The Archimedean Property: New Horizons and Perspectives
In memoriam: Bjarni Jónsson

Antonio Ledda[1], Francesco Paoli[1], Constantine Tsinakis[2]
[1]Università di Cagliari; [2]Vanderbilt University

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Abstract

Although there have been repeated attempts to define the concept of an Archimedean algebra for individual classes of residuated lattices, there is no all-purpose definition that suits the general case. We suggest as a possible candidate the notion of a normal-valued and e-cyclic residuated lattice that has the *zero radical compact property* — namely, a normal-valued and e-cyclic residuated lattice in which every principal convex subuniverse has a trivial radical (understood as the intersection of all its maximal convex subuniverses). We characterize the Archimedean members in the variety of e-cyclic residuated lattices, as well as in various special cases of interest. A theorem to the effect that each Archimedean and prelinear GBL-algebra is commutative, subsuming as corollaries several analogous results from the recent literature, is grist to the mill of our proposal's adequacy. Finally, we revisit the concept of a *hyper-Archimedean* residuated lattice, another notion with which researchers have engaged from disparate angles, and investigate some of its properties.

1 Introduction

If we are given two real numbers a, b such that $0 < a < b$, we know that there must be a positive integer n such that $na > b$ — in other words, $\mathbb{R}$ contains no *infinitesimal elements*. This *Archimedean property* is one of the most distinctive and useful features of the field of the reals; it grounds both the theory of magnitudes and classical, as opposed to nonstandard, analysis. Given its crucial role in mathematics, it is desirable to inquire whether it can be extended beyond its original domain of significance, so as to be applicable to more general ordered structures, such as *residuated lattices* [4, 28, 33].

Indeed, for individual classes of residuated lattices – such as ℓ-groups [2, Ch. 2] or MV-algebras [7, §§ 3.6, 6.3], just to name two examples – there exist well-worked definitions of an infinitesimal element and of an Archimedean algebra, that capture the idea of such algebras "having no infinitesimals". In the general case, however, it is by no means obvious how to express this intuitive concept in rigorous terms, all the more so if we aim (as in [14, 8]) at encompassing under a common umbrella all the extant definitions. In the literature, two main viewpoints have been adopted according to whether the notion of Archimedean algebra or the notion of infinitesimal element is viewed as more fundamental:

- Some authors have suggested to identify Archimedean residuated lattices with *semisimple* algebras, letting infinitesimal elements be the non-unit elements of an algebra's normal radical (see below).
- In other areas, instead, it is more customary to seek a handy characterization of infinitesimal (or of non-infinitesimal) element, which is subsequently employed to yield a definition of Archimedean residuated lattice as a residuated lattice containing no infinitesimals in precisely this sense.

It will be clear from what follows that neither viewpoint is general enough. In fact, one can find residuated lattices that should count as Archimedean by all reasonable standards but elude the former definition, while it is hard to come up with a definition of infinitesimal element that can be applied across the board. Before advancing an alternative proposal, however, let us review these perspectives in some more detail.

Let us recall that a semisimple algebra is an algebra that is isomorphic to a subdirect product of simple algebras. Equivalently, an algebra $\mathbf{A}$ is semisimple just in case the intersection of its maximal congruences is the identity relation Δ. Since all varieties of residuated lattices are ideal-determined[1], such an equivalence assumes in this context an especially attractive form: a residuated lattice $\mathbf{L}$ is semisimple if the intersection of all the maximal normal convex subuniverses of $\mathbf{L}$ — in other words, its *normal radical* — is the singleton of its unit element $\{e\}$. The circumstance that, at least in some paradigmatic cases, the members of the normal radical can be described via an appropriate "non-Archimedean" condition invites the identification of infinitesimal elements with the non-unit elements of the normal radical of $\mathbf{L}$, and consequently, the identification of Archimedean and semisimple residuated lattices [7, § 3.6].

This viewpoint has its allure and provides an intriguing bridge to the classical notion, especially in varieties of residuated lattices whose strongly simple members can be embedded into algebras over the real numbers. However, it does not work in general — the obvious pitfall being varieties of residuated lattices, such as ℓ-groups, whose members need not have maximal normal convex subuniverses. For example, let X be a Stone space with nonmeasurable cardinality and with no isolated points, and let $D(X)$ be the set of continuous, almost finite real-valued functions on X [2, Ex. E21]. $D(X)$ can be made into the universe of an ℓ-group which is Archimedean in the ℓ-group sense, but not semisimple.

[1] Let us remark that the property of being an ideal variety is a Maltsev property, whence it carries over from a variety $\mathcal{V}$ to its subvarieties.

The alternative viewpoint, as we have observed, focuses on the notion of an *infinitesimal element*. For example, an element $a < e$ of an ℓ-group $\mathbf{L}$ is said to be infinitesimal just in case there exists $b < e$ in L such that for all positive integers n, $a^n > b$, and an Archimedean ℓ-group is usually defined as an ℓ-group with no infinitesimals. However, this definition wears thin as soon as we trespass onto varieties of (pointed) residuated lattices that can contain non-trivial idempotent members. For example, if we apply this definition to the class of Boolean algebras, it turns out, against anyone's better judgment, that no non-trivial Boolean algebra is Archimedean.

For integral and involutive[2] pointed residuated lattices, there is a way around this problem. Let $\mathbf{L}$ be such a residuated lattice, and let $a \in L$ be such that $a < e$. Then a is said to be infinitesimal just in case for all positive integers n, $a^n \geq a \backslash f$, and Archimedean residuated lattices are those residuated lattices that contain no such infinitesimals [19]. Interestingly enough, in the case of pseudo-MV algebras (see below), these are just the semisimple algebras: [16], [7, Prop. 3.6.4]. But again, although this concept works well in integral and pointed structures, it is ill-suited for non-pointed residuated lattices, and certainly its failure to account for Archimedean ℓ-groups represents a major drawback.

A recurrent theme that spans the literature on Archimedean residuated lattices is provided by results to the effect that, given some class $\mathcal{K}$ of residuated lattices, and given a definition of Archimedean algebra that suits $\mathcal{K}$, all Archimedean members of $\mathcal{K}$ are commutative: [2, Thm. 2.2], [16, Thm. 4.2], [18, Thm. 3.3], [30, Cor. 5.2]. On the other hand, examples of noncommutative residuated lattices, usually taken from outside the class of GBL-algebras, that are Archimedean under some of the above acceptations have been discussed as well [8]. Against this backdrop, a natural challenge is to find a definition of Archimedean residuated lattice that, unlike the above-mentioned suggestions, subsumes all the existing concepts, and to single out a large enough class $\mathcal{K}$ for which being Archimedean in this sense implies being commutative. This is what we set out to do in the present paper.

The way we intend to approach this problem is inspired by the work of P.F. Conrad, who, in the 1960s, launched a general program for the investigation of ℓ-groups [9, 10, 11, 12], aimed at capturing relevant information about these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups, as well as at showing that many significant properties of ℓ-groups are, in essence, either purely lattice-theoretic, or at least such that the underlying group structure does not play a predominant role. A natural continuation of Conrad's original program consists in extending it from ℓ-groups to more comprehensive domains, *in primis* residuated lattices. This *extended Conrad program* has fueled some of the recent developments in the theory of residuated structures, leading to promising results e.g. in the study of semilinear and Hamiltonian varieties [5], in the investigation of normal-valued residuated lattices [6] and in the description of projectable objects [25, 31].

[2] In [8], a generalization of this definition to the case of not necessarily involutive residuated lattices is introduced and discussed.

What in our view is the most far-reaching attempt to capture the Archimedean property so far, due to Jorge Martinez [32], is indeed consistent with Conrad's approach. An algebraic distributive lattice $\mathbf{L}$, where the set of compact elements is closed under finite meets, has the *zero radical compact property*[3] if for each compact element $c \in L$, the meet of all the maximal elements in the interval $[\bot, c]$ in $\mathbf{L}$ is $\bot$. Martinez observed that an Abelian ℓ-group is Archimedean if and only if its lattice of convex subuniverses has the zero radical compact property [32, p. 249]. In this special case, therefore, the Archimedean property is fully captured in the lattices of convex subuniverses of the ℓ-groups in question. However, this is no longer true, even in the ℓ-group setting, once we bring non-commutative algebras to the fore, divorcing thereby the notion of convex subuniverse from the notion of *normal* convex subuniverse. Our hopes of pinning down the Archimedean property in purely lattice-theoretic terms are readily dashed once we consider that there exist Archimedean ℓ-groups and non-normal-valued ℓ-groups whose lattices of convex subuniverses are isomorphic [2, Ex. E53].

We advance hereby a suggestion to the effect that a residuated lattice is *Archimedean* just in case (1) it is e-cyclic; (2) it is normal-valued; and (3) its lattice of convex subuniverses has the zero radical compact property. The restriction to e-cyclic residuated lattices is motivated by the fact that only in the e-cyclic case lattices of convex subuniverses are known to behave in the appropriate way – for example, they are distributive lattices [5]. The *desideratum* (3) is clearly inspired by Martinez's characterization of Archimedean Abelian ℓ-groups. Finally, (2) supplements the lattice-theoretic toolbox with the needed extra information for fully describing the property. In Definition 8, we will state the property of being normal-valued in such a way that it includes e-cyclicity, a convention that we assume to take effect hereafter.

A stronger notion of a *hyper-Archimedean* residuated lattice has been paid some attention in the literature: [18], [39], [7, § 6.3]. A hyper-Archimedean (pointed) residuated lattice is at times defined as a residuated lattice $\mathbf{L}$ such that for all $a \in L$ there exists some $n \geq 1$ such that $a^n = a^{n+1}$ (this n need not be the same for all elements, unlike in *n-potent* residuated lattices, namely, residuated lattices satisfying the equation $x^n \approx x^{n+1}$ for some $n \geq 1$). This condition is quite restrictive, since all GBL-algebras that are hyper-Archimedean in this sense are necessarily integral and Hamiltonian [18]; moreover, it does not mix well with cancellativity. In other contexts, for example in the ℓ-group case, hyper-Archimedean residuated lattices are characterized as residuated lattices whose *quotients* modulo any normal convex subuniverse are Archimedean. These two different perspectives can be reconciled by defining a hyper-Archimedean residuated lattice as a normal-valued residuated lattice $\mathbf{L}$ whose prime (meet-irreducible) convex subuniverses are maximal – that is, they form an anti-chain. Observe that this definition requires no additional foray outside the borders of the lattice-theoretic territory, other than those that are

[3]Martinez calls such lattices *Archimedean*. We prefer to use a brand new label in order to avoid conflicts with the notion of an Archimedean residuated lattice.

already demanded by the notion of an Archimedean lattice.

Let us now illustrate the article's discourse. In Section 2, we dispatch some preliminaries on residuated lattices and their convex subuniverses. In Section 3, we initially consider normal-valued residuated lattices with a strong order unit, giving a description of their maximal convex subuniverses and, consequently, of their radicals. This characterization is put to good use once we have introduced the notion of an Archimedean residuated lattice. Given a residuated lattice $\mathbf{L}$, an element $a \in L$ and a sequence $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ of iterated conjugation maps in $\mathbf{L}$, let

$$\pi_\gamma^u (a) = \prod_{j=1}^{m} \gamma_j (|a|) \backslash u \wedge e,$$

and, for $n \in \mathbb{Z}^+$,

$$\pi_n^u (a) = |a|^n \backslash u \wedge e.$$

The main results of this section (whose full terminology and notation will be duly explained there) are as follows:

Theorem A (see Theorem 18) *Let* $\mathbf{L}$ *be a non-trivial normal-valued residuated lattice. The following statements are equivalent:*

1. $\mathbf{L}$ *is Archimedean.*
2. *For all* $u < e$ *in* L *and all elements* $a \neq e$ *in* $\mathrm{C}[u]$*, there exists* $m \in \mathbb{Z}^+$ *such that* $\pi_n^u(\pi_m^u (a)) \neq e$*, for all* $n \in \mathbb{Z}^+$*.*
3. *For all* $u < e$ *in* L *and all elements* $a \neq e$ *in* $\mathrm{C}[u]$*, there exists an* m*-tuple* $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ *of iterated conjugation maps in* $\mathbf{L}$ *such that* $\pi_\delta^u(\pi_\gamma^u (a)) \neq e$*, for all* n*-tuples* $\delta = \langle \delta_1, ..., \delta_n \rangle$*.*

Theorem B (see Theorem 24) *Let* $\mathbf{L}$ *be a non-trivial normal-valued, prelinear, and cancellative residuated lattice. The following statements are equivalent:*

1. $\mathbf{L}$ *is Archimedean.*
2. *For all* $u < e$ *in* L *and all elements* $a < e$ *in* $\mathrm{C}[u]$*, there exists* $m \in \mathbb{Z}^+$ *such that* $u \not\leq a^m$*.*

Next, we consider *generalized BL-algebras* (or *GBL-algebras*: [28]): basically, divisible residuated lattices. In Section 5, we connect a number of results scattered in the literature and observe that GBL-algebras are amenable to a generalization of Hőlder's theorem for ℓ-groups, to the effect that any strongly simple totally ordered GBL-algebra (i.e., any totally ordered GBL-algebra with no non-trivial convex subuniverse) is isomorphic to either a subalgebra of $\mathbb{R}$, or a subalgebra of $\mathbb{R}^-$, or a subalgebra of the standard MV-algebra on $[0, 1]$. The main result of this section is a theorem that subsumes as special cases most of the above-mentioned results on the commutativity of Archimedean residuated lattices:

Theorem C (see Theorem 32) *Any Archimedean and prelinear GBL-algebra is commutative.*

Since *generalized MV-algebras* [23] are prelinear, we obtain in particular that:

Theorem D (see Theorem 34) *Any Archimedean GMV-algebra is commutative.*

In the final section, we lay the groundwork for an investigation of hyper-Archimedean residuated lattices. Several equivalent characterizations of this class are available for individual classes of residuated lattices, such as pseudo-MV-algebras or ℓ-groups. These characterizations generalize as follows to GMV-algebras (again, the full terminology and notation will be explained in Section 6).

Theorem E (see Theorem 41) *For any normal-valued GMV-algebra* $\mathbf{L}$ *the following conditions are equivalent:*

(1) $\mathbf{L}$ *is hyper-Archimedean.*

(2) *The interval* $[H, L]$ *in* $\mathcal{C}(\mathbf{L})$ *has the zero radical compact property, for every* $H \in \mathcal{C}(\mathbf{L})$.

(3) *The prime subuniverses of every principal convex subuniverse* H *of* $\mathbf{L}$ *are maximal convex subuniverses of* H.

(4) $\mathbf{L} = \mathrm{C}\,[a] \otimes a^{\perp}$, *for all* $a \in L$.

(5) *For all* $a, b \in L^{-}$, *there exists a natural number* m *such that* $b \vee a^{m} = b \vee a^{m+1}$.

2 Preliminaries

2.1 Residuated lattices

We refer the reader to [4, 28, 33, 22] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

A binary operation $\cdot$ on a partially ordered set $\mathbf{A} = (A, \leq)$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on A such that for all $a, b, c \in A$,

$$(\mathrm{Res})\ a \cdot b \leq c \ \text{ iff } \ a \leq c/b \ \text{ iff } \ b \leq a\backslash c.$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$, if present. $\prod_{j=1}^{n} x_j$ is shorthand notation for $x_1 \cdots x_n$. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing xy by yx and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element e and the partial order $\leq$ is a lattice order. In this case, we add the monoid unit and the lattice operation symbols to the similarity type and refer to the resulting structure $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ as a *residuated lattice*. The class of residuated lattices forms a variety (see e.g. [33, Prop. 4.5]) that we denote throughout this paper by $\mathcal{RL}$.

A *pointed residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e, f)$ such that $(L, \wedge, \vee, \cdot, \backslash, /, e)$ is a residuated lattice and f is a designated element of L. The variety of pointed residuated lattices will be denoted by $\mathcal{PRL}$, and analogously, given any class of residuated lattices, we prepend a "P" to its name to denote the corresponding class of pointed residuated lattices.

A subvariety of $\mathcal{RL}$ of particular interest is the variety $\mathcal{CRL}$ of commutative residuated lattices, which satisfies the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. We always think of this variety as a subvariety of $\mathcal{RL}$, but we slightly abuse notation by listing only one occurrence of the operation $\backslash$ in describing its members. A restricted form of commutativity is e-cyclicity. A residuated lattice $\mathbf{L}$ is said to be *e-cyclic* if $a\backslash e = e/a$ for all $a \in L$. It will become clear below that the structure theory of residuated lattices is especially smooth in the e-cyclic case.

Given a residuated lattice $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$, an element $a \in L$ is said to be *integral* if $e = a\backslash e = e/a$, and $\mathbf{L}$ itself is said to be *integral* if every member of it is integral. We denote by $\mathcal{IRL}$ the variety of all integral residuated lattices. The *negative cone* $L^- = \{x \in L : x \leq e\}$ of a a residuated lattice $\mathbf{L}$ is the universe of an integral residuated lattice $\mathbf{L}^-$; the monoid and lattice operations of $\mathbf{L}^-$ are just the restrictions of the operations of $\mathbf{L}$, while the residuals are given by $a\backslash^{\mathbf{L}^-} b = a\backslash^{\mathbf{L}} b \wedge e$ and $b/^{\mathbf{L}^-} a = b/^{\mathbf{L}} a \wedge e$, for all $a, b \in L^-$. Throughout this article we will sometimes abbreviate by $x\backslash_- y$ (respectively, by $y/_- x$) the term $x\backslash y \wedge e$ (respectively, $y/x \wedge e$).

An element $a \in L$ is said to be *invertible* if $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The residuated lattices in which every element is invertible are precisely the ℓ-groups. Perhaps a word of caution is appropriate here. An *ℓ-group* is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, {}^{-1}, e)$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, {}^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the equation $(e/x)x \approx e$; the term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}, x\backslash y = x^{-1}y$. Throughout this paper, the members of this subvariety will be simply referred to as ℓ-groups. It follows from the preceding comments that negative cones of ℓ-groups are residuated lattices as well. We denote by $\mathcal{LG}^-$ the class, which is indeed a variety [3], of negative cones of ℓ-groups.

2.2 Convex subuniverses

In this subsection, we review some relevant properties of the lattice of convex subuniverses of an e-cyclic residuated lattice. An extensive study of related topics can be found in [5].

A subset C of a poset $\mathbf{P} = (P, \leqslant)$ is *order-convex* (or simply *convex*) in $\mathbf{P}$ if for every $a, b, c \in P$, whenever $a, c \in C$ with $a \leqslant b \leqslant c$, then $b \in C$. For a residuated lattice $\mathbf{L}$, we write $\mathcal{C}(\mathbf{L})$ for the algebraic closure system of all convex subuniverses of $\mathbf{L}$, partially ordered by set-inclusion. It is noted in Theorem 4 below that $\mathcal{C}(\mathbf{L})$ is a distributive lattice.

For any $S \subseteq L$, we let $\mathrm{C}[S]$ denote the smallest convex subuniverse of $\mathbf{L}$ containing S, as well as the corresponding algebra. As is customary, we call $\mathrm{C}[S]$ the convex subuniverse *generated* by S and let $\mathrm{C}[a] = \mathrm{C}[\{a\}]$. We refer to $\mathrm{C}[a]$ as the *principal* convex subuniverse of $\mathbf{L}$ generated by the element a. The principal convex subuniverses in $\mathcal{C}(\mathbf{L})$ are the compact members of $\mathcal{C}(\mathbf{L})$, as by Lemma 3.(3) below, every finitely generated convex subuniverse of $\mathbf{L}$ is principal.

An important concept in the theory of ℓ-groups is the notion of absolute value. This idea can be fruitfully generalized in the context of residuated lattices [5, 37].

Definition 1

1. *The* absolute value *of an element x in a residuated lattice $\mathbf{L}$ is the element*

$$|x| = x \wedge e/x \wedge e.$$

2. *If $X \subseteq L$, we set $|X| = \{|x| : x \in X\}$.*

The proof of the following lemma is routine:

Lemma 2 [5] *Let $\mathbf{L}$ be an e-cyclic residuated lattice, $x \in L$, and $a \in L^-$. The following conditions hold:*

1. *$x \leqslant e$ if and only if $|x| = x$;*
2. *$|x| \leqslant x \leqslant |x| \backslash e$;*
3. *$|x| = e$ if and only if $x = e$;*
4. *$a \leqslant x \leqslant a \backslash e$ if and only if $a \leqslant |x|$; and*
5. *if $H \in \mathcal{C}(\mathbf{L})$, then $x \in H$ if and only if $|x| \in H$.*

In what follows, given a subset S of a residuated lattice $\mathbf{L}$, we write $\langle S \rangle$ for the submonoid of $\mathbf{L}$ generated by S. Thus, $x \in \langle S \rangle$ if and only if there exist elements $s_1, \ldots, s_n \in S$ such that $x = s_1 \cdots s_n$.

The next lemma provides an intrinsic description of the convex subuniverse generated by a subset of an e-cyclic residuated lattice.

Lemma 3 *[5] Let* $\mathbf{L}$ *be an e-cyclic residuated lattice.*

1. *For* $S \subseteq L$,

$$\mathrm{C}[S] = \mathrm{C}[|S|] = \{x \in L : h \leqslant x \leqslant h \backslash e, \text{ for some } h \in \langle |S| \rangle\}$$
$$= \{x \in L : h \leqslant |x|, \text{ for some } h \in \langle |S| \rangle\}.$$

2. *For* $a \in L$,

$$\mathrm{C}[a] = \mathrm{C}[|a|] = \{x \in L : |a|^n \leqslant x \leqslant |a|^n \backslash e, \text{ for some } n \in \mathbb{N}\}$$
$$= \{x \in L : |a|^n \leqslant |x|, \text{ for some } n \in \mathbb{N}\}.$$

3. *For* $a, b \in L$, $\mathrm{C}[a] \cap \mathrm{C}[b] = \mathrm{C}[|a| \vee |b|]$ *and* $\mathrm{C}[a] \vee \mathrm{C}[b] = \mathrm{C}[|a| \wedge |b|] = \mathrm{C}[|a|\,|b|]$.

4. *If* H *is a convex subuniverse of* $\mathbf{L}$*, then* $H = \mathrm{C}[H^-]$.

Lemma 3 yields the following results.

Theorem 4 *[5, Thm. 3.8] If* $\mathbf{L}$ *is an e-cyclic residuated lattice, then:*

1. $\mathcal{C}(\mathbf{L})$ *is an algebraic distributive lattice.*

2. *The poset* $\mathcal{K}(\mathcal{C}(\mathbf{L}))$ *of compact elements of* $\mathcal{C}(\mathbf{L})$ *– consisting of the principal convex subuniverses of* $\mathbf{L}$ *– is a sublattice of* $\mathcal{C}(\mathbf{L})$.

In view of the preceding theorem, the lattice $\mathcal{C}(\mathbf{L})$ of convex subuniverses of a residuated lattice $\mathbf{L}$ is an algebraic distributive lattice. As such, it satisfies the join-infinite distributive law

$$X \cap \bigvee_{i \in I} Y_i = \bigvee_{i \in I} (X \cap Y_i),$$

and hence it is relatively pseudocomplemented. That is, for all $X, Y \in \mathcal{C}(\mathbf{L})$, the relative pseudocomplement $X \to Y$ of X relative to Y exists:

$$X \to Y = \max\{Z \in \mathcal{C}(\mathbf{L}) : X \cap Z \subseteq Y\}.$$

The next lemma provides an element-wise description of $X \to Y$ in terms of the absolute value, and in particular one for the pseudocomplement $X^\perp = X \to \{e\}$ of X.

Lemma 5 *[5] If* $\mathbf{L}$ *is an e-cyclic residuated lattice, then* $\mathcal{C}(\mathbf{L})$ *is a relatively pseudocomplemented lattice. Specifically, given* $X, Y \in \mathcal{C}(\mathbf{L})$,

$$X \to Y = \{a \in L : |a| \vee |x| \in Y, \text{ for all } x \in X\},$$

and in particular,

$$X^\perp = X \to \{e\} = \{a \in L : |a| \vee |x| = e, \text{ for all } x \in X\}.$$

We can define $X^\perp$ for any non-empty subset $X \subseteq L$. It is not hard to show that $X^\perp = \mathrm{C}[X]^\perp$, so $X^\perp$ is always a convex subuniverse. We refer to $X^\perp$ as the *polar* of X; in case $X = \{x\}$, we write $x^\perp$ instead of $\{x\}^\perp$ (or $\mathrm{C}[x]^\perp$) and refer to it as the *principal polar* of x.

The map $^\perp : \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{L})$ is a self-adjoint inclusion-reversing map, while the map sending $H \in \mathcal{C}(\mathbf{L})$ to its double polar $H^{\perp\perp}$ is an intersection-preserving closure operator on $\mathcal{C}(\mathbf{L})$. By a classic result due to Glivenko, the image of this closure operator is a (complete) Boolean algebra $\mathbf{B}_{\mathcal{C}(\mathbf{L})}$ with least element $\{e\}$ and largest element L. The complement of H in $\mathbf{B}_{\mathcal{C}(\mathbf{L})}$ is precisely $H^\perp$, whereas, for any pair of convex subuniverses $H, K \in \mathbf{B}_{\mathcal{C}(\mathbf{L})}$,

$$H \vee^{\mathbf{B}_{\mathcal{C}(\mathbf{L})}} K = (H^\perp \cap K^\perp)^\perp = (H \cup K)^{\perp\perp}.$$

On the other hand, meets in $\mathbf{B}_{\mathcal{C}(\mathbf{L})}$ are just intersections.

This is a convenient place to discuss briefly congruence relations of residuated lattices. It is proved in [4] (see also [22, Thm. 3.47]), that the congruences of any residuated lattice $\mathbf{L}$ are completely determined by its normal convex subuniverses. In more detail, given $u \in L$, we define

$$\lambda_u(x) = u\backslash xu \wedge e \quad \text{and} \quad \rho_u(x) = ux/u \wedge e,$$

for all $x \in L$. We refer to λ_u and ρ_u as *left conjugation* and *right conjugation* by u. A convex subuniverse H of $\mathbf{L}$ is said to be *normal* if it is closed under all left and right conjugation maps. In other words, for all $x \in H$ and $u \in L$, $\lambda_u(x), \rho_u(x) \in H$. An *iterated conjugation map* in $\mathbf{L}$ is a composition $\gamma = \gamma_1 \circ ... \circ \gamma_n$, where each γ_i is either a right conjugation or a left conjugation by an element $a_i \in L$. It is clear that H is normal if and only if it is closed under all iterated conjugation maps. If $X \subseteq L$, we denote by $\Gamma_{\mathbf{L}}[X]$, or simply by $\Gamma[X]$, the set of all iterated conjugates of elements of X in $\mathbf{L}$, letting as usual $\Gamma[a]$ be short for $\Gamma[\{a\}]$. In the next lemma, we describe the normal convex subuniverse NC[S] generated by $S \subseteq L$.

Lemma 6 *[4] Let* $\mathbf{L}$ *be a residuated lattice. For* $S \subseteq L$,

$$\begin{aligned} \mathrm{NC[S]} = \mathrm{NC[|S|]} &= \{x \in L : h \leqslant x \leqslant h\backslash e, \text{ for some } h \in \langle \Gamma[S] \rangle\} \\ &= \{x \in L : h \leqslant |x|, \text{ for some } h \in \langle \Gamma[S] \rangle\}. \end{aligned}$$

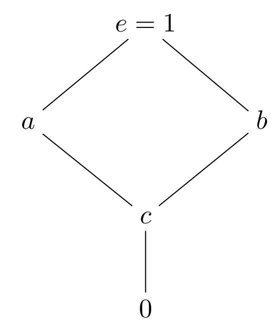
Let us remark that, by [4, Thm. 4.12], given a normal convex subuniverse $\mathbf{H}$ of $\mathbf{L}$, the relation

$$\Theta_{\mathbf{H}} = \{(x, y) \in L^2 : x\backslash y \wedge y\backslash x \wedge e \in H\}$$

is a congruence of $\mathbf{L}$. Conversely, given a congruence relation Θ of $\mathbf{L}$, the equivalence class $[e]_\Theta$ is a normal convex subuniverse of $\mathbf{L}$. Further, this correspondence establishes an isomorphism between the congruence lattice of $\mathbf{L}$ and the lattice of its normal convex subuniverses. In what follows, if H is a normal convex subuniverse of $\mathbf{L}$, we write $\mathbf{L}/H$ for the quotient algebra $\mathbf{L}/\Theta_{\mathbf{H}}$, and denote the equivalence class of an element $x \in L$ by $[x]_H$.

We close this subsection by introducing a key concept for our considerations. A residuated lattice $\mathbf{L}$ is said to be *strongly simple* in case its only convex subuniverses are L and $\{e\}$. Clearly, if $\mathbf{L}$ is *Hamiltonian* (that is, every convex subuniverse is normal), then it is strongly simple if and only if it is simple. In general, however, there are simple residuated lattices having non-trivial convex subuniverses, as the following example shows.

Example 7 *[5, Ex. 6.1] The residuated lattice with the Hasse diagram displayed below:*



and whose multiplication table is

$\cdot$	**0**	**a**	**b**	**c**	**e**
0	0	0	0	0	0
a	0	a	0	0	a
b	0	c	b	c	b
c	0	c	0	0	c
e	0	a	b	c	e

is simple, but has two non-trivial convex subuniverses, $\{e, a\}$ *and* $\{e, b\}$*.*

3 Archimedean residuated lattices

This section discusses the class of *Archimedean residuated lattices*. Most of the results below will be essentially devoted to buttressing up the identification of Archimedean residuated lattices with normal-valued residuated lattices whose lattices of convex subuniverses have the zero radical compact property. We will do so by showing that, in all cases of special interest, the latter enjoy many of the properties one reasonably expects from Archimedean objects.

We proceed with a couple of pertinent definitions. Recall that $H \in \mathcal{C}(\mathbf{L})$ is said to be *prime* if it is meet-irreducible in $\mathcal{C}(\mathbf{L})$. That is, whenever $X, Y \in \mathcal{C}(\mathbf{L})$, and $X \cap Y = H$, then $X = H$ or $Y = H$. In view of the fact that $\mathcal{C}(\mathbf{L})$ is distributive (Theorem 4), H is meet-prime in $\mathcal{C}(\mathbf{L})$, that is, whenever $X, Y \in \mathcal{C}(\mathbf{L})$, and $X \cap Y \subseteq H$, then $X \subseteq H$ or $Y \subseteq H$.

The definitions of a normal-valued residuated lattice requires the related concept of a completely meet-irreducible convex subuniverse. Recall that H

is completely meet-irreducible in $\mathcal{C}(\mathbf{L})$ if whenever $\{X_i : i \in I\}$ is a family of convex subuniverses of $\mathbf{L}$ such that $H = \bigwedge_{i \in I} X_i$, then $H = X_i$ for some $i \in I$. Each completely meet-irreducible convex subalgebra H has a unique cover $H^\sharp$ in $\mathcal{C}(\mathbf{L})$.

Definition 8 *A residuated lattice* $\mathbf{L}$ *is said to be* normal-valued *provided it is e-cyclic and each completely meet-irreducible convex subuniverse* H *of* $\mathbf{L}$ *is normal in its cover* $H^\sharp$ *in* $\mathcal{C}(\mathbf{L})$ *(more precisely, the subalgebra of* $\mathbf{L}$ *whose universe is* $H^\sharp$*).*

The term *normal-valued* is borrowed from the theory of ℓ-groups. In our setting, *a value* of a non-identity element $a \in L$ is a convex subuniverse H of $\mathbf{L}$ that is maximal with respect to not containing the element a. The convex subuniverse H, whose existence is guaranteed by Zorn's Lemma, is easily seen to be a completely meet-irreducible convex subuniverse of $\mathbf{L}$. This is actually a lattice-theoretic concept. A value of a compact element $c \neq \bot$ in an algebraic lattice is an element that is maximal with respect to not exceeding c. Such an element is necessarily completely meet-irreducible. Thus, given an element a of an e-cyclic residuated lattice $\mathbf{L}$, a subuniverse H of $\mathbf{L}$ is a value of a if and only if H is a value of the principal convex subuniverse $C[a]$ in the algebraic lattice $\mathcal{C}(\mathbf{L})$ of convex subuniverses of $\mathbf{L}$.

It has been known for some time that normal-valued ℓ-groups form a variety [40], and that this is the largest proper variety of ℓ-groups [27]. It is an open question as to whether the class of all e-cyclic normal-valued residuated lattices is a variety. However, an extension of this characterization to a subclass of normal-valued residuated lattices, see Theorem 21 below, has recently been established in [5] and will be essential for the results of Section 4.

Moving to the zero radical compact property, we first present it as a purely lattice-theoretic property, and then, by extension, as a property that is applicable to residuated lattices via their lattices of convex subuniverses.

Definition 9 *An algebraic distributive lattice* $\mathbf{L}$*, whose set of compact elements is closed under finite meets, has the* zero radical compact property *if for every compact element* $c \in L$*, the meet of all maximal elements in* $\downarrow c$ *is the bottom element* $\bot$ *of* $\mathbf{L}$*.*

Definition 10 *A residuated lattice* $\mathbf{L}$ *is said to have the* zero radical compact property *if the lattice* $\mathcal{C}(\mathbf{L})$ *has the zero radical compact property.*

Finally, we spell out our definition of an Archimedean residuated lattice.

Definition 11 *A residuated lattice* $\mathbf{L}$ *is said to be* Archimedean *if it is normal-valued and has the zero radical compact property.*

A characterization of Archimedean residuated lattices calls for some work. For a start, we will stay in the setting of *unital* residuated lattices, a concept that is recalled hereafter.

Definition 12 *A residuated lattice* $\mathbf{L}$ *is called* unital *if there exists* $u \in L^-$ *(called a* strong unit *of* $\mathbf{L}$*) such that* $\mathrm{C}[u] = L$.

In view of Lemma 3, $\mathbf{L}$ is unital if and only if, for all $a \in L^-$, $u^n \leq a$ for some positive integer n.

Definition 13 *If* $\mathbf{L}$ *is unital, we denote by* $\mathcal{M}(\mathbf{L})$ *the set of maximal convex subuniverses of* $\mathbf{L}$*; the set* $\bigcap \mathcal{M}(\mathbf{L})$ *will be called the* radical *of* $\mathbf{L}$ *and denoted by* $\mathrm{Rad}(\mathbf{L})$.

Observe that $\mathcal{M}(\mathbf{L})$ and $\mathrm{Rad}(\mathbf{L})$, being defined for unital objects, are always non-empty.

With an eye to describing the maximal convex subuniverses in the case of normal-valued and unital residuated lattices, we need the following result whose simple proof is left to the reader:

Lemma 14 *Let* $\mathbf{L}$ *and* $\mathbf{M}$ *be e-cyclic residuated lattices, let* $\phi : \mathbf{L} \to \mathbf{M}$ *be a surjective homomorphism, let* $H = \{x \in L : \phi(x) = e\}$ *be the kernel of* ϕ*, and let* $[H, L]$ *denote the principal order-filter determined by* H *in* $\mathcal{C}(\mathbf{L})$*. Then the maps* $\phi[] : [H, L] \to C(\mathbf{M})$ *and* $\phi^{-1}[] : C(\mathbf{M}) \to [H, L]$ *– defined by* $\phi[K] = \{\phi(x) : x \in K\}$ *and* $\phi^{-1}[N] = \{x \in L : \phi(x) \in N\}$ *– are mutually inverse isomorphisms between* $[H, L]$ *and* $\mathcal{C}(\mathbf{M})$.

We make a note of an important preliminary observation that will be useful in the rest of the section. Whenever $\mathbf{L}$ is a normal-valued residuated lattice with strong unit u, the lattices $\mathcal{C}(\mathbf{L})$ of convex subuniverses of $\mathbf{L}$ and $\mathcal{NC}(\mathbf{L})$ of normal convex subuniverses of $\mathbf{L}$ have exactly the same maximal members, because maximal convex subuniverses, being values of u, are normal in their cover $\mathbf{L}$.

Let $\mathbf{L}$ be a residuated lattice and $a, u \in L$. Given a sequence $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ of iterated conjugation maps, we define:

$$\pi_\gamma^u(a) = \prod_{j=1}^{m} \gamma_j(|a|) \backslash_- u, \tag{1}$$

and, for $n \in \mathbb{Z}^+$,

$$\pi_n^u(a) = |a|^n \backslash_- u. \tag{2}$$

The results below generalize several theorems variously scattered in the literature: e.g. [7, Prop. 3.6.4], [22, Lm. 11.4], [38], [24, Prop. 3.12].

Lemma 15 *Let* $\mathbf{L}$ *be a normal-valued and unital residuated lattice with strong unit* u*, and let* F *be a proper normal convex subuniverse of* $\mathbf{L}$*. Then the following are equivalent:*

(1) $F \in \mathcal{M}(\mathbf{L})$.

(2) *for all* $a \in L$, *either* $a \in F$ *or* $\pi_n^u(a) \in F$, *for some positive integer* n.

(3) *for all* $a \in L$, *either* $a \in F$ *or* $\pi_\gamma^u(a) \in F$, *for some tuple* γ *of iterated conjugation maps.*

Proof.

(1) $\Rightarrow$ (2). Suppose that F is maximal and that there exists $a \in L - F$ such that, for all positive integers n, $\pi_n^u(a) \notin F$. Under this assumption, $[|a|]_F \neq [e]_F \neq [\pi_n^u(a)]_F$, for all such n. However, by Lemma 3.(2),

$$\mathrm{C}[[|a|]_F] = \left\{ [x]_F \in L/F : [|a|]_F^m \leq [x]_F \text{, for some } m \in \mathbb{Z}^+ \right\}.$$

As $[e]_F \neq [\pi_n^u(a)]_F$, for all $n \geq 1$, it follows that $[u]_F \notin \mathrm{C}[[|a|]_F]$. Therefore, $\mathrm{C}[[|a|]_F]$ is not $\{e\}$, because $[|a|]_F \neq [e]_F$, and, moreover, it differs from L/F, for $[u]_F \notin \mathrm{C}[[|a|]_F]$. But this contradicts the fact that, by Lemma 14, $\mathbf{L}/F$ is strongly simple, because of the maximality of F.

(2) $\Rightarrow$ (3) Under the hypothesis in (2), if $a \in L - F$, then there is a positive integer n such that $\pi_n^u(a) = |a|^n \backslash_- u \in F$. Upon taking γ to be an n-tuple of iterations of the identity conjugation map, it follows that $\pi_\gamma^u(a) \in F$.

(3) $\Rightarrow$ (1). Let $a \notin F$. Then, by virtue of statement (3), there is a sequence γ of iterated conjugation maps such that $\pi_\gamma^u(a) \in F$. This means that

$$\left[\prod_{j=1}^{m} \gamma_j\left(|a|\right) \right]_F \backslash_- [u]_F = [e]_F\,,$$

and so $[u]_F \in \mathrm{NC}\left[[a]_F\right]$. In sum, for any $a \in L$ such that $[a]_F \neq [e]_F$, we have that $\mathrm{NC}\left[[a]_F\right] = L/F$. In other words, $\mathbf{L}/F$ is simple and non-trivial, whence F is maximal in $\mathcal{NC}(\mathbf{L})$ and hence in $\mathcal{C}(\mathbf{L})$. ■

Lemma 16 *Let* $\mathbf{L}$ *be a normal-valued and unital residuated lattice with strong unit* u, *and let* $a \in L$, $a \neq e$. *The following statements are equivalent:*

(1) *The element* a *is in* $\mathrm{Rad}(\mathbf{L})$.

(2) *For every* $m \in \mathbb{Z}^+$, *there exists* $n \in \mathbb{Z}^+$ *such that* $\pi_n^u(\pi_m^u(a)) = e$.

(3) *For every* m*-tuple* $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ *of iterated conjugation maps in* $\mathbf{L}$, *there exists another such* n*-tuple* $\delta = \langle \delta_1, ..., \delta_n \rangle$ *such that* $\pi_\delta^u(\pi_\gamma^u(a)) = e$.

Proof. (1) $\Rightarrow$ (2). Suppose that (2) fails. Then there exists $m \in \mathbb{Z}^+$ such that, for all $n \in \mathbb{Z}^+$, we have that $\pi_n^u\left(\pi_m^u(a)\right) \neq e$. In view of Lemma 2, we may assume that $a \in L^-$. It follows that $(a^m \backslash_- u)^n \not\leq u$. Let $z = a^m \backslash_- u$, and consider the principal convex subuniverse $\mathrm{C}[z]$ of $\mathbf{L}$. By Lemma 3, $u \notin \mathrm{C}[z]$. Therefore, by Zorn's Lemma, $\mathrm{C}[z]$ can be extended to a value H of u, which is clearly in $\mathcal{M}(\mathbf{L})$.

We first claim that $\pi_n^u\left(\pi_m^u(a)\right) \notin H$, for all $n \in \mathbb{Z}^+$. Indeed, suppose that $\pi_n^u\left(\pi_m^u(a)\right) \in H$, for some $n \in \mathbb{Z}^+$. Then

$$[e]_H = [\pi_n^u\left(\pi_m^u(a)\right)]_H = [z]_H^n \backslash_- [u]_H = [e]_H \backslash_- [u]_H = [u]_H,$$

as $z^n \in H$. This implies that $H = L$, which clashes with the maximality of H. Next, we prove that $a^m \notin H$. Suppose on the contrary that $a^m \in H$. Then,

$$\begin{aligned}[\pi_n^u(\pi_m^u(a))]_H &= ([a]_H^m \backslash_- [u]_H)^n \backslash_- [u]_H \\ &= ([e]_H \backslash_- [u]_H)^n \backslash_- [u]_H \\ &= [u]_H^n \backslash_- [u]_H \\ &= [e]_H.\end{aligned}$$

This yields the contradiction that $\pi_n^u(\pi_m^u(a)) \in H$. Hence, $a \notin \mathrm{Rad}(\mathbf{L})$.

(2) $\Rightarrow$ (3) Take an m-tuple γ of conjugation maps, and consider the element $z = \prod_{i \leq m} \gamma_i(|a|)$. Then, by hypothesis, there exists an appropriate n-tuple δ of iterations of the identity conjugation map such that $\pi_\delta^u(\pi_1^u(z)) = \pi_\delta^u(\pi_\gamma^u(a)) = e$.

(3) $\Rightarrow$ (1). Let $a \notin \mathrm{Rad}(\mathbf{L})$, that is, $a \notin F$ for some $F \in \mathcal{M}(\mathbf{L})$. Now, since F is normal, Lemma 15 implies the existence of an m-tuple $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ of iterated conjugation maps such that $\pi_\gamma^u(a) \in F$. Let now $\delta = \langle \delta_1, ..., \delta_n \rangle$ be any n-tuple of iterated conjugation maps. We compute:

$$\begin{aligned}\left[\pi_\delta^u(\pi_\gamma^u(a))\right]_F &= \left[\prod_{i=1}^n \delta_i \left(\prod_{j=1}^m \gamma_j(|a|) \backslash_- u\right) \backslash_- u\right]_F \\ &= \left[\prod_{i=1}^n \delta_i \left(\prod_{j=1}^m \gamma_j(|a|) \backslash_- u\right)\right]_F \backslash_- [u]_F \\ &= \prod_{i=1}^n [\delta_i]_F \left(\left[\prod_{j=1}^m \gamma_j(|a|) \backslash_- u\right]_F\right) \backslash_- [u]_F \\ &= \prod_{i=1}^n [\delta_i]_F ([e]_F) \backslash_- [u]_F \\ &= [e]_F \backslash_- [u]_F = [u]_F.\end{aligned}$$

It follows that $\pi_\delta^u(\pi_\gamma^u(a)) \neq e$, and this establishes the claim. ■

The considerations above imply the following second-order characterization of unital and normal-valued residuated lattices that are subdirect products of strongly simple residuated lattices.

Theorem 17 *Let* $\mathbf{L}$ *be a non-trivial normal-valued and unital residuated lattice. The following statements are equivalent:*

(1) $\mathbf{L}$ *is a subdirect product of strongly simple residuated lattices.*

(2) $\mathrm{Rad}(\mathbf{L}) = \{e\}$.

(3) *For all* $a \neq e$ *in* L, *there exists* $m \in \mathbb{Z}^+$ *such that* $\pi_n^u(\pi_m^u(a)) \neq e$, *for all* $n \in \mathbb{Z}^+$.

(4) *For all $a \neq e$ in L, there exists an m-tuple $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ of iterated conjugation maps in* **L** *such that $\pi^u_\delta(\pi^u_\gamma(a)) \neq e$, for all n-tuples $\delta = \langle \delta_1, ..., \delta_n \rangle$.*

As a consequence of the preceding result, Archimedean residuated lattices admit an element-wise description, in analogy with the case of ℓ-groups.

Theorem 18 *Let* **L** *be a non-trivial normal-valued residuated lattice. The following statements are equivalent:*

(1) **L** *is Archimedean.*

(2) *For all $u < e$ in L and all elements $a \neq e$ in $\mathrm{C}[u]$, there exists $m \in \mathbb{Z}^+$ such that $\pi^u_n(\pi^u_m(a)) \neq e$, for all $n \in \mathbb{Z}^+$.*

(3) *For all $u < e$ in L and all elements $a \neq e$ in $\mathrm{C}[u]$, there exists an m-tuple $\gamma = \langle \gamma_1, ..., \gamma_m \rangle$ of iterated conjugation maps in* **L** *such that $\pi^u_\delta(\pi^u_\gamma(a)) \neq e$, for all n-tuples $\delta = \langle \delta_1, ..., \delta_n \rangle$.*

4 Prelinearity and the Archimedean property

In this section we specialize our study of the Archimedean property to the class of normal-valued, prelinear and cancellative residuated lattices. A byproduct of our theory is the characterization of this property for the well-studied classes of ℓ-groups and MV-algebras. In particular, it will become evident that our usage of the term "Archimedean" does not clash with the established meaning this phrase possesses in the theory of ℓ-groups.

A residuated lattice **L** is said to be *left prelinear* if it satisfies the equation $(x\backslash y \wedge e) \vee (y\backslash x \wedge e) \approx e$, and *right prelinear* if it satisfies the equation $(y/x \wedge e) \vee (x/y \wedge e) \approx e$. Although these identities are not equivalent, all the properties of concern in the present paper hold for left prelinear residuated lattices if and only if they hold for right prelinear ones. In the sequel, therefore, we will call *prelinear* a residuated lattice that satisfies either law. The reader is also reminded that a residuated lattice is called *cancellative*, if its monoid reduct is cancellative. It is shown in [3] that the class $\mathcal{C}an\mathcal{RL}$ of all cancellative residuated lattices is a variety with defining equations $xy/y \approx x \approx y\backslash yx$.

When **L** is prelinear and e-cyclic, the prime convex subuniverses have special properties, which are summarized in the following lemma.

Lemma 19 *[5] Let* **L** *be a prelinear e-cyclic residuated lattice. For a convex subuniverse H of* **L**, *the following are equivalent:*

(1) *H is a prime convex subuniverse of* **L**.

(2) *For all $a, b \in L$, if $|a| \vee |b| \in H$, then $a \in H$ or $b \in H$.*

(3) *For all $a, b \in L$, if $|a| \vee |b| = e$, then $a \in H$ or $b \in H$.*

(4) *For all* $a, b \in L$, $a\backslash b \wedge e \in H$ *or* $b\backslash a \wedge e \in H$.

(5) *The set of all convex subuniverses exceeding* H *is a chain under set-inclusion.*

We also have the following important consequence:

Corollary 20 *[5] Let* $\mathbf{L}$ *be a prelinear e-cyclic residuated lattice. If* $\mathcal{C}(\mathbf{L})$ *is totally ordered, then so is* $\mathbf{L}$*. In particular, if* P *is a prime convex subuniverse that is normal, then* $\mathbf{L}/P$ *is totally ordered.*

Furthermore, the following result generalizes the corresponding result for normal-valued ℓ-groups.

Theorem 21 *[6] The class of normal-valued and prelinear residuated lattices is a variety.*

Lemma 22 *Let* $\mathbf{L}$ *be non-trivial normal-valued, prelinear and cancellative unital residuated lattice with a strong unit* u*. The following statements are equivalent for an element* $a \in L$*.*

(1) *The element* a *is in* $\mathrm{Rad}(\mathbf{L})$*.*

(2) *For all* $k \in \mathbb{Z}^+$, $u < |a|^k$*.*

Proof. (1) $\Rightarrow$ (2). Let $a \in \mathrm{Rad}(\mathbf{L}) = \bigcap \mathcal{M}(\mathbf{L})$. Note that $a \in \mathrm{Rad}(\mathbf{L})$ if and only if $|a| \in \mathrm{Rad}(\mathbf{L})$, if and only if $|a|^k \in \mathrm{Rad}(\mathbf{L})$, for all $k \in \mathbb{Z}^+$. Thus, it will suffice to assume that $a \leq e$ and prove that $u < a$. Suppose that $u \not< a$. Then, $u\backslash_- a = u\backslash a \wedge e \neq e$. Let V be a value of $u\backslash_- a$. We observe that $u \notin V$, as $u\backslash_- a \neq e$. By Zorn's Lemma, V can be extended to a value H of u, which belongs to $\mathcal{M}(\mathbf{L})$. By prelinearity, $(u\backslash_- a) \vee (a\backslash_- u) = e \in V \subseteq H$. From the fact that $u\backslash_- a \notin V$, we deduce from Lemma 19 that $a\backslash_- u \in V \subseteq H$. As $a \in \mathrm{Rad}(\mathbf{A})$, a fortiori $a \in H$. But, as we noticed, also $a\backslash_- u \in H$. Then, from the fact that $a\backslash_- u \leq a\backslash u$ we obtain that $a(a\backslash_- u) \leq u = |u| \in H$, by Lemma 6, a contradiction. Thus, $u < a$, as was to be shown.

(2) $\Rightarrow$ (1). Assume that $u < |a|^k$, for all $k \in \mathbb{Z}^+$. To simplify notation, we may assume without loss of generality that $a < e$. Suppose that there exists $m \in \mathbb{Z}^+$ such that $(a^m\backslash_- u)^n\backslash_- u \neq e$, for all $n \in \mathbb{Z}^+$. Then, $a^m\backslash_- u$ is not a strong unit of $\mathbf{L}$. For otherwise, there is $n \in \mathbb{Z}^+$ such that $(a^m\backslash_- u)^n \leq u$, and this violates the assumption $(a^m\backslash_- u)^n\backslash_- u \neq e$, for all $n \in \mathbb{Z}^+$. Consequently, by Lemma 3, $u \notin \mathrm{C}[a^m\backslash_- u]$. Therefore, Zorn's Lemma can be used to extend $\mathrm{C}[a^m\backslash_- u]$ to a value H of u. By hypothesis, $u < a^{m+1}$. As $\mathbf{L}$ is cancellative, $a^m\backslash_- u \leq a^m\backslash_- a^{m+1} = a$. It follows that a, and hence a^m, are in H. But then, $a^m(a^m\backslash_- u) \leq a^m(a^m\backslash u) \leq u \in H$, contradicting the fact that H is a value of u. As a consequence, for any $m \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that $\pi_u^n(\pi_u^m(a)) = e$, which implies, by Lemma 16-(2), that $a \in \mathrm{Rad}(\mathbf{L})$. ■

Lemma 22 implies the following refined version of Theorem 17 for prelinear and cancellative residuated lattices.

Theorem 23 *Let* $\mathbf{L}$ *be a non-trivial normal-valued, prelinear and cancellative unital residuated lattice with a strong unit* u*. The following statements are equivalent:*

(1) $\mathbf{L}$ *is a subdirect product of strongly simple residuated lattices.*

(2) $\mathrm{Rad}\,(\mathbf{L}) = \{e\}$

(3) *For all* $a \neq e$ *in* L*, there exists* $m \in \mathbb{Z}^+$ *such that* $u \not\leq |a|^m$*.*

The preceding considerations imply the following result.

Theorem 24 *Let* $\mathbf{L}$ *be a non-trivial normal-valued, prelinear and cancellative residuated lattice. The following statements are equivalent:*

(1) $\mathbf{L}$ *is Archimedean.*

(2) *For all* $u < e$ *in* L *and all elements* $a < e$ *in* $\mathrm{C}[u]$*, there exists* $m \in \mathbb{Z}^+$ *such that* $u \not\leq a^m$*.*

(3) *Whenever* u, a *are in* L^- *and* $u \leq a^n$*, for all* $n \in \mathbb{Z}^+$*, then* $a = e$*.*

Proof. The equivalence of (1) and (2) is a direct consequence of Theorem 23. The proof of the equivalence of (2) and (3) is quite simple. Indeed, suppose first that (3) fails. Then there exist $u, a < e$ in L such that $u \leq a^n$, for all $n \in \mathbb{Z}^+$. It follows that $a \in \mathrm{C}[u]$, and hence (2) is not satisfied. Conversely, suppose that (2) is not satisfied. Then there exist $u < e$ and $a < e$ in $\mathrm{C}[u]$ such that $u \leq a^n$, for all $n \in \mathbb{Z}^+$. But then (3) fails. We have proved that (2) and (3) are equivalent. ■

We close this section with two observations. The first observation concerns ℓ-groups. Recall that an ℓ-group is Archimedean if it satisfies Condition (3) of Theorem 24. It is almost evident from this theorem that our present usage of the term "Archimedean" does not clash with the established meaning this phrase possesses in the theory of ℓ-groups. However, it may be appropriate to dignify this observation as a full-fledged proposition.

Proposition 25 *An* ℓ*-group* $\mathbf{L}$ *is Archimedean in the traditional sense (that is, it satisfies condition (3) of Theorem 24) iff it is Archimedean in the sense of this paper (Definition 11).*

Proof. It will suffice to prove that any non-trivial ℓ-group that satisfies Condition (3) of Theorem 24 is cancellative, prelinear and normal-valued. The first two properties, however, hold for any ℓ-group, while the proof of third property can be found in any book on ℓ-groups, for example, in [2, p. 31]. ■

Second, we show how Condition (2) in Lemma 16 simplifies to a more familiar characterization of the radical in an especially well-behaved case. Indeed, let $\mathbf{L}$ be an Abelian ℓ-group with strong unit u. It is well-known that the interval $[u, e]$ in $\mathbf{L}$ is the universe of an MV-algebra $\Gamma\,(\mathbf{L}, u)$, whose operations are defined as follows:

- $x \cdot^{\Gamma(\mathbf{L},u)} y = x \cdot^{\mathbf{L}} y \vee u$;
- $x \backslash^{\Gamma(\mathbf{L},u)} y = x \backslash^{\mathbf{L}} y \wedge e$;
- $e^{\Gamma(\mathbf{L},u)} = e$.

If we further let $\neg x = x \backslash^{\Gamma(\mathbf{L},u)} u \wedge e$, a quick calculation yields that

$$\pi_n^u(\pi_m^u(a)) = \neg(\neg(a^m)^n).$$

In [36, Thm. 6.2, Cor. 6.3], it is shown that if $\mathbf{L}$ is an MV-algebra, then $a \in \mathrm{Rad}(\mathbf{L})$ iff for every $m \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that $\neg\left((\neg(a^m))^n\right) = e$, and that this description of the radical is equivalent to the usual one, given in [20] or [7, Prop. 3.6.4]. It follows from the previous considerations that this result may be viewed as a corollary of Lemma 16.

5 Commutativity and the Archimedean property

In this section we explore the Archimedean property for GBL-algebras and GMV-algebras, and prove theorems to the effect that all Archimedean and prelinear GBL-algebras (in particular, all Archimedean GMV-algebras) are commutative. A crucial step in this process consists in showing that any strongly simple prelinear GBL-algebra is totally ordered. Then, we apply a generalized version of Hölder's Theorem which completely characterizes these algebras either as subalgebras of $\mathbb{R}$, or of $\mathbb{R}^-$, or of the standard MV-algebra on the real interval $[0,1]$.

5.1 Preliminaries on GBL-algebras

A residuated lattice $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ is a *generalized basic logic algebra* (for short, *GBL-algebra*) if it satisfies the equation

E1 $y\,(y \backslash (x \wedge y)) \approx x \wedge y$,

as well as its mirror image. In case $\mathbf{L}$ is a pointed residuated lattice satisfying the same identities, we call it a *pointed* GBL-algebra. In both cases, if $\mathbf{L}$ is integral, **E1** reduces to

E2 $y\,(y \backslash x) \approx x \wedge y$,

and its mirror image undergoes an analogous simplification. We use the symbols $\mathcal{GBL}$ and $\mathcal{IGBL}$ to denote the varieties of GBL-algebras and integral GBL-algebras, respectively. The same equations also axiomatize $\mathcal{PGBL}$ and $\mathcal{PIGBL}$ relative to $\mathcal{PRL}$. The variety of *pseudo-BL-algebras* is term equivalent to the subvariety $\mathcal{P}s\mathcal{BL}$ of $\mathcal{PIGBL}$ whose equational basis relative to $\mathcal{PIGBL}$ is given by the equation:

E3 $f \leq x$.

Commutative and semilinear pseudo-BL-algebras go under the name of *BL-algebras*. It is well-known that both ℓ-groups (which form prominent examples of GBL-algebras) and BL-algebras are both e-cyclic and distributive as lattices. One may then wonder whether these properties are shared by all GBL-algebras. The next proposition confirms that this is indeed the case.

Proposition 26 *[23, Lemmas 2.7 and 2.9] Any GBL-algebra is e-cyclic and has a distributive lattice reduct.*

The study of GBL-algebras is facilitated by Theorem 28 below, whose statement requires the next definition.

Definition 27 *[29] A residuated lattice* $\mathbf{L}$ *is said to be the* inner direct product *of its subalgebras* $\mathbf{B}$ *and* $\mathbf{C}$ *– in symbols,* $\mathbf{L} = \mathbf{B} \otimes \mathbf{C}$ *– if* $\mathbf{B} \vee \mathbf{C} = \mathbf{L}$*, where the join is taken in the lattice of subuniverses of* $\mathbf{L}$ *and the map* $(b, c) \mapsto bc$ *is an isomorphism from* $\mathbf{B} \times \mathbf{C}$ *to* $\mathbf{L}$*. In other words: (i) every* $a \in L$ *can be written uniquely as a product* bc*, for some* $b \in B$ *and* $c \in C$*; (ii) each element in* B *commutes with every element in* C*; and (iii)* $b_1 c_1 \leq b_2 c_2$ *– with* $b_1, b_2 \in B$ *and* $c_1, c_2 \in C$ *– if and only if* $b_1 \leq b_2$ *and* $c_1 \leq c_2$*.*

Theorem 28 *[23, Thm. 5.2] Every* $\mathbf{L} \in \mathcal{GBL}$ *can be decomposed as an inner direct product of its* ℓ*-group subalgebra of invertible elements and of its integral GBL-subalgebra of integral elements.*

A crucial construction in the investigation of integral GBL-algebras is provided by *ordinal sums* of integral totally ordered GBL-algebras [21, 1, 17]. Let $(I, \leq)$ be a totally ordered set, and, for any $i \in I$, let $\mathbf{L}_i$ be an integral totally ordered GBL-algebra. Suppose, in addition, that for $i \neq j$ we have that $L_i \cap L_j = \{e\}$. The *ordinal sum* of the family $\{\mathbf{L}_i\}_{i \in I}$ is the algebra

$$\bigoplus_{i \in I} \mathbf{L}_i = \left(\bigcup_{i \in I} L_i, \wedge, \vee, \cdot, \backslash, /, e \right),$$

where:

- For $a \in L_i - \{e\}$, $b \in L_j - \{e\}$, $a \leq b$ iff either $i < j$ or $i = j$ and $a \leq^{\mathbf{L}_i} b$. Further, for all $a \in \bigcup_{i \in I} L_i$, $a \leq e$.
- If $a, b \in L_i$, then $a \cdot b = a \cdot^{\mathbf{L}_i} b$, while if $i < j$, $a \in L_i - \{e\}$ and $b \in L_j - \{e\}$, then $a \cdot b = a$.

These clauses uniquely determine the behavior of meet, join, and the residuals in the following terms:

- If $a, b \in L_i$, then $a \circ b = a \circ^{\mathbf{L}_i} b$ for $\circ \in \{\wedge, \vee\}$, while if $i < j$, $a \in L_i - \{e\}$ and $b \in L_j - \{e\}$, then $a \wedge b = a$ and $a \vee b = b$.

- If $a, b \in L_i$, then $a \circ b = a \circ^{\mathbf{L}_i} b$ for $\circ \in \{\backslash, /\}$, while if $i < j$, $a \in L_i - \{e\}$ and $b \in L_j - \{e\}$, then $a\backslash b = e = b/a$ and $b\backslash a = a = a/b$.

It can be seen that $\bigoplus_{i\in I}\mathbf{L}_i$ is an integral totally ordered GBL-algebra— actually, it is a totally ordered pseudo-BL-algebra if $(I, \leq)$ has a bottom element $\bot$ and $\mathbf{L}_\bot$ is a totally ordered pseudo-BL-algebra. Conversely, we say that an integral totally ordered GBL-algebra $\mathbf{L}$ is *ordinally irreducible* if whenever $\mathbf{L} \simeq \bigoplus_{i\in I}\mathbf{L}_i$, for some family of subalgebras $\{\mathbf{L}_i\}_{i\in I}$, there exists $j \in I$ such that $\mathbf{L} \simeq \mathbf{L}_j$, while for $i \neq j$, $L_i = \{e\}$. We have that:

Theorem 29 *([17, Thm. 4.1]; see also [34, 21, 1]) Let $\mathbf{L}$ be an integral totally ordered GBL-algebra. Then there exists a unique family $\{\mathbf{L}_i\}_{i\in I}$ of ordinally irreducible integral totally ordered GBL-algebras (with I totally ordered) such that $\mathbf{L} \simeq \bigoplus_{i\in I}\mathbf{L}_i$.*

If an integral totally ordered GBL-algebra fails to be ordinally irreducible, it must have at least one non-trivial convex subalgebra. Moreover, there is not much leeway in the class of ordinally irreducible integral totally ordered GBL-algebras:

Theorem 30 *[17, Prop. 3.7] Every ordinally irreducible integral totally ordered GBL-algebra $\mathbf{L}$ is either a totally ordered pseudo-MV-algebra (see below) or the negative cone of a totally ordered group.*

An especially important subvariety of $\mathcal{GBL}$ is given by *GMV-algebras* [23], simultaneous generalizations of *MV-algebras* [7] to the noncommutative, unbounded and non-integral case. The variety $\mathcal{GMV}$ of GMV-algebras is axiomatized relative to $\mathcal{RL}$ by the equations

E4 $\quad x/\left((x \vee y)\backslash x\right) \approx x \vee y \approx \left(x/\left(x \vee y\right)\right)\backslash x.$

In the context of the other residuated lattice identities, E4 implies E1, whence all GMV-algebras are GBL-algebras. The variety $\mathcal{IGMV}$ of integral GMV-algebras, of course, is axiomatized relative to $\mathcal{IRL}$ by the equation E4, which in this context simplifies to

E5 $\quad x/\left(y\backslash x\right) \approx x \vee y \approx \left(x/y\right)\backslash x.$

The class $\mathcal{LG}^-$ of negative cones of ℓ-groups is a subvariety of $\mathcal{IGMV}$, axiomatized relative to $\mathcal{IGMV}$ [3, Thm. 6.2] by the cancellativity equations

E6 $\; x\backslash xy \approx y \approx yx/x.$

The same equations also axiomatize the linguistic expansion of $\mathcal{PGMV}$ and $\mathcal{PIGMV}$ relative to $\mathcal{PRL}$. In this context, the variety of *pseudo-MV-algebras* is term equivalent to the subvariety $\mathcal{P}s\mathcal{MV}$ of $\mathcal{PIGMV}$ whose equational basis relative to $\mathcal{PIGMV}$ is given by the equation E3. Finally, the variety of *MV algebras* is term equivalent to the subvariety $\mathcal{MV}$ of $\mathcal{P}s\mathcal{MV}$ whose equational basis relative to $\mathcal{P}s\mathcal{MV}$ is given by the equation:

E7 $xy \approx yx$.

GMV-algebras are thoroughly studied in [23]. The reader is referred to this article for a detailed account of their properties.

5.2 Strongly simple totally ordered GBL-algebras

Around the turn of last century, O. Hölder proved that any strongly simple totally ordered group is isomorphic to an ℓ-subgroup of the additive reals. As noted in [13, p. 145], this fundamental achievement has "enormous importance in the theory of ℓ-groups, being the ultimate basis for most of the representations we encounter". Analogous representations of simple totally ordered algebras in terms of subalgebras of the real numbers are available for other subvarieties of $\mathcal{GBL}$; for example, any simple (hence totally ordered) MV-algebra is isomorphic to a subalgebra of the standard MV-algebra with universe $[0, 1]$ [7, Theorem 3.5.1]. As we presently show, this is no accident, for an appropriate version of Hölder's theorem can be generalized to the whole of $\mathcal{GBL}$.

Connecting a number of results in the literature, we make the following observation (for a different proof, see [6]):

Theorem 31 *A totally ordered GBL-algebra* $\mathbf{L}$ *is strongly simple if and only if it is isomorphic to one of the following algebras:*

1. *a subalgebra of* $\mathbb{R}$ *(viewed as a member of* $\mathcal{LG}$*);*
2. *a subalgebra of* $\mathbb{R}^-$ *(viewed as a member of* $\mathcal{LG}^-$*); or*
3. *a subalgebra of the standard MV-algebra on* $[0, 1]$.

Proof. All the algebras on the real numbers mentioned in the above statement are readily seen to be simple, commutative, and hence strongly simple. Indeed, any convex subuniverse would be normal (because of commutativity). Conversely, let $\mathbf{L}$ be a strongly simple totally ordered GBL-algebra. In particular, $\mathbf{L}$ is directly indecomposable, whence by Theorem 28 $\mathbf{L}$ is either a strongly simple totally ordered group, or a strongly simple integral totally ordered GBL-algebra. If the former, then we apply Hölder's theorem to obtain the required embedding into $\mathbb{R}$. If the latter, we observe that $\mathbf{L}$ is ordinally irreducible. In fact, suppose otherwise — then, as remarked immediately after Theorem 29, $\mathbf{L}$ would admit a non-trivial convex subalgebra, a contradiction. Thus, by Theorem 30, $\mathbf{L}$ is either a totally ordered pseudo-MV-algebra or the negative cone of a totally ordered group — and, again, we go through a case-splitting argument.

(1) Suppose **L** is a totally ordered pseudo-MV-algebra. Since any strongly simple totally ordered pseudo-MV-algebra is commutative [15, Theorem 4.3], **L** is in fact a simple totally ordered MV-algebra, and therefore, as already recalled, it is isomorphic to a subalgebra of the standard MV-algebra on $[0, 1]$.

(2) Suppose, on the other hand, that **L** is the negative cone of some totally ordered group **G**. Since the sole convex subuniverses of **L** are the trivial ones, and given that the convex subuniverses of an ℓ-group are uniquely determined by their negative cones, it follows that **G** is itself a strongly simple ℓ-group. Consequently, by Hölder's theorem, **G** is isomorphic to a subalgebra **H** of $\mathbb{R}$, and the restriction of such an isomorphism to the negative elements of **G** provides the required embedding of **L** into $\mathbb{R}^-$. ∎

5.3 Archimedean GBL-algebras

With Theorem 31 at our disposal, we can show that any Archimedean and prelinear GBL-algebra is commutative. This result, which generalizes analogous theorems for special classes of GBL-algebras mentioned in the introduction, can be seen as a benchmark for the adequacy of our definition as an umbrella notion encompassing the various concepts of Archimedean residuated lattices available in the literature.

Theorem 32 *Any Archimedean and prelinear GBL-algebra* **L** *is commutative.*

Proof. Let **L** be as in the statement of the theorem. To prove that **L** is commutative, it will suffice to show that all its principal convex subalgebras are commutative. Indeed, any two elements $a, b \in L^-$ belong to the convex subuniverse $\mathrm{C}[ab]$ of **L**, and hence the commutativity of $\mathrm{C}[ab]$ will imply in particular that $ab = ba$. By assumption, $\mathrm{C}[ab]$ has a zero radical and, by Lemma 16 and Theorem 17, it is a subdirect product of strongly simple algebras. Thus, what we need to show is that every strongly simple, normal-valued and prelinear GBL-algebra is commutative. To this end, let **H** be such an algebra. Using the prelinearity assumption, Corollary 20 yields that **H** is totally ordered. Consequently, Theorem 31 can be invoked to show that **H** is either $\mathbb{R}$, $\mathbb{R}^-$ or the standard MV-algebra on $[0, 1]$, hence commutative. ∎

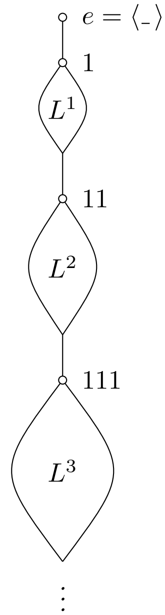
Since all GMV-algebras are prelinear, the preceding results simplify as follows for $\mathcal{GMV}$:

Lemma 33 *Every strongly simple GMV-algebra* **L** *is totally ordered and commutative.*

Theorem 34 *Any Archimedean GMV-algebra* **L** *is commutative.*

Outside $\mathcal{GBL}$, of course, there can be no presumption to the effect that Archimedean residuated lattices are commutative. The next example, indeed, shows that there are non-commutative Archimedean, cancellative and integral residuated lattices.

Example 35



Let $\mathbf{L} = (\{0,1\}, \wedge, \vee)$ *be the* 2*-element lattice with greatest element* 1*, and let* $\mathbf{L}^* = (L^*, \cdot)$ *be the free monoid over* L*. We will define a cancellative residuated lattice whose multiplicative reduct is that of* $\mathbf{L}^*$ *and whose lattice reduct contains* $\mathbf{L}$ *as a sublattice. We let* $l(w)$ *denote the length of* $w \in L^*$*. We define now an order on* L^* *by setting* $u \leq v$ *if and only if*

(i) $l(u) > l(v)$*, or*

(ii) $l(u) = l(v)$ *and* $u_i \leq v_i$ *for* $i = 1, \ldots, n$*, where* $u = u_1 \ldots u_n$ *and* $v = v_1 \ldots v_n$*.*

Then the empty word e *is the greatest element of* L^* *and, for any* $n \in N$*, the words of length* n *form a sublattice isomorphic to the direct power* $(L^n, \wedge, \vee)$*. If* $u \leq v$*, then clearly* $u \backslash v = e$*. If* $u \not\leq v$ *with* $u = u_1 \ldots u_m$ *and* $v = v_1 \ldots v_n$*, then* $m \leq n$ *and*

$$u \backslash v = \begin{cases} v_{m+1} \ldots v_n & \text{if } m < n \text{ and } u_i \leq v_i \text{ for } i = 1, \ldots, m, \\ 1^k & \text{otherwise, where } k = n - m + 1. \end{cases}$$

The operation / *has a very similar characterization. It follows easily that the residuated lattice* $(L^*, \wedge, \vee, \cdot, \backslash, /, e)$ *is cancellative and that* $\mathbf{L}$ *is a sublattice of its underlying lattice structure. It can be seen that* $\mathbf{L}^*$ *is an Archimedean integral cancellative residuated lattice whose monoidal operation is not commutative.*

6 Hyper-Archimedean GMV-algebras

As we recalled in our introduction, both ℓ-group theorists and researchers who are interested in integral subclasses of $\mathcal{RL}$ have often brought hyper-Archimedean algebras to the spotlight. We also observed that there is no consensus on a general definition that encompasses all the existing proposals. The purpose of this final section is to advance a positive suggestion to this effect, as well as to stack up as much evidence as possible in its favor.

Here goes the suggestion:

Definition 36 *A normal-valued residuated lattice* $\mathbf{L}$ *is said to be* hyper-Archimedean *if all its prime convex subuniverses are trivially ordered (that is, they are maximal convex subuniverses).*

In the literature on individual varieties of residuated lattices, it is not infrequent to encounter theorems that provide several equivalent characterizations of hyper-Archimedean algebras (e.g. [7, Thm. 6.3.2]). Once again it has to be noticed that, more often than not, these results involve an admixture of properties that depend on peculiar features of the algebras in question, and of purely lattice-theoretic properties of their lattices of convex subuniverses. From the point of view of the extended Conrad's program, it is crucial to understand how much headway we can make with just plain lattice theory. This is what we are going to do next, in the context of GMV-algebras. We will see that *some* of the equivalent descriptions one can find of hyper-Archimedean algebras in $\mathcal{GMV}$ (or subclasses of such) are purely lattice-theoretic – on the other hand, to get the complete list, one has to resort to the full power of the structure theory of $\mathcal{GMV}$.

Lemma 37 *If* $\mathbf{L}$ *is an algebraic distributive lattice, and* $\bot \neq a \in L$*, then the poset* $\mathbf{P}_a$ *of meet-prime elements in the downset* $\downarrow a$ *is isomorphic to the poset* $\tilde{\mathbf{P}}_a$ *of meet-prime elements of* L *that do not exceed* a*. The isomorphism is implemented by the mutually inverse maps* $f_a : P_a \to \tilde{P}_a$ *and* $\tilde{f}_a : \tilde{P}_a \to P_a$ *defined by* $f_a(x) = a \to x$ *and* $\tilde{f}_a(x) = a \wedge x$*, respectively.*

Proof. Let $p \in P_a$. As p is a meet-prime element of $\downarrow a$, $p \neq a$. Hence $a \not\leq p$ and $a \not\leq a \to p$. Suppose next that $b, c \in L$ are such that $b \wedge c \leq a \to p$. Then, $(a \wedge b) \wedge (a \wedge c) = a \wedge b \wedge c \leq p$. So, $a \wedge b \leq p$ or $a \wedge c \leq p$, and then $b \leq a \to p$ or $c \leq a \to p$. This shows that $a \to p \in \tilde{P}_a$ and that f_a is well-defined. Conversely, let $q \in \tilde{P}_a$. We claim that $a \wedge q \in P_a$. Note that $a \wedge q < a$, as $a \not\leq q$. Let now $b, c \in L$ be such that $b \wedge c \leq a \wedge q$. Then $b \wedge c \leq q$ and so, $b \leq q$ or $c \leq q$. This trivially implies that $b \leq a \wedge q$ or $c \leq a \wedge q$. Hence, $a \wedge q \in P_a$ and, moreover, f_a is well-defined.

It is clear that both f_a and $\tilde{f}_a$ are order-preserving. Further, for all $p \in P_a$, $\tilde{f}_a f_a(p) = \tilde{f}_a(a \to p) = \tilde{f}_a(p) = a \wedge p = p$; and likewise, $f_a \tilde{f}_a(q) = q$, for all $q \in \tilde{P}_a$. Thus f_a and $\tilde{f}_a$ are inverses of each other and order-isomorphisms. ■

The preceding lemma immediately implies the following result:

Lemma 38 *[5] Let* $\mathbf{L}$ *be an algebraic distributive lattice. If* $a \in \mathcal{K}(\mathbf{L})$*, and* $\bot \neq a$*, then the preceding isomorphism restricts to a bijection between the values of* a *in* L *and the co-atoms of* $\downarrow a$*.*

We will also need the following result, stated here without a proof.

Lemma 39 *[5] Let* $\mathbf{L}$ *be an algebraic distributive lattice such that* $\mathcal{K}(\mathbf{L})$ *forms a sublattice, and let* $a \in \mathcal{K}(\mathbf{L})$ *be such that* $a \to b \leq p$*, for* $b \in L$*. Then, there exists a meet-irreducible element* q *such that* $a \not\leq q$*, and* $b \leq q \leq p$*.*

We now collect in the next lemma the purely lattice-theoretic implications of the property that prime convex subuniverses are maximal. Even though most of these equivalences appear in [32], we provide a streamlined proof for the reader's benefit.

Lemma 40 *Let* $\mathbf{L}$ *be an algebraic distributive lattice whose join-semilattice* $\mathcal{K}(\mathbf{L})$ *of compact elements is a sublattice. Then Conditions* (1)-(4) *below are equivalent:*

(1) *The meet-irreducible (equivalently, meet-prime) elements of* $\mathbf{L}$ *are maximal.*

(2) *The meet-irreducible elements of* $\downarrow a$ *are maximal, whenever* a *is a non-zero element of* $\mathcal{K}(\mathbf{L})$.

(3) *Every compact element of the lattice* $\downarrow a$ *has a complement, whenever* $a \in \mathcal{K}(\mathbf{L})$.

(4) *Every element of* $\mathcal{K}(\mathbf{L})$ *has a complement.*

Moreover, these conditions imply that the interval $[h, \top]$ *in* $\mathbf{L}$ *has the zero radical compact property, for all* $h \in L$.

Proof.

(1) $\Rightarrow$ (2) Let us assume that (1) is satisfied, and let $a \in \mathcal{K}(\mathbf{L})$. By assumption, the set of meet-irreducible elements of $\mathbf{L}$ that do not exceed a is precisely the set of values of a. Hence, by Lemmas 37 and 38, all meet-irreducible elements in $\downarrow a$ are maximal.

(2) $\Rightarrow$ (3) Suppose that (2) is satisfied and let $a \in \mathcal{K}(\mathbf{L})$. Note that $b \in \mathcal{K}(\downarrow a)$ if and only if $b \in \mathcal{K}(L) \cap \downarrow a$. Let b be such an element and let $b^{\perp_a}$ denote its pseudo-complement in $\downarrow a$. We need to prove that $b \vee b^{\perp_a} = a$. Suppose, on the contrary that $b \vee b^{\perp_a} < a$. Then, there is a meet-irreducible element p such that $b, b^{\perp_a} \leq b \vee b^{\perp_a} \leq p$. By Lemma 39, there exists a meet-irreducible element $q \in \downarrow a$ such that $b \not\leq q$ and $q \leq p$. As $b \not\leq q$ and $b \leq p$, it must be that $q < p$. Hence, there is a meet-irreducible element in $\mathcal{K}(\downarrow a)$ which is not maximal, contradicting (2).

(3) $\Rightarrow$ (4) Assume that (3) holds. Let $b \in \mathcal{K}(\mathbf{L})$ and let $b^{\perp}$ denote the pseudo-complement of b in $\mathbf{L}$. We proceed to show that $b \vee b^{\perp} = \top$. Observe first that for each $a \in \mathcal{K}(\mathbf{L})$ with $b \leq a$, $b^{\perp_a} = b^{\perp} \wedge a$. Clearly, $\top = \bigvee\{a \in \mathcal{K}(\mathbf{L}) : b \leq a\}$. Then, for each such a, $a = b \vee b^{\perp_a} = b \vee (b^{\perp} \wedge a) = (b \vee b^{\perp}) \wedge (b \vee a) = (b \vee b^{\perp}) \wedge a$. Thus, $b \vee b^{\perp} \geq a$ and $\top = b \vee b^{\perp}$.

(4) $\Rightarrow$ (1) Suppose that (1) fails, and let $p < q$, with p, q meet-irreducible elements of $\mathbf{L}$. Then, there is an element $b \in \mathcal{K}(\mathbf{L})$ such that $b \not\leq p$ and $b \leq q$. Then, as $\perp = b \wedge b^{\perp} \leq p$, and $b \not\leq p$, we have that $b^{\perp} \leq p < q$. Hence, $b \vee b^{\perp} \leq q < \top$, which shows that (4) fails.

Lastly, suppose that $\mathbf{L}$ satisfies the preceding equivalent conditions. Let $h \in L$. We claim that the interval $[h, \top]$ has the zero radical compact property. Note first that $[h, \top]$ is an algebraic distributive lattice. An element $a \in [h, \top]$ is compact if and only if $a = h \vee c$ for some $c \in \mathcal{K}(\mathbf{L})$. Observe that as the meet-irreducible elements of $\mathbf{L}$ are maximal, then the same is true for those of the interval $[h, \top]$. Consider $a \in \mathcal{K}([h, \top])$. (As one might expect, the proof actually works for an arbitrary $a \in [h, \top]$.) Denoting by P_a the set of co-atoms of the lattice $[h, a]$, we need to prove that $\bigwedge P_a = h$. Let P denote the set of meet-prime elements of $[h, \top]$, let $P_1 = \{p \in P : p \geq a\}$ and let $P_2 = \{p \in P : p \not\geq a\}$. It is clear that $P = P_1 \cup P_2$. In view of Lemma 37, $P_a = \{p \wedge a : p \in P_2\}$. Hence, $h = h \wedge a = \bigwedge P = \bigwedge P_1 \wedge \bigwedge P_2 \wedge a = (\bigwedge P_1 \wedge a) \wedge (\bigwedge P_2 \wedge a) = a \wedge \bigwedge P_a = \bigwedge P_a$. This completes the proof of the claim. ■

Lattice theory cannot take us beyond this point. To extend the list of equivalent characterizations of hyper-Archimedean algebras in subvarieties of $\mathcal{RL}$, we must use specific properties of these subvarieties. For the variety $\mathcal{GMV}$ of GMV-algebras, the preceding considerations lead to the following theorem.

Theorem 41 *For any normal-valued GMV-algebra* $\mathbf{L}$ *the following conditions are equivalent:*

(1) $\mathbf{L}$ *is hyper-Archimedean.*

(2) *The interval* $[H, L]$ *in* $\mathcal{C}(\mathbf{L})$ *has the zero radical compact property, for every* $H \in \mathcal{C}(\mathbf{L})$.

(3) *The prime subuniverses of every principal convex subuniverse* H *of* $\mathbf{L}$ *are maximal convex subuniverses of* H.

(4) $\mathbf{L} = \mathrm{C}[a] \otimes a^{\perp}$, *for all* $a \in L$.

(5) *For all* $a, b \in L^{-}$, *there exists a natural number* m *such that* $b \vee a^{m} = b \vee a^{m+1}$.

Proof. Let us first recall that the variety of GMV-algebras satisfies the quasi-equation $x \vee y = e \Rightarrow xy = x \wedge y$ [23]. The proof for the variety $\mathcal{IGMV}$ of integral GMV-algebras – and hence also for the variety of GMV-algebras with the use of Theorem 28 – is simple. Indeed, let x, y be orthogonal elements of an integral GMV-algebra $\mathbf{G}$, that is $x \vee y = e$. The equation E5 implies that $y/(x\backslash y) = x \vee y = e$, and so $x\backslash y \leq y$. Hence, employing E1, we get that $x \wedge y = x(x\backslash y) \leq xy$. As the inequality $xy \leq x \wedge y$ holds trivially, we have $xy = x \wedge y$.

Proceeding with the proof, we note that by [26, Proposition 18], $\mathrm{C}[a] \vee a^{\perp} = \mathrm{C}[a] \otimes a^{\perp}$, for all $a \in L$. Thus, by Lemma 40, (1), (3) and (4) are equivalent and imply (2).

(2) $\Rightarrow$ (3) Suppose (2) holds. $\mathbf{L}$ is Archimedean because $[\{e\}, L] = \mathcal{C}(\mathbf{L})$ has the zero radical compact property. By Theorem 34, $\mathbf{L}$ is commutative. Let M be a proper prime convex subuniverse in $\mathcal{C}(\mathbf{L})$. It follows from Corollary 20 that $\mathbf{L}/M$ is totally ordered. By (2), $\mathbf{L}/M$ is also Archimedean. Therefore, by Theorem 28 $\mathbf{L}/M$ is either an ℓ-group, a negative cone of an ℓ-group, or a pseudo-MV-algebra. In either of these cases $\mathbf{L}/M$ is strongly simple, and so M is maximal.

(4) $\Rightarrow$ (5) Suppose that (4) is satisfied and let $a, b \in L^{-}$. By (4), $b = b_1 b_2$, for some $b_1 \in \mathrm{C}[a]$ and $b_2 \in a^{\perp}$. Thus, there exists a positive integer m such that $a^m \leq b_1$. As b_1 and b_2 (respectively, a^m and b_2) are orthogonal,

$$b \vee a^m = b_1 b_2 \vee a^m = (b_1 \wedge b_2) \vee a^m = \\ (b_1 \vee a^m) \wedge (b_2 \vee a^m) = b_1 \wedge (b_2 \vee a^m) = b_1 \wedge e = b_1.$$

Likewise, the orthogonality of a^{m+1} and b_2 implies that $b \vee a^{m+1} = b_1 = b \vee a^m$.

(5) $\Rightarrow$ (4) Suppose (5) holds. We first prove that the implication holds for integral GMV-algebras. Let $a \in L$. We need to prove that $\mathbf{L} = \mathrm{C}[a] \otimes a^{\perp}$.

More precisely, we need to prove that any $b \in L$ can be written as $b = b_1 b_2$, for some $b_1 \in \mathrm{C}[a]$ and $b_2 \in a^{\perp}$. Consider such a b. By (5), there exists a positive integer m such that $b \vee a^m = b \vee a^{m+1}$. We claim that (i) $b_1 = b \vee a^m \in \mathrm{C}[a]$; (ii) $b_2 = a^m \backslash b \in a^{\perp}$; and (iii) $b = b_1 b_2$.

Condition (i) trivially holds. For (ii), we have to show that $e \leq a \vee (a^m \backslash b)$, or equivalently that $e \leq (a^m \backslash b)/(a \backslash (a^m \backslash b))$. The later inequality is in turn equivalent to $a^{m+1} \backslash b = a \backslash (a^m \backslash b) \leq a^m \backslash b$. We verify the last inequality (in fact, equality) by making use of (5): $a^{m+1} \backslash b = [b/(a^{m+1} \backslash b)] \backslash b = [b \vee a^{m+1}] \backslash b = [b \vee a^m] \backslash b = [b/(a^m \backslash b)] \backslash b = a^m \backslash b$.
For (iii), as

$$b_1 b_2 = (b \vee a^m)(a^m \backslash b) = (b/(a^m \backslash b))(a^m \backslash b),$$

we have that $b_1 b_2 \leq b$. For the other direction, using the fact that b_1 and b_2 are orthogonal, we get that $b \leq (b \vee a^m) \wedge (a^m \backslash b) = b_1 \wedge b_2 = b_1 b_2$. (The first inequality is valid due to the integrality of $\mathbf{L}$.) This completes the proof of (iii) for integral GMV-algebras.

Finally, using the representation Theorem of GMV-algebras (Theorem 28), and the fact that the claim is true for ℓ-groups [13, Theorem 55.1], the result follows. ■

It would be interesting to see whether the full list of equivalences continues to hold for some notable proper superclass of $\mathcal{GMV}$, but this problem must be deferred to another paper.

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