\mathcal{C} \preccurlyeq \mathcal{A}$. Whence, for the particular case of the trivial partition $\{L\}$, we have $\mathbf{L}_{\{L\}} = \mathbf{L}/L^{\perp} = \mathbf{L}/\{e\} \cong \mathbf{L}$. Therefore, there exists an embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$, more specifically the composition of the isomorphism $\mathbf{L} \cong \mathbf{L}/\{e\}$ with the embedding $\overline{\phi}_{\{L\}}$. In Theorem 39 below, we prove that this embedding is dense in the sense of the next definition.

Definition 38. An embedding $\phi : \mathbf{L} \to \mathbf{L}'$ between residuated lattices is *dense* if for every $p \in L'$, $p < e$, there exists $a \in L$ such that $p \leqslant \phi(a) < e$.

Recall that every element of $\mathcal{O}(\mathbf{L})$ has a proxy at some partition $\mathcal{C}$. That is, given an element $p \in \mathcal{O}(\mathbf{L})$ there exists a partition $\mathcal{C}$ and an element $x \in \mathbf{L}_{\mathcal{C}}$ such that $\overline{\phi}_{\mathcal{C}}(x) = p$. Let us note that, for any partition $\mathcal{C}$, if an element p has a proxy x at $\mathcal{C}$, then x is unique, since $\overline{\phi}_{\mathcal{C}}$ is an embedding. Clearly, an element of $\mathcal{O}(\mathbf{L})$ is different from the identity if and only if all its proxies, at the different partitions in which they exist, are different from the identity.

Theorem 39. *Any e-cyclic semilinear residuated lattice* $\mathbf{L}$ *can be densely embedded into* $\mathcal{O}(\mathbf{L})$.

Proof. As noted above, the map $\overline{\phi} : \mathbf{L} \xrightarrow{\cong} \mathbf{L}_{\{L\}} \xrightarrow{\overline{\phi}_{\{L\}}} \mathcal{O}(\mathbf{L})$ is an embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$. For every $a \in L$, $\overline{\phi}(a) = [\overline{a}]_{\sim}$, where $\overline{a} = (\overline{a}_{\mathcal{C}} \mid \mathcal{C} \in \mathbb{D}(\mathbf{L}))$, and for every partition $\mathcal{C}$, $\overline{a}_{\mathcal{C}} = ([a]_{C^{\perp}} \mid C \in \mathcal{C})$.

In order to establish the density of $\overline{\phi}$, consider $p \in \mathcal{O}(\mathbf{L})$ such that $p < e_{\mathcal{O}(\mathbf{L})}$. Let $x = ([x_C]_{C^{\perp}} \mid C \in \mathcal{C})$ be a proxy of p at some partition $\mathcal{C}$. Then, for every $C \in \mathcal{C}$, $[x_C]_{C^{\perp}} \leqslant [e]_{C^{\perp}}$, and hence without loss of generality we can assume that all the representatives x_C are negative. Since $p \neq e_{\mathcal{O}(\mathbf{L})}$, there exists a $C \in \mathcal{C}$ such that $[x_C]_{C^{\perp}} \neq [e]_{C^{\perp}}$. Therefore, by Lemma 34.(v), $x_C^{\perp\perp} \cap C \neq \{e\}$ and we can choose an element $b \in x_C^{\perp\perp} \cap C$, such that $b < e$. By the convexity of the polars, $a = x_C \vee b \in x_C^{\perp\perp} \cap C$. If $a = e$, then, since $b \in x_C^{\perp\perp}$, $b \in x_C^{\perp} \cap x_C^{\perp\perp} = \{e\}$, and therefore $b = e$, contradicting the hypothesis that $b < e$. Hence, $x_C \leqslant a < e$. Since $a \in C$, $a^{\perp\perp} \subseteq C^{\perp\perp} = C$, and hence, since polars in $\mathcal{C}$ are pairwise disjoint, for every $D \in \mathcal{C}$, $C \neq D$ implies $a^{\perp\perp} \cap D = \{e\}$, and therefore $[a]_{D^{\perp}} = [e]_{D^{\perp}}$. Thus, $\overline{a}_{\mathcal{C}} = ([a]_{C^{\perp}} \mid C \in \mathcal{C})$ has only one component different from the identity, which is $[a]_{C^{\perp}}$. Moreover $x_C \leqslant a$ implies $[x_C]_{C^{\perp}} \leqslant [a]_{C^{\perp}}$. Hence $x \leqslant \overline{a}_{\mathcal{C}} < e$, and then $p = \overline{\phi}_{\mathcal{C}}(x) \leqslant \overline{\phi}_{\mathcal{C}}(\overline{a}_{\mathcal{C}}) = \overline{\phi}(a) < e_{\mathcal{O}(\mathbf{L})}$. $\square$

Definition 40. Two elements $a, b < e$ of a residuated lattice $\mathbf{L}$ are said to be *disjoint* if $a \vee b = e$. An non-empty subset $X \subseteq L$ is called *disjoint* provided any two distinct elements of it are disjoint. A residuated lattice is said to be *laterally complete* if all its disjoint subsets have an infimum.

Remark 41. *Let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of elements of $\mathbf{L}_{\mathcal{C}}^-$, for some partition $\mathcal{C}$, which have pairwise disjoint supports:* $\mathrm{Supp}(x_\lambda) \cap \mathrm{Supp}(x_\mu) = \emptyset$ *if $\lambda \neq \mu$. Then the meet $\bigwedge_\Lambda^{\mathbf{L}_\mathcal{C}} x_\lambda$ exists. Actually, $\bigwedge_\Lambda^{\mathbf{L}_\mathcal{C}} x_\lambda = z = ([z_C]_{C^\perp} \mid C \in \mathcal{C})$, where*

$$z_C = \begin{cases} (x_\lambda)_C & \textit{if } C \in \mathrm{Supp}(x_\lambda), \textit{ for some (unique) } \lambda \in \Lambda; \\ e & \textit{otherwise.} \end{cases}$$

In the remainder of the section we prove that, given a family of disjoint elements $S \subseteq \mathcal{O}(\mathbf{L})$, there exists a partition $\mathcal{E}$ such that (i) every element of the disjoint family has a proxy at $\mathcal{E}$, (ii) the proxies at $\mathcal{E}$ of the elements in S have disjoint support, and (iii) their meet is a proxy of the meet of S.

Remark 42. *Given a proxy $x \in \mathbf{L}_\mathcal{C}$ of an element $p \in \mathcal{O}(\mathbf{L})$, exactly one of the following situations occurs for every $C \in \mathcal{C}$:*

(i) $x_C^{\perp\perp} \cap C = \{e\}$,

(ii) $C \subseteq x_C^{\perp\perp}$, *or*

(iii) $x_C^{\perp\perp} \cap C \neq \{e\}$ *and* $C \nsubseteq x_C^{\perp\perp}$.

By virtue of Lemma 34, (i) is equivalent to $[x_C]_{C^\perp} = [e]_{C^\perp}$, that is, $C \notin \mathrm{Supp}(x)$, while (ii) implies that $C \in \mathrm{Supp}(x)$.

Definition 43. Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice and $\mathcal{C}$ a partition of the Boolean algebra of polars of $\mathbf{L}$. An element $x \in \mathbf{L}_\mathcal{C}$ is said to be *canonical* if for every $C \in \mathrm{Supp}(x)$, $C \subseteq x_C^{\perp\perp}$.

Notice that canonicity is a well-defined notion, since it does not depend on the representatives: if $[a]_{C^\perp} = [b]_{C^\perp}$ then, by virtue of Lemma 35, $a^{\perp\perp} \cap C = b^{\perp\perp} \cap C$, and therefore $C \subseteq a^{\perp\perp}$ if and only if $C \subseteq b^{\perp\perp}$. It is also important to note, and easy to prove, that canonicity is preserved by refinements: if $x \in \mathbf{L}_\mathcal{C}$ is canonical and $\mathcal{C} \preccurlyeq \mathcal{A}$, then $\phi_{\mathcal{C}\mathcal{A}}(x)$ is also canonical.

We now prove two fairly technical lemmas that will be useful in subsequent proofs. Note that Lemma 45 shows that Condition (iii) in Remark 42 is avoidable: proxies can be chosen so that their coordinates satisfy either (i) or (ii).

Lemma 44. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice and let* $\mathcal{C}, \mathcal{A}$ *be two partitions such that* $\mathcal{C} \preccurlyeq \mathcal{A}$*. Then, whenever* $y \in \mathbf{L}_{\mathcal{A}}$ *and* $\mathrm{Supp}(y) \subseteq \mathcal{C}$*, then there is a (unique)* $x \in \mathbf{L}_{\mathcal{C}}$ *such that* $\phi_{\mathcal{C}\mathcal{A}}(x) = y$*. Furthermore,* $\mathrm{Supp}(x) = \mathrm{Supp}(y)$*.*

Proof. Let $y = ([y_A]_{A^\perp} \mid A \in \mathcal{A}) \in \mathbf{L}_{\mathcal{A}}$ such that $\mathrm{Supp}(y) \subseteq \mathcal{C}$. For every $C \in \mathcal{C}$, we define $x_C = y_C$ if $C \in \mathrm{Supp}(y)$, and $x_C = e$ otherwise, and set $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C}) \in \mathbf{L}_{\mathcal{C}}$. Then obviously $\mathrm{Supp}(x) = \mathrm{Supp}(y)$.

We claim that $\phi_{\mathcal{C}\mathcal{A}}(x) = y$, which will establish the statement of the lemma. Let $\phi_{\mathcal{C}\mathcal{A}}(x) = ([t_A]_{A^\perp} \mid A \in \mathcal{A})$. Recall that for each $A \in \mathcal{A}$, we can choose $t_A = x_C$, where C is the unique element in $\mathcal{C}$ such that $A \subseteq C$. Consider A and C as in the previous sentence. If $C \in \mathrm{Supp}(y)$, which by assumption is a subset of $\mathcal{C}$, then $A \subseteq C \in \mathcal{A}$ implies $A = C \in \mathrm{Supp}(y)$, since polars in $\mathcal{A}$ are pairwise disjoint. Thus, if $A \notin \mathrm{Supp}(y)$, then $C \notin \mathrm{Supp}(y) = \mathrm{Supp}(x)$, and therefore $[t_A]_{A^\perp} = [e_A]_{A^\perp} = [y_A]_{A^\perp}$. On the other hand, if $A \in \mathrm{Supp}(y)$, then $C = A$, because $A \in \mathcal{C} \supseteq \mathrm{Supp}(y)$, and therefore $t_A = x_A = y_A$. Thus, we have shown that $\phi_{\mathcal{C}\mathcal{A}}(x) = y$, as required. □

Lemma 45. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice, and let* $\mathcal{O}(\mathbf{L})$ *be the direct limit of the family* $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A}\}$*. Consider an arbitrary partition* $\mathcal{C}$ *and an element* $p \neq e$ *in* $\mathcal{O}(\mathbf{L})$*. Then:*

1. *If* x *is a proxy of* p *at* $\mathcal{C}$*, then there is a refinement* $\mathcal{A}$ *of* $\mathcal{C}$ *such that* $y = \phi_{\mathcal{C}\mathcal{A}}(x)$ *is canonical.*

2. *If* x *is a proxy of* p *at* $\mathcal{C}$ *and* $\mathcal{B}$ *is any partition such that* $\mathrm{Supp}(x) \subseteq \mathcal{B}$*, then* p *has a proxy* z *at* $\mathcal{B}$ *and* $\mathrm{Supp}(z) = \mathrm{Supp}(x)$*. Moreover, if* x *is canonical, then so is* z*.*

Proof.
1. Let x be a proxy of p at $\mathcal{C}$ and consider the set $\mathcal{E}_x = \{x_C^{\perp\perp} \cap C \mid C \in \mathrm{Supp}(x)\}$. Since $\mathcal{C}$ is a disjoint family of polars, so is $\mathcal{E}_x$. Moreover $\{e\} \notin \mathcal{E}_x$, and therefore it can be extended to a partition $\bar{\mathcal{E}}_x$. Consider the common refinement $\mathcal{A} = \mathcal{C} \vee \bar{\mathcal{E}}_x$ (see Equation (5)) of both $\mathcal{C}$ and $\bar{\mathcal{E}}_x$. Notice that $\mathcal{E}_x \subseteq \mathcal{A}$ because, if $E \in \mathcal{E}_x$, then there is $C \in \mathcal{C}$ such that $\{e\} \neq E = x_C^{\perp\perp} \cap C$, whence $E = C \cap E \in \mathcal{C} \vee \bar{\mathcal{E}}_x$.

Let $y = \phi_{\mathcal{C}\mathcal{A}}(x)$. As usual, we choose the representatives of y as follows: $y_A = x_C$, where for every $A \in \mathcal{A}$, C is the unique polar such that $A \subseteq C \in \mathcal{C}$. In order to prove the canonicity of y, consider an arbitrary $A \in \mathcal{A}$. If

$A \in \mathcal{E}_x$, then there is $C \in \mathrm{Supp}(x)$ such that $A = x_C^{\perp\perp} \cap C \subseteq x_C^{\perp\perp} = y_A^{\perp\perp}$. If $A \notin \mathcal{E}_x$, let C the unique polar in $\mathcal{C}$ such that $A \subseteq C$. If $C \notin \mathrm{Supp}(x)$, then $y_A^{\perp\perp} \cap A \subseteq x_C^{\perp\perp} \cap C = \{e\}$, whence $A \notin \mathrm{Supp}(y)$. If $C \in \mathrm{Supp}(x)$, then $x_C^{\perp\perp} \cap C$ and A are two distinct elements of $\mathcal{A}$ (since $A \notin \mathcal{E}_x$), and so $(x_C^{\perp\perp} \cap C) \cap A = \{e\}$, since all polars in $\mathcal{A}$ are pairwise disjoint. Hence

$$y_A^{\perp\perp} \cap A = x_C^{\perp\perp} \cap (C \cap A) = (x_C^{\perp\perp} \cap C) \cap A = \{e\},$$

showing that $A \notin \mathrm{Supp}(y)$. Therefore, $\mathrm{Supp}(y) = \mathcal{E}_x$, and then for every $A \in \mathrm{Supp}(y)$, $A \subseteq y_A^{\perp\perp}$.

2. Suppose now that x is a proxy of p at $\mathcal{C}$ and that $\mathcal{B}$ is a partition such that $\mathrm{Supp}(x) \subseteq \mathcal{B}$. Consider $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$ and $y = \phi_{\mathcal{CA}}(x)$, where we choose the representatives of y as usual. Since $\mathrm{Supp}(x) \subseteq \mathcal{B}$ and $\mathrm{Supp}(x) \subseteq \mathcal{C}$, then obviously $\mathrm{Supp}(x) \subseteq \mathcal{A}$, whence it follows that $\mathrm{Supp}(y) = \mathrm{Supp}(x)$. By virtue of Lemma 44, there is $z \in \mathbf{L}_{\mathcal{B}}$ such that $\phi_{\mathcal{B}}(z) = y$ and $\mathrm{Supp}(z) = \mathrm{Supp}(y) = \mathrm{Supp}(x)$. Moreover, if x is canonical then y is canonical, and by the way we constructed z, we deduce also the canonicity of z. $\square$

We'll make use of the following simple lemma:

Lemma 46. *Let $p \in \mathcal{O}(\mathbf{L})$ and let x be a proxy of p at $\mathcal{C}$. Then*

1. $\mathrm{Supp}(x) = \mathrm{Supp}(|x|)$*;*

2. *The element $|x|$ is the proxy of $|p|$ at $\mathcal{C}$, and $|x|$ is canonical whenever x is.*

Proof. 1. Let x be a proxy of p at $\mathcal{C}$. Then, by Lemmas 8, 34, and 35, for any $C \in \mathcal{C}$, $C \in \mathrm{Supp}(x)$ iff $[x_C]_{C^\perp} \neq [e]_{C^\perp}$ iff $x^{\perp\perp} \cap C \neq \{e\}$ iff $|x|^{\perp\perp} \cap C \neq \{e\}$ iff $[|x_C|]_{C^\perp} \neq [e]_{C^\perp}$ iff $C \in \mathrm{Supp}(|x|)$. 2. It is clear that $|x|$ is the proxy of $|p|$ at $\mathcal{C}$. If $C \in \mathrm{Supp}(|x_C|)$, then $C \in \mathrm{Supp}(x_C)$, by item (1). Then, $C \subseteq x_C^{\perp\perp} = |x_C|^{\perp\perp}$. $\square$

The next lemma is the missing piece that we need to prove Theorem 48. We have already seen that we can choose proxies in a canonical way and that, under certain conditions, we can move them from one partition to another. Intuitively, all the information about an element is included in the coordinates of its support, and this information transfers from a partition to any other partition that contains its support. In Lemma 47 below, we show that elements in $\mathcal{O}(\mathbf{L})$ are disjoint if and only if the polars in their supports are a disjoint family.

Lemma 47. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice,* $p, q \in \mathcal{O}(\mathbf{L})$*, and* x *and* y *canonical proxies of* p *and* q *at some partitions* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively. Then,* $|p|$ *and* $|q|$ *are disjoint elements of* $\mathcal{O}(\mathbf{L})$ *if and only if* $\mathrm{Supp}(x) \cup \mathrm{Supp}(y)$ *is a disjoint set of polars of* $\mathbf{L}$*.*

Proof. Without loss of generality, by virtue of Lemma 46, we can assume that p, q are negative, and therefore we can take all the representatives of x and y negative. Let $\mathcal{A} = \mathcal{C} \vee \mathcal{D}$, $s = \phi_{\mathcal{C}\mathcal{A}}(x)$ and $t = \phi_{\mathcal{D}\mathcal{A}}(y)$, where the representatives of s and t are chosen in the usual way.

Suppose that $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are such that $A = C \cap D \neq \{e\}$. Notice that $(s_A \vee t_A)^{\perp\perp} = (x_C \vee y_D)^{\perp\perp} = x_C^{\perp\perp} \cap y_D^{\perp\perp}$, by virtue of Lemma 21. One can easily see that the result follows from the fact that:

$$A \in \mathrm{Supp}(s \vee t) \quad \Leftrightarrow \quad C \in \mathrm{Supp}(x) \text{ and } D \in \mathrm{Supp}(y).$$

The implication $(\Rightarrow)$ is clear. For the reverse implication $(\Leftarrow)$ we make use of the canonicity of x and y. Indeed, if $C \in \mathrm{Supp}(x)$ and $D \in \mathrm{Supp}(y)$, then $C \subseteq x_C^{\perp\perp}$ and $D \subseteq y_D^{\perp\perp}$, and hence $\{e\} \neq A = C \cap D \subseteq x_C^{\perp\perp} \cap y_D^{\perp\perp} = (s_A \vee t_A)^{\perp\perp}$. □

We now have all the tools we need to prove that $\mathcal{O}(\mathbf{L})$ is actually laterally complete. The proof proceeds as follows: for any disjoint set of negative elements S of $\mathcal{O}(\mathbf{L})$, canonical proxies with pairwise disjoint supports are chosen. Then, these supports are collected into a partition that possesses proxies of the elements of S. Lastly, it is shown that the infimum of these proxies exists and is a proxy of the infimum of the original family.

Theorem 48. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathcal{O}(\mathbf{L})$ *is laterally complete.*

Proof. Let $\{p_\lambda \mid \lambda \in \Lambda\}$ be a disjoint subset of $\mathcal{O}(\mathbf{L})$, and for every $\lambda \in \Lambda$, let x_λ be a canonical proxy of p_λ at some partition $\mathcal{C}_\lambda$. Then, by Lemma 47, the set $\bigcup_\Lambda \mathrm{Supp}(x_\lambda)$ is a disjoint set of polars of $\mathbf{L}$ and can be extended to a partition $\mathcal{E}$. Now, for every $\lambda \in \Lambda$, $\mathcal{E}$ is a partition containing $\mathrm{Supp}(x_\lambda)$, and then by virtue of Lemma 45, p_λ has a canonical proxy x'_λ at $\mathcal{E}$ and $\mathrm{Supp}(x'_\lambda) = \mathrm{Supp}(x_\lambda)$. It follows that the supports of the elements x'_λ at $\mathcal{E}$ are all disjoint, and therefore their meet $z = \bigwedge_\Lambda x'_\lambda$ in $\mathbf{L}_\mathcal{E}$ exists, by Remark 41.

We complete the proof by showing that $\bigwedge_\Lambda p_\lambda$ exists and z is its proxy at $\mathcal{E}$. Since $z \leqslant x'_\lambda$ for all $\lambda \in \Lambda$, $\overline{\phi}_\mathcal{E}(z) \leqslant \overline{\phi}_\mathcal{E}(x'_\lambda) = p_\lambda$. Suppose now that $q \in \mathcal{O}(\mathbf{L})$ is a lower bound of $\{p_\lambda \mid \lambda \in \Lambda\}$, let y be a proxy of q at some partition $\mathcal{C}$, and let $\mathcal{A}$ be a refinement of $\mathcal{E}$ and $\mathcal{C}$. Set $y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(x'_\lambda)$, for every $\lambda \in \Lambda$. It can be seen that $\bigwedge_\Lambda y_\lambda$ exists in $\mathbf{L}_\mathcal{A}$, and actually $\bigwedge_\Lambda y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(z)$: clearly, $\phi_{\mathcal{E}\mathcal{A}}(z) \leqslant y_\lambda$, for every $\lambda \in \Lambda$. Suppose now that $s \in \mathbf{L}_\mathcal{A}$ and for every $\lambda \in \Lambda$, $s \leqslant y_\lambda$. Fix $A \in \mathcal{A}$ and let $E \in \mathcal{E}$ be the unique element in $\mathcal{E}$ such that $A \subseteq E$. Since all the supports of the x'_λ are disjoint, then there is at most one $\lambda_0 \in \Lambda$ such that $[(x'_{\lambda_0})_E]_{E^\perp} \neq [e]_{E^\perp}$, in which case $[z_E]_{E^\perp} = [(x'_{\lambda_0})_E]_{E^\perp}$, and therefore $[s_A]_{A^\perp} \leqslant [(x'_{\lambda_0})_A]_{A^\perp} = [z_A]_{A^\perp}$. Otherwise, $[z_E]_{E^\perp} = [e]_{E^\perp}$, whence $[s_A]_{A^\perp} \leqslant [z_A]_{A^\perp}$. Thus, it follows that $s \leqslant \phi_{\mathcal{E}\mathcal{A}}(z)$.

Further, for every $\lambda \in \Lambda$,

$$\overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y) \wedge y_\lambda) = \overline{\phi}_\mathcal{C}(y) \wedge \overline{\phi}_\mathcal{A}(y_\lambda) = q \wedge p_\lambda = q = \overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y)).$$

Therefore, due to the injectivity of $\overline{\phi}_\mathcal{A}$, $\phi_{\mathcal{C}\mathcal{A}}(y) \wedge y_\lambda = \phi_{\mathcal{C}\mathcal{A}}(y)$, that is to say, $\phi_{\mathcal{C}\mathcal{A}}(y) \leqslant y_\lambda$. This implies that $\phi_{\mathcal{C}\mathcal{A}}(y) \leqslant \bigwedge_\Lambda y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(z)$, and therefore $q = \overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y)) \leqslant \overline{\phi}_\mathcal{A}(\phi_{\mathcal{E}\mathcal{A}}(z)) = \overline{\phi}_\mathcal{E}(z)$. This establishes the proof of $\overline{\phi}_\mathcal{E}(z) = \bigwedge_\Lambda p_\lambda$. □

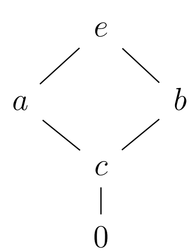
Finally, we have the main result of the section:

Theorem 49. *Any algebra* $\mathbf{L}$ *in a variety* $\mathcal{V}$ *of e-cyclic semilinear residuated lattices is densely embeddable in a laterally complete member of* $\mathcal{V}$.

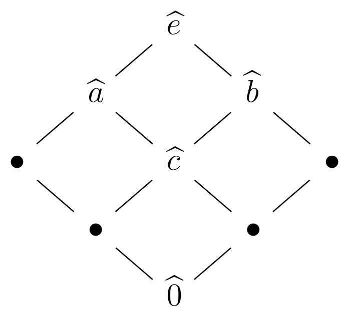
Proof. It is an immediate consequence of Theorems 39 and 48, and the fact that $\mathcal{O}(\mathbf{L})$ is a direct limit of products of quotients of $\mathbf{L}$. □

As we already mentioned, $\mathcal{O}(\mathbf{L})$ cannot be "much larger" than $\mathbf{L}$, since $\mathbf{L}$ is dense in $\mathcal{O}(\mathbf{L})$. We could then inquire into the minimality of $\mathcal{O}(\mathbf{L})$. That is, we can ask whether $\mathcal{O}(\mathbf{L})$ is the smallest laterally complete residuated lattice in which $\mathbf{L}$ is densely embeddable. The answer is no in general, and it is not difficult to find a counterexample:

Example 50. Consider the Heyting algebra $\mathbf{L}$ given by the following Hasse diagram:



It can be easily seen that $\mathbf{L}$ is an integral semilinear residuated lattice (Gödel algebra). The Boolean algebra of polars of $\mathbf{L}$ is $\mathrm{Pol}(\mathbf{L}) = \{\{e\}, a^{\perp\perp}, b^{\perp\perp}, L\}$, with $a^{\perp\perp} = \{e, a\}$ and $b^{\perp\perp} = \{e, b\}$. Hence, the set of partitions of $\mathrm{Pol}(\mathbf{L})$ is $\mathbb{D}(\mathbf{L}) = \big\{\{L\}, \{a^{\perp\perp}, b^{\perp\perp}\}\big\}$. Let us denote the non trivial partition of $\mathbf{L}$ by $\mathcal{C}$. $\mathbb{D}(\mathbf{L})$ is a directed set with a top element, namely $\mathcal{C}$, and therefore the limit of the directed system $\big\{\phi_{\{L\}\mathcal{C}} : \mathbf{L}_{\{L\}} \to \mathbf{L}_{\mathcal{C}}\big\}$ is $\mathbf{L}_{\mathcal{C}}$ itself. It is not difficult to see that $\mathbf{L}/a^{\perp}$ is a chain with three elements $[0]_{a^{\perp}} < [a]_{a^{\perp}} < [e]_{a^{\perp}}$, and analogously $\mathbf{L}/b^{\perp}$. Then $\mathcal{O}(\mathbf{L})$ is the Heyting algebra:



where we have named the images of the embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$. We note that since $\mathbf{L}$ is finite, it is trivially laterally complete. This implies that the theory developed in this section does not produce a "minimal" laterally complete extension. In Section 9, we prove the existence of minimal such extensions in the class of GMV algebras.

8. Projectablility of $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}_{<\omega}(\mathbf{L})$

As the title of the section suggests, the main result of this section is Theorem 55, which asserts that $\mathcal{O}(\mathbf{L})$ is strongly projectable for any semilinear residuated lattice $\mathbf{L}$. Thus, in view of the results of Section 7, every member of a variety $\mathcal{V}$ of e-cyclic semilinear residuated lattices can be densely embedded in a strongly projectable member of $\mathcal{V}$. We, in fact, show that there is another strongly projectable residuated lattice, denoted by $\mathcal{O}_{<\omega}(\mathbf{L})$, which is generally smaller than $\mathcal{O}(\mathbf{L})$, is obtained via a direct limit construction analogous to the one for $\mathcal{O}(\mathbf{L})$, and contains $\mathbf{L}$ as a dense subalgebra.

Definition 51. An e-cyclic residuated lattice $\mathbf{L}$ is said to be *projectable* if every principal polar is a complemented element of $\mathcal{C}(\mathbf{L})$. That is, for all $a \in L$,

$$L = a^{\perp} \vee^{\mathcal{C}(\mathbf{L})} a^{\perp\perp}.$$

It is called *strongly projectable* if for every convex subalgebra $H \in \mathcal{C}(\mathbf{L})$,

$$\mathbf{L} = H^{\perp} \vee^{\mathcal{C}(\mathbf{L})} H^{\perp\perp}.$$

Thus strong projectability of $\mathbf{L}$ is equivalent to the lattice $\mathcal{C}(\mathbf{L})$ being a Stone algebra.

We start by exploring the relationship of the lattices of convex subalgebras of an e-cyclic residuated lattice and any dense extension of it. This discussion concludes with Proposition 53, which asserts that the Boolean algebras of their polars are isomorphic. This result will be needed in this section and plays a key role in Section 9.

Let $\mathbf{L}, \mathbf{H}$ be e-cyclic residuated lattices such that $\mathbf{L}$ is a subalgebra of $\mathbf{H}$. Define the order-homomorphisms $\mu : \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{H})$ and $\nu : \mathcal{C}(\mathbf{H}) \to \mathcal{C}(\mathbf{L})$ as follows: for all $A \in \mathcal{C}(\mathbf{L})$ and $B \in \mathcal{C}(\mathbf{H}), \mu(A) = C_{\mathbf{H}}[A]$, the convex subalgebra of $\mathbf{H}$ generated by A, and $\nu(B) = B \cap L$. We first note that (μ, ν) is an adjunction, since for all $A \in \mathcal{C}(\mathbf{L})$ and every $B \in \mathcal{C}(\mathbf{H})$,

$$A \subseteq \nu(B) \quad \Leftrightarrow \quad A \subseteq B \cap L \quad \Leftrightarrow \quad A \subseteq B \quad \Leftrightarrow \quad C_{\mathbf{H}}[A] \subseteq B \quad \Leftrightarrow \quad \mu(A) \subseteq B.$$

Furthermore, ν is surjective, and hence μ is injective. Indeed, let $A \in \mathcal{C}(\mathbf{L})$ and let $S = \{h \in H \mid a \leqslant h \leqslant e, \text{ for some } a \in A\}$. If $B = C_{\mathbf{H}}[S]$, then $S = B^{-}$ (see Lemma 10) and $\nu(B) = B \cap L = A$. Lastly, μ preserves finite meets since for every $A_1, A_2 \in \mathcal{C}(\mathbf{L})$ and every $x \in C_{\mathbf{H}}[A_1] \cap C_{\mathbf{H}}[A_2]$, if $x \leqslant e$ then there are $a_i \in A_i$ such that $a_i \leqslant x$, for $i = 1, 2$, and therefore $a_1 \vee a_2 \leqslant x \leqslant e$, whence $x \in C_{\mathbf{H}}[A_1 \cap A_2]$, and the other inclusion is trivial. Thus, μ is an injective lattice homomorphism preserving arbitrary joins.

When an e-cyclic residuated lattice $\mathbf{H}$ is an extension of $\mathbf{L}$, we use $(\,)^*$ to denote the polars of $\mathbf{H}$ and $(\,)^{\perp}$ to denote those of $\mathbf{L}$. In the event $\mathbf{L}$ is dense in $\mathbf{H}$, we can say more about the adjunction (μ, ν):

Lemma 52. *Let $\mathbf{L}$ and $\mathbf{H}$ be two e-cyclic residuated lattices such that $\mathbf{L}$ is a dense subalgebra of $\mathbf{H}$. The following hold with respect to the adjunction (μ, ν) defined above:*

(i) For every $B \in \mathcal{C}(\mathbf{H}), B^* = (\mu\nu(B))^*$.

(ii) For every $A \in \mathcal{C}(\mathbf{L})$, $\nu(\mu(A)^*) = (\nu\mu(A))^{\perp}$.

(iii) The map ν preserves pseudocomplements, that is, for every $B \in \mathcal{C}(\mathbf{H})$, $\nu(B^*) = \nu(B)^{\perp}$.

Proof.
(i) Let $B \in \mathcal{C}(\mathbf{H})$. We need to show that $B^* = (\mu\nu(B))^*$. For the left-to-right inclusion, notice that $\mu\nu(B) \cap B^* \subseteq B \cap B^* = \{e\}$, and so $B^* \subseteq (\mu\nu(B))^*$. For the other inclusion, suppose that $D \in \mathcal{C}(\mathbf{H})$ is such that $\mu\nu(B) \cap D = \{e\}$. Then,

$$\{e\} = \nu(\mu\nu(B) \cap D) = \nu\mu\nu(B) \cap \nu(D) = \nu(B) \cap \nu(D) = \nu(B \cap D) = (B \cap D) \cap L.$$

Since $\mathbf{L}$ is dense in $\mathbf{H}$, we have that $B \cap D = \{e\}$, whence $D \subseteq B^*$. Therefore, $(\mu\nu(B))^* \subseteq B^*$.

(ii) Let $A \in \mathcal{C}(\mathbf{L})$. We need to prove that $\nu(\mu(A)^*) = (\nu\mu(A))^\perp$. One inclusion is just the observation that $\nu(\mu(A)^*) \cap \nu(\mu(A)) = \nu(\mu(A)^* \cap \mu(A)) = \nu(\{e\}) = \{e\}$, whence $\nu(\mu(A)^*) \subseteq (\nu\mu(A))^\perp$. For the other inclusion, suppose that $D \in \mathcal{C}(\mathbf{L})$ is such that $\nu\mu(A) \cap D = \{e\}$. Then, $\mu(A) \cap \mu(D) = \mu\nu\mu(A) \cap \mu(D) = \mu(\nu\mu(A) \cap D) = \mu(\{e\}) = \{e\}$, and hence $\mu(D) \subseteq \mu(A)^*$. Since ν is onto, we have $D = \nu\mu(D) \subseteq \nu(\mu(A)^*)$.

(iii) Let $B \in \mathcal{C}(\mathbf{H})$. We have in view of the preceding discussion that $\nu(B^*) = \nu((\mu\nu(B))^*) = (\nu(\mu\nu(B)))^\perp = \nu(B)^\perp$, as we wanted to prove. $\square$

The adjunction (μ, ν) is put to use in the proof of the next proposition.

Proposition 53. *If* $\mathbf{L}$ *is a dense subalgebra of an e-cyclic residuated lattice* $\mathbf{H}$*, then the Boolean algebras* $\mathrm{Pol}(\mathbf{L})$ *and* $\mathrm{Pol}(\mathbf{H})$ *are isomorphic. The isomorphism is implemented by the maps* $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$ *and* $\hat{\nu} : \mathrm{Pol}(\mathbf{H}) \to \mathrm{Pol}(\mathbf{L})$*, defined by* $\hat{\mu}(A) = (\mu(A^\perp))^*$ *and* $\hat{\nu}(B) = \nu(B)$*, for all* $A \in \mathrm{Pol}(\mathbf{L})$ *and* $B \in \mathrm{Pol}(\mathbf{H})$*.*

Proof. Let $\hat{\mu}$ and $\hat{\nu}$ be as in the statement of the proposition. In light of Lemma 52.(ii), $\nu(B) \in \mathrm{Pol}(\mathbf{L})$, for each $B \in \mathrm{Pol}(\mathbf{H})$. Note that $\hat{\nu}$ is surjective, since ν is surjective and preserves pseudocomplements. We claim that $\hat{\mu}$ is the left adjoint of $\hat{\nu}$. Note first that if $A \in \mathrm{Pol}(\mathbf{L})$, there exists $C \in \mathrm{Pol}(\mathbf{H})$ such that $A = \nu(C)$. Then, by invoking both conditions of Lemma 52, we get $\hat{\mu}(A) = (\mu(A^\perp))^* = (\mu(\nu(C)^\perp))^* = (\mu\nu(C^*))^* = C^{**} = C$.

Now let $A \in \mathrm{Pol}(\mathbf{L})$ and $B \in \mathrm{Pol}(\mathbf{H})$. We need to prove that $\hat{\mu}(A) \subseteq B$ if and only if $A \subseteq \hat{\nu}(B)$. Suppose first that $\hat{\mu}(A) \subseteq B$ and let $C \in \mathrm{Pol}(\mathbf{H})$ such that $A = \nu(C)$. Then $C \subseteq B$, and so $A = \nu(C) \subseteq B$. On the other hand, if $A \subseteq B$, then $\nu(B)^\perp \subseteq A^\perp$, which, combined with Lemma 52.(iii) and the momotonicity of μ, implies that $\mu\nu(B^*) \subseteq \mu(A^\perp)$. Then another application of Lemma 52.(i) yields $\hat{\mu}(A) = (\mu(A^\perp))^* \subseteq (\mu\nu(B^*))^* = B^{**} =$

B. We have verified that $(\hat{\mu}, \hat{\nu})$ is an adjunction, and hence, in particular, that $\hat{\mu}$ is injective. Lastly, $\hat{\mu}$ is surjective, since for $B \in \mathrm{Pol}(\mathbf{H})$, $\hat{\mu}(\nu(B)) = (\mu(\nu(B)^{\perp}))^* = (\mu(\nu(B^*)))^* = B^{**} = B$, by Lemma 52.(iii). We have shown that $\hat{\mu}$ is a lattice (and hence a Boolean) isomorphism with inverse $\hat{\nu}$. $\square$

We continue by providing a description of all the polars of $\mathcal{O}(\mathbf{L})$ in terms of the supports of their elements. Following the practice above, given $S \subseteq \mathcal{O}(\mathbf{L})$, we denote by S^* the polar of S in $\mathcal{O}(\mathbf{L})$. For $X \subseteq L$, we abuse notation and write X^* for $(\overline{\phi}(X))^*$; in other words, we identify the elements of $\mathbf{L}$ with their images in $\mathcal{O}(\mathbf{L})$ (see Theorem 39).

Lemma 54. *Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, $A \in \mathrm{Pol}(\mathbf{L})$, and $p \in \mathcal{O}(\mathbf{L})$. Let $\mathcal{C}$ be a refinement of the partition $\{A, A^{\perp}\}$ so that $\mathbf{L}_{\mathcal{C}}$ contains a canonical proxy x of p. Then:*

1. *$p \in A^*$[12] if and only if $C \subseteq A^{\perp}$, for every $C \in \mathrm{Supp}(x)$.*
2. *$p \in A^{**}$ if and only $C \subseteq A$, for every $C \in \mathrm{Supp}(x)$.*

Proof. Without loss of generality, we may assume that p is negative, since $p \in A^*$ if and only if $|p| \in A^*$ (Lemma 10), and $\mathrm{Supp}(x) = \mathrm{Supp}(|x|)$ (Lemma 46). For every $C \in \mathcal{C}$, there are only two mutually exclusive possibilities: $C \subseteq A$ or $C \subseteq A^{\perp}$.

1. ($\Leftarrow$) Suppose that $C \subseteq A^{\perp}$, for every $C \in \mathrm{Supp}(x)$. Consider an arbitrary element $a \in A^-$ and write $\overline{a} = \overline{\phi}(a)$ for its image in $\mathcal{O}(\mathbf{L})$. Note that $\overline{a}_{\mathcal{C}} = ([a]_{C^{\perp}} \mid C \in \mathcal{C})$ is the proxy of $\overline{a}$ at $\mathcal{C}$. Under the assumption, for every $C \in \mathrm{Supp}(x)$, $C \subseteq A^{\perp}$, and so $a^{\perp\perp} \cap C = \{e\}$. But then, by Lemma 34, $[a]_{C^{\perp}} = [e]_{C^{\perp}}$. Thus,

$$[x_C]_{C^{\perp}} \vee [a]_{C^{\perp}} = \begin{cases} [e]_{C^{\perp}} \vee [a]_{C^{\perp}} & \text{if } C \subseteq A \\ [x_C]_{C^{\perp}} \vee [e]_{C^{\perp}} & \text{if } C \subseteq A^{\perp} \end{cases} = [e]_{C^{\perp}}.$$

Therefore, x and $\overline{a}_{\mathcal{C}}$ are disjoint, whence p and $\overline{a}$ are disjoint too. Since $a \in A^-$ is arbitrary, it follows that $p \in A^*$.

($\Rightarrow$) Suppose there exists $C \in \mathrm{Supp}(x)$ such that $C \subseteq A$, and let $a \in C$, $a < e$. By the canonicity of x and our choice of C, $a^{\perp\perp} \subseteq C \subseteq x_C^{\perp\perp}$, and

[12] There is minor abuse of notation here. A more precise notation in accordance with the preceding discussion would have been $p \in \mu(A)^*$.

hence

$$(a \vee x_C)^{\perp\perp} \cap C = a^{\perp\perp} \cap x_C^{\perp\perp} \cap C = a^{\perp\perp} \neq \{e\}.$$

Therefore, x and $\overline{a}_{\mathcal{C}}$ are not disjoint, and thus p and $\overline{a}$ are not disjoint either. We have shown that $p \notin A^*$.

1. Is a consequence of (1) and the fact that $A^{\perp *} = A^{**}$, or more precisely, $\mu(A^{\perp})^* = \mu(A)^{**}$. Indeed, $\mu(A)^{**} = (\mu(A)^*)^* = (\hat{\mu}(A^{\perp}))^* = (\hat{\mu}(A))^{**}$ (since $\hat{\mu}$ is a homomorphism) $= \hat{\mu}(A) = \mu(A^{\perp})^*$. □

We are now in the position to prove that $\mathcal{O}(\mathbf{L})$ is strongly projectable. In fact, we prove more:

Theorem 55. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice. Then* $\mathcal{O}(\mathbf{L})$ *is strongly projectable. Moreover, for all* $B \in \mathrm{Pol}(\mathcal{O}(\mathbf{L}))$, $B \vee^{\mathcal{C}(\mathcal{O}(\mathbf{L}))} B^* = B \otimes B^* = \mathcal{O}(\mathbf{L})$.

Proof. Let B be an arbitrary polar of $\mathcal{O}(\mathbf{L})$, and let $A = \{a \in L \mid \overline{a} \in B\}$. Note that in view of Proposition 53, $A = \hat{\nu}(B)$, $A \in \mathrm{Pol}(\mathbf{L})$, and $B = A^{**}$. Let $p \in \mathcal{O}(\mathbf{L})$ and let $\mathcal{C}$ be a partition of $\mathbf{L}$ that refines $\mathcal{A} = \{A, A^{\perp}\}$ and p has a canonical proxy x at $\mathcal{C}$. We define, for every $C \in \mathcal{C}$:

$$z_C = \begin{cases} e & \text{if } C \subseteq A^{\perp}, \\ x_C & \text{if } C \subseteq A, \end{cases} \qquad t_C = \begin{cases} x_C & \text{if } C \subseteq A^{\perp}, \\ e & \text{if } C \subseteq A. \end{cases}$$

Thus, taking $z = ([z_C]_{C^{\perp}} \mid C \in \mathcal{C})$, $t = ([t_C]_{C^{\perp}} \mid C \in \mathcal{C})$, we can easily see that both z and t are canonical, since x is canonical, and $zt = x$. Thus, if $q_1 = \overline{\phi}_{\mathcal{C}}(z)$ and $q_2 = \overline{\phi}_{\mathcal{C}}(t)$, we have

$$p = \overline{\phi}_{\mathcal{C}}(x) = \overline{\phi}_{\mathcal{C}}(z \cdot t) = \overline{\phi}_{\mathcal{C}}(z) \cdot \overline{\phi}_{\mathcal{C}}(t) = q_1 \cdot q_2.$$

Moreover, in view of Lemma 54, $q_1 \in A^{**} = B$ and $q_2 \in A^* = B^*$. In order to establish the uniqueness of the decomposition of p as a product of an element in B and an element in B^*, suppose that we have two such decompositions:

$$q_1 \cdot q_2 = p = q_1' \cdot q_2'.$$

Let $\mathcal{C}$ be a partition that refines $\mathcal{A}$ and contains canonical proxies x, z, t, z', and t' for the elements p, q_1, q_2, q_1', and q_2', respectively. Hence, $z \cdot t = x = z' \cdot t'$, because proxies are unique at each partition. Note that, since $q_1, q_1' \in B$, for every $C \in \mathcal{C}$, if $C \subseteq A^{\perp}$, $[z_C]_{C^{\perp}} = [e]_{C^{\perp}} = [z_C']_{C^{\perp}}$, by Lemma 54. And analogously, if $C \subseteq A$, $[t_C]_{C^\perp} = [e]_{C^\perp} = [t'_C]_{C^\perp}$, since $q_2, q'_2 \in B^*$, and therefore:

$$\begin{aligned}[z_C]_{C^\perp} &= [z_C]_{C^\perp} \cdot [e]_{C^\perp} = [z_C]_{C^\perp} \cdot [t_C]_{C^\perp} = [x_C]_{C^\perp} = [z'_C]_{C^\perp} \cdot [t'_C]_{C^\perp} \\ &= [z'_C]_{C^\perp} \cdot [e]_{C^\perp} = [z'_C]_{C^\perp}.\end{aligned}$$

Hence, $z = z'$. Analogously, $t = t'$, and therefore $q_1 = q'_1$ and $q_2 = q'_2$.

Next we have to prove that if $q_1 \cdot q_2 \leqslant q'_1 \cdot q'_2$, with $q_1, q'_1 \in B$ and $q_2, q'_2 \in B^*$, then $q_1 \leqslant q'_1$ and $q_2 \leqslant q'_2$. Arguing as and retaining the notations of the preceding paragraph, it is easily shown that $t \leqslant t'$ and $z \leqslant z'$ in $\mathcal{C}$. Hence $q_1 \leqslant q'_1$ and $q_2 \leqslant q'_2$ in $\mathcal{O}(\mathbf{L})$.

Lastly, we can appeal to Lemma 54 once more and argue as above to conclude that every element of B commutes with every element of B^*. This completes the proof of the theorem. □

Definition 56. An e-cyclic residuated lattice $\mathbf{L}$ is said to be *orthocomplete* if it is both laterally complete and strongly projectable.

We readily obtain the following result:

Corollary 57. *If $\mathbf{L}$ is an algebra in a variety $\mathcal{V}$ of e-cyclic semilinear residuated lattices, then $\mathcal{O}(\mathbf{L})$ is an orthocomplete dense extension of $\mathbf{L}$ that belongs to $\mathcal{V}$.*

Proof. It is an immediate consequence of Theorems 39, 48, and 55. □

Given an e-cyclic semilinear residuated lattice, we denote by $\mathbb{D}_{<\omega}(\mathbf{L})$ the set of all finite partitions of $\mathrm{Pol}(\mathbf{L})$. If $\mathcal{C}, \mathcal{D} \in \mathbb{D}_{<\omega}(\mathbf{L})$, then the refinement of $\mathcal{C}$ and $\mathcal{D}$ is also finite (see Equation (5)), and thus the set $\mathbb{D}_{<\omega}(\mathbf{L})$ is also a directed set. Let $\mathcal{O}_{<\omega}(\mathbf{L})$ denote the direct limit of the directed system $\left\{\phi_{\mathcal{CA}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A} \text{ in } \mathbb{D}_{<\omega}(\mathbf{L})\right\}$. Notice that, since $\mathbb{D}_{<\omega}(\mathbf{L})$ is a subposet of $\mathbb{D}(\mathbf{L})$, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is embeddable in $\mathcal{O}(\mathbf{L})$.

In Section 7, infinite partitions were only required in the proof of the lateral completeness of $\mathcal{O}(\mathbf{L})$. In fact, the set $\{p_\lambda \mid \lambda \in \Lambda\}$ chosen at the beginning of the proof of Theorem 48 may be infinite, in which case the partition $\mathcal{E}$ in the proof may be infinite as well. Thus, all we proved for $\mathcal{O}(\mathbf{L})$ is also true for $\mathcal{O}_{<\omega}(\mathbf{L})$, except for Theorem 48 and Corollary 57. Lemma 54 is also true if we take $p \in \mathcal{O}_{<\omega}(\mathbf{L})$ and B a polar in $\mathcal{O}_{<\omega}(\mathbf{L})$. Therefore, the following result holds.

Theorem 58. *If $\mathcal{V}$ is a variety of e-cyclic semilinear residuated lattices and $\mathbf{L} \in \mathcal{V}$, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is a strongly projectable dense extension of $\mathbf{L}$ that belongs to $\mathcal{V}$.*

9. Laterally Complete Hull and Projectable Hull

We have established that every e-cyclic semilinear residuated lattice $\mathbf{L}$ can be densely embedded in one that is both laterally complete and strongly projectable, and hence in particular projectable. In this section we explore conditions under which such an extension is minimal with respect to each property in the following sense.

Definition 59. A *laterally complete hull* of a residuated lattice $\mathbf{L}$ is a laterally complete residuated lattice $\mathbf{H}$ containing $\mathbf{L}$ as subalgebra, such that (i) no proper subalgebra of $\mathbf{H}$ containing $\mathbf{L}$ is laterally complete; and (ii) $\mathbf{L}$ is dense in $\mathbf{H}$. A *projectable hull* and a *strongly projectable hull* are defined in an analogous manner.

The main result of this section is Theorem 65, which asserts that any algebra in a variety $\mathcal{V}$ of semilinear GMV algebras has a unique, up to isomorphism, semilinear laterally complete hull that belongs to $\mathcal{V}$. Further, in Theorem 74 we establish the existence and uniqueness of strongly projectable, projectable and orthocomplete hulls in any variety of semilinear GMV algebras. Lastly, in Theorem 76 we show that given a variety $\mathcal{V}$ of semilinear GMV algebras and $\mathbf{L} \in \mathcal{V}$, $\mathcal{O}_{<\omega}(\mathbf{L})$ is the unique, up to isomorphism, strongly projectable hull of $\mathbf{L}$. Combining Lemma 5 with Theorem 25, we obtain:

Proposition 60. *A variety $\mathcal{V}$ of GMV algebras is semilinear if and only if for every $\mathbf{L} \in \mathcal{V}$, all (principal) polars in $\mathbf{L}$ are normal.*

A first step in establishing the existence and uniqueness of a laterally complete hull in a variety $\mathcal{V}$ of semilinear GMV algebras is to show that any algebra $\mathbf{L}$ in $\mathcal{V}$ has a minimal laterally complete extension inside $\mathcal{O}(\mathbf{L})$. A reasonable course of action would be to take the intersection of all laterally complete subalgebras of $\mathcal{O}(\mathbf{L})$ containing $\mathbf{L}$. The next lemma reveals why this approach will work in the setting of GMV algebras, but not necessarily in general.

Lemma 61. *Let $\mathbf{L}$ be a dense subalgebra of a GMV algebra $\mathbf{H}$. For any subset X of L^-, if $\bigwedge^{\mathbf{L}} X$ exists, then so does $\bigwedge^{\mathbf{H}} X$ and they are equal.*

Proof. Let $a = \bigwedge^{\mathbf{L}} X$. Then, a is a lower bound of X in $\mathbf{L}$, and therefore in $\mathbf{H}$. Suppose that b is a lower bound of X in $\mathbf{H}$. Then clearly $a \leqslant a \vee b \leqslant e$, since all elements of X are negative, and therefore $a = e\backslash a \leqslant (a \vee b)\backslash a \leqslant a\backslash a = e$.

Suppose that $(a \vee b)\backslash a < e$. Then by the density of $\mathbf{L}$ in $\mathbf{H}$, there exists a $c \in L$ such that

$$a \leqslant (a \vee b)\backslash a \leqslant c < e.$$

Hence, $a/c \leqslant a/((a \vee b)\backslash a) = a \vee b$, and therefore a/c is a lower bound of X. Now, since $a/c \in L$, we obtain $a/c \leqslant \bigwedge^{\mathbf{L}} X = a$. Whence $e = a\backslash a \leqslant (a/c)\backslash a = (a/c \wedge e)\backslash a = a \vee c$, since a/c is negative. But $a \leqslant c$, and therefore $c = a \vee c = e$, against the choice of $c < e$. Therefore, $(a \vee b)\backslash a = e$, which implies $b \leqslant a \vee b \leqslant a$. Since b is an arbitrary lower bound of X in $\mathbf{H}$, we deduce that $\bigwedge^{\mathbf{H}} X$ exists and $\bigwedge^{\mathbf{H}} X = a$, as we wanted to prove. □

Corollary 62. *If* $\mathbf{H}$ *is a GMV algebra and* $\{\mathbf{L}_i \mid i \in I\}$ *is a nonempty family of subalgebras of* $\mathbf{H}$ *that are laterally complete and dense in* $\mathbf{H}$*, then* $\bigcap_I \mathbf{L}_i$ *is laterally complete.*

Proof. Let $\mathbf{L} = \bigcap_I \mathbf{L}_i$. In order to prove that $\mathbf{L}$ is laterally complete, suppose that $X \subseteq L^-$ is a disjoint subset. Then, for every $i \in I$, $X \subseteq L_i^-$, and therefore $\bigwedge^{\mathbf{L}_i} X$ exists. Since $\mathbf{L}_i$ is dense in $\mathbf{H}$, by Lemma 61, $\bigwedge^{\mathbf{H}} X$ exists and $\bigwedge^{\mathbf{H}} X = \bigwedge^{\mathbf{L}_i} X \in L_i$. Thus, $\bigwedge^{\mathbf{H}} X$ is in every $\mathbf{L}_i$, and hence $\bigwedge^{\mathbf{L}} X$ exists and coincides with $\bigwedge^{\mathbf{H}} X$. □

We have seen in Proposition 53 that if $\mathbf{L}$ is a dense subalgebra of an e-cyclic residuated lattice $\mathbf{H}$, then the Boolean algebras of polars of $\mathbf{L}$ and $\mathbf{H}$ are isomorphic. This result will be employed in Corollary 63 to establish an isomorphism between the directed sets of partitions of $\mathbf{L}$ and $\mathbf{H}$, and pave the way for the proof of Proposition 64, which states that $\mathcal{O}(\mathbf{L})$ can be embedded into $\mathcal{O}(\mathbf{H})$.

Corollary 63. *If* $\mathbf{L}$ *is a dense subalgebra of an e-cyclic residuated lattice* $\mathbf{H}$*, the map*

$$\mathcal{C} \mapsto \overline{\mathcal{C}} = \{\hat{\mu}(C) \mid C \in \mathcal{C}\}$$

is an order isomorphism between the join-semilattice $\langle \mathbb{D}(\mathbf{L}), \preccurlyeq \rangle$ *of partitions of* $\mathrm{Pol}(\mathbf{L})$ *and the join-semilattice* $\langle \mathbb{D}(\mathbf{H}), \preccurlyeq \rangle$ *of partitions of* $\mathrm{Pol}(\mathbf{H})$.

Proof. Using the fact that $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$ and $\hat{\nu} : \mathrm{Pol}(\mathbf{H}) \to \mathrm{Pol}(\mathbf{L})$ are isomorphisms, and Lemma 27, it is easy to see that the map is well defined, and actually a bijection. If $\mathcal{C} \preccurlyeq \mathcal{A}$, and $E \in \overline{\mathcal{A}}$, then there exists a unique $A \in \mathcal{A}$ such that $\hat{\mu}(A) = E$, and a unique $C \in \mathcal{C}$ such that $A \subseteq C$. Therefore $E = \hat{\mu}(A) \subseteq \hat{\mu}(C) \in \overline{\mathcal{C}}$, and it is straightforward that $\hat{\mu}(C)$ is the only element of $\overline{\mathcal{C}}$ containing E. That is, $\overline{\mathcal{C}} \preccurlyeq \overline{\mathcal{A}}$. □

Proposition 64. *Let* $\mathbf{L}$ *be a dense subalgebra of an e-cyclic residuated lattice* $\mathbf{H}$*, and let* $\alpha : \mathbf{L} \to \mathcal{O}(\mathbf{L})$*,* $\beta : \mathbf{H} \to \mathcal{O}(\mathbf{H})$ *be the canonical embeddings. Then there is an embedding* $\tau : \mathcal{O}(\mathbf{L}) \to \mathcal{O}(\mathbf{H})$ *rendering commutative the following diagram:*

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\beta"] & \mathcal{O}(\mathbf{H}) \\ \mathbf{L} \arrow[u, hook, "i"] \arrow[r, "\alpha"'] & \mathcal{O}(\mathbf{L}) \arrow[u, hook, "\tau"'] \end{tikzcd} \tag{13}$$

Proof. By Proposition 53, there exists an isomorphism $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$, with inverse $\hat{\nu}$. If $C \in \mathrm{Pol}(\mathbf{L})$, then the assignment $f_{C^\perp} : \mathbf{L}/C^\perp \to \mathbf{H}/\hat{\mu}(C)^*$ – defined by $f_{C^\perp}([a]_{C^\perp}) = [a]_{\hat{\mu}(C)^*}$, for all $a \in L$ – is an injective homomorphism. Indeed, just note that $\hat{\mu}(C)^* \cap L = \hat{\mu}(C^\perp) \cap L = \hat{\nu}\hat{\mu}(C^\perp) = C^\perp$. This produces the family of homomorphisms $\{f_{C^\perp}\pi_{C^\perp} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}_{\hat{\mu}(C)^*} \mid C \in \mathcal{C}\}$. Therefore, recalling that $\overline{\mathcal{C}} = \{\hat{\mu}(C) \mid C \in \mathcal{C}\}$, the couniversal property of the product $\mathbf{H}_{\overline{\mathcal{C}}}$ induces a homomorphism $\tau_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}_{\overline{\mathcal{C}}}$ such that $\pi_{\hat{\mu}(C)^*}\,\tau_{\mathcal{C}} = f_{C^\perp}\pi_{C^\perp}$, for all $C \in \mathcal{C}$. Note that, for every $x = ([x_C]_{C^\perp} \mid C \in \mathcal{C})$ in $\mathbf{L}_{\mathcal{C}}$, $\tau_{\mathcal{C}}(x) = ([x_C]_{\hat{\mu}(C)^*} \mid \hat{\mu}(C) \in \overline{\mathcal{C}})$. Further, $\tau_{\mathcal{C}}$ is an embedding, since each $f_{C^\perp}$, $C \in \mathcal{C}$, is an embedding.

It can be readily seen that for every $\mathcal{C} \preccurlyeq \mathcal{A}$ in $\mathbb{D}(\mathbf{L})$, $\overline{\mathcal{C}} \preccurlyeq \overline{\mathcal{A}}$ (by Corollary 63), and the bottom square of the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(\mathbf{L}) \arrow[rrr, dashed, "\tau"] & & & \mathcal{O}(\mathbf{H}) \\ & \mathbf{L}_{\mathcal{C}} \arrow[ul, bend left, "\overline{\phi}_{\mathcal{C}}"'] \arrow[r, "\tau_{\mathcal{C}}"] \arrow[d, "\phi_{\mathcal{C}\mathcal{A}}"'] & \mathbf{H}_{\overline{\mathcal{C}}} \arrow[ur, bend right, "\overline{\phi}_{\overline{\mathcal{C}}}"] \arrow[d, "\phi_{\overline{\mathcal{C}}\,\overline{\mathcal{A}}}"] & \\ & \mathbf{L}_{\mathcal{A}} \arrow[uul, bend left=40, "\overline{\phi}_{\mathcal{A}}"] \arrow[r, "\tau_{\mathcal{A}}"] & \mathbf{H}_{\overline{\mathcal{A}}} \arrow[uur, bend right=40, "\overline{\phi}_{\overline{\mathcal{A}}}"'] & \end{tikzcd}$$

Therefore, since $\mathcal{O}(L)$ is the direct limit, there exists a unique τ rendering the whole diagram commutative. Furthermore, τ is an embedding, since if $p, q \in \mathcal{O}(\mathbf{L})$ are such that $\tau(p) = \tau(q)$, and x, y are proxies of p, q at $\mathcal{C}$, then $\overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(x) = \tau(\overline{\phi}_{\mathcal{C}}(x)) = \tau(p) = \tau(q) = \tau(\overline{\phi}_{\mathcal{C}}(y)) = \overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(y)$. The equality $\overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(x) = \overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(y)$ shows that $\tau_{\mathcal{C}}(x)$ and $\tau_{\mathcal{C}}(y)$ are proxies of $\tau(p)$ at $\overline{\mathcal{C}}$. It follows that $\tau_{\mathcal{C}}(x) = \tau_{\mathcal{C}}(y)$, and hence $x = y$ by the injectivity of $\tau_{\mathcal{C}}$.

Finally, taking α and β the embeddings of $\mathbf{L}$ and $\mathbf{H}$ into $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}(\mathbf{H})$, respectively, we readily see that the following diagram commutes, where i is the inclusion of $\mathbf{L}$ into $\mathbf{H}$:

$$\begin{tikzcd}
\mathbf{H} \arrow[r, "\cong"'] \arrow[rr, bend left, "\beta"] & \mathbf{H}_{\{H\}} \arrow[r, "\overline{\phi}_{\{H\}}"'] & \mathcal{O}(\mathbf{H}) \\
\mathbf{L} \arrow[u, hook, "i"] \arrow[r, "\cong"] \arrow[rr, bend right, "\alpha"'] & \mathbf{L}_{\{L\}} \arrow[u, "\tau_{\{L\}}"] \arrow[r, "\overline{\phi}_{\{L\}}"] & \mathcal{O}(\mathbf{L}) \arrow[u, "\tau"']
\end{tikzcd}$$ □

Now we have all we need to prove one of the main results of this section, namely that every GMV algebra possesses a laterally complete hull, which is unique up to isomorphism.

Theorem 65. *Any algebra* $\mathbf{L}$ *in a variety* $\mathcal{V}$ *of semilinear GMV algebras has a unique, up to isomorphism, laterally complete hull that belongs to* $\mathcal{V}$*.*

Proof. Let $\mathbf{L}$ be a semilinear GMV algebra. In view of Theorems 39 and 48, $\mathcal{O}(\mathbf{L})$ is laterally complete and the isomorphic copy $\alpha[\mathbf{L}]$ of $\mathbf{L}$ under the canonical embedding $\alpha : \mathbf{L} \to \mathcal{O}(\mathbf{L})$ is a dense subalgebra of $\mathcal{O}(\mathbf{L})$. It is clear that any subalgebra of $\mathcal{O}(\mathbf{L})$ that contains $\alpha[\mathbf{L}]$ is a dense subalgebra. Let $\mathbf{K}$ be the intersection of all (necessarily dense) subalgebras of $\mathcal{O}(\mathbf{L})$ that are laterally complete and contain $\alpha[\mathbf{L}]$. Hence, $\mathbf{K}$ is laterally complete by Corollary 62. Further, $\mathcal{O}(\mathbf{L})$ and $\mathbf{K}$ belong to $\mathcal{V}$. Combining these facts, we conclude that $\mathbf{L}$ has a laterally complete hull $\mathbf{K}$ in $\mathcal{V}$.

Suppose that $\mathbf{H}$ is another laterally complete hull of $\mathbf{L}$ in $\mathcal{V}$. We can apply Proposition 64 to find an embedding τ that renders Diagram (13) commutative. Note that, since $\mathbf{L}$ is dense in $\mathbf{H}$ and β is a dense embedding, we have that $\beta[\mathbf{L}]$ is dense in $\mathcal{O}(\mathbf{H})$. Hence $\tau[\mathcal{O}(\mathbf{L})]$ is dense in $\mathcal{O}(\mathbf{H})$, since $\beta[\mathbf{L}] = \tau\alpha[\mathbf{L}] \leqslant \tau[\mathcal{O}(\mathbf{L})] \leqslant \mathcal{O}(\mathbf{H})$. Therefore, $\tau[\mathcal{O}(\mathbf{L})]$ and $\beta[\mathbf{H}]$ are both laterally complete and dense in $\mathcal{O}(\mathbf{H})$, and hence $\tau[\mathcal{O}(\mathbf{L})] \cap \beta[\mathbf{H}]$ is laterally complete by Corollary 62. Therefore, $\beta[\mathbf{H}] \cap \tau[\mathcal{O}(\mathbf{L})] = \beta[\mathbf{H}]$, since $\beta[\mathbf{H}]$ is a laterally complete hull of $\beta[\mathbf{L}]$, and $\beta[\mathbf{H}] \cap \tau[\mathcal{O}(\mathbf{L})]$ is a lateral complete subalgebra of $\beta[\mathbf{H}]$ containing $\beta[\mathbf{L}]$. Thus, $\beta[\mathbf{H}] \leqslant \tau[\mathcal{O}(\mathbf{L})]$, and we can take $\mathbf{H}' = \tau^{-1}\beta[\mathbf{H}] \leqslant \mathcal{O}(\mathbf{L})$. Then, $\alpha[\mathbf{L}] = \tau^{-1}\beta[\mathbf{L}] \leqslant \mathbf{H}'$ and $\mathbf{H}'$ is laterally complete, and therefore $\mathbf{K} \leqslant \mathbf{H}'$. But, since $\mathbf{H}$ is a laterally complete hull of $\mathbf{L}$, then $\mathbf{H}'$ is a laterally complete hull of $\alpha[\mathbf{L}]$, and therefore $\mathbf{H}' = \mathbf{K}$. Hence, $\mathbf{H} \cong \mathbf{K}$. □

We next prove that any laterally complete projectable GMV algebra is strongly projectable (Proposition 70) by showing that any polar in a laterally complete GMV algebra is principal (Corollary 69). The proofs of these results require some preliminary lemmas.

Lemma 66. *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice and let* C *be a polar of* $\mathbf{L}$. *We have the following:*

1. *C contains a maximal disjoint subset.*
2. *If X a maximal disjoint subset of the residuated lattice $\mathbf{C}$, then $C = X^{\perp\perp}$ in $\mathcal{C}(\mathbf{L})$.*

Proof. 1. The proof of this part is a standard application of Zorn's Lemma. We consider the poset $\mathcal{X}$ of all disjoint subsets of C ordered by inclusion. Then, the union of every chain of $\mathcal{X}$ is obviously disjoint, and therefore, by Zorn's Lemma, $\mathcal{X}$ possesses a maximal element. If X is a such an element, then $X^{\perp} \cap C = \{e\}$, because otherwise there would be some $a \in C \cap X^{\perp}$ and $a < e$ disjoint to all the elements of X. But then $X \cup \{a\}$ would be in $\mathcal{X}$, contradicting the maximality of X.

2. Since X is a maximal disjoint subset of $\mathbf{C}$, $C \cap X^{\perp} = \{e\}$, and so $C \subseteq X^{\perp\perp}$. Then $C^{\perp} \vee X^{\perp\perp} = L$, where the join $\vee$ is taken in the Boolean algebra of polars. On the other hand, $X \subseteq C$ implies that $X^{\perp\perp} \subseteq C^{\perp\perp} = C$. Thus $C = X^{\perp\perp}$ as was to be shown. □

The proofs of the next three results make use of the hypothesis that the algebras under consideration are GMV algebras.

Lemma 67. *If* $\mathbf{L}$ *is a GMV algebra and* $a, b \in L^{-}$, *then* $a \vee b = e$ *if and only if* $a\backslash b \wedge e = b$.

Proof. First note that $a\backslash b \wedge e = a\backslash b \wedge b\backslash b = (a \vee b)\backslash b$. Therefore, if $a \vee b = e$, then $a\backslash b \wedge e = (a \vee b)\backslash b = e\backslash b = b$. Conversely, if $a\backslash b \wedge e = b$, then $a \vee b = b/((a \vee b)\backslash b) = b/((a\backslash b) \wedge (b\backslash b)) = b/((a\backslash b) \wedge e) = b/b = e$. □

Lemma 68. *Let* $\mathbf{L}$ *is a GMV algebra and let* $X \subseteq L^{-}$ *such that* $\bigwedge X = a$ *exists. Then* $X^{\perp} = a^{\perp}$.

Proof. Obviously, if $|y| \vee a = e$, then for every $x \in X$, $|y| \vee x = e$, since $a \leqslant x \leqslant e$. Thus, by Equation (3), $a^{\perp} \subseteq X^{\perp}$. In order to prove the other inclusion, let us suppose that $y \in X^{\perp}$. Then, again by Equation (3), $|y| \vee x = e$, for all $x \in X$, and so, $|y| \backslash x \wedge e = x$ by Lemma 67. Hence

$$|y| \backslash a \wedge e = \big(|y| \backslash \bigwedge X\big) \wedge e = \big(\bigwedge_{x \in X} (|y| \backslash x)\big) \wedge e = \bigwedge_{x \in X} \big((|y| \backslash x) \wedge e\big) = \bigwedge_{x \in X} x = a,$$

which implies, again by Lemma 67, that $|y| \vee a = e$. Thus $X^{\perp} \subseteq a^{\perp}$, by Equation (3). □

Corollary 69. *Any polar in a laterally complete GMV algebra is principal.*

Proposition 70. *If a GMV algebra is laterally complete and projectable, then it is orthocomplete.*

Proof. This is an immediate consequence of Corollary 69. □

The following result is of independent interest.

Proposition 71. *A GMV algebra* $\mathbf{L}$ *is (strongly) projectable if and only if* $\mathbf{G}(\mathbf{L})$ *and* $\mathbf{I}(\mathbf{L})$ *are (strongly) projectable.*

Proof. We first consider the strongly projectable case. Let $H \in \mathcal{C}(\mathbf{L})$. In view of Lemma 12, there exist $H_1 \in \mathcal{C}(\mathbf{G}(\mathbf{L}))$ and $H_2 \in \mathcal{C}(\mathbf{I}(\mathbf{L}))$ are such that $H = H_1 \otimes H_2$. Using Lemmas 12 and 22, we have:

$$\begin{aligned}
H^{\perp_{\mathbf{L}}} \vee^{\mathcal{C}(\mathbf{L})} H^{\perp\perp_{\mathbf{L}}} &= (H_1 \otimes H_2)^{\perp_{\mathbf{L}}} \vee^{\mathcal{C}(\mathbf{L})} (H_1 \otimes H_2)^{\perp\perp_{\mathbf{L}}} \\
&= (H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \otimes H_2^{\perp_{\mathbf{I}(\mathbf{L})}}) \vee^{\mathcal{C}(\mathbf{L})} (H_1^{\perp\perp_{\mathbf{G}(\mathbf{L})}} \otimes H_2^{\perp\perp_{\mathbf{I}(\mathbf{L})}}) \\
&= (H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_2^{\perp_{\mathbf{I}(\mathbf{L})}}) \vee^{\mathcal{C}(\mathbf{L})} (H_1^{\perp\perp_{\mathbf{G}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_2^{\perp\perp_{\mathbf{I}(\mathbf{L})}}) \\
&= (H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_1^{\perp\perp_{\mathbf{G}(\mathbf{L})}}) \vee^{\mathcal{C}(\mathbf{L})} (H_2^{\perp_{\mathbf{I}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_2^{\perp\perp_{\mathbf{I}(\mathbf{L})}}) \\
&= (H_1^{\perp_{\mathbf{G}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_1^{\perp\perp_{\mathbf{G}(\mathbf{L})}}) \otimes (H_2^{\perp_{\mathbf{I}(\mathbf{L})}} \vee^{\mathcal{C}(\mathbf{L})} H_2^{\perp\perp_{\mathbf{I}(\mathbf{L})}})
\end{aligned}$$

It is now clear $\mathbf{L}$ is strongly projectable if and only if $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$ are strongly projectable.

The proof for projectability would be entirely analogous, taking into account that principal polars of $\mathbf{L}$ decompose as a inner direct product of principal polars of $\mathbf{G}$ and $\mathbf{I}$, in view of Lemma 24. □

Corollary 72. *1. In a projectable GMV algebra* $\mathbf{L}$*,*

$$L = a^{\perp} \vee^{\mathcal{C}(\mathbf{L})} a^{\perp\perp} = a^{\perp} \otimes a^{\perp\perp},$$

for all $a \in L$*.*

2. In a strongly projectable GMV algebra $\mathbf{L}$*,*

$$L = A^{\perp} \vee^{\mathcal{C}(\mathbf{L})} A^{\perp\perp} = A^{\perp} \otimes A^{\perp\perp},$$

for all $A \in \mathcal{C}(\mathbf{L})$*.*

Proof. This is an immediate consequence of Proposition 18. □

The next lemma is essential in proving the uniqueness of projectable and strongly projectable hulls in the setting of semilinear GMV algebras.

Lemma 73. *Let* $\mathbf{A}$ *be a (strongly) projectable GMV algebra,* $\mathbf{B}$ *a dense subalgebra of* $\mathbf{A}$*, and* $\{\mathbf{H}_i \mid i \in I\}$ *a family of (strongly) projectable subalgebras of* $\mathbf{A}$ *that contain* $\mathbf{B}$*. Then,* $\mathbf{H} = \bigcap_{i\in I} \mathbf{H}_i$ *is (strongly) projectable.*

Proof. To begin with, notice that since for every $i \in I$, $\mathbf{B} \subseteq \mathbf{H}_i$ and $\mathbf{B}$ is dense in $\mathbf{A}$, then so are $\mathbf{H}_i$, for every $i \in I$, and $\mathbf{H} = \bigcap_{i\in I} \mathbf{H}_i$.

Fix an arbitrary $i \in I$. Since $\mathbf{H}_i$ is a dense subalgebra of $\mathbf{A}$, $\hat{\nu}_i : \mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{H}_i)$ determined by $\hat{\nu}_i(F) = F \cap H_i$ is an isomorphism of Boolean algebras, by virtue of Theorem 53. Obviously, if $X \subseteq H_i$, we have that $\hat{\nu}_i(X^{\perp^{\mathbf{A}}}) = X^{\perp^{\mathbf{A}}} \cap H_i = X^{\perp^{\mathbf{H}_i}}$, even though X is not a polar. Therefore, $X^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H_i = \hat{\nu}_i(X^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}) = \hat{\nu}_i(X^{\perp^{\mathbf{A}}})^{\perp^{\mathbf{H}_i}} = X^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}}$.

We consider an arbitrary element $h \in H$ and will see that $\mathbf{H} = h^{\perp^{\mathbf{H}}} \otimes h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$, the case of strong projectability being entirely analogous. Every $x \in H$ admits a unique decomposition $x = x_1x_2$ as an element of $\mathbf{H}_i = h^{\perp^{\mathbf{H}_i}} \otimes h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}}$, since $\mathbf{H}_i$ is projectable. But, $h^{\perp^{\mathbf{H}_i}} = h^{\perp^{\mathbf{A}}} \cap H_i$ and $h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}} = h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H_i$, as we mentioned before. Therefore, $x = x_1x_2$ is the unique decomposition of x in $\mathbf{A} = h^{\perp^{\mathbf{A}}} \otimes h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}$.

Since $i \in I$ was arbitrarily chosen, then all the decompositions of x as an element of $\mathbf{H}_i = h^{\perp^{\mathbf{H}_i}} \otimes h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}}$, for every $i \in I$, actually coincide among them, as they coincide with the decomposition of x as an element of $\mathbf{A} = h^{\perp^{\mathbf{A}}} \otimes h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}$, whence $x_1, x_2 \in H = \bigcap_I H_i$. Therefore, $x_1 \in h^{\perp^{\mathbf{A}}} \cap H = h^{\perp^{\mathbf{H}}}$, and $x_2 \in h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H = h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$. It is also obvious now that if $x = x_1x_2$ and $y = y_1y_2$ are unique decompositions of x and y as elements of $\mathbf{H} = h^{\perp^{\mathbf{H}}} \otimes h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$, then $x \leqslant y$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. □

Theorem 74. *Any algebra* $\mathbf{L}$ *in a variety* $\mathcal{V}$ *of semilinear GMV algebras has a unique, up to isomorphism, projectable hull, strongly projectable hull, and orthocomplete hull that belongs to* $\mathcal{V}$*.*

Proof. By Theorem 58, if $\mathbf{L}$ is a semilinear GMV algebra, then it can be densely embedded in the strongly projectable residuated lattice $\mathcal{O}_{<\omega}(\mathbf{L})$. Therefore, by Lemma 73, $\mathbf{L}$ has a projectable hull and a strongly projectable hull, which are the intersection of all the (strongly) projectable subalgebras of $\mathcal{O}_{<\omega}(\mathbf{L})$ containing $\mathbf{L}$.

Moreover in view of Corollary 57, $\mathbf{L}$ is densely embeddable in an orthocompete GMV algebra, namely $\mathcal{O}(\mathbf{L})$. Therefore, by Corollary 62 and Lemma 73, $\mathbf{L}$ has an orthocomplete hull, which is the intersection of all the orthocomplete subalgebras of $\mathcal{O}(\mathbf{L})$ containing $\mathbf{L}$.

An argument similar to the one in the proof of Theorem 65 shows that these hulls are unique up to isomorphism. □

We close this section by showing $\mathcal{O}_{<\omega}(\mathbf{L})$ is the strongly projectable hall of $\mathbf{L}$. We start with a technical lemma.

Lemma 75. *If $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a partition of a projectable GMV algebra $\mathbf{H}$, and $a_i \in C_i$, for $i = 1, \ldots, n$, are such that $a_1 \cdots a_n = e$, then for all i, $a_i = e$.*

Proof. We proceed by induction in n. If $n = 1$, there is nothing to prove. Suppose that $n > 1$. Since $\mathcal{C}$ is a partition and $a_i \in C_i$, we have that for every $j \neq i$, $a_i \in C_j^{\perp}$. Therefore,

$$e = a_1 \cdots a_n = a_1(a_2 \cdots a_n) \in C_1 \otimes C_1^{\perp},$$

whence we obtain that $a_1 = e$ and $a_2 \cdots a_n = e$. By the induction hypothesis, $a_2 = \cdots = a_n = e$, as was to be proved. □

Theorem 76. *If $\mathbf{L}$ is an algebra in a variety $\mathcal{V}$ of semilinear GMV algebras, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is the strongly projectable hull of $\mathbf{L}$ in $\mathcal{V}$.*

Proof. We only need show that if $\mathbf{L}$ is a dense subalgebra of a strongly projectable algebra $\mathbf{H}$ in $\mathcal{V}$, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is embeddable in $\mathbf{H}$.

In order to do so, we start by defining for every $C \in \mathrm{Pol}(\mathbf{L})$ a homomorphism $f_C : \mathbf{L} \to \mathbf{H}$, using the decomposition $\mathbf{H} = C^* \otimes C^{**}$, for every $x \in L$, $f_C(x) = x_1$ is the unique element of C^* such that there is $x_2 \in C^{**}$ such that $x = x_1 \cdot x_2$. The map f_C is well defined and a homomorphism. We notice that if $x \in C$, then $f_C(x) = e_{\mathbf{H}}$, and hence $C \subseteq \ker f_C$, whence we obtain a homomorphism $\widetilde{f}_C : \mathbf{L}/C \to \mathbf{H}$.

Consider now a finite partition $\mathcal{C}$ of $\mathbf{L}$, and the map $\psi_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}$ determined by:

$$\psi_{\mathcal{C}}([x_1]_{C_1^{\perp}}, \dots, [x_n]_{C_n^{\perp}}) = \widetilde{f}_{C_1^{\perp}}([x_1]) \cdots \widetilde{f}_{C_n^{\perp}}([x_n]).$$

The map $\psi_{\mathcal{C}}$ is trivially a homomorphism in view of Proposition 18 and moreover, it is injective by virtue of Lemma 75. This defines a family $\{\psi_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H} \mid \mathcal{C} \in \mathbb{D}_{<\omega}(\mathbf{L})\}$ of injective homomorphisms, which moreover is compatible with the system $\{\phi_{\mathcal{C}\mathcal{A}} \mid \mathcal{C} \preccurlyeq \mathcal{A},\ \mathcal{C}, \mathcal{A} \in \mathbb{D}_{<\omega}(\mathbf{L})\}$, in the sense that for every $\mathcal{C} \preccurlyeq \mathcal{A}$ in $\mathbb{D}_{<\omega}(\mathbf{L})$, $\psi_{\mathcal{A}}\phi_{\mathcal{C}\mathcal{A}} = \psi_{\mathcal{C}}$. Thus, there is a unique homomorphism $\psi : \mathcal{O}_{<\omega}(\mathbf{L}) \to \mathbf{L}$ rendering commutative the diagram:

$$\begin{tikzcd}
\mathcal{O}_{<\omega}(\mathbf{L}) \arrow[rr, hook, dashed, "\psi"] & & \mathbf{H} \\
& \mathbf{L}_{\mathcal{C}} \arrow[ul, hook, "\phi_{\mathcal{C}}"] \arrow[ur, hook, "\psi_{\mathcal{C}}"'] &
\end{tikzcd}$$

Since all the involved homomorphisms are injective, we have that $\psi_{\mathcal{C}}$ is an embedding of $\mathcal{O}_{<\omega}(\mathbf{L})$ into $\mathbf{H}$, as we wanted to prove. $\square$

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