# THE FAILURE OF AMALGAMATION PROPERTY FOR SEMILINEAR VARIETIES OF RESIDUATED LATTICES

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This work is dedicated to Antonio Di Nola, on the occasion of his 65th birthday, for his valuable and multifaceted contributions to mathematics.

Abstract. The amalgamation property (AP) is of particular interest in the study of residuated lattices due to its relationship with various syntactic interpolation properties of substructural logics. There are no examples to date of non-commutative varieties of residuated lattices that satisfy the AP. The variety $\mathcal{S}em\mathcal{RL}$ of semilinear residuated lattices is a natural candidate for enjoying this property, since most varieties that have a manageable representation theory and satisfy the AP are semilinear. However, we prove that this is not the case, and in the process we establish that the same is true for the variety $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ of semilinear cancellative residuated lattices. In addition, we prove that the variety whose members have a distributive lattice reduct and satisfy the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the AP.

## 1. Introduction

The word "amalgamation" refers to the process of combining a pair of algebras in such a way as to preserve a common subalgebra. This is made precise in the following definitions. Let $\mathcal{K}$ be a class of algebras of the same signature. A *V-formation in* $\mathcal{K}$ is a quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $i, j$ are embeddings of $\mathbf{A}$ into $\mathbf{B}, \mathbf{C}$, respectively. Given a $V$-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$, $(\mathbf{D}, h, k)$ is said to be an *amalgam* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$ if $\mathbf{D} \in \mathcal{K}$ and $h, k$ are embeddings of $\mathbf{B}, \mathbf{C}$, respectively, into $\mathbf{D}$ such that the compositions $hi$ and $kj$ coincide:

*Date*: November 8, 2013.

2000 *Mathematics Subject Classification.* Primary: 06F05, Secondary: 06D35, 06F15, 03G10.

*Key words and phrases.* Amalgamation property, Residuated lattices, Semilinear residuated lattices, Lattice-ordered groups.

$$\begin{tikzcd}
& \mathbf{C} \arrow[dr, tail, dashed, "k"] & \\
\mathbf{A} \arrow[ur, tail, "j"] \arrow[dr, tail, "i"'] & & \mathbf{D} \\
& \mathbf{B} \arrow[ur, tail, dashed, "h"'] &
\end{tikzcd}$$

$\mathcal{K}$ *has the amalgamation property* (AP) if each $V$-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}$.

Amalgamations were first considered for groups by Schreier [26] in the form of amalgamated free products. The general form of the AP was first formulated by Fraïsse [7], and the significance of this property to the study of algebraic systems was further demonstrated in Jónsson's pioneering work on the topic [13, 14, 15, 16, 17]. The added interest in the AP for algebras of logic is due to its relationship with various syntactic interpolation properties. We refer the reader to [20] for relevant references and an extensive discussion of these relationships; see also [21] and [18].

There are no results to date of non-commutative varieties of residuated lattices enjoying the AP. The variety $\mathcal{S}em\mathcal{RL}$ of *semilinear* (*representable*) residuated lattices, i.e., the variety generated by all totally ordered residuated lattices, seems like a natural candidate for enjoying this property, since most varieties that have a manageable representation theory and satisfy the AP are semilinear. An indication that this may not be the case comes from the fact that the variety $\mathcal{R}ep\mathcal{LG}$ of representable lattice-ordered groups fails the AP. Indeed, we prove that both $\mathcal{S}em\mathcal{RL}$ and the variety $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ of semilinear cancellative residuated lattices fail the AP (Theorem 4.3). In addition, we prove (Theorem 4.2) that the much larger variety $\mathcal{U}$ of all residuated lattices with distributive lattice reduct and satisfying the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the AP. In fact, we show that any subvariety of this variety fails the AP, as long as its intersection with the variety of lattice-ordered groups fails the AP.

There are two key ingredients in the proofs of these results. First, the fact that the specific $V$-formations that demonstrate the failure of the AP for the variety $\mathcal{R}ep\mathcal{LG}$ of representable lattice-ordered groups ([25], [5]; see Theorem 3.1 and [5, Theorem B]) also demonstrate its failure for $\mathcal{S}em\mathcal{RL}$ and $\mathcal{S}em\mathcal{C}an\mathcal{RL}$. The second key element in the proofs is the fact that each algebra in these varieties has a representation in terms of residuated maps of a chain ([22], [1]; see Lemma 4.1). For the convenience of the reader, and the fact that large portions of the theory of lattice-ordered groups are not very familiar to researchers working with algebras of logic, we have strived for a reasonably self-contained presentation. In particular, in Section 3 we present parts of the original proof of the failure of the AP for $\mathcal{R}ep\mathcal{LG}$.

## 2. Basic notions

In this section we briefly recall basic facts about the varieties of residuated lattices, referring to [4], [12], [8], and [21] for further details. These varieties provide algebraic semantics for substructural logics, and encompass other important classes of algebras such as lattice-ordered groups.

A *residuated lattice* is an algebra $\mathbf{L} = (L, \cdot, \backslash, /, \vee, \wedge, \mathrm{e})$ satisfying:

(a) $(L, \cdot, \mathrm{e})$ is a monoid;
(b) $(L, \vee, \wedge)$ is a lattice with order $\leq$; and
(c) $\backslash$ and $/$ are binary operations satisfying the residuation property:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y.$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write $xy$ for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$.

Throughout this paper, the class of residuated lattices will be denoted by $\mathcal{RL}$. It is easy to see that the equivalences that define residuation can be captured by finitely many equations and thus $\mathcal{RL}$ is a finitely based variety (see [4], [3]).

The existence of residuals has the following basic consequences, which will be used in the remainder of the paper without explicit reference.

**Lemma 2.1.** *Let* $\mathbf{L}$ *be a residuated lattice.*

(1) *The multiplication preserves all existing joins in each argument; i.e., if* $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq L$, *then* $\bigvee_{x\in X, y\in Y}(xy)$ *exists and*

$$\Big(\bigvee X\Big)\Big(\bigvee Y\Big) = \bigvee_{x\in X, y\in Y}(xy).$$

(2) *The residuals preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e. if* $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq L$, *then for any* $z \in L$, $\bigwedge_{x\in X}(x\backslash z)$ *and* $\bigwedge_{y\in Y}(z\backslash y)$ *exist and*

$$\Big(\bigvee X\Big)\Big\backslash z = \bigwedge_{x\in X}(x\backslash z) \quad \text{and} \quad z\Big\backslash\Big(\bigwedge Y\Big) = \bigwedge_{y\in Y}(z\backslash y).$$

(3) *The following identities (and their mirror images)*[1] *hold in* $\mathbf{L}$.
   (a) $(x\backslash y)z \leq x\backslash yz$
   (b) $x\backslash y \leq zx\backslash zy$
   (c) $(x\backslash y)(y\backslash z) \leq x\backslash z$

[1](1) and (2) are expressed as inequalities, but are clearly equivalent to identities.

(d) $xy\backslash z = y\backslash(x\backslash z)$
(e) $x\backslash(y/z) = (x\backslash y)/z$
(f) $x(x\backslash x) = x$
(g) $(x\backslash x)^2 = x\backslash x$

A residuated lattice is said to be *integral* if its top element is e. A residuated lattice is *commutative* if it satisfies the equation $xy \approx yx$, in which case, $x\backslash y$ and $y/x$ coincide and are denoted by $x \to y$.

An element $a \in L$ is said to be *invertible* if $(\mathrm{e}/a)a = \mathrm{e} = a(a\backslash \mathrm{e})$. This is of course true if and only if $a$ has a (two-sided) inverse $a^{-1}$, in which case $\mathrm{e}/a = a^{-1} = a\backslash\mathrm{e}$. The structures in which every element is invertible are therefore precisely the lattice-ordered groups ($\ell$-groups). It should be noted that an $\ell$-group is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, ^{-1}, \mathrm{e})$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, ^{-1}, \mathrm{e})$ is a group, and multiplication is order preserving – or, equivalently, it distributes over the lattice operations (see [2], [9]). The variety of $\ell$-groups is term equivalent to the subvariety of $\mathcal{RL}$ defined by the equations $(\mathrm{e}/x)x \approx \mathrm{e} \approx x(x\backslash\mathrm{e})$; the term equivalence is given by $x^{-1} = \mathrm{e}\backslash x$ and $x/y = xy^{-1}$, $x\backslash y = x^{-1}y$. We denote by $\mathcal{LG}$ the aforementioned subvariety and refer to its members as *$\ell$-groups*.

Let $\mathbf{L}$ be a residuated lattice. If $F \subseteq L$, we write $F^-$ for the set of "negative" elements of $F$; i.e., $F^- = \{x \in F : x \leq \mathrm{e}\}$. The *negative cone* of $\mathbf{L}$ is the algebra $\mathbf{L}^-$ with domain $L^-$ and lattice operations and the monoid operation of $\mathbf{L}^-$, the restrictions to $\mathbf{L}^-$ of the corresponding operations in $\mathbf{L}$. The residuals $\backslash_-$ and $/_-$ are defined by

$$x\backslash_- y = (x\backslash y) \wedge \mathrm{e} \qquad \text{and} \qquad y/_- x = (y/x) \wedge \mathrm{e}$$

where $\backslash$ and $/$ denote the residuals in $\mathbf{L}$.

Given a class $\mathcal{V}$ of residuated lattices, we denote the class of the negative cones of algebras of $\mathcal{V}$ by $\mathcal{V}^-$. We state the following result from [3, Thm. 7.1] for future reference:

**Lemma 2.2.** *If $\mathcal{V}$ is a variety of $\ell$-groups, then ~~so is~~ $\mathcal{V}^-$ is a variety of residuated lattices. Moreover, $\mathcal{V}$ and $\mathcal{V}^-$ are isomorphic as categories.*

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. Recall that a map $f : \mathbf{P} \to \mathbf{Q}$ is *residuated* if there exists a map $f^* : \mathbf{Q} \to \mathbf{P}$ such that for any $a \in P$ and any $b \in Q$, $f(a) \leq^{\mathbf{Q}} b$ iff $a \leq^{\mathbf{P}} f^*(b)$. In this case, we say that $f$ and $f^*$ form a *residuated pair*, and that $f^*$ is a *residual* of $f$. Note that a binary map is residuated in the preceding sense if and only if all translates of the map are residuated in the preceding sense.

## 3. Failure of the AP for representable $\ell$-groups

Throughout this work, the varieties of representable lattice-ordered groups ($\ell$-groups) and semilinear residuated lattices will be denoted by $\mathcal{R}ep\mathcal{LG}$ and

$\mathcal{S}em\mathcal{RL}$, respectively. $\mathcal{S}em\mathcal{RL}$ is the variety of $\ell$-groups generated by all totally ordered groups, and, as noted above, $\mathcal{S}em\mathcal{RL}$ is the variety of residuated lattices generated by all totally ordered residuated lattices. Our proof of the failure of the AP for $\mathcal{S}em\mathcal{RL}$ extends the techniques used in [25] to establish the failure of the AP in $\mathcal{R}ep\mathcal{LG}$. We therefore start with a review of the latter:

Let us first note that, for every positive integer $n$, any representable $\ell$-group satisfies the quasi-equation

$$x^n = y^n \Rightarrow x = y. \tag{$\star$}$$

The plan is to construct a $V$-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{R}ep\mathcal{LG}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ totally ordered and $i, j$ inclusions. For a given positive integer $n$, the $\ell$-groups $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ will contain elements $a, b, c$, respectively, such that $b^n = c^n \in A$, but $a^b = bab^{-1}$ and $a^c = cac^{-1}$ are distinct elements of $A$. Therefore, the $V$-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ cannot be amalgamable in $\mathcal{R}ep\mathcal{LG}$, since otherwise the amalgam would contain distinct elements $b$ and $c$ such that $b^n = c^n$, falsifying $(\star)$.

To construct the $V$-formation in question, let us first consider a totally ordered group $\mathbf{A}$, its group of order-automorphisms of the underlying total order of $\mathbf{A}$, and $\alpha \in \mathbf{Aut}(\mathbf{A})$. The *cyclic extension* of $\mathbf{A}$ by $\alpha$ is the totally ordered group $\mathbf{A}(\alpha)$ whose universe is

$$\{(a, \alpha^n) : a \in A \text{ and } n \in \mathbb{Z}\}$$

with operations defined by

$$(a, \alpha^m)(b, \alpha^n) = (a\alpha^m(b), \alpha^{m+n}),$$

and order defined by

$$(a, \alpha^n) \leq (b, \alpha^m) \text{ iff } n < m \text{ or } n = m \text{ and } a \leq b.$$

It is convenient to regard $\mathbf{A}$ as a subalgebra $\mathbf{A}(\alpha)$ by identifying $a$ with $(a, id)$; we also identify $(1, \alpha)$ with $\alpha$. Let now $\alpha, \beta, \gamma$ be order-automorphisms of the totally ordered group $\mathbf{A}$ such that, for a positive integer $n$, $\alpha = \beta^n = \gamma^n$, but $\beta \neq \gamma$. Note that $\mathbf{A}(\alpha)$ is a subalgebra of both $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$. Note further that for all $a \in A$, $a^\beta = \beta(a)$ in $\mathbf{A}(\beta)$ and $a^\gamma = \gamma(a)$ in $\mathbf{A}(\gamma)$, and, of course $a^\beta$ and $a^\gamma$ are elements of $A$.

We claim that any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ fails the AP. More specifically, the $V$-formation $(\mathbf{A}(\alpha), \mathbf{A}(\beta), \mathbf{A}(\gamma), i, j)$ – with $i, j$ inclusions – does not have an amalgam in $\mathcal{R}ep\mathcal{LG}$. Indeed, in any $\ell$-group $\mathbf{G}$ that contains $\mathbf{A}(\alpha)$ and $\mathbf{A}(\beta)$ as $\ell$-subgroups, the equality $\beta^n = \gamma^n$ is satisfied. On the other hand, there is $a \in A$ such that $\beta(a) \neq \gamma(a)$, and hence there is $a \in A$ such that $a^\gamma$ and $a^\beta$ are distinct elements of $A$ and hence of $\mathbf{G}$. This shows that $\beta \neq \gamma \in \mathbf{G}$, and so $\mathbf{G}$ cannot be representable.

Thus, we have the following result from [25] (see also [10]):

**Lemma 3.1.** *Let* $\mathbf{A}$ *be a totally ordered group, and let* $\alpha, \beta, \gamma$ *be distinct order automorphisms of the lattice-reduct of* $\mathbf{A}$ *such that* $\alpha = \beta^n = \gamma^n$*, for some integer* $n \geq 2$*. Then any subvariety of* $\mathcal{R}ep\mathcal{LG}$ *that contains* $\mathbf{A}(\beta)$ *and* $\mathbf{A}(\gamma)$ *fails the* AP.

There is a natural way of constructing totally ordered groups such as $\mathbf{A}(\alpha), \mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$. Let us look at the case $n = 2$. Set $I = \mathbb{Z} \overleftarrow{\times} \mathbb{Z}$, ordered anti-lexicogaphically, and define $\bar{\beta}, \bar{\gamma} \in \mathbf{Aut}(\mathbf{I})$ by:

$$\bar{\beta}(x,y) = \begin{cases} (x+1, y+1), & \text{if } y \text{ is even;} \\ (x, y+1), & \text{otherwise.} \end{cases}$$

$$\bar{\gamma}(x,y) = \begin{cases} (x, y+1), & \text{if } y \text{ is even;} \\ (x+1, y+1), & \text{otherwise.} \end{cases}$$

The following lemma follows straightforwardly from the definitions.

**Lemma 3.2.** *Let* $I$*,* $\bar{\beta}, \bar{\gamma}$ *as above. Then:*

(i) $\bar{\beta} \neq \bar{\gamma}$;
(ii) $\bar{\beta}^2 = \bar{\gamma}^2 = \bar{\alpha}$;
(iii) *for all* $a \in I$*,* $\bar{\alpha}(a) > a$.

Set $\mathbf{A}$ to be $\overleftarrow{\bigoplus}_{i \in I} \mathbb{Z}_i$ the direct sum of copies of the integers, anti-lexicogaphically ordered, indexed by $I$. Let us remark that each order automorphism $\bar{\delta}$ of $I$ determines an automorphism $\delta$ on $\mathbf{A}$ defined by

$$\delta((x_i)_{i \in I}) = (x_{\bar{\delta}(i)})_{i \in I}.$$

As a consequence, both $\bar{\beta}, \bar{\gamma}$ induce distinct order-automorphisms $\beta, \gamma$ of the underlying chain of $\mathbf{A}$ such that $\beta^2 = \gamma^2$. In light of the preceding discussion, any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ fails the AP.

The careful analysis in [25] shows much more than the failure of the AP for $\mathcal{R}ep\mathcal{LG}$. Let us denote by $\mathcal{M}$ the variety of $\ell$-groups generated by $\mathbb{Z}\,\mathtt{wr}\,\mathbb{Z}$, where $\mathbb{Z}$ denotes the $\ell$-group of integers. $\mathcal{M}$ is a cover of the variety of abelian $\ell$-groups [19]. It can be proved – refer to [25] for details – that the totally ordered groups constructed in the preceding paragraph actually belong to $\mathcal{M}$. This leads to the following result of [25].

**Theorem 3.1.** *If* $\mathcal{V}$ *is a subvariety of* $\mathcal{R}ep\mathcal{LG}$ *containing* $\mathcal{M}$*, then* $\mathcal{V}$ *fails the* AP.

It should be noted that, by results in [6], the interval $[\mathcal{M}, \mathcal{R}ep\mathcal{LG}]$ is uncountable. Thus, in light of Theorem 3.1, there are uncountably many subvarieties of $\mathcal{R}ep\mathcal{LG}$ that fail the AP.

## 4. Failure of the AP for semilinear residuated lattices

In this section, we prove the main results of this article. Namely, we show that the variety $\mathcal{S}em\mathcal{RL}$ of semilinear residuated lattices and the variety $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ of semilinear cancellative residuated lattices fail the AP. In addition, we prove that the variety $\mathcal{U}$ consisting of all residuated lattices that have a distributive lattice reduct and satisfy the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the AP.

We start with the definition of an $\ell$-monoid. An *$\ell$-monoid* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \mathrm{e})$ of type $(2, 2, 2, 0)$ such that

(i) $(L, \wedge, \vee)$ is a lattice;
(ii) $(L, \cdot, \mathrm{e})$ is a monoid; and
(iii) $\mathbf{L}$ satisfies the following equations:

$$\begin{aligned} a\,(b \vee c)\,d &\approx abd \vee acd \\ a\,(b \wedge c)\,d &\approx abd \wedge acd \end{aligned}$$

Homomorphisms of $\ell$-monoids are referred to as *$\ell$-homomorphisms*, and, in the preceding equations and in what follows, we use plain juxtaposition in place of "$\cdot$".

Now, given any chain $\mathbf{\Omega}$, the set $\mathcal{R}es(\mathbf{\Omega})$ of all residuated maps on $\mathbf{\Omega}$ is (the universe of) a monoid with respect to function composition, and a lattice with respect to pointwise join and meet; moreover, it is easily checked that $\mathcal{R}es(\mathbf{\Omega})$ is the universe of an $\ell$-monoid whose lattice reduct is *distributive*. By abuse of notation, we denote such an $\ell$-monoid by the same label $\mathcal{R}es(\mathbf{\Omega})$. Also $\mathcal{A}ut(\mathbf{\Omega})$, the set of all *order-automorphisms* of $\mathbf{\Omega}$, is the universe of an $\ell$-monoid which is actually an $\ell$-group. We make use of the following result in [22] (see also [1]), which generalizes Holland's Embedding Theorem ([11]).

**Theorem 4.1.** *A residuated lattice* $\mathbf{A}$ *can be embedded as an $\ell$-monoid into* $\mathcal{R}es(\mathbf{\Omega})$*, for some chain* $\mathbf{\Omega}$*, if and only if it satisfies the equations*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$*; and*
(2) $x(y \wedge z)w \approx xyw \wedge xzw$.

This representation afforded by the preceding result will play a key role in the proofs of the results below.

**Lemma 4.1.** *If* $\mathbf{D}$ *is a residuated lattice that satisfies the equations*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$*; and*
(2) $x(y \wedge z)w \approx xyw \wedge xzw$,

*then the set* $Inv(D)$ *of invertible elements of* $\mathbf{D}$ *is the universe of a subalgebra of* $\mathbf{D}$ *that is an $\ell$-group.*

*Proof.* Let $Inv(D)$ be the set of invertible elements of $\mathbf{D}$. We have to prove that it is closed under the operations of $\mathbf{D}$. It is obvious that $Inv(D)$ is closed under products and contains e. Further, it is easy to see that if $a, b \in Inv(D)$, then $a\backslash_{\mathbf{D}} b = a^{-1}b$ and $a/_{\mathbf{D}} b = ab^{-1}$. Let us just verify the equality $a\backslash_{\mathbf{D}} b = a^{-1}b$, as the other is analogous. Clearly $a(a^{-1}b) = b \leq b$. If $ac \leq b$, then $c = a^{-1}ac \leq a^{-1}b$. As for the lattice operations, we note first that in virtue of Theorem 4.1 there exists a chain $\mathbf{\Omega}$ such that $\mathbf{D}$ is an $\ell$-submonoid of $\mathcal{R}es(\mathbf{\Omega})$. Since $\mathbf{\Omega}$ is a chain, $\mathcal{R}es(\mathbf{\Omega})$ is distributive and the product distributes over joins and meets. Moreover, the invertible elements of $\mathcal{R}es(\mathbf{\Omega})$ are the order automorphisms of $\mathbf{\Omega}$, and given an order automorphism $a : \mathbf{\Omega} \to \Omega$, it is easy to see that $a \wedge a^{-1} \leqslant \mathrm{e} \leqslant a \vee a^{-1}$. Therefore, for every pair of invertible elements $g, h \in \mathcal{R}es(\mathbf{\Omega})$, $g^{-1}h \wedge h^{-1}g \leq \mathrm{e} \leq g^{-1}h \vee h^{-1}g$. Thus, if $g, h \in Inv(D)$, then

$$\begin{aligned}(g^{-1} \vee h^{-1})(g \wedge h) &= g^{-1}(g \wedge h) \vee h^{-1}(g \wedge h) \\ &= (g^{-1}g \wedge g^{-1}h) \vee (h^{-1}g \wedge h^{-1}h) \\ &= (\mathrm{e} \wedge g^{-1}h) \vee (h^{-1}g \wedge \mathrm{e}) \\ &= \mathrm{e} \wedge (g^{-1}h \vee h^{-1}g) = \mathrm{e},\end{aligned}$$

and analogously, $(g^{-1} \wedge h^{-1})(g \vee h) = \mathrm{e}$, which shows that $(g \wedge h)$ and $(g \vee h)$ are invertible, as we wanted to prove. Therefore, $Inv(D)$ is the universe of a subalgebra of $\mathbf{D}$, and obviously every element in $Inv(D)$ has an inverse, and hence it is an $\ell$-group. $\square$

It is already known (see [23], [5] or [24]) that the variety of all $\ell$-groups fails the `AP`. We remark that [24] contains an improved presentation of the original proof in [23], while the recent paper [5] shows that the $\ell$-groups $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ and $\mathbb{Z}^n$, for $n \geq 3$, are not an *amalgamation base* of $\mathcal{LG}$. This means that there exist $V$-formations $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ – with $\mathbf{A} = \mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ or $\mathbb{Z}^n, n \geq 3$ – that do not have an amalgam in $\mathcal{LG}$.

We can use these results to prove that any variety of residuated lattices satisfying equations (1) and (2) of the previous lemma and containing the variety of $\ell$-groups fails the `AP`. More generally, we have:

**Theorem 4.2.** *Let $\mathcal{V}$ is a variety of residuated lattices satisfying the following equations:*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$

(2) $x(y \wedge z)w \approx xyw \wedge xzw$

*If $\mathcal{V} \cap \mathcal{LG}$ fails the* AP*, then so does $\mathcal{V}$.*

*Proof.* Let $\mathbf{B}$, $\mathbf{C}$ in $\mathcal{V} \cap \mathcal{LG}$, and $\mathbf{A}$ a common subalgebra. Suppose that a $V$-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ has an amalgam $(h, k, \mathbf{D})$ in $\mathcal{V}$. We may assume that all maps $i, j, h$ and $k$ are inclusions. Then, by Lemma 4.1, $Inv(\mathbf{D})$ is a subalgebra of $\mathbf{D}$, which obviously contains $\mathbf{B}$ and $\mathbf{C}$, because every element of $B \cup C$ is invertible in $\mathbf{D}$. Furthermore $Inv(\mathbf{D})$ is an $\ell$-group which is also in $\mathcal{V}$. Hence, $Inv(\mathbf{D})$ would be an amalgam in $\mathcal{V} \cap \mathcal{LG}$ of the $V$-formation, which does not exist in general. $\square$

As a consequence of this theorem and Theorem 3.1, we have:

**Theorem 4.3.** *The variety $\mathcal{SemRL}$ of semilinear residuated lattices and the variety $\mathcal{SemCanRL}$ of semilinear cancellative residuated lattices fail the* AP.

*Proof.* The variety $\mathcal{LG} \cap \mathcal{SemRL}$ is the variety $\mathcal{RepLG}$, which we know by Theorem 3.1 that fails the AP. Likewise for $\mathcal{SemCanRL}$. □

**Corollary 4.3.1.** *The varieties $\mathcal{LG}^-$ and $\mathcal{RepLG}^-$ fail the* AP.

*Proof.* As was noted above, the varieties $\mathcal{LG}$ and $\mathcal{RepLG}$ fail the AP. Thus, the result follows from Lemma 2.2.

Another way of proving the failure of the AP for $\mathcal{RepLG}^-$ directly is the following: Let us first note that Condition ($\star$) holds in $\mathcal{RepLG}^-$. Suppose that there exists a $\mathbf{D}$ in $\mathcal{RepLG}^-$ amalgamating the negative cones of $\mathbf{A}(\alpha)$, $\mathbf{A}(\beta)$, $\mathbf{A}(\gamma)$. Consider the elements $((0), \beta^{-1})$, $((0), \gamma^{-1})$, defined as above. Clearly, $((0), \beta^{-1}), ((0), \gamma^{-1}), ((0), \alpha^{-1})$ are in the negative cone. Moreover, $((0), \beta^{-1})^2 = ((0), \gamma^{-1})^2 = ((0), \alpha^{-1})$. However, $((0), \beta^{-1}) \neq ((0), \gamma^{-1})$. This contradicts Condition ($\star$), which is impossible. □

## 5. Open problems

The techniques of the present paper do not seem to be adequate to determine whether the variety $\mathcal{SemIRL}$ of semilinear integral residuated lattices and the variety $\mathcal{SemCanIRL}$ of semilinear cancellative integral residuated lattices fail the AP. Hence, we propose the next two open problems:

**Problem 1:** Does the variety $\mathcal{SemIRL}$ of semilinear integral residuated lattices fail the AP.

**Problem 2:** Does the variety $\mathcal{SemCanIRL}$ of semilinear cancellative integral residuated lattices fail the AP.

A substantially harder open problem, which is connected to the long-standing question of embedding a totally ordered group into a divisible one, is the following:

**Problem 3:** Let $\mathbf{A}$ be an arbitrary (not necessarily commutative) totally ordered group. Do all $V$-formations of the form $(\mathbb{Z}, \mathbb{Q}, \mathbf{A}, i, j)$ have an amalgam in $\mathcal{RepLG}$? Here, $\mathbb{Z}$ and $\mathbb{Q}$ denote the totally ordered groups of integers and rationals, respectively.

More generally we can ask:

**Problem 4:** Let $\mathbf{A}, \mathbf{B}$ be arbitrary totally ordered groups. Do all $V$-formations of the form $(\mathbb{Z}, \mathbf{A}, \mathbf{B}, i, j)$ have an amalgam in $\mathcal{RepLG}$? In other words, is $\mathbb{Z}$ an amalgamation base of $\mathcal{RepLG}$?

As has already been remarked, all subvarieties of $\mathcal{RL}$ that are known to satisfy the AP are commutative.

**Problem 5:** Is there a non-commutative variety of residuated lattices that satisfies the $\mathsf{AP}$? In particular, does the variety $\mathcal{RL}$ of all residuated lattices satisfy the $\mathsf{AP}$?

Three open problems that may have affirmative answers are the following:

**Problem 6:** Does the variety $\mathcal{C}an\mathcal{CRL}$ of cancellative commutative residuated lattices have the $\mathsf{AP}$?

**Problem 7:** Does the variety $\mathcal{S}em\mathcal{C}an\mathcal{CRL}$ of semilinear cancellative commutative residuated lattices have the $\mathsf{AP}$?

**Problem 8:** Does the variety $\mathcal{S}em\mathcal{CRL}$ of semilinear commutative residuated lattices have the $\mathsf{AP}$?

**Acknowledgements.** The first author acknowledges the support of the grants MTM2008-01139 of the Spanish Ministry of Science and Innovation (which includes EUs FEDER funds), and 2009SGR-01433 of the Catalan Government. The second author acknowledges the support of the Italian Ministry of Scientific Research within the FIRB project "Structures and dynamics of knowledge and cognition," Cagliari: F21J12000140001. The third author gratefully acknowledges the Visiting Professors Program, sponsored by Regione Autonoma della Sardegna, for enabling him to visit the University of Cagliari.

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