EQUIVALENCE OF CONSEQUENCE RELATIONS: AN ORDER-THEORETIC AND CATEGORICAL PERSPECTIVE

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Abstract. Equivalences and translations between consequence relations abound in logic. The notion of equivalence can be defined syntactically, in terms of translations of formulas, and order-theoretically, in terms of the associated lattices of theories. W. Blok and D. Pigozzi proved in [2] that the two definitions coincide in the case of an algebraizable sentential deductive system. A refined treatment of this equivalence was provided by W. Blok and B. Jónsson in [3]. Other authors have extended this result to the cases of k-deductive systems and of consequence relations on associative, commutative, multiple conclusion sequents. Our main result subsumes all existing results in the literature and reveals their common character. The proofs are of order-theoretic and categorical nature.

1. introduction

The aim of the present paper is to propose an order-theoretic and categorical framework for various constructions and concepts connected with the study of logical consequence relations. Our approach places under a common umbrella a number of existing results regarding the equivalence of consequence relations and provides a road map for future research in this area.

A consequence relation is defined relative to an algebraic signature $\mathcal{L}$. The set Fm of $\mathcal{L}$-formulas is the universe of the term algebra $\mathbf{Fm}$ of signature $\mathcal{L}$ over a countably infinite set of variables. Throughout this paper, we identify the algebra $\mathbf{Eq}$ of $\mathcal{L}$-equations with the algebra $\mathbf{Fm} \times \mathbf{Fm}$, and denote by $\mathbf{\Sigma}$ the monoid of substitutions of $\mathbf{Fm}$.

W. Blok and D. Pigozzi proved in [2] that a substitution invariant, finitary consequence relation $\vdash$ on $\mathbf{Fm}$ is algebraizable if and only if there exists an algebraic consequence relation $\models$ on $\mathbf{Eq}$ such that the lattices $\mathbf{Th}_{\vdash}$ and $\mathbf{Th}_{\models}$ of the theories corresponding to $\vdash$ and $\models$ are isomorphic under a map that commutes with inverse substitutions. A refined treatment of this equivalence was provided by W. Blok and B. Jónsson in [3]. They observed that the definition of algebraizability of $\vdash$, given in [2], can be rephrased as follows:

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there exist (i) an algebraic consequence relation $\models$ on $\mathbf{Eq}$ and (ii) finitary maps $\tau : Fm \to \mathcal{P}(Eq)$, and $\rho : Eq \to \mathcal{P}(Fm)$ (referred to as *translators*), which commute with substitutions, such that for all $\Psi \cup \{\phi\} \in \mathcal{P}(Fm)$ and $\varepsilon \in Eq$,

(1) $\Psi \vdash \phi$ iff $\tau[\Psi] \models \tau(\phi)$, and
(2) $\varepsilon \mathrel{=\!\mid\!\models} \tau\rho(\varepsilon)$.

In addition, they extended the previously mentioned result in [2] in the setting of M-sets.

Our approach, which owes considerable intellectual debt to the cited work of Blok and Jónsson, is more general and places the aforementioned considerations on solid algebraic and categorical ground. Starting with the concrete situation above, we note that there exists a natural action of $\mathbf{\Sigma}$ on $\mathbf{Fm}$ that extends to an action of the corresponding power sets. The power set $\mathcal{P}(\Sigma)$ is a ringlike object (to be precise a semiring with identity) – in which set-union plays the role of addition and complex product serves as multiplication. On the other hand, $\mathcal{P}(Fm)$ is a structure corresponding to an abelian group (to be precise a commutative monoid), with set-union playing again the role of addition. The latter action possesses the critical property of being residuated, which, in this particular instance, means that it preserves arbitrary unions in each coordinate. Analogous comments hold for the action of $\mathbf{\Sigma}$ on $\mathbf{Eq}$.

This concrete situation leads naturally to the general concept of a (left) module. The *scalars* of such a structure are the elements of a complete residuated lattice $\mathbf{A}$. The *vectors* form a complete lattice $\mathbf{P}$. The scalar multiplication $\star : \mathbf{A} \times \mathbf{P} \to \mathbf{P}$ is a bi-residuated map (i.e., a residuated map in each coordinate) that satisfies the usual properties of a monoid action. For a given complete residuated lattice $\mathbf{A}$, all $\mathbf{A}$-modules constitute the objects of a category, ${}_{\mathbf{A}}\mathcal{M}$, whose morphisms are residuated maps that preserve scalar multiplication.

The category ${}_{\mathbf{A}}\mathcal{M}$ provides an ideal environment to abstract the aforementioned concepts and identify their categorical properties. For example, the structural consequence relations on an object $\mathbf{P}$ correspond bijectively to the epimorphic images of $\mathbf{P}$. Thus, such relations may be identified with objects of this category. Not surprisingly then, we stipulate that two structural consequence relations are equivalent if the $\mathbf{A}$-modules corresponding to them are isomorphic. For the particular case where $\mathbf{P}$ is the powerset of formulas and $\mathbf{A}$ the powerset of substitutions, the module associated with a consequence relation is the lattice of theories enriched by inverse substitutions; the isomorphism of the modules then captures exactly the fact that the enriched (with inverse substitutions) lattices of theories are isomorphic. On the other hand, we can define equivalence of structural consequence relations by abstracting the second condition for algebraizability stated above, namely the existence of syntactic translators with the appropriate properties.

The second definition always implies the first. The main result of this work identifies categorically the modules for which the two definitions coincide: they are precisely the *projective* objects of this category. For projective modules $\mathbf{P}$ and $\mathbf{Q}$, the result reads as follows. Let $\vdash_\gamma$ and $\vdash_\delta$ be two structural consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively, and let γ and δ be the structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$ that correspond to $\vdash_\gamma$ and $\vdash_\delta$. Then, for every isomorphism f between the modules of theories $\mathbf{P}_\gamma = \mathbf{Th}_{\vdash_\gamma}$ and $\mathbf{Q}_\delta = \mathbf{Th}_{\vdash_\delta}$, there exist translators (i.e., module morphisms) $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ such that $\delta\tau = f\gamma$ and $\gamma\rho = f^{-1}\delta$. This result subsumes the cases considered in [3], as well as those involving the equivalence of structural consequence relations on sequents.

More specifically, we prove that the $\mathcal{P}(\mathbf{\Sigma})$-modules $\mathcal{P}(\mathbf{Fm})$ of formulas and $\mathcal{P}(\mathbf{Eq})$ of equations are projective (Corollary 5.9). Each of these modules is cyclic, i.e., it is generated by a single element. An interesting additional result is Theorem 5.7, which presents several characterizations of projective cyclic $\mathbf{A}$-modules.

Let *Seq* be a set of *sequents* (single conclusion, multiple conclusion or non-associative, multi-sequents or hypersequents; refer to [11], [7] or [1]). Unless all elements in *Seq* have the same length, the $\mathcal{P}(\mathbf{\Sigma})$-module $\mathcal{P}(\mathbf{Seq})$ is not cyclic (Proposition 5.10), but we prove that it is projective (Theorem 5.13). This result is proved by noting that $\mathcal{P}(\mathbf{Seq})$ is a coproduct of cyclic projective modules.

J. Rebagliato and V. Verdú [15] have defined the notion of equivalence of two consequence relations on (associative) sequents. The results in [3] do not cover the case of sequents, but it follows from Corollary 6.17 that the isomorphism of the modules of theories is equivalent to the definition of Rebagliato and Verdú [15] and to the one of Raftery [14].

Lastly, Corollary 6.16 guarantees that under additional natural assumptions the desired translators τ and ρ are finitary; i.e., they send compact elements to compact elements. In the case of powersets, this means that they map finite sets to finite sets.

In Section 2, after we review the case of an algebraizable consequence relation we give an equivalent formulation of the definition in terms of translators, which extends to the situation of consequence relations over sequents. Then we characterize the extensions of these maps to powersets and provide the necessary intuition leading to the definition of a module in the more general setting of complete lattices in Section 3. In Section 4 we review all the necessary background on residuation theory, closure operators and consequence relations, and develop the elementary theory of modules that will be necessary for the rest of the paper.

Section 5 makes use of the residuation setting to give characterizations of the notions of similarity and equivalence of consequence relations introduced there, while Section 6 puts things in a categorical setting by identifying the modules for which equivalences (or structural representations) are induced by translators with the projective modules in the appropriate category. At the same time, cyclic and cyclic projective modules are characterized, while the consequence relations on the set of formulas are shown to be particular cases of cyclic projective modules. The case of sequents is handled by appealing to coproducts in the category. Finally, in Section 7 it is shown that the assumption of finitarity can be safely added to the preceding study. By working in the appropriate 'finitary' subcategory it is proved that the inducing translators can be taken to be finitary if all the other objects involved are assumed finitary. This involves the identification of the notion of regular modules, which are shown to be projective now in the said subcategory.

2. Consequence relations and translations

2.1. **Algebraizability.** As usual, by a *propositional* (or *algebraic*) *language* we mean a pair $\mathcal{L} = \langle L, \alpha \rangle$ consisting of a set L and a map α from L to the natural numbers. The elements of L are called *(primitive) connectives* (or *operation symbols*) and the image of a connective under α is called the *arity* of the connective.

An $\mathcal{L}$-*algebra* is a pair $\mathbf{A} = \langle A, Op[L] \rangle$, where A is a set, Op is a map that assigns an operation $Op(f) = f^{\mathbf{A}}$ on A of arity $\alpha(f)$ to every operation symbol f of L; often the map Op is considered understood for a given algebra $\mathbf{A}$. If L is finite, we usually list the elements of $Op[L]$ in the expression $\langle A, Op[L] \rangle$.

We denote by $Fm_{\mathcal{L}}$ the set of *(propositional) formulas* (or *terms*) over the language $\mathcal{L}$ and a countably infinite set Var of *propositional variables.* Also, $\mathbf{Fm}_{\mathcal{L}}$ denotes the associated $\mathcal{L}$-algebra. We denote by $\mathbf{\Sigma}_{\mathcal{L}}$ the endomorphism monoid of $\mathbf{Fm}_{\mathcal{L}}$ and refer to its elements as *substitutions.*

An *(asymmetric) consequence relation over the set* $Fm_{\mathcal{L}}$ is a subset $\vdash$ of $\mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ satisfying the following conditions, for all subsets $\Phi \cup \Psi \cup \{\phi, \psi, \chi\}$ of $Fm_{\mathcal{L}}$:

(1) if $\phi \in \Phi$, then $\Phi \vdash \phi$; and
(2) if $\Phi \vdash \psi$, for all $\psi \in \Psi$, and $\Psi \vdash \chi$, then $\Phi \vdash \chi$.

Usually, we write $\phi \vdash \psi$ for $\{\phi\} \vdash \psi$. A consequence relation $\vdash$ over $Fm_{\mathcal{L}}$ is called *finitary*, if for all subsets $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$, whenever $\Phi \vdash \phi$, there exists a finite subset Φ_0 of Φ such that $\Phi_0 \vdash \phi$. It is called *substitution invariant* or *structural*, if for every substitution $\sigma \in \Sigma_{\mathcal{L}}$, and for all subsets $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$, $\Phi \vdash \phi$ implies $\sigma[\Phi] \vdash \sigma(\phi)$.

The deducibility (or provability) relation of a Hilbert system with finitely many rule schemes (we consider axiom schemes as special cases of rule schemes) is a finitary and substitution invariant consequence relation. For example, the deducibility (or provability) relation $\vdash_{CPL}$ of Classical Propositional Logic (CPL) is a finitary and substitution invariant consequence relation over $Fm_{\mathcal{L}}$, where $\mathcal{L}$ is the language of CPL.

Associated with a consequence relation $\vdash$ on $\mathbf{Fm}_{\mathcal{L}}$ is a closure operator $\gamma_{\vdash}$ on $\mathbf{Fm}_{\mathcal{L}}$, defined by $\gamma_{\vdash}(\Phi) = \{\psi \in Fm_{\mathcal{L}} \mid \Phi \vdash \psi\}$. Conversely, a closure operator $\mathbf{Fm}_{\mathcal{L}}$ gives rise to a consequence relation. We discuss this connection in a more general setting in Section 3.

By an *equation* over $\mathcal{L}$ we mean a pair of elements $s, t \in Fm_{\mathcal{L}}$ and we usually denote it by the expression $s \approx t$. We denote by $\mathbf{Eq}_{\mathcal{L}}$ the $\mathcal{L}$-algebra $(\mathbf{Fm}_{\mathcal{L}})^2$ of equations over $\mathcal{L}$. A substitution invariant, finitary consequence relation over $\mathbf{Eq}_{\mathcal{L}}$ is defined by analogy to the previous case. If $\mathbf{A}$ is an $\mathcal{L}$-algebra, $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ is a homomorphism and $(s \approx t) \in Eq_{\mathcal{L}}$, then we denote by $h(s \approx t)$ the pair $(h(s), h(t))$ and we refer to it as an *equality*; we say that the equality is true if $h(s) = h(t)$.

If $\mathcal{K}$ is a class of $\mathcal{L}$-algebras, and $E \cup \{\varepsilon\}$ is a subset of $Eq_{\mathcal{L}}$, $E \models_{\mathcal{K}} \varepsilon$ means that for all $\mathbf{A} \in \mathcal{K}$ and all homomorphisms $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, if $h[E]$ is a set of true equalities, then $h(\varepsilon)$ is a true equality. It is clear that $\models_{\mathcal{K}}$ is a substitution invariant consequence relation over $Eq_{\mathcal{L}}$. It is well known , see e.g. [14], that $\models_{\mathcal{K}}$ is finitary iff $\mathcal{K}$ is closed under ultraproducts.

Our discussion in the remainder of this section draws heavily from [3]. According to Blok and Pigozzi [2], a *deductive system* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is a substitution invariant, finitary consequence relation over $Fm_{\mathcal{L}}$.

A deductive system $\langle \mathcal{L}, \vdash \rangle$ is called *algebraizable* ([2]), if there exists a class of $\mathcal{L}$-algebras $\mathcal{K}$, a finite set of equations $u_i \approx v_i$, $i \in I$, on a single variable and a finite set of binary definable connectives Δ_j, $j \in J$, such that for every subset $\Psi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ and for every equation $s \approx t$ over $Fm_{\mathcal{L}}$,

(1) $\Psi \vdash \phi$ iff $\{u_i(\psi) \approx v_i(\psi) \mid \psi \in \Psi\} \models_{\mathcal{K}} u_i(\phi) \approx v_i(\phi)$, for all $i \in I$, and

(2) $s \approx t \mathrel{=\!\mid\models}_{\mathcal{K}} \{u_i(s\, \Delta_j\, t) \approx v_i(s\, \Delta_j\, t) \mid i \in I, j \in J\}$.

The class $\mathcal{K}$ is called an *equivalent algebraic semantics* for $\langle \mathcal{L}, \vdash \rangle$.

It can be shown that the combination of (1) and (2) above is equivalent to the condition that for every set of equations $E \cup \{s \approx t\}$ over $Fm_{\mathcal{L}}$ and for every $\phi \in Fm_{\mathcal{L}}$,

(3) $E \models_{\mathcal{K}} s \approx t$ iff $\{u\, \Delta_j\, v \mid u \approx v \in E, j \in J\} \vdash s\, \Delta_j\, t$, for all $j \in J$.

(4) $\phi \dashv\vdash \{u_i(\phi)\, \Delta_j\, v_i(\phi) \mid i \in I, j \in J\}$.

If we define the maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$ by $\tau(\phi) = \{u_i(\phi) \approx v_i(\phi) | i \in I\}$ and $\rho(s \approx t) = \{s\, \Delta_j\, t | j \in J\}$, then conditions (1) and (2) take the more elegant form

(1) $\Psi \vdash \phi$ iff $\tau[\Psi] \models_{\mathcal{K}} \tau(\phi)$, and

(2) $\varepsilon \dashv\models_{\mathcal{K}} \tau\rho(\varepsilon)$.

Next, we identify conditions under which arbitrary maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$ are of the form above. First of all, for all $\phi \in Fm_{\mathcal{L}}$ and all $\varepsilon \in Eq_{\mathcal{L}}$, both $\tau(\phi)$ and $\rho(\varepsilon)$ are finite sets; we will call maps that have this property *finitary*. Also, if $\phi \in Fm_{\mathcal{L}}$, $\varepsilon \in Eq_{\mathcal{L}}$ and $\sigma \in \Sigma_{\mathcal{L}}$ is a substitution, then $\sigma[\tau(\phi)] = \tau(\sigma(\phi))$ and $\sigma[\rho(\varepsilon)] = \rho(\sigma(\varepsilon))$; we will call such maps *substitution invariant*. The following result is implicit in [3].

Lemma 2.1. *For maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$, the following conditions are equivalent.*

(1) *τ, ρ are finitary and substitution invariant maps.*
(2) *There exists a finite set of equations $u_i \approx v_i$, $i \in I$, on a single variable, and a finite set of binary definable connectives Δ_j, $j \in J$, satisfying the relations $\tau(\phi) = \{u_i(\phi) \approx v_i(\phi) \mid i \in I\}$ and $\rho(s \approx t) = \{s\, \Delta_j\, t \mid j \in J\}$.*

Proof. We will show that (1) implies (2). Let x, y be distinct variables in Var and assume that $\tau(x) = \{u_i \approx v_i \mid i \in I\}$ and $\rho(x \approx y) = \{t_j \mid j \in J\}$. Since τ and ρ are finitary, it follows that I and J are finite.

If $\phi \in Fm_{\mathcal{L}}$, let $\kappa_\phi \in \Sigma_{\mathcal{L}}$ be the substitution that sends all variables to ϕ. Since τ is substitution invariant, we have $\kappa_x[\tau(x)] = \tau(\kappa_x(x)) = \tau(x)$, for every variable x. In other words, if we replace all variables in $\tau(x)$ by x, we get back $\tau(x)$; i.e., all the equations $u_i \approx v_i$ contain single variable. Moreover, for all $\phi \in Fm_{\mathcal{L}}$, we have $\tau(\phi) = \tau(\kappa_\phi(x)) = \kappa_\phi[\tau(x)] = \{\kappa_\phi(u_i(x) \approx v_i(x)) \mid i \in I\} = \{u_i(\phi) \approx v_i(\phi) \mid i \in I\}$.

Let Var_1 and Var_2 be two sets that partition the set Var of all variables in a way that $x \in Var_1$ and $y \in Var_2$. For all $(s \approx t) \in Eq_{\mathcal{L}}$, let $\kappa_{s\approx t} \in \Sigma_{\mathcal{L}}$ be the substitution that sends all variables in Var_1 to s and all variables in Var_2 to t. Since τ is substitution invariant, we have $\kappa_{x\approx y}[\rho(x \approx y)] = \rho(\kappa_{x\approx y}(x \approx y)) = \rho(x \approx y)$. In other words, the terms t_j are binary and depend only on the variables x and y; we set $t_j = x\Delta_j y$. We have, for all $(s \approx t) \in Eq_{\mathcal{L}}$, $\rho(s \approx t) = \rho(\kappa_{s\approx t}(x \approx y)) = \kappa_{s\approx t}[\rho(x \approx y)] = \{\kappa_{s\approx t}(x\Delta_j y) \mid j \in J\} = \{s\Delta_j t \mid i \in I\}$. □

Corollary 2.2. *A deductive system $\langle \mathcal{L}, \vdash \rangle$ is algebraizable iff there exist finitary and substitution invariant maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$, and a class of $\mathcal{L}$-algebras $\mathcal{K}$ such that, for every subset $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ and $\varepsilon \in Eq_{\mathcal{L}}$,*

(1) *$\Psi \vdash \phi$ iff $\tau[\Psi] \models_{\mathcal{K}} \tau(\phi)$, and*
(2) *$\varepsilon \dashv\models_{\mathcal{K}} \tau\rho(\varepsilon)$.*

Obviously, the maps τ and ρ extend to maps $\tau' : \mathcal{P}(Fm_{\mathcal{L}}) \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho' : \mathcal{P}(Eq_{\mathcal{L}}) \to \mathcal{P}(Fm_{\mathcal{L}})$, defined by $\tau'(\Phi) = \tau[\Phi]$ and $\rho'(E) = \rho[E]$, for $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$ and $E \in \mathcal{P}(Eq_{\mathcal{L}})$. Moreover, $\tau'(\Phi)$ and $\rho'(E)$ are finite, if $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$ and $E \in \mathcal{P}(Eq_{\mathcal{L}})$ are finite; we will call such maps *finitary*. Also, if $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$, $E \in \mathcal{P}(Eq_{\mathcal{L}})$ and $\sigma \in \Sigma_{\mathcal{L}}$, then $\sigma[\tau'(\Phi)] = \tau'(\sigma[\Phi])$ and $\sigma[\rho'(E)] = \rho'(\sigma[E])$; we will call such maps *substitution invariant.* Clearly, τ' and ρ' stem from maps τ and ρ iff they preserve unions.

Example 2.3. Let $\vdash_{BCK}$ be the least substitution invariant consequence relation on $\mathbf{Fm}_{\{\to\}}$ satisfying the following properties for all $x, y, z \in \mathbf{Fm}_{\{\to\}}$.

(B) $\vdash_{BCK} (x \to y) \to ((y \to z) \to (x \to z))$
(C) $\vdash_{BCK} (x \to (y \to z)) \to (y \to (x \to z))$
(I) $\vdash_{BCK} x \to x$
(K) $\vdash_{BCK} x \to (y \to y)$
(MP) $\{x, x \to y\} \vdash_{BCK} y$

Actually, (I) is redundant, but we include it for later reference. It is shown in [2] that $\vdash_{BCK}$ is algebraizable and the $\{\to\}$-subreducts of commutative integral residuated lattices form an algebraic semantics for it. (Refer to [5] or [12] for a short introduction to residuated lattices, and to [9] for a comprehensive treatment of these structures.) The corresponding maps τ and ρ are given by $\tau(\phi) = \{\phi \approx (\phi \to \phi)\}$ and $\rho(u \approx v) = \{u \to v, v \to u\}$. An extension of this correspondence is obtained by the algebraizability of substructural logics via residuated lattices; see [10]. $\square$

A *theory* of a consequence relation $\vdash$ over $Fm_{\mathcal{L}}$ is a subset T of $Fm_{\mathcal{L}}$ closed under $\vdash$; i.e., for all $\phi \in Fm_{\mathcal{L}}$, $T \vdash \phi$ implies $\phi \in T$. The set of theories of $\vdash$ forms a lattice that we denote by $\mathbf{Th}_{\vdash}$. Likewise we define the lattice of theories $\mathbf{Th}_{\models}$ of a consequence relation $\models$ over $Eq_{\mathcal{L}}$. The notions of finitarity and substitution invariance have analogues for closure operators and lattices of theories. We discuss the connections between consequence relations, closure operators and lattices of theories in a more general setting in Section 3.

The following characterization of algebraizability of a deductive system is proved in [2].

Theorem 2.4. [2] *A deductive system $\langle \mathcal{L}, \vdash \rangle$ is algebraizable with equivalent algebraic semantics a quasivariety $\mathcal{K}$ iff there exists an isomorphism between $\mathbf{Th}_{\vdash}$ and $\mathbf{Th}_{\models_{\mathcal{K}}}$ that commutes with inverse substitutions.*

We will extend this result in a more general setting and provide a categorical reason for its validity.

2.2. **Consequence relations on sets of sequents.** In this section, we consider one more example of a consequence relation.

If m, n are non-negative integers (not both equal to zero), by a *(multiple conclusion, associative) sequent over $\mathcal{L}$ of type* (m, n), we understand a pair (Γ, Δ) of a sequence $\Gamma = (\phi_1, \phi_2, \ldots, \phi_m)$ of $\mathcal{L}$-formulas of length m and a sequence $\Delta = (\psi_1, \psi_2, \ldots, \psi_n)$ of $\mathcal{L}$-formulas of length n. We usually write $\phi_1, \phi_2, \ldots, \phi_m \Rightarrow \psi_1, \psi_2, \ldots, \psi_n$ for (Γ, Δ). These sequents are used in the formulation of substructural logics over **FL**; see, for example, [9]. Variants of this notion of sequent have been considered in the literature; refer to [11], [1], [7], and Section 5.

We usually consider sets of sequents *closed under type*, i.e., sets of sequents such that, for all m, n, if they contain an (m, n)-sequent, then they contain all (m, n)-sequents. If Seq is a set of sequents closed under type, then $Tp(Seq)$ denotes the set of all types of the sequents in Seq.

The set of formulas can be identified with the set of all $(0, 1)$-sequents, and the set of equations can be identified with the set of all $(1, 1)$-sequents.

If $s = \phi_1, \phi_2, \ldots, \phi_m \Rightarrow \psi_1, \psi_2, \ldots, \psi_n$ is a sequent and $\sigma \in \Sigma_{\mathcal{L}}$ is a substitution, $(\sigma(\phi_1), \sigma(\phi_2), \ldots, \sigma(\phi_m) \Rightarrow \sigma(\psi_1), \sigma(\psi_2), \ldots, \sigma(\psi_n))$ is denoted by $\sigma(s)$. If Seq is a set of sequents closed under type, then a (finitary, substitution invariant) consequence relation over Seq is defined as in the case of $Fm_{\mathcal{L}}$ and $Eq_{\mathcal{L}}$.

The notion of algebraizability of a set Seq of sequents closed under type has been defined by Rebagliato and Verdú [15]. If Seq_1 and Seq_2 are sets of sequents over $\mathcal{L}$ closed under type, and $\vdash_1$ and $\vdash_2$ are two consequence relations over Seq_1 and Seq_2, respectively, a *translation* between Seq_1 and Seq_2 is a set $\boldsymbol{\tau} = \{\boldsymbol{\tau}_{(m,n)} \mid (m, n) \in Tp(Seq_1)\}$, where $\boldsymbol{\tau}_{(m,n)}$ is a finite subset of Seq_2 in (at most) $m + n$ variables. If $s \in Seq_1$ is an (m, n)-sequent, $\boldsymbol{\tau}(s) = \boldsymbol{\tau}_{(m,n)}(s)$ denotes the result of replacing the $m+n$ formulas of s for the variables in $\boldsymbol{\tau}_{(m,n)}$.

Two consequence relations $\vdash_1$ and $\vdash_2$ over Seq_1 and Seq_2, respectively, are called *equivalent in the sense of Rebagliato and Verdú*, if there are translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ between Seq_1 and Seq_2 such that for all subsets $S_1 \cup \{s_1\}$ of Seq_1 and all subsets $S_2 \cup \{s_2\}$ of Seq_2,

(1) $S_1 \vdash_1 s_1$ iff $\boldsymbol{\tau}[S_1] \vdash_2 \boldsymbol{\tau}(s_1)$, and
(2) $s_2 \dashv\vdash_2 \boldsymbol{\tau}\boldsymbol{\rho}(s_2)$.

It follows that

(3) $S_2 \vdash_2 s_2$ iff $\boldsymbol{\rho}[S_2] \vdash_1 \boldsymbol{\rho}(s_2)$, and
(4) $s_1 \dashv\vdash_1 \boldsymbol{\rho}\boldsymbol{\tau}(s_1)$.

Lemma 2.5. *Consider maps $\tau : \mathcal{P}(Seq_1) \to \mathcal{P}(Seq_2)$ and $\rho : \mathcal{P}(Seq_2) \to \mathcal{P}(Seq_1)$. The following are equivalent.*

(1) *The maps τ, ρ are finitary, substitution invariant and preserve unions.*
(2) *There exist translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ between Seq_1 and Seq_2 such that $\tau(s_1) = \boldsymbol{\tau}(s_1)$ and $\rho(s_2) = \boldsymbol{\rho}(s_2)$ for all $s_1 \in Seq_1$ and $s_2 \in Seq_2$.*

Proof. The proof is based on the ideas in the proof of Lemma 2.1. The lemma is also a consequence of more general results that we prove later; see Theorem 5.13. $\square$

It will follow from our analysis that the analogue of Theorem 2.4 holds in the case of sequents, as well. A. Pynko [13] proves the result for finitary consequence relations and J. Raftery [14] for the general case of associative sequents.

Example 2.6. An *single conclusion, associative, commutative sequent* on a set A is a pair (Γ, ϕ), where $\Gamma \cup \{\phi\}$ is a multiset on A; traditionally, the sequent (Γ, ϕ) is denoted by $\Gamma \ \Rightarrow \ \phi$. We denote by $Seq_{Iac}(A)$ the set of all single conclusion, associative, commutative sequents on A. The deducibility relation $\vdash_{\mathbf{FL_{ei}}^{\{\rightarrow\}}}$ of the $\{\rightarrow\}$-fragment $\mathbf{FL_{ei}}^{\{\rightarrow\}}$ of the system $\mathbf{FL_{ei}}$ – see [11] for details – is the least structural consequence relation on $Seq_{Iac}(Fm_{\{\rightarrow\}})$ that satisfies the following conditions for all $\Gamma, \Pi, \Sigma \cup \{\alpha, \beta, \delta\} \subseteq Fm_{\{\rightarrow\}}$.

$$\frac{}{\alpha \Rightarrow \alpha}(\mathrm{id}) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Sigma, \alpha, \Pi \Rightarrow \delta}{\Sigma, \Gamma, \Pi \Rightarrow \delta}(\mathrm{cut}) \qquad \frac{\Gamma, \Sigma \Rightarrow \delta}{\Gamma, \alpha, \Sigma \Rightarrow \delta}(\mathrm{i})$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \Gamma, \alpha \rightarrow \beta, \Sigma \Rightarrow \delta}(\rightarrow\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}(\Rightarrow\rightarrow)$$

Here we adopt the convention that the fraction notation $\frac{S}{s}$ means $S \vdash_{\mathbf{FL_{ei}}^{\{\rightarrow\}}} s$, where $S \cup \{s\}$ is a subset of $Seq_{Iac}(Fm_{\{\rightarrow\}})$.

It is shown in [11] that $\vdash_{\mathbf{FL_{ei}}^{\{\rightarrow\}}}$ is equivalent in the sense of Rebagliato and Verdú to $\vdash_{BCK}$; see Example 2.3. Moreover, the consequence relation $\vdash_{\mathbf{FL_{e}}^{\{\rightarrow\}}}$, obtained by removing Rule (i) from $\mathbf{FL_{ei}}^{\{\rightarrow\}}$, is equivalent in the sense of Rebagliato and Verdú to the consequence relation $\vdash_{BCI}$, which is obtained from $\vdash_{BCK}$ by removing Axiom (K). Nevertheless, the relations $\vdash_{\mathbf{FL_{e}}^{\{\rightarrow\}}}$ and $\vdash_{BCI}$ are not algebraizable; see [2]. □

2.3. **Consequence relations on powersets.** So far we have defined consequence relations on the sets $\mathcal{P}(Fm_{\mathcal{L}})$, $\mathcal{P}(Eq_{\mathcal{L}})$ and, more generally, on $\mathcal{P}(Seq)$, where Seq is a set of sequents closed under type. Before we give the general definition in the case of complete lattices, we give a preview in the case of powersets. Our presentation is based on ideas developed in [3].

The definition of a (finitary) consequence relation for the three examples is a special case of the following well known definition.

Let S be a set. An *asymmetric consequence relation over* S is a subset $\vdash$ of $\mathcal{P}(S) \times S$ such that, for all subsets $X \cup Y \cup \{x, y, z\}$ of S,

(1) if $x \in X$, then $X \vdash x$, and
(2) if $X \vdash y$, for all $y \in Y$, and $Y \vdash z$, then $X \vdash z$.

An asymmetric consequence relation over S is called *finitary*, if for all subsets $X \cup \{x\}$ of S, if $X \vdash x$, then there is a finite subset X_0 of X such that $X_0 \vdash x$.

A *symmetric consequence relation over* S is a binary relation $\vdash$ on $\mathcal{P}(S)$ that satisfies, for all $X, Y, Z \in \mathcal{P}(S)$,

(1) if $Y \subseteq X$, then $X \vdash Y$

(2) if $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.
(3) $X \vdash \bigcup_{X \vdash Y} Y$.

Note that $\vdash$ satisfies the first two conditions iff it is a pre-order on $\mathcal{P}(S)$ that contains the relation $\supseteq$. A symmetric consequence relation over S is called *finitary*, if for all $X, Y \in \mathcal{P}(S)$, if $X \vdash Y$ and Y is finite, then there is a finite subset X_0 of X such that $X_0 \vdash Y$.

Given an asymmetric consequence relation $\vdash$, we define its symmetric counterpart $\vdash^s$, by $X \vdash^s Y$, for $X, Y \in \mathcal{P}(S)$, to mean $X \vdash y$, for all $y \in Y$. Conversely, given a symmetric consequence relation $\vdash$, we define its asymmetric counterpart $\vdash^a$, by $X \vdash^a x$ iff $X \vdash \{x\}$, for $X \in \mathcal{P}(S)$ and $x \in S$. It is well known that asymmetric consequence relations are equivalent to symmetric ones and the notions of finitarity and invariance that we will define, correspond. We will work with symmetric consequence relations, as they are amenable to generalization to arbitrary lattices.

The generalization of the notion of substitution invariance to arbitrary powersets requires a new notion of substitution. Note that the monoid of substitutions $\mathbf{\Sigma}_{\mathcal{L}}$ acts on both $Fm_{\mathcal{L}}$ and $Eq_{\mathcal{L}}$ – more generally on a set Seq of sequents over $\mathcal{L}$ closed under type – in the sense that for all $\sigma_1, \sigma_2 \in \Sigma_{\mathcal{L}}$, and s in either $Fm_{\mathcal{L}}$, $Eq_{\mathcal{L}}$ or Seq,

(1) $(\sigma_1 \sigma_2)(s) = \sigma_1(\sigma_2(s))$
(2) $Id_{\mathbf{\Sigma}_{\mathcal{L}}}(s) = s$.

We say that a monoid $\mathbf{\Sigma} = \langle \Sigma, \cdot, e \rangle$ *acts* on a set S, if there exists a map $\star : \Sigma \times S \to S$ such that for all $\sigma_1, \sigma_2 \in \Sigma$, and $s \in S$,

(1) $(\sigma_1 \cdot \sigma_2) \star s = \sigma_1 \star (\sigma_2 \star s)$
(2) $e \star s = s$.

A consequence relation $\vdash$ on $\mathcal{P}(S)$ is called $\mathbf{\Sigma}$-*invariant*, if for all $X \cup \{y\} \subseteq S$ and $\sigma \in \Sigma$, $X \vdash y$ implies $\{\sigma \star x \mid x \in X\} \vdash \sigma \star y$.

Actually, if $\mathbf{\Sigma}$ acts on S, then $\mathcal{P}(\mathbf{\Sigma})$ acts on $\mathcal{P}(S)$, as well, i.e., there exists a map $\star : \mathcal{P}(\Sigma) \times \mathcal{P}(S) \to \mathcal{P}(S)$ such that for all $A_1, A_2 \in \mathcal{P}(\Sigma)$ and $X \in \mathcal{P}(S)$,

(1) $(A_1 \cdot A_2) \star X = A_1 \star (A_2 \star X)$
(2) $\{e\} \star X = X$,

where $A_1 \star X = \{a \star x | a \in A_1, x \in X\}$ and $A_1 \cdot A_2 = \{a_1 \cdot a_2 | a_1 \in A_1, a_2 \in A_2\}$. In this case, $\vdash$ on $\mathcal{P}(S)$ is called $\mathcal{P}(\mathbf{\Sigma})$-*invariant*, if for all $X, Y \in \mathcal{P}(S)$ and $A \in \mathcal{P}(\mathbf{\Sigma})$, $X \vdash Y$ implies $A \star X \vdash A \star Y$.

Moreover, for all $A \in \mathcal{P}(\Sigma)$ and $X, Y \in \mathcal{P}(S)$ we have

$$A \star X \subseteq Y \text{ iff } A \subseteq Y /_{\star} X \text{ iff } X \subseteq A \backslash_{\star} Y,$$

where

$$Y /_{\star} X = \{a \in \Sigma \mid \{a\} \star X \subseteq Y\} \text{ and } A \backslash_{\star} Y = \{x \in S \mid A \star \{x\} \subseteq Y\};$$

equivalently $\star$ preserves arbitrary unions. If all of the above conditions are satisfied, we say that $\mathcal{P}(S)$ is a $\mathcal{P}(\mathbf{\Sigma})$-*module*. For example $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$,

$\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ and $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}})$, where Seq is a set of sequents closed under type, are all $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules.

A map $\tau : \mathcal{P}(S_1) \to \mathcal{P}(S_2)$ is called $\mathcal{P}(\mathbf{\Sigma})$-*invariant* or *structural*, if for all $A \in \mathcal{P}(\Sigma)$ and $X \in \mathcal{P}(S)$, we have $A \star \tau(X) = \tau(A \star X)$.

Assume that S_1 and S_2 are sets, and that $\vdash_1$ and $\vdash_2$ are consequence relations on $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$, respectively. Further, assume that there exist maps $\tau : \mathcal{P}(S_1) \to \mathcal{P}(S_2)$ and $\rho : \mathcal{P}(S_2) \to \mathcal{P}(S_1)$ that preserve unions such that for every subset $X \cup \{x\}$ of S_1 and $y \in S_2$,

(1) $X \vdash_1 x$ iff $\tau(X) \vdash_2 \tau(x)$,
(2) $y \dashv\vdash_2 \tau\rho(y)$.

Then we say that $\vdash_1$ and $\vdash_2$ are *similar via τ and ρ*. We will show in Lemma 4.5 that, in this case, $\vdash_2$ and $\vdash_1$ are similar via ρ and τ, as well.

Assume further that $\mathbf{\Sigma}$ is a monoid and that $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ are $\mathcal{P}(\mathbf{\Sigma})$-modules. If $\vdash_1$ and $\vdash_2$ are similar via τ and ρ, and both τ and ρ are $\mathcal{P}(\mathbf{\Sigma})$-invariant, then we say that $\vdash_1$ and $\vdash_2$ are *equivalent via τ and ρ*.

It is easy to see that a consequence relation $\vdash$ on $\mathcal{P}(Seq)$, where Seq is a set of $\mathcal{L}$-sequents closed under type, is algebraizable in the sense of Rebagliato and Verdú (or in the sense of Blok and Pigozzi in the case when $Seq = Fm_{\mathcal{L}}$) iff there exists a class $\mathcal{K}$ of $\mathcal{L}$-algebras such that $\vdash$ and $\models_{\mathcal{K}}$ are equivalent via finitary, $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-invariant maps $\tau : \mathcal{P}(Seq) \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : \mathcal{P}(Eq_{\mathcal{L}}) \to \mathcal{P}(Seq)$ that preserve unions.

3. Consequence relations, theories and closure operators on modules

In this section we introduce the notion of a consequence relation on an arbitrary complete lattice, and show that consequence relations on a given lattice are in bijective correspondence with closure operators on it. Next we discuss the appropriate notion of substitution invariance for both consequence relations and for closure operators in the setting where the lattice is endowed with the additional structure of a module. This will provide the required background for formulating the definition of equivalence of two consequence relations and to prove our main theorem.

3.1. Consequence relations.

Symmetric consequence relations are binary relations on the powerset of a set. We generalize their definition to complete lattices. We note that the definitions and results of this section extend easily to arbitrary posets.

Let $\mathbf{P}$ be a complete lattice. A *(symmetric) consequence relation on* $\mathbf{P}$ is a binary relation $\vdash$ on $\mathbf{P}$ that satisfies the following conditions, for all $x, y, z \in \mathbf{P}$.

(1) if $y \leq x$, then $x \vdash y$
(2) if $x \vdash y$ and $y \vdash z$, then $x \vdash z$
(3) $x \vdash \bigvee_{x \vdash y} y$, for all $x \in P$

Note that $\vdash$ satisfies the first two conditions iff it is a pre-order on $\mathbf{P}$ that contains the relation $\geq$.

3.2. **Residuated maps on complete lattices.** Let S_1, S_2 be arbitrary sets, and let $\vdash_1$, $\vdash_2$ be consequence relations on $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$, respectively. We have seen that the maps $\tau : \mathcal{P}(S_1) \rightarrow \mathcal{P}(S_2)$ and $\rho : \mathcal{P}(S_2) \rightarrow \mathcal{P}(S_1)$ involved in the definition of similarity of $\vdash_1$ and $\vdash_2$ were assumed to preserve unions. We have noted that this is a necessary and sufficient condition for these maps to extend maps from the sets S_1 and S_2 to the powersets $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ respectively. The generalization of this notion in the setting of complete lattices is that of a map that preserves arbitrary joins. We will find it convenient, however, to work with the equivalent concept of a residuated map.

Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices. A map $\tau : P \rightarrow Q$ is called *residuated*, if there exists a map $\tau_* : Q \rightarrow P$, called the *residual* of τ, such that for all $x \in P$ and $y \in Q$,

$$\tau(x) \leq y \ \Leftrightarrow \ x \leq \tau_*(y).$$

Note that a binary map is residuated, in the sense of the previous subsection, if and only if all its unary translates (sections) are residuated in the preceding sense. We will often write $\tau : \mathbf{P} \rightarrow \mathbf{Q}$ for $\tau : P \rightarrow Q$, to indicate the dependency of the residuation property on the order structure of $\mathbf{P}$ and $\mathbf{Q}$. It is clear that the residual of a residuated map is uniquely defined by

$$\tau_*(y) = max\{x \in P \mid \tau(x) \leq y\}.$$

We will always denote it by τ_*. The following lemma states well known facts from residuation theory; for example, see [6], [4].

Lemma 3.1. *Assume that $\tau : \mathbf{P} \rightarrow \mathbf{Q}$ and $\rho : \mathbf{Q} \rightarrow \mathbf{R}$ are residuated maps.*

(1) *τ preserves all arbitrary joins in $\mathbf{P}$ and τ_* preserves all arbitrary meets in $\mathbf{Q}$.*
(2) *$\tau\tau_* \leq I_Q$ and $\tau_*\tau \geq I_P$.*
(3) *The composition $\rho\tau$ is residuated, as well, with residual $(\rho\tau)_* = \tau_*\rho_*$.*

We note again that for complete lattices $\mathbf{P}$ and $\mathbf{Q}$, $\tau : \mathbf{P} \rightarrow \mathbf{Q}$ is residuated iff it preserves arbitrary joins.

Example 3.2. Let A and B be sets and let $R \subseteq A \times B$ be a binary relation from A to B. The map $\tau_R : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$, defined by $\tau_R(X) = R[X] = \{y \in B \mid R(x, y)$, for some $x \in X\}$, is residuated and its residual is given by $(\tau_R)_*(Y) = R^{-1}[Y] = \{x \in A \mid R(x, y)$, for some $y \in Y\}$. $\square$

Note that if $\tau : A \rightarrow \mathcal{P}(B)$ is defined by $\tau(x) = \{y \mid (x, y) \in R\}$, then $\tau_R : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is what we called τ' in Section 2.1.

3.3. **Closure operators.** Recall that a *closure operator* γ on a complete lattice $\mathbf{P}$ is an *expanding* ($x \leq \gamma(x)$), *monotone* ($x \leq y \ \Rightarrow\ \gamma(x) \leq \gamma(y)$) and *idempotent* ($\gamma(\gamma(x)) = \gamma(x)$) map on P; an *interior operator* γ on $\mathbf{P}$ is an *contracting* ($\gamma(x) \leq x$), *monotone* and *idempotent* map on P. If $\gamma : P \to P$ is a map, we denote by P_γ the image $\gamma[P]$ of P under γ and by $\mathbf{P}_\gamma$ the subposet of $\mathbf{P}$ with carrier P_γ. The elements of P_γ are known as the *fixed points* of γ or the γ-*closed elements* of $\mathbf{P}$.

A subset Q of P is said to be *completely meet-closed*, if whenever $X \subseteq Q$, then $\bigwedge^{\mathbf{P}} X \in Q$. We define $\gamma_Q(x) = \bigwedge^{\mathbf{P}}(\uparrow x \cap Q)$.

The following is in the folklore of the area and can be found in [6], [4].

Lemma 3.3. *Let $\mathbf{P}$ be a complete lattice, γ be a closure operator on $\mathbf{P}$ and Q a completely meet-closed subset of $\mathbf{P}$. Then the following hold.*

(1) *P_γ is a completely meet-closed subset of $\mathbf{P}$.*
(2) *γ_Q is a closure operator on $\mathbf{P}$.*
(3) *$\gamma_{P_\gamma} = \gamma$ and $P_{\gamma_Q} = Q$.*
(4) *$\mathbf{P}_\gamma$ is a complete lattice, and a complete meet-subsemilattice of $\mathbf{P}$, with join $\bigvee^{\mathbf{P}_\gamma} \gamma[X] = \gamma(\bigvee^{\mathbf{P}} \gamma[X]) = \gamma(\bigvee^{\mathbf{P}} X)$ and meet $\bigwedge^{\mathbf{P}_\gamma} \gamma[X] = \bigwedge^{\mathbf{P}} \gamma[X]$.*

It is easy to see that $\gamma : P \to P$ is a closure operator on $\mathbf{P}$ iff the map $\gamma' : P \to P_\gamma$, defined by $\gamma'(x) = \gamma(x)$, for all $x \in P$, is residuated and the inclusion map $In_{P_\gamma} : P_\gamma \to P$ is its residual. We will often identify γ and γ', with the understanding that only γ' is residuated and only γ is a closure operator.

Note that, in view of Lemma 3.3, $\gamma : \mathbf{P} \to \mathbf{P}_\gamma$ preserves arbitrary joins. Also, closure operators are determined by their fixed points.

Lemma 3.4. *Assume that $\tau : \mathbf{P} \to \mathbf{Q}$ is a residuated map between the complete lattices $\mathbf{P}$ and $\mathbf{Q}$.*

(1) *$\tau_*\tau$ is a closure operator on $\mathbf{P}$ and $\tau\tau_*$ is an interior operator on $\mathbf{Q}$.*
(2) *$\tau\tau_*\tau = \tau$ and $\tau_*\tau\tau_* = \tau_*$.*
(3) *$\mathbf{P}_{\tau_*\tau}$ is isomorphic to $\mathbf{Q}_{\tau\tau_*}$.*

If f and g are both maps from $\mathbf{P}$ to $\mathbf{Q}$, we write $f \leq g$, if $f(x) \leq g(x)$ for all $x \in P$. It is obvious that if h is a map from $\mathbf{Q}$ to a complete lattice $\mathbf{R}$ and k is a monotone map from a poset $\mathbf{T}$ to $\mathbf{P}$, then $f \leq g$ implies $fh \leq gh$ and $kf \leq kg$. Note that if γ is a closure operator on $\mathbf{P}$ and δ is an interior operator on $\mathbf{P}$, then $\delta \leq I_P \leq \gamma$, where I_P is the identity map on P.

Given a consequence relation $\vdash$ on a complete lattice $\mathbf{P}$, we define the map $\gamma_\vdash : P \to P$, by $\gamma_\vdash(x) = \bigvee_{x \vdash y} y$. Also given a closure operator $\gamma : P \to P$, we define the binary relation $\vdash_\gamma$ on P, by $x \vdash_\gamma y$ iff $y \leq \gamma(x)$.

Lemma 3.5. *Consequence relations on a complete lattice $\mathbf{P}$ are in bijective correspondence with closure operators on $\mathbf{P}$ via the maps $\vdash \ \mapsto \gamma_\vdash$ and $\gamma \mapsto \ \vdash_\gamma$.*

3.4. **Theories.** As we have seen closure operators and consequence relations are interdefinable. Also, the properties of being structural and finitary (to be defined later) are preserved under this correspondence. Here we discuss yet another way of looking at the same properties.

Let $\vdash$ be a consequence relation on a complete lattice $\mathbf{P}$. An element t of P is called a *theory* of $\vdash$ if $t \vdash x$ implies $x \leq t$. Note that if t is a theory, then $x \leq t$ and $x \vdash y$ imply $y \leq t$. We denote the set of theories of $\vdash$ by $Th_{\vdash}$.

Lemma 3.6. *If* $\vdash$ *is a consequence relation on the complete lattice* $\mathbf{P}$*, then* $Th_{\vdash} = P_{\gamma_{\vdash}}$.

Proof. Let $t \in Th_{\vdash}$ and set $\gamma = \gamma_{\vdash}$. We will show that $t \in P_{\gamma}$, i.e., that $\gamma(t) = t$. We have $\gamma(t) \leq \gamma(t)$, so $t \vdash \gamma(t)$. Since t is a theory, $\gamma(t) \leq t$. The other inequality holds because γ is extensive.

Conversely, assume that $\gamma(t) = t$, and let $x \in P$ such that $t \vdash x$. Then $x \leq \gamma(t) = t$. □

We define the *lattice of theories* $\mathbf{Th}_{\vdash}$ of $\vdash$ to be the complete lattice $\mathbf{P}_{\gamma_{\vdash}}$. Lemma 3.3 shows that the lattices of theories can be characterized abstractly and from it we can recover the corresponding consequence relation or closure operator.

3.5. **Modules over complete lattices and invariance under the action.** To define substitution invariance of a consequence relation on a complete lattice, we need to assume that the latter is endowed with a module structure. Therefore we define modules in the case of arbitrary complete lattices.

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be complete lattices. A map $\star : A \times B \to C$ (viewed as a binary map) is called *residuated* provided there exist maps $\backslash_{\star} : A \times C \to B$ and $/_{\star} : C \times B \to A$, called the *residuals* of $\star$, such that for all $x \in A$, $y \in B$ and $z \in C$,

$$x \star y \leq z \;\Leftrightarrow\; x \leq z /_{\star} y \;\Leftrightarrow\; y \leq x \backslash_{\star} z.$$

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and the operation $\cdot$ is residuated with residuals $\backslash$ and $/$.

Let $\mathbf{A}$ be a complete residuated lattice, $\mathbf{P}$ a complete lattice and $\star : A \times P \to P$ a map. We say that $\langle \mathbf{P}, \star\rangle$ is a *(left)* $\mathbf{A}$*-module*, or a *(left) module over* $\mathbf{A}$, if for all $x \in P$ and $a, b \in A$,

(M1) $1 \star x = x$,
(M2) $a \star (b \star x) = ab \star x$, and
(M3) $\star$ is residuated (we denote the residuals by $\backslash_{\star}$ and $/_{\star}$).

In what follows we will often suppress $\star$ in $\langle \mathbf{P}, \star\rangle$, and simply write $\mathbf{P}$ instead. Clearly, $\mathbf{A}$ is itself an $\mathbf{A}$-module. We assume that $\star$ has priority over the *division operations* $\backslash_\star$ and $/_\star$; so $a \star x /_\star y$ is short for $(a \star x) /_\star y$. In the expressions $y \backslash_\star x$ and $x /_\star y$, x is called the *numerator* and y the *denominator*

Note that if $P = \mathcal{P}(S)$, then we obtain the notion of a module for powersets.

Let $\mathbf{P}$, $\mathbf{Q}$ be $\mathbf{A}$-modules. A map $\tau : P \to Q$ is called *structural* if $a \star \tau(x) = \tau(a \star x)$, for all $x \in P$ and $a \in A$. Obviously, structural maps on the $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules of the previous sections are exactly the substitution invariant maps.

A *module morphism* $\tau : \mathbf{P} \to \mathbf{Q}$ from $\mathbf{P}$ to $\mathbf{Q}$ is a structural residuated map. We will often use the term *translator* for such a morphism. For a fixed complete residuated lattice $\mathbf{A}$, we will denote by ${}_{\mathbf{A}}\mathcal{M}$ the category of all $\mathbf{A}$-modules and module morphisms (translators).

Lemma 3.7. *The following properties hold for every $\mathbf{A}$-module $\langle \mathbf{P}, \star \rangle$, $a \in A$ and $x, y \in P$.*

1. *The operation $\star$ preserves arbitrary joins in both coordinates. In particular, it is order-preserving in both coordinates.*
2. *The operations $\backslash_\star$ and $/_\star$ preserve arbitrary meets in the numerator; moreover, they convert arbitrary joins in the denominator into meets. In particular, they are both order-preserving in the numerator and order reversing in the denominator.*
3. $(x /_\star y) \star y \leq x$
4. $a \star (a \backslash_\star x) \leq x$
5. $x \leq a \backslash_\star (a \star x)$ *and* $a \leq (a \star x) /_\star x$
6. $(a \backslash_\star x) /_\star y = a \backslash (x /_\star y)$
7. $[(x /_\star y) \star y] /_\star y = x /_\star y$
8. $1 \leq x /_\star x$
9. $(x /_\star x) \star x = x$

The proof of the lemma is a straightforward application of the definitions and is therefore omitted. Note that some of the above (in)equalities are in $\mathbf{P}$, like the first in item (5), and some are in $\mathbf{A}$, like the second of item (5). We use the same symbol $\leq$ for inequality in both $\mathbf{A}$ and in $\mathbf{P}$ and rely on the context telling them apart.

A consequence relation $\vdash$ on the $\mathbf{A}$ module $\mathbf{P}$ is called *structural*, if $x \vdash y$ implies $a \star x \vdash a \star y$, for all $x, y \in P$ and $a \in A$.

Note that in the case where $P = \mathcal{P}(S)$ the notions of structurality and of substitution invariance of a consequence relation coincide.

A closure operator on an $\mathbf{A}$-module $\mathbf{P}$ is called *structural*, if the corresponding consequence relation is structural; i.e., $a \star \gamma(x) \leq \gamma(a \star x)$, for all $x \in P$ and $a \in A$. Note that Lemma 3.9 below reconciles the two notions of structurality for a closure operator $\gamma : P \to P$ that is viewed as a residuated map $\gamma : P \to P_\gamma$.

We give four more characterizations of a structural closure operator on a module.

Lemma 3.8. *Let* $\mathbf{P}$ *be an* $\mathbf{A}$*-module and let* γ *be a closure operator on* $\mathbf{P}$*. The following are equivalent*

(1) γ *is structural.*
(2) $\gamma(a \star \gamma(x)) = \gamma(a \star x)$*, for all* $a \in A$ *and* $x \in P$*.*
(3) $\gamma(x) /_{\star}\, y = \gamma(x) /_{\star}\, \gamma(y)$*, for all* $x, y \in P$*.*
(4) $\gamma(a \backslash_{\star} x) \leq a \backslash_{\star} \gamma(x)$*, for all* $a \in A$ *and* $x \in P$*.*
(5) $a \backslash_{\star} \gamma(x) \in P_{\gamma}$*, for all* $a \in A$ *and* $x \in P$*.*

Proof. It is clear that (1) is equivalent to (2). To show that (1) implies (3), let $x, y \in P$. The inequality $\gamma(x) /_{\star}\, \gamma(y) \leq \gamma(x) /_{\star}\, y$ follows from the fact that $y \leq \gamma(y)$. For the reverse inequality, by the structurality of γ, we have

$$[\gamma(x) /_{\star}\, y] \star \gamma(y) \leq \gamma([\gamma(x) /_{\star}\, y] \star y) \leq \gamma(\gamma(x)) = \gamma(x);$$

we used Lemma 3.7(3) and the monotonicity of γ. So $\gamma(x) /_{\star}\, y \leq \gamma(x) /_{\star}\, \gamma(y)$.

For the converse implication, let $a \in A$ and $x \in P$. Since $a \star x \leq \gamma(a \star x)$, we have $a \leq \gamma(a \star x) /_{\star}\, x = \gamma(a \star x) /_{\star}\, \gamma(x)$. Thus, $a \star \gamma(x) \leq \gamma(a \star x)$.

For the equivalence of (1) and (4), let $a \in A$ and $x \in P$. We have $a \star \gamma(a \backslash_{\star} x) \leq \gamma(a \star (a \backslash_{\star} x)) \leq \gamma(x)$, by Lemma 3.7(4). Conversely, $a \star \gamma(x) \leq a \star \gamma(a \backslash_{\star} a \star x) \leq a \star [a \backslash_{\star} \gamma(a \star x)] \leq \gamma(a \star x)$, by Lemma 3.7(5,4).

To show that (1) implies (5), let $a \in A$ and $x \in P$. It suffices to show that $\gamma(a \backslash_{\star} \gamma(x)) \leq a \backslash_{\star} \gamma(x)$; i.e., $a \star \gamma(a \backslash_{\star} \gamma(x)) \leq \gamma(x)$. Indeed,

$$a \star \gamma(a \backslash_{\star} \gamma(x)) \leq \gamma(a \star (a \backslash_{\star} \gamma(x))) \leq \gamma(\gamma(x)) \leq \gamma(x).$$

For the converse implication, let $a \in A$ and $x \in P$. Since $a \star x \leq \gamma(a \star x)$, we have $x \leq a \backslash_{\star} \gamma(a \star x)$. By the hypothesis, it follows that $\gamma(x) \leq a \backslash_{\star} \gamma(a \star x)$, hence $a \star \gamma(x) \leq \gamma(a \star x)$. □

Lemma 3.9. *Let* $\mathbf{P}$ *be an* $\mathbf{A}$*-module and let* γ *be a structural closure operator on* $\mathbf{P}$*. Then* $\langle \mathbf{P}_{\gamma}, \star_{\gamma} \rangle$ *is an* $\mathbf{A}$*-module, where* $\star_{\gamma} : A \times P_{\gamma} \to P_{\gamma}$ *is the map defined by* $a \star_{\gamma} x = \gamma(a \star x)$*. As usual, we write* $\mathbf{P}_{\gamma}$ *for* $\langle \mathbf{P}_{\gamma}, \star_{\gamma} \rangle$*. Moreover,* $\gamma : \mathbf{P} \to \mathbf{P}_{\gamma}$ *is a module morphism.*

Proof. It is clear that the first two conditions in the definition of a module are satisfied. To show that $\star_{\gamma}$ is residuated, note that for all $a \in A$ and $x, y \in P_{\gamma}$, we have

$$a \star_{\gamma} x \leq y \;\Leftrightarrow\; \gamma(a \star x) \leq y \;\Leftrightarrow\; a \star x \leq y \;\Leftrightarrow\; x \leq a \backslash_{\star} y.$$

By Lemma 3.8(5), $a \backslash_{\star} y \in P_{\gamma}$, so $\star_{\gamma}$ is left residuated with left division $\backslash_{\gamma}$ the restriction of $\backslash_{\star}$ to P_{γ}.

Furthermore, we have

$$a \star_{\gamma} x \leq y \;\Leftrightarrow\; \gamma(a \star x) \leq y \;\Leftrightarrow\; a \star x \leq y \;\Leftrightarrow\; a \leq y /_{\star}\, x.$$

Thus, $\star_{\gamma}$ is right residuated and $/_{\gamma}$ is the restriction of $/_{\star}$ to P_{γ}.

The fact that $\gamma : \mathbf{P} \to \mathbf{P}_{\gamma}$ is a module morphism follows from the definition of $\star_{\gamma}$ and the fact that γ is residuated. □

Remark 3.10. Condition (5) of Lemma 3.8, in the special case of $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules, states that the lattice of theories is closed under inverse substitutions. Indeed, if $P = \mathcal{P}(S)$, where S is the set of formulas, equations or sequents, then $\mathbf{P}_\gamma$ is the lattice of theories of $\vdash_\gamma$. Note that condition (5), for a a set of substitutions, is equivalent to its restriction, where a ranges only over singletons, by Theorem 3.7(2). So, condition (5) is equivalent to the statement that $\{\sigma\} \backslash_\star T = \sigma^{-1}[T]$ is a theory, for every substitution σ and theory T, namely that the set of theories is closed under inverse substitutions.

It follows from the proof of Lemma 3.9 that, for all $a \in A$, the map $x \mapsto a \star_\gamma x$ on $\mathbf{P}_\gamma$ is residuated and $x \mapsto a \backslash_\star x$ is its residual. As a map and its residual determine each other uniquely, one can enrich the lattice $\mathbf{P}_\gamma$ of theories with either type of maps. We opted for adding the first type of maps (namely adding a module structure). This is the opposite but equivalent to the choice made in [2], where the lattice of theories is enriched with inverse substitutions, which correspond to adding the residual maps, as discussed above. □

4. Similarity and equivalence of two consequence relations

In this section we define the notions of representation, similarity and equivalence between two closure operators or two consequence relations. Our development generalizes the corresponding notions in [3].

4.1. **Representation.** Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively. A *representation* of γ in δ is a residuated order embedding $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$; i.e., a residuated map satisfying $x \leq y$ iff $f(x) \leq f(y)$, for all $x, y \in P_\gamma$. (Equivalently, a representation can be defined as a residuated and one-to-one mapping, as such a mapping is automatically an order-embedding.) Clearly, if R and S are completely meet-closed subsets of the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively, we define a representation of R in S to be a residuated order embedding $f : R \to S$. A representation $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ of γ in δ is said to be *induced* by the residuated map $\tau : \mathbf{P} \to \mathbf{Q}$, if $f\gamma = \delta\tau$.

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\tau} & \mathbf{Q} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathbf{P}_\gamma & \xrightarrow{f} & \mathbf{Q}_\delta \end{array}$$

In view of the correspondence between consequence relations and closure operators, we will denote an arbitrary consequence relation on a poset $\mathbf{P}$ by $\vdash_\gamma$ with the understanding that γ is the associated closure operator.

We say that a consequence relation $\vdash_\gamma$ is *represented* in the consequence relation $\vdash_\delta$ if the associated closure operator γ is represented in δ; the representation of $\vdash_\gamma$ in $\vdash_\delta$ is *induced* by a residuated map $\tau : \mathbf{P} \to \mathbf{Q}$, if the representation of the corresponding closure operators is induced by τ. Corollary 4.4 shows that $\vdash_\gamma$ is represented in $\vdash_\delta$ via τ if and only if for all $x, y \in P$,

$$x \vdash_\gamma y \text{ iff } \tau(x) \vdash_\delta \tau(y).$$

Lemma 4.1. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, $\tau : \mathbf{P} \to \mathbf{Q}$ be a residuated map and δ a closure operator on $\mathbf{Q}$.*

(1) *The map $\delta^\tau = \tau_*\delta\tau : P \to P$ is a closure operator on $\mathbf{P}$.*
(2) *If $\mathbf{P}$ and $\mathbf{Q}$ are $\mathbf{A}$-modules, and τ and δ are structural then so is δ^τ.*

Proof. Note that $\delta : \mathbf{Q} \to \mathbf{Q}_\delta$ is residuated with residual the inclusion map In_{Q_δ}, so $\delta\tau : \mathbf{P} \to \mathbf{Q}_\delta$ is residuated, as well, with residual $\tau_* In_{Q_\delta} = \tau_{*|_{\mathbf{Q}_\delta}}$, by Lemma 3.1(3).

$$\begin{tikzcd}
\mathbf{P} \arrow[r, shift left, "\tau"] & \mathbf{Q} \arrow[l, shift left, "\tau_*"] \arrow[d, shift left, "\delta"] \\
& \mathbf{Q}_\delta \arrow[ul, "\tau_{*|_{\mathbf{Q}_\delta}}"] \arrow[u, hook, shift left]
\end{tikzcd}$$

Therefore, $\delta^\tau = \tau_*\delta\tau = \tau_{*|_{\mathbf{Q}_\delta}}\delta\tau : P \to P$ is a closure operator on $\mathbf{P}$.

For all $a \in A$ and $x \in P$, by using Lemma 3.1(2), we have

$$\tau(a \star \delta^\tau(x)) = a \star \tau\delta^\tau(x) = a \star \tau\tau_*\delta\tau(x) \leq a \star \delta\tau(x) \leq \delta(a \star \tau(x)) = \delta\tau(a \star x),$$

so $a \star \delta^\tau(x) \leq \tau_*\delta\tau(a \star x) = \delta^\tau(a \star x)$. □

We will call δ^τ the *τ-transform* of δ. Similarly, we can define the τ-transform of a consequence relation $\vdash$ on $\mathbf{Q}$ to be the relation $\vdash^\tau$ on $\mathbf{P}$ defined by $x \vdash^\tau y$ iff $\tau(x) \vdash \tau(y)$, for all $x, y \in P$. Also, we define the τ-transform of a completely meet-closed subset R of $\mathbf{Q}$ to be the subposet $\tau^{-1}[R]$ of $\mathbf{P}$. The following lemma shows that the τ-transform of a consequence relation (completely meet-closed subset) is a consequence relation (completely meet-closed subset, respectively) and the associated closure operator is the τ-transform of the original relation (meet-closed subset, respectively).

Lemma 4.2. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, $\tau : \mathbf{P} \to \mathbf{Q}$ a residuated map and γ, δ closure operators on $\mathbf{P}$ and $\mathbf{Q}$, respectively. The following statements are equivalent*

(1) $\gamma = \delta^\tau$
(2) *for all $x, y \in P$, $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$.*
(3) $P_\gamma = \tau_*[Q_\delta]$

Proof. Assume (1) holds; then for all $x, y \in P$, we have $x \vdash_{\delta^\tau} y$ iff $y \leq \tau_*\delta\tau(x)$ iff $\tau(y) \leq \delta\tau(x)$ iff $\tau(x) \vdash_\delta \tau(x)$. Conversely, for all $x, y \in P$, we have $y \leq \gamma(x)$ iff $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$ iff $\tau(y) \leq \delta\tau(x)$ iff $y \leq \tau_*\delta\tau(x)$. Consequently, $\gamma = \delta^\tau$.

For the equivalence of (1) and (3), first note that $x \in \tau_*[Q_\delta]$ iff $x = \tau_*(\delta(z))$, for some $z \in Q$. We claim that this is further equivalent to $\delta^\tau(x) =$

x. Indeed, the backward direction follows by choosing $z = \tau(x)$. For the forward direction, we have $\delta^\tau(x) = \delta^\tau(\tau_*(\delta(z))) = \tau_*\delta\tau\tau_*\delta(z) \leq \tau_*\delta\delta(z) = \tau_*\delta(z) = x$. We have shown that $\tau_*[Q_\delta] = P_{\delta^\tau}$. Therefore, (3) claims that $P_\gamma = P_{\delta^\tau}$, namely that γ and δ^τ have the same fixed elements. By Lemma 3.3, this is equivalent to $\gamma = \delta^\tau$. □

Lemma 4.3. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be complete lattices,* $\tau : \mathbf{P} \to \mathbf{Q}$ *a residuated map, and* δ *a closure operator on* $\mathbf{Q}$*.*

1. *The map* $f = \delta\tau_{|\mathbf{P}_{\delta^\tau}} : \mathbf{P}_{\delta^\tau} \to \mathbf{Q}_\delta$ *is residuated with residual* $f_* = \tau_{*|\mathbf{Q}_\delta} = \delta^\tau\tau_{*|\mathbf{Q}_\delta} : \mathbf{Q}_\delta \to \mathbf{P}_{\delta^\tau}$*.*
2. f *is a representation of* δ^τ *in* δ *induced by* τ*.*
3. δ^τ *is the only closure operator on* $\mathbf{P}$ *that is represented in* δ *under a representation induced by* τ*.*
4. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* $\mathbf{A}$*-modules,* $\tau : \mathbf{P} \to \mathbf{Q}$ *is a module morphism and* δ *is a structural closure operator on* $\mathbf{Q}$*, then* f *is structural.*

Proof. (1) We first show that $\tau_{*|\mathbf{Q}_\delta} = \delta^\tau\tau_{*|\mathbf{Q}_\delta}$. Indeed, $I_P \leq \delta^\tau$, since δ^τ is a closure operator on $\mathbf{P}$, so $\tau_{*|\mathbf{Q}_\delta} \leq \delta^\tau\tau_{*|\mathbf{Q}_\delta}$. Conversely, $\tau\tau_* \leq I_Q$, by Lemma 3.1(2), so $\tau\tau_* In_{\mathbf{Q}_\delta} \leq I_{\mathbf{Q}} In_{\mathbf{Q}_\delta}$, that is $\tau\tau_{*|\mathbf{Q}_\delta} \leq In_{\mathbf{Q}_\delta}$. By the monotonicity of $\tau_*\delta$, we have $\tau_*\delta\tau\tau_{*|\mathbf{Q}_\delta} \leq \tau_*\delta In_{\mathbf{Q}_\delta}$; i.e., $\delta^\tau\tau_{*|\mathbf{Q}_\delta} \leq \tau_{*|\mathbf{Q}_\delta}$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, shift left, "\tau"] \arrow[d, shift left, "\delta^\tau"] & \mathbf{Q} \arrow[l, shift left, "\tau_*"] \arrow[d, shift left, "\delta"] \\
\mathbf{P}_{\delta^\tau} \arrow[u, hook, shift left] \arrow[r, shift left, "f"] & \mathbf{Q}_\delta \arrow[l, shift left, "f_*"] \arrow[u, hook, shift left]
\end{tikzcd}$$

Recall that $\delta\tau : \mathbf{P} \to \mathbf{Q}_\delta$ is residuated with residual $\tau_{*|\mathbf{Q}_\delta}$. For all $x \in \mathbf{P}_{\delta^\tau}$ and $y \in \mathbf{Q}_\delta$, we have

$$f(x) \leq y \;\Leftrightarrow\; \delta\tau(x) \leq y \;\Leftrightarrow\; x \leq \tau_{*|\mathbf{Q}_\delta}(y) = \delta^\tau\tau_{*|\mathbf{Q}_\delta}(y).$$

Since the range of $\delta^\tau\tau_{*|\mathbf{Q}_\delta}$ is in $\mathbf{P}_{\delta^\tau}$, it follows that f is residuated and its residual is $f_* = \delta^\tau\tau_{*|\mathbf{Q}_\delta}$.

(2) Since f is residuated with residual f_*, both f and f_* preserve order. To show that f is a representation it suffices to show that it reflects the order. Note that

$$f_*f = \tau_{*|\mathbf{Q}_\delta}\delta\tau_{|\mathbf{P}_{\delta^\tau}} = \tau_* I_{Q_\delta}\delta\tau I_{P_{\delta^\tau}} = \tau_*\delta\tau I_{P_{\delta^\tau}} = \delta^\tau I_{P_{\delta^\tau}} = I_{P_{\delta^\tau}}.$$

Now, for all $x, y \in P_{\delta^\tau}$, if $f(x) \leq f(y)$, then $f_*f(x) \leq f_*f(y)$, so $x \leq y$.

Moreover $f\delta^\tau = \delta\tau_{|\mathbf{P}_{\delta^\tau}}\delta^\tau = \delta\tau In_{\mathbf{P}_{\delta^\tau}}\delta^\tau = \delta\tau\delta^\tau = \delta\tau$. The last equality holds because $\delta\tau \leq \delta\tau\delta^\tau$ (since δ^τ is a closure operator) and $\delta\tau\delta^\tau = \delta\tau\tau_*\delta\tau \leq \delta\delta\tau = \delta\tau$ (since $\tau\tau_*$ is an interior operator). Consequently, f is induced by τ.

(3) Let γ be a closure operator on $\mathbf{P}$ that is represented in δ by a representation f induced by τ. We will show that $\gamma = \delta^\tau$.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "\tau"] \arrow[d, two heads, "\gamma"'] & \mathbf{Q} \arrow[d, two heads, "\delta"] \\
\mathbf{P}_\gamma \arrow[r, "f"] & \mathbf{Q}_\delta
\end{tikzcd}$$

Since γ is a closure operator on $\mathbf{P}$, we have $\gamma_* = In_{P_\gamma}$, so $\gamma_*\gamma = \gamma$; for the same reason, we have $\delta_*\delta = \delta$. Consequently, by Lemma 3.1(3),

$$\delta^\tau = \tau_*\delta\tau = \tau_*\delta_*\delta\tau = (\delta\tau)_*\delta\tau = (f\gamma)_*f\gamma = \gamma_*\gamma = \gamma$$

The equation $(f\gamma)_*f\gamma = \gamma_*\gamma$ follows directly from the fact that f is order reflecting, since $x \leq (f\gamma)_*f\gamma(y)$ iff $f\gamma(x) \leq f\gamma(y)$ iff $\gamma(x) \leq \gamma(y)$ iff $x \leq \gamma_*\gamma(y)$, for all $x, y \in P$.

(4) For all $a \in A$ and $x \in P_{\delta^\tau}$, we have $f(a \star_{\mathbf{P}_{\delta^\tau}} x) = f\delta^\tau(a \star_{\mathbf{P}} x) = \delta\tau(a \star_{\mathbf{P}} x) = \delta(a \star_{\mathbf{Q}} \tau(x)) = \delta(a \star_{\mathbf{Q}} \delta\tau(x)) = a \star_{\mathbf{Q}_\delta} \delta\tau(x) = a \star_{\mathbf{Q}_\delta} f(x)$. □

Corollary 4.4. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, and let $\vdash_\gamma$ and $\vdash_\delta$ be consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively. Then, $\vdash_\gamma$ is represented in $\vdash_\delta$ via a residuated map $\tau : \mathbf{P} \to \mathbf{Q}$ if and only if for all $x, y \in P$, we have $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$.*

Proof. The corollary is a direct consequence of Lemma 4.2 and Lemma 4.3(3). □

It is easy to see that $\vdash_\gamma$ is represented in $\vdash_\delta$ by $f : \mathbf{Th}_{\vdash_\gamma} \to \mathbf{Th}_{\vdash_\delta}$ means that f is residuated and for all $x, y \in P$,

$$x \vdash_\gamma y \text{ iff } f\gamma(x) \vdash_\delta f\gamma(y).$$

Indeed, if $\vdash_\gamma$ is represented in $\vdash_\delta$ by f, then $x \vdash_\gamma y$ iff $y \leq \gamma(x)$ iff $f(y) \leq f(\gamma(x))$ (since f preserves and reflects order) iff $f(y) \leq \delta f(\gamma(x))$ iff $f\gamma(x) \vdash_\delta f\gamma(y)$. Conversely, to show that f reflects order, let $f\gamma(y) \leq f\gamma(x)$. Then $f\gamma(y) \leq \delta f\gamma(x)$, that is, $f\gamma(x) \vdash_\delta f\gamma(y)$; so $x \vdash_\gamma y$ that is $\gamma(y) \leq \gamma(x)$.

4.2. **Similarity.** Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively. A *similarity* between γ and δ is an isomorphism $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$. If there exists a similarity between γ and δ, then γ and δ are called *similar*. A similarity f between γ and δ is said to be *induced* by the residuated maps $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$, if $f\gamma = \delta\tau$ and $f^{-1}\delta = \gamma\rho$. In this case we will say that γ and δ are similar *via* τ and ρ.

$$\begin{tikzcd}
\mathbf{P} \arrow[r, shift left, "\tau"] \arrow[d, two heads, "\gamma"'] & \mathbf{Q} \arrow[l, shift left, "\rho"] \arrow[d, two heads, "\delta"] \\
\mathbf{P}_\gamma \arrow[r, shift left, tail, two heads, "f"] & \mathbf{Q}_\delta \arrow[l, shift left, tail, two heads, "f^{-1}"]
\end{tikzcd}$$

It is clear that f is a similarity between γ and δ iff f is a representation of γ in δ, f is a bijection and f^{-1} is a representation of δ in γ.

A consequence relation $\vdash_\gamma$ is called *similar* to the consequence relation $\vdash_\delta$ (via a residuated map τ) if γ is similar to δ (via τ).

Lemma 4.5. *Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively, and let $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ be residuated maps. The following statements are equivalent.*

(1) *γ and δ are similar via (a similarity induced by) τ and ρ.*
(2) *$\gamma = \delta^\tau$ and $\delta\tau\rho = \delta$.*
(3) *$\delta = \gamma^\rho$ and $\gamma\rho\tau = \gamma$.*

Proof. We will show the equivalence of the first two statements; the equivalence of the first to the third will follow by symmetry. The forward direction follows from Lemma 4.3(3) and the definition of similarity ($\delta\tau\rho = f\gamma\rho = ff^{-1}\delta = \delta$). For the converse, assume that $\gamma = \delta^\tau$ and $\delta\tau\rho = \delta$. Let f be the representation of $\gamma = \delta^\tau$ in δ given in Lemma 4.3(1). We have $f\gamma = \delta\tau$, by Lemma 4.3(2).

$$\begin{tikzcd}
\mathbf{P} \arrow[r, shift left, "\tau"] \arrow[d, "\gamma"'] & \mathbf{Q} \arrow[l, shift left, "\rho"] \arrow[d, "\delta"] \\
\mathbf{P}_\gamma \arrow[r, shift left, "f"] & \mathbf{Q}_\delta \arrow[l, shift left, dashed, "f^{-1}"]
\end{tikzcd}$$

To show that f is onto, let $y \in Q_\delta$ and set $x = \gamma\rho(y) \in P_\gamma$. We have $f(x) = f\gamma\rho(y) = \delta\tau\rho(y) = \delta(y) = y$. Consequently, f is an order-isomorphism and γ and δ are similar. To show that the similarity f is induced by τ and ρ, we need only prove that $f^{-1}\delta = \gamma\rho$, or equivalently that $\delta = f\gamma\rho$. This is true, because $\delta = \delta\tau\rho$ and $f\gamma = \delta\tau$. $\square$

Corollary 4.6. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices and let $\vdash_\gamma$ and $\vdash_\delta$ be consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively. Then, $\vdash_\gamma$ is similar to $\vdash_\delta$ via the residuated maps $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ if and only if the following conditions hold:*

(1) *for all $x, y \in P$, $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$; and*
(2) *for all $z \in Q$, $z \dashv\vdash_\delta \tau\rho(z)$.*

Proof. It is easy to see that $\delta\tau\rho = \delta$ iff for all $z \in Q$, $z \dashv\vdash_\delta \tau\rho(z)$. Now, the corollary follows from of Lemma 4.5(2) and Corollary 4.4. $\square$

4.3. **Equivalence.** Let $\mathbf{P}$ and $\mathbf{Q}$ be $\mathbf{A}$-modules and let γ and δ be structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$, respectively. An *equivalence* between γ and δ is a module isomorphism $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$. Note that an equivalence is just a structural similarity. Moreover, f^{-1} is also structural. If such an isomorphism exists then γ and δ are called *equivalent*. If the equivalence is induced by module morphisms $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$, then γ and δ are called *equivalent via τ and ρ*.

Theorem 4.7. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be* $\mathbf{A}$*-modules and let* γ *and* δ *be structural closure operators on* $\mathbf{P}$ *and* $\mathbf{Q}$*, respectively. If* γ *and* δ *are similar via the translators (i.e., module morphisms)* τ *and* ρ*, then they are equivalent via* τ *and* ρ*.*

Proof. It suffices to show that the similarity f of γ in δ is structural. Indeed, for all $a \in A$ and $x \in P_\gamma$, we have

$$\begin{aligned} f(a \star_\gamma x) &= f\gamma(a \star x) = \delta\tau(a \star x) = \delta(a \star \tau(x)) \\ &= \delta(a \star \delta\tau(x)) = a \star_\delta \delta\tau(x) = a \star_\delta f\gamma(x) \\ &= a \star_\delta f(x), \end{aligned}$$

since $\gamma(x) = x$. □

5. Equivalences induced by translators

Theorem 4.7 shows that every similarity between structural closure operators induced by translators is structural. A natural question to ask is whether every equivalence of consequence relations is induced by translators. Example 5.8 shows that this is not always true. Nevertheless, we will show that this is the case for all standard situations including the powersets of formulas, equations and sequents.

Having developed the fundamentals of the theory of $\mathbf{A}$-modules and reformulated the isomorphism of the enriched lattices of theories into the setting of $\mathbf{A}$-modules, we are ready to prove a result that provides the key categorical insight. We will show that the modules in the category ${}_{\mathbf{A}}\mathcal{M}$ for which equivalences are induced by translators coincide with the projective modules in this category. More specifically, we will prove that an $\mathbf{A}$-module $\mathbf{P}$ is projective iff for any $\mathbf{A}$-module $\mathbf{Q}$ and structural closure operators γ and δ on $\mathbf{P}$ and $\mathbf{Q}$ respectively, every structural representation $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ of γ in δ is induced by a translator.

5.1. **Projective objects.** Recall that by ${}_{\mathbf{A}}\mathcal{M}$ we denote the category of $\mathbf{A}$-modules and translators (module morphisms). Every structural closure operator γ on the $\mathbf{A}$-module $\mathbf{P}$ is a translator from $\mathbf{P}$ to $\mathbf{P}_\gamma$. Assume that $\mathbf{P}$ and $\mathbf{Q}$ are $\mathbf{A}$-modules, γ and δ are structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$ respectively, and $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ is a structural representation of γ in δ. We want to find a translator $\tau : \mathbf{P} \to \mathbf{Q}$ that induces f; i.e., $\delta\tau = f\gamma$. In other words we want a morphism τ in the category ${}_{\mathbf{A}}\mathcal{M}$ that completes the square.

(S)
$$\begin{tikzcd} \mathbf{P} \arrow[r, dashed, "\tau"] \arrow[d, "\gamma"'] & \mathbf{Q} \arrow[d, "\delta"] \\ \mathbf{P}_\gamma \arrow[r, tail, "f"] & \mathbf{Q}_\delta \end{tikzcd}$$

It turns out that the objects $\mathbf{P}$ of the category ${}_{\mathbf{A}}\mathcal{M}$ for which such square can be completed are precisely the projective objects of ${}_{\mathbf{A}}\mathcal{M}$. An object

$\mathbf{P}$ of ${}_{\mathbf{A}}\mathcal{M}$ is called *projective (relative to onto maps)*, if whenever there are modules $\mathbf{Q}$ and $\mathbf{R}$ and module morphisms $g : \mathbf{Q} \to \mathbf{R}$ and $k : \mathbf{P} \to \mathbf{R}$, with g onto, then there exists a morphism $h : \mathbf{P} \to \mathbf{Q}$, such that $k = gh$.

(T)
$$\begin{tikzcd}
\mathbf{P} \arrow[r, dashed, "h"] \arrow[dr, "k"'] & \mathbf{Q} \arrow[d, two heads, "g"] \\
 & \mathbf{R}
\end{tikzcd}$$

Theorem 5.1. *The objects $\mathbf{P}$ of the category ${}_{\mathbf{A}}\mathcal{M}$ for which all squares of type (S) can be completed are exactly the projective objects of ${}_{\mathbf{A}}\mathcal{M}$.*

Proof. Obviously, if $\mathbf{P}$ is projective, then the square (S) can be completed, since we can chose $\mathbf{R} = \mathbf{Q}_\delta$, $k = f\gamma$ and $g = \delta$ in the triangle (T).

Conversely, assume that $\mathbf{P}$ is such that every square (S) can be completed and consider the triangle (T), where h is to be determined.

$$\begin{tikzcd}
\mathbf{P} \arrow[rr, dashed, "h=\tau"] \arrow[dr, "k"] \arrow[dd, two heads, "k_*k"'] & & \mathbf{Q} \arrow[dl, two heads, "g"'] \arrow[dd, two heads, "g_*g"] \\
 & \mathbf{R} & \\
\mathbf{P}_{k_*k} \arrow[ur, tail, "k'"] \arrow[rr, tail, "f"] & & \mathbf{Q}_{g_*g} \arrow[ul, two heads, "g'"']
\end{tikzcd}$$

We know by Lemma 3.4 that k_*k is a closure operator on $\mathbf{P}$ and that $\mathbf{P}_{k_*k}$ is isomorphic to $k[\mathbf{P}]$ via the map $k' = k_{|\mathbf{P}_{k_*k}}$. Therefore, the map k factors as $k = k'(k_*k)$. Likewise, we have $g = g'(g_*g)$, where $g' = g_{|\mathbf{Q}_{g_*g}}$. Moreover, k' is an embedding and g' is an isomorphism, so the map $f = (g')^{-1}k'$ is an embedding. Since the outer square can be completed, we have $fk_*k = g_*gh$, so $g'fk_*k = g'g_*gh$, hence $k'k_*k = gh$; thus $k = gh$ and the upper triangle commutes. □

5.2. **Cyclic Modules.** We will show that the $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules discussed in Sections 2.2 and 2.3 are projective. Consequently, in view of the preceding theorem, all equivalences on these modules are induced by translators. More generally, we will identify a set of intrinsic conditions that describe cyclic projective modules. The $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules of formulas and of equations are cyclic and projective. The $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-module of sequents is not cyclic, but we prove that it is a coproduct of cyclic projective modules, and hence it is projective.

Let $\mathbf{A}$ be a complete residuated lattice. An $\mathbf{A}$-module $\mathbf{P}$ is called *cyclic*, if there exists an element $v \in P$, called a *generator* of $\mathbf{P}$, such that $P = A \star v = \{a \star v \mid a \in A\}$.

Lemma 5.2. *An* $\mathbf{A}$*-module* $\mathbf{P}$ *is cyclic with generator* v *iff* $(x/_\star v) \star v = x$*, for all* $x \in P$*.*

Proof. If v is a generator, then for all $x \in P$, there exists an $a \in A$ such that $x = a \star v$; so $a \leq (x/_\star v)$. We have $x = a \star v \leq (x/_\star v) \star v \leq x$, by Lemma 3.7. So, $(x/_\star v) \star v = x$. The converse, is obvious. □

Recall the construction of the module $\mathbf{P}_\gamma$, where $\mathbf{P}$ is a module and γ a structural closure operator on $\mathbf{P}$, from Lemma 3.9. Also recall that $\mathbf{A}$ itself is an $\mathbf{A}$-module. From now on we will make use of this structure, which relies on the residuation of $\mathbf{A}$.

Lemma 5.3. *If* $\mathbf{A}$ *is a complete residuated lattice and* $\gamma : A \to A$ *is a structural closure operator, then the* $\mathbf{A}$*-module* $\langle \mathbf{A}_\gamma, \cdot_\gamma \rangle$ *is cyclic with generator* $\gamma(1)$*.*

Proof. Obviously, $\gamma(1) \in A_\gamma$. Also, for all $\gamma(a) \in A_\gamma$, $a \cdot_\gamma \gamma(1) = \gamma(a \cdot 1) = \gamma(a)$. □

Lemma 5.4. *Let* $\langle \mathbf{P}, \star \rangle$ *be an* $\mathbf{A}$*-module,* $v \in P$ *and* $A \star v = \{a \star v \mid a \in A\}$*.*

(1) *Then* $\mathbf{A} \star v = \langle A \star v, \star \rangle$ *is an* $\mathbf{A}$*-module in which joins coincide with those in* $\mathbf{P}$*. The residual of the operation* $\star$ *in* $\mathbf{A} \star v$ *is given by* $a \backslash_{\mathbf{A}\star v} q = [(a \backslash_\star q)/_\star v] \star v$*.*
(2) *The map* $\gamma_v : A \to A$*, defined by* $\gamma_v(a) = a \star v/_\star v$ *is a structural closure operator.*
(3) $\mathbf{A} \star v$ *is isomorphic to* $\mathbf{A}_{\gamma_v}$*.*

Consequently, an $\mathbf{A}$*-module is cyclic if and only if it is isomorphic to a module* $\mathbf{A}_\gamma$*, for a structural closure operator* $\gamma : \mathbf{A} \to \mathbf{A}$*.*

Proof. (1) First note that if $a \in A$ and $q \in A \star v$, then $q = b \star v$, for some $b \in A$, so $a \star (b \star v) = ab \star v \in A \star v$. Moreover, if $r = c \star v \in A \star v$, where $c \in A$, then $a \star r \leq q$ iff $a \star (c \star v) \leq q$ iff $c \leq (a \backslash_\star q)/_\star v$ iff $c \star v \leq [(a \backslash_\star q)/_\star v] \star v$. The last equivalence follows from Lemma 3.7(7).

Clearly, $\bigvee^{\mathbf{P}}_{i \in I}(a_i \star v) = (\bigvee^{\mathbf{A}}_{i \in I} a_i) \star v \in A \star v$. Therefore, $A \star v$ is closed under $\bigvee^{\mathbf{P}}$, and is therefore a complete lattice.

(2) We have $a \leq \gamma_v(a)$; if $a \leq b$, then $\gamma_v(a) \leq \gamma_v(b)$ and $\gamma_v(\gamma_v(a)) = \gamma_v(a)$, by Lemma 3.7(7). Also, $a\gamma_v(b) \star v = a[(b \star v)/_\star v] \star v \leq a \star (b \star v) = ab \star v$, by Lemma 3.7(3), so $a\gamma_v(b) \leq \gamma_v(ab)$. Thus, γ_v is structural.

(3) Let $f(a) = a \star v$ and $g(x) = x/_\star v$. Note that $f : A_{\gamma_v} \to A \star v$ and $g : A \star v \to A_{\gamma_v}$, since $f(a) = a \star v \in A \star v$ and $g(a \star v) = (a \star v)/_\star v \in A_{\gamma_v}$. For all $x \in A \star v$, we have $f(g(x)) = (x/_\star v) \star v = x$, because of cyclicity. Also, for all $a \in A_{\gamma_v}$, $g(f(a)) = \gamma_v(a) = a$. So, $f^{-1} = g$. Moreover, both f and g are order-preserving, so they are order reflecting as well. □

Corollary 5.5. *If* $\mathbf{A}$ *is a complete residuated lattice and* $u \in A$*, then* $\mathbf{A}u = \langle Au, \cdot \rangle$ *is a cyclic* $\mathbf{A}$*-module isomorphic to* $\mathbf{A}_{\gamma_u}$*.*

Lemma 5.6. *Let* $\mathbf{A}$ *be a complete residuated lattice,* $\gamma : A \to A$ *a structural closure operator and* $u \in A$*. The following are equivalent.*

(1) $\gamma(u) = \gamma(1)$, *and* $\gamma(a)u = au$, *for all* $a \in A$.
(2) $\gamma = \gamma_u$ *and* $u = u^2$

Proof. The fact that (2) implies (1) is easy to check. Conversely, from $\gamma(a)u = au$, we obtain $\gamma(a) \leq au/u = \gamma_u(a)$, for all $a \in A$. Also, from $\gamma(u) = \gamma(1)$, we obtain for all $b \in A$, $\gamma(bu) = b \star_\gamma \gamma(u) = b \star_\gamma \gamma(1) = \gamma(b1) = \gamma(b)$. We have the following implications.

$$\gamma_u(a)u \leq au \;\Rightarrow\; \gamma(\gamma_u(a)u) \leq \gamma(au) \;\Rightarrow\; \gamma(\gamma_u(a)) \leq \gamma(a) \;\Rightarrow\; \gamma_u(a) \leq \gamma(a)$$

Moreover, since $\gamma = \gamma_u$, we have $\gamma_u(u) = \gamma_u(1)$, so $uu/u = u/u$, hence $(u^2/u)u = (u/u)u$. From this we obtain $u^2 = u$, because $(u/u)u = u$, by Lemma 3.7(9), and $u^2 = uu \leq (u^2/u)u \leq u^2$, by Lemma 3.7(5,3). □

Theorem 5.7. *For an* $\mathbf{A}$*-module* $\langle \mathbf{P}, \star \rangle$*, the following conditions are equivalent.*

(1) $u \star v = v$, $[(a \star v)/_\star\, v]u = au$, *for all* $a \in A$, *and* $\mathbf{P} = \mathbf{A} \star v$, *for some* $v \in P$ *and* $u \in A$.
(2) $\gamma_v(a)u = au$, *for all* $a \in A$, $\gamma_v(u) = \gamma_v(1)$, *and* $\mathbf{P} = \mathbf{A} \star v$, *for some* $v \in P$ *and* $u \in A$.
(3) $\gamma_v = \gamma_u$, $u^2 = u$ *and* $\mathbf{P} = \mathbf{A} \star v$, *for some* $v \in P$ *and* $u \in A$.
(4) $\mathbf{P}$ *is isomorphic to* $\mathbf{A}u$ *and* $u^2 = u$, *for some* $u \in A$.
(5) $\mathbf{P}$ *is cyclic and projective.*

Moreover, the elements u *and* v *can be taken to be the same in all statements in which they appear.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the fact that $\gamma_v(u) = \gamma_v(1)$ iff $u \star v/_\star\, v = v/_\star\, v$ iff $u \star v = v$, by using Lemma 3.7. The implication (2) $\Rightarrow$ (3) follows from the preceding lemma. The implication (3) $\Rightarrow$ (4) follows from the facts $\mathbf{A} \star v \cong \mathbf{A}_{\gamma_v}$ (Lemma 5.4), $\mathbf{A}u \cong \mathbf{A}_{\gamma_u}$ (Corollary 5.5), and $\gamma_u = \gamma_v$. Furthermore,(4) $\Rightarrow$ (1) follows from the fact that if $u^2 = u$, then $\mathbf{A}u$ satisfies (1) with $v = u$.

For the equivalence of (4) and (5), note first that every cyclic module is of the form $\mathbf{A}_\gamma$ for some structural closure operator $\gamma : A \to A$, by Lemma 5.4. Suppose $\mathbf{A}_\gamma$ is projective. We will verify condition (4). Since $\mathbf{A}_\gamma$ is projective, there exists a module morphism f that completes the diagram below.

$$\begin{tikzcd} \mathbf{A}_\gamma \arrow[r, dashed, "f"] \arrow[dr, "Id"'] & \mathbf{A} \arrow[d, "\gamma"] \\ & \mathbf{A}_\gamma \end{tikzcd}$$

Let $u = f(\gamma(1))$. For all $a \in A$, we have $\gamma(a) = \gamma(a1) = \gamma(a\gamma(1)) = a \cdot_\gamma \gamma(1)$, so $f(\gamma(a)) = a \cdot f(\gamma(1)) = au$. Consequently, $f[A_\gamma] = Au$. Moreover, f is injective, by the diagram, so $\mathbf{A}_\gamma \cong \mathbf{A}u$. We will show that $u^2 = u$. Indeed, $u^2 = f(\gamma(1))f(\gamma(1)) = f(f(\gamma(1)) \cdot_\gamma \gamma(1)) = f(\gamma(f(\gamma(1)))) = f(\gamma(1)) = u$, because $\gamma f = Id$. We have established condition (4). To show that a module satisfying condition (4), obviously cyclic, is projective, consider the diagram

$$\begin{tikzcd} \mathbf{A}u \arrow[r, dashed, "h"] \arrow[dr, "k"'] & \mathbf{Q} \arrow[d, "g"] \\ & \mathbf{R} \end{tikzcd}$$

and let $q \in Q$ be such that $g(q) = k(u)$. Then the it is straightforward to show that the unique morphism determined by $h(u) = q$ completes the diagram. $\square$

Example 5.8. Let $\mathbf{A}$ be the residuated lattice on the set $A = \{\bot, a, 1, \top\}$, where $\bot < a < 1 < \top$, $\bot$ is an absorbing element, 1 is the neutral element, $a^2 = \bot$, $a\top = \top a = a$ and $\top^2 = \top$; $\mathbf{A}$ is denoted by $\mathbf{T}_1$ in [8]. Consider the cyclic module $\mathbf{P} = \mathbf{A} \cdot a$, where $P = \{\bot, a\}$, and note that a is the only $x \in A$ such that $\mathbf{A} \cdot x$ is isomorphic to $\mathbf{P}$; indeed, $A \cdot \top = \{\bot, a, \top\}$, $A \cdot 1 = A$ and $A \cdot \bot = \{\bot\}$. As a is not idempotent, $\mathbf{P}$ is a cyclic module that is not projective, by Theorem 5.7. $\square$

Corollary 5.9. *$\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$ and $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ are projective cyclic $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules.*

Proof. We will make use of Theorem 5.7. In the case of the module $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$, we let $v = \{x\}$, where x is a variable, and $u = \{\kappa_x\}$. Recall that κ_x is the substitution that maps all variables to x. We have $u \star v = \{\kappa_x(x)\} = \{x\} = v$. Also, for a set a of substitutions, we have $a \star v = \{\sigma(x) : \sigma \in a\}$ and $\tau \in (a \star v)/_{\star}\, v$ iff $\tau(x) = \sigma(x)$, for some $\sigma \in a$. For such τ and for every variable z, we have $\tau\kappa_x(z) = \sigma(x) = \sigma\kappa_x(z)$, for some $\sigma \in a$, therefore $[(a \star v)/_{\star}\, v]u = au$.

For the module $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$, we can take $v = \{x \approx y\}$ and $u = \{\kappa_{x\approx y}\}$, where x, y are distinct variables. Here we assume that we have partitioned the set of all variables in two disjoint sets V_x, V_y with $x \in V_x$ and $y \in V_y$, and that $\kappa_{x\approx y}$ is the substitution that sends all of V_x to x and all of V_y to y. The verification of property (1) of Theorem 5.7 is straightforward. $\square$

5.3. **Coproducts.** The preceding results do not cover the case of the $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module of sequents, as we show in the following proposition. Even though this module is not cyclic, we prove that it is a coproduct of cyclic projective modules, and hence it is projective.

Proposition 5.10. *The $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module $\mathcal{P}(\mathbf{Seq})$ of sequents is not cyclic, for every set of sequents with more than one type.*

Proof. By way of contradiction assume that a set v of sequents is a generator of $\mathcal{P}(\mathbf{Seq})$. As the application of a substitution to a sequent does not change its type, it is easy to see that for every set a of substitutions, v contains a sequent of a given type iff $a \star v$ contains a sequent of the same type. Now, if v omits a given type, then $a \star v$ will omit the same type, for all a, a contradiction as v was assumed to be a generator of $\mathcal{P}(\mathbf{Seq})$. Likewise, if v contains sequents from all types, then so does $a \star v$, for all a, also a contradiction since there are sets in $\mathcal{P}(\mathbf{Seq})$ that omit certain sequent types. $\square$

We start by defining coproducts in the category of $\mathbf{A}$-modules. Let $(\mathbf{P}_i | i \in I)$ be a family of $\mathbf{A}$-modules. The *coproduct* of this family is an $\mathbf{A}$-module $\mathbf{P}$, denoted by $\coprod_{i\in I} \mathbf{P}_i$, together with a family of injective morphisms $(\sigma_i : \mathbf{P}_i \to \mathbf{P} \mid i \in I)$ such that for every $\mathbf{A}$-module $\mathbf{Q}$ and every family of morphisms $(\tau_i : \mathbf{P}_i \to \mathbf{Q} \mid i \in I)$, there exists a unique morphism $\tau : \mathbf{P} \to \mathbf{Q}$ such that $\tau\sigma_i = \tau_i$.

We remark that if the coproduct of a family $(\mathbf{P}_i \mid i \in I)$ of $\mathbf{A}$-modules exists, then the associated module morphisms σ_i are injective, and $\bigcup_{i\in I} \sigma_i(P_i)$ generates $\mathbf{P}$ as an $\mathbf{A}$-module.

It is clear that whenever the coproduct of a family of $\mathbf{A}$-modules exists, it is unique up to isomorphism. The next result guarantees that it always exists.

Lemma 5.11. *Let $(\mathbf{P}_i \mid i \in I)$ be a family of $\mathbf{A}$-modules. The $\mathbf{A}$-module $\coprod_{i\in I} \mathbf{P}_i$ in the definition of coproduct is the direct product $\prod_{i\in I} \mathbf{P}_i$ (with scalar multiplication defined component-wise). The associated injective module morphisms $\sigma_i : \mathbf{P}_i \to \prod_{i\in I} \mathbf{P}_i$ are defined, for each $i \in I$, by $\sigma_i(p) = (x_j)_{j\in I}$, where $x_i = p$ and $x_j = \bot$, if $j \neq i$.*

Proof. Note that the maps $\sigma_i : \mathbf{P}_i \to \prod_{i\in I} \mathbf{P}_i$ are module morphisms. If $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ are module morphisms, then the map $\tau : \prod_{i\in I} P_i \to Q$, defined by $\tau((x_i)_{i\in I}) = \bigvee \tau_i(x_i)$, is residuated and its residual is $\tau_*(y) = ((\tau_i)_*(y))_{i\in I}$. It also preserves scalar multiplication, and hence it is a module morphism. $\square$

The following standard categorical result shows why we are interested in coproducts.

Lemma 5.12. *The coproduct of a family of projective $\mathbf{A}$-modules is a projective $\mathbf{A}$-module.*

Proof. Assume that $(\mathbf{P}_i \mid i \in I)$ is a family of projective $\mathbf{A}$-modules, let $\mathbf{Q}, \mathbf{R}$ be $\mathbf{A}$-modules, and let $g : \mathbf{Q} \to \mathbf{R}$, $k : \coprod_{i\in I} \mathbf{P}_i \to \mathbf{R}$ be module morphisms such that g is onto. Let $\sigma_i : \mathbf{P}_i \to \coprod_{i\in I} \mathbf{P}_i$ be the injective module morphisms associated with the coproduct. Set $k_i = k\sigma_i$. Since each $\mathbf{P}_i$ is projective, there exists a module morphisms $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ such that $k_i = g\tau_i$. It follows that there exists a module morphism $\tau : \coprod_{i\in I} \mathbf{P}_i \to \mathbf{Q}$

such that $\tau_i = \tau I_i$.

$$\begin{tikzcd}
 & & \mathbf{Q} \arrow[dd, two heads, "g"] \\
\coprod_{i\in I} \mathbf{P}_i \arrow[urr, dashed, "\tau"] \arrow[drr, "k"'] & \mathbf{P}_i \arrow[l, tail] \arrow[ur, dashed, "\tau_i"'] \arrow[dr, "k_i"] & \\
 & & \mathbf{R}
\end{tikzcd}$$

Consequently, $k\sigma_i = k_i = g\tau_i = g\tau\sigma_i$ for all $i \in I$. Since for each $i \in I$ both of these morphisms are from $\mathbf{P}_i$ to $\mathbf{R}$, by the definition of the coproduct, there exists a unique morphism from $\coprod_{i\in I} \mathbf{P}_i$ to $\mathbf{R}$ such that these morphisms factor through $\mathbf{P}_i$. Since both k and $g\tau$ serve this purpose, they are equal. □

One can define different kinds of sequents. We saw single conclusion, associative commutative sequents in Example 2.6 and we discussed multiple conclusion, associative sequents. For non-associative sequents see [11], hypersequents see [1], and multi-sequents see [7]. The powersets of all these will be shown to be coproducts of cyclic projective modules.

Inspired by Pynko [13], given an algebraic language $\mathcal{L}$ and a set $\mathcal{P}$ of predicate symbols, we consider atomic formulas in the language $\mathcal{L} \cup \mathcal{P}$ and we call them *$\mathcal{LP}$-sequents*. As an example, we mention that to represent associative (multiple conclusion) sequents, for every pair (m, n) of not simultaneously zero natural numbers, we introduce a $(m + n)$-ary predicate symbol $P_{(m,n)}$. Then, $P_{(m,n)}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$ is defined as the sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$, where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ are $\mathcal{L}$-terms.

For every predicate symbol P in $\mathcal{P}$ of arity n, and a substitution σ on the terms algebra over $\mathcal{L}$, we define $\sigma(P(x_1, \dots, x_n)) = P(\sigma(x_1), \dots, \sigma(x_n))$. If $Seq_{\mathcal{LP}}$ denotes the set of the above general sequents, then clearly, $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ is a $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module.

Theorem 5.13. *The $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ is a coproduct of cyclic projective modules. Consequently it is projective.*

Proof. As in the proof of Corollary 5.9, for every such atomic formula $P(x_1, \dots, x_n)$, we chose a partition $V_{x_1}, \dots, V_{x_n}$ of the set of variables, with $x_i \in V_{x_i}$, and let $\kappa_{(x_1,\dots,x_n)}$ be the substitution that sends all of V_{x_i} to x_i, for all i. The elements u_P are the singletons containing the substitutions $\kappa_{(x_1,\dots,x_n)}$.

It is easy to see that each v_P generates a cyclic $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module $\mathcal{P}(\mathbf{P}_P)$ that is also projective, by verifying property (1) of Theorem 5.7. Moreover, the powerset of all the sequents $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ becomes the coproduct of these modules. This is simply because $\mathbf{Seq}_{\mathcal{LP}} = \bigcup_{P\in\mathcal{P}} \mathbf{P}_P$, hence $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}}) = \mathcal{P}(\bigcup_{P\in\mathcal{P}} \mathbf{P}_P)$, which is isomorphic to $\prod_{P\in\mathcal{P}} \mathcal{P}(\mathbf{P}_P)$. In light of Lemma 5.11, the latter – together with the associated injections – is the coproduct of the family $(\mathcal{P}(\mathbf{P}_P) : P \in \mathcal{P})$. □

6. Finitary Translators

In the last section of the paper, we identify conditions under which an equivalence of finitary consequence relations is induced by finitary translators. We start with the definitions of the pertinent notions.

Recall that, given a complete lattice $\mathbf{P}$, a subset X of P is called *(upward) directed* in $\mathbf{P}$, if for all $x, y \in X$, there exists a $z \in X$ such that $x, y \leq z$. An element x of a complete lattice $\mathbf{P}$ is called *compact*, if, for all directed $Y \subseteq P$, $x \leq \bigvee Y$ implies $x \leq y$, for some $y \in Y$. Equivalently, x is compact if for all $Z \subseteq P$ if $x \leq \bigvee Z$, then there is a finite subset Z_0 of Z such that $x \leq \bigvee Z_0$. For every subset Q of P, we denote by $K_{\mathbf{P}}(Q)$ the set of compact elements of $\mathbf{P}$ that are contained in Q. We write $K_{\mathbf{P}}$ for $K_{\mathbf{P}}(P)$. By a *finitary lattice* we understand a complete lattice in which every element is a join of compact elements; in particular, $x = \bigvee K_{\mathbf{P}}(\downarrow x)$, for all $x \in P$. Note that $K_{\mathbf{P}}$ and $K_{\mathbf{P}}(\downarrow x)$ are directed sets, as the finite join of compact elements is compact.

A consequence relation on a finitary lattice $\mathbf{P}$ is called *finitary*, if for all $x, y \in P$, if $x \vdash y$ and y is compact, then there exists a compact element $x_0 \in P$ such that $x_0 \leq x$ and $x_0 \vdash y$. As the compact elements of the powerset $\mathcal{P}(S)$ are exactly the finite subsets of S, the notion of a finitary consequence relation generalizes the one defined for powersets. A closure operator γ on a finitary lattice $\mathbf{P}$ is called *finitary*, if the corresponding consequence relation $\vdash_\gamma$, given in Lemma 3.5, is finitary. In other words, γ is finitary if for all $x, y \in P$, whenever $y \leq \gamma(x)$ and y is compact, there exists a compact element $x_0 \leq x$ such that $y \leq \gamma(x_0)$.

It should be noted that our choice of the terms "finitary closure operator" and "finitary lattice" is dictated by other uses of "algebraic" in this area. The most commonly used terms for these concepts in universal algebra are *algebraic closure operator* and *algebraic lattice*, respectively.

Lemma 6.1. *Let γ be a closure operator on a finitary lattice $\mathbf{P}$. For every compact element y of $\mathbf{P}_\gamma$, there exists a compact element x of $\mathbf{P}$ such that $y = \gamma(x)$. Therefore, $K_{\mathbf{P}_\gamma} \subseteq \gamma[K_{\mathbf{P}}]$.*

Proof. Let y be a compact element of $\mathbf{P}_\gamma$. Then $y = \gamma(z)$ for some $z \in P$ and $\gamma(z) = \gamma(\bigvee^{\mathbf{P}} K_{\mathbf{P}}(\downarrow z)) = \bigvee^{\mathbf{P}_\gamma} \gamma[K_{\mathbf{P}}(\downarrow z)]$. Since y is compact in $\mathbf{P}_\gamma$ and $y \leq \bigvee^{\mathbf{P}_\gamma} \gamma[K_{\mathbf{P}}(\downarrow z)]$, we have $y \leq \gamma(x)$, for some $x \in K_{\mathbf{P}}(\downarrow z)$. Thus, $y \leq \gamma(x) \leq \bigvee^{\mathbf{P}_\gamma} \gamma[K_{\mathbf{P}}(\downarrow z)] = y$, and $y = \gamma(x)$. □

Lemma 6.2. *Let $\mathbf{P}$ be a finitary lattice and γ a closure operator on $\mathbf{P}$. The following statements are equivalent.*

(1) *γ is finitary.*
(2) *γ preserves directed joins. That is, for every directed $X \subseteq P$, $\gamma(\bigvee^{\mathbf{P}} X) = \bigvee^{\mathbf{P}} \gamma[X]$.*
(3) *Arbitrary directed joins in $\mathbf{P}_\gamma$ coincide with the corresponding joins in $\mathbf{P}$. That is, $\bigvee^{\mathbf{P}_\gamma} Y = \bigvee^{\mathbf{P}} Y$, for every directed $Y \subseteq P_\gamma$.*

(4) $\gamma(x) = \bigvee^{\mathbf{P}} \gamma[K_{\mathbf{P}}(\downarrow x)]$, *for all* $x \in P$.
(5) *For every compact element* x *of* $\mathbf{P}$, $\gamma(x)$ *is compact in* $\mathbf{P}_\gamma$.
(6) $K_{\mathbf{P}_\gamma} = \gamma[K_{\mathbf{P}}]$.

If the above conditions hold, then $\mathbf{P}_\gamma$ *is finitary.*

Proof. To show that (1) $\Rightarrow$ (2), let X be a directed subset of P. Since γ is finitary, for every compact element y with $y \leq \gamma(\bigvee^{\mathbf{P}} X)$, there exists a compact element $x_0 \leq \bigvee^{\mathbf{P}} X$, such that $y \leq \gamma(x_0)$. Since $x_0 \leq \bigvee^{\mathbf{P}} X$, X is directed and x_0 is compact, there exists $x \in X$ such that $x_0 \leq x$. So, $y \leq \gamma(x_0) \leq \gamma(x) \leq \bigvee^{\mathbf{P}} \gamma[X]$. Therefore, $y \leq \bigvee^{\mathbf{P}} \gamma[X]$, for all compact elements $y \leq \gamma(\bigvee^{\mathbf{P}} X)$. As $\mathbf{P}$ is finitary, we have $\gamma(\bigvee^{\mathbf{P}} X) \leq \bigvee^{\mathbf{P}} \gamma[X]$.

(2) $\Rightarrow$ (3) is obvious.

For (3) $\Rightarrow$ (4), we have $x = \bigvee^{\mathbf{P}} K_{\mathbf{P}}(\downarrow x)$, as $\mathbf{P}$ is finitary, so $\gamma(x) = \gamma(\bigvee^{\mathbf{P}} K_{\mathbf{P}}(\downarrow x)) = \bigvee^{\mathbf{P}_\gamma} \gamma[K_{\mathbf{P}}(\downarrow x))] = \bigvee^{\mathbf{P}} \gamma[K_{\mathbf{P}}(\downarrow x))]$, by the formula for $\bigvee^{\mathbf{P}_\gamma}$ given in Lemma 3.3 and the assumption (3).

For (4) $\Rightarrow$ (1), let y be compact with $y \leq \gamma(x)$, for some $x \in P$. Then, $y \leq \bigvee^{\mathbf{P}} \gamma[K_{\mathbf{P}}(\downarrow x)]$, so $y \leq \gamma(x_0)$, for some $x_0 \in K_{\mathbf{P}}(\downarrow x)$, i.e., for some compact x_0 with $x_0 \leq x$.

For (3) $\Rightarrow$ (5), assume that x is a compact element of $\mathbf{P}$ and let $\gamma(x) \leq \bigvee^{\mathbf{P}_\gamma} Y$, for some directed subset Y of $\mathbf{P}_\gamma$. We have $x \leq \gamma(x) \leq \bigvee^{\mathbf{P}_\gamma} Y = \bigvee^{\mathbf{P}} Y$, by (3). Since x is compact, there is a $y \in Y$, such that $x \leq \gamma(y)$; hence, $\gamma(x) \leq \gamma(y)$. Consequently, $\gamma(x)$ is compact in $\mathbf{P}_\gamma$.

For (5) $\Rightarrow$ (1), let y be a compact element of $\mathbf{P}$ and $y \leq \gamma(x)$ for some $x \in P$. We have $\gamma(y) \leq \gamma(x) = \gamma(\bigvee^{\mathbf{P}} K_{\mathbf{P}}(\downarrow x)) = \bigvee^{\mathbf{P}_\gamma} \gamma[K_{\mathbf{P}}(\downarrow x)]$, by the fact that $\gamma : \mathbf{P} \to \mathbf{P}_\gamma$ preserves joins. Since y is compact in $\mathbf{P}$, $\gamma(y)$ is compact in $K_{\mathbf{P}_\gamma}$, so $y \leq \gamma(y) \leq \gamma(x_0)$ for some $x_0 \in K_{\mathbf{P}}$ with $x_0 \leq x$.

The equivalence of (5) and (6) holds because of Lemma 6.1. $\square$

We will make free use of the above equivalent statements for a given closure operator, without explicit reference to the lemma.

It is easy to see that the condition that $\mathbf{P}_\gamma$ is finitary is not enough to guarantee that γ is finitary. For example, let $\mathbf{N}^\infty$ be the poset of natural numbers, under the natural ordering, with an extra top element ∞. Then the closure operator γ on $\mathcal{P}(\mathbf{N}^\infty)$ that sends a set X to the downset $\downarrow(\bigvee X)$ has image $\mathcal{P}(\mathbf{N}^\infty)_\gamma$, consisting exactly of the empty set and the principal downsets of $\mathbf{N}^\infty$, which is a finitary lattice. However, γ is not finitary as it sends the compact element $\{\infty\}$ to the non-compact element $\downarrow \infty$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be finitary lattices. A residuated map $\tau : \mathbf{P} \to \mathbf{Q}$ is called *finitary*, if the image of every compact element is compact. The following corollary, which restates the equivalence of (1) and (5) of the preceding lemma, shows that the two definitions of finitarity coincide for a map viewed as a closure operator or as a residuated map.

Corollary 6.3. *Let γ be a closure operator on a finitary lattice $\mathbf{P}$. Then $\gamma : \mathbf{P} \to \mathbf{P}$ is finitary as a closure operator iff $\gamma : \mathbf{P} \to \mathbf{P}_\gamma$ is finitary as a residuated map.*

Lemma 6.4. *If $k : \mathbf{P} \to \mathbf{Q}$ is a finitary residuated map between finitary lattices, then k_*k is a finitary closure operator on $\mathbf{P}$.*

Proof. Given a directed subset Y of P_{k_*k}, we will show that $\bigvee^{\mathbf{P}_{k_*k}} Y \leq \bigvee^{\mathbf{P}} Y$. As $\mathbf{P}$ is finitary, it is enough to show that every compact element of $\mathbf{P}$ less or equal to $\bigvee^{\mathbf{P}_{k_*k}} Y$ is also less or equal to $\bigvee^{\mathbf{P}} Y$. Let $x \in K_{\mathbf{P}}$ with $x \leq \bigvee^{\mathbf{P}_{k_*k}} Y$. Then $k(x) \leq k(\bigvee^{\mathbf{P}_{k_*k}} Y) = k(k_*k(\bigvee^{\mathbf{P}} Y)) = k(\bigvee^{\mathbf{P}} Y) = \bigvee^{\mathbf{Q}} k[Y]$. As k is finitary and x is compact in $\mathbf{P}$, $k(x)$ is compact in $\mathbf{Q}$. Hence, since $k[Y]$ is directed, $k(x) \leq k(y)$, for some $y \in Y$. Consequently, $x \leq k_*k(x) \leq k_*k(y) = y \leq \bigvee^{\mathbf{P}_{k_*k}} Y$. □

Lemma 6.5. *Let $\mathbf{P}$ and $\mathbf{Q}$ be finitary lattices, $\tau : \mathbf{P} \to \mathbf{Q}$ a finitary residuated map, and δ a finitary closure operator on $\mathbf{Q}$.*

(1) *The closure operator $\delta^\tau = \tau_*\delta\tau : P \to P$ is finitary.*
(2) *The map $f = \delta\tau_{|\mathbf{P}_{\delta^\tau}} : \mathbf{P}_{\delta^\tau} \to \mathbf{Q}_\delta$ is finitary.*

Proof. (1) If $y \leq \delta^\tau(x)$, for some compact element y, then $y \leq \tau_*\delta\tau(x)$, so $\tau(y) \leq \delta\tau(x)$. Since τ is finitary and y is compact, $\tau(y)$ is compact. Furthermore, since δ is finitary, there is a compact element $x' \leq \tau(x)$ such that $\tau(y) \leq \delta(x')$. Since $\mathbf{P}$ is finitary, $x = \bigvee K_{\mathbf{P}}(\downarrow x)$, so $\tau(x) = \bigvee \tau[K_{\mathbf{P}}(\downarrow x)]$, by Lemma 3.1(1). Since $x' \leq \tau(x)$, there exists a compact element $x_0 \leq x$ such that $x' \leq \tau(x_0)$. Consequently, $\tau(y) \leq \delta\tau(x_0)$, hence $y \leq \tau_*\delta\tau(x_0) = \delta^\tau(x_0)$, for some compact element $x_0 \leq x$. Thus, δ^τ is finitary.

(2) Let x be a compact element of $\mathbf{P}_{\delta^\tau}$; we will show that $f(x)$ is compact in $\mathbf{Q}_\delta$. By Lemma 6.1, there exists a compact element y of $\mathbf{P}$ such that $x = \delta^\tau(y)$. By the finitarity of τ and δ, we have that $f(x) = f(\delta^\tau(y)) = \delta(\tau(y)$ is compact, in view of Lemma 6.2. □

A *finitary residuated lattice* is a finitary lattice in which the identity is a compact element, and the product of any two compact elements is compact.

A *finitary module* is an $\mathbf{A}$-module $\mathbf{P}$ such that (i) $\mathbf{A}$ is a finitary residuated lattice; (ii) $\mathbf{P}$ is a finitary lattice; and (iii) if a, v are compact elements of $\mathbf{A}$ and $\mathbf{P}$, respectively, then $a \star v$ is a compact element of $\mathbf{P}$.

For a fixed finitary residuated lattice $\mathbf{A}$, we will denote by ${}_{\mathbf{A}}\mathcal{FM}$ the category of finitary $\mathbf{A}$-modules and finitary module morphisms (finitary translators). Recall that such a morphism maps compact elements to compact elements.

Note that the notion of projectivity depends on the category ${}_{\mathbf{A}}\mathcal{M}$ or ${}_{\mathbf{A}}\mathcal{FM}$. Recall the definitions of the triangle (S) and the square (T) preceding Theorem 5.1. In view Corollary 6.3 finitary structural closure operators on finitary modules can be identified with morphisms in the category ${}_{\mathbf{A}}\mathcal{FM}$, so the square (T) makes sense. We verify the analogue of the Theorem 5.1 where (S) and (T) (projectivity) are considered in ${}_{\mathbf{A}}\mathcal{FM}$.

Theorem 6.6. *The objects* $\mathbf{P}$ *of the category* ${}_{\mathbf{A}}\mathcal{FM}$ *for which all squares of type (S) can be completed are exactly the projective objects of* ${}_{\mathbf{A}}\mathcal{FM}$*.*

Proof. We will show that the proof of Theorem 5.1 extends to the current setting. In particular, we assume that all objects and morphisms are finitary and show that the derived objects and morphisms are also finitary. In particular, k_*k is finitary, as a closure operator on $\mathbf{P}$, by Lemma 6.4, and as a module morphism $k_*k : \mathbf{P} \to \mathbf{P}_{k_*k}$ by Lemma 6.2. $\mathbf{P}_{k_*k}$ is finitary by Lemma 6.3. To see that k' is finitary, note that for $x \in K_{\mathbf{P}_{k_*k}}$, $k'(x) = k'(k_*k(x)) = k(k_*k(x)) = k(x)$, which is compact in $\mathbf{R}$. Likewise, we show that g_*g, g' and $\mathbf{Q}_{g_*g}$ are finitary. Finally, f is finitary, being the composition of two finitary maps. □

Corollary 6.7. *Suppose* $\mathbf{P}$ *is an object in* ${}_{\mathbf{A}}\mathcal{FM}$*, and* γ *a finitary structural operator on* $\mathbf{P}$*. Then* $\mathbf{P}_\gamma$ *is finitary as an* $\mathbf{A}$*-module.*

Proof. By Lemma 6.2, $\mathbf{P}_\gamma$ is finitary as a lattice. To show that it is a finitary module, we need to verify that scalar multiplication preserves compactness. Let $a \in K_{\mathbf{A}}$ and $\gamma(x) \in \mathbf{P}_\gamma$. By Lemma 6.1, $x \in \mathbf{P}$ can be taken to be compact. As $\mathbf{P}$ is finitary $a \star x$ is compact in $\mathbf{P}$. Also, since γ is finitary, $\gamma(a \star x) = a \star_\gamma \gamma(x)$ is compact in $\mathbf{P}_\gamma$. □

Recall that by Theorem 5.7 the cyclic projective in ${}_{\mathbf{A}}\mathcal{M}$ modules are up to isomorphism exactly the ones of the form $\mathbf{A}u$, where u is idempotent. If further u is compact in $\mathbf{A}$, we will refer to such a module as *regular*. Note that since the joins in $\mathbf{A}u$ coincide with the joins in $\mathbf{A}$ by Lemma 5.4, if u is compact in $\mathbf{A}$, then it is compact in $\mathbf{A}u$.

Lemma 6.8. *The* $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$*-modules* $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$ *and* $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ *are regular.*

Proof. It was noted in the proof of Corollary 5.9 that $u = \{\kappa_x\}$ for $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$ and $u = \{\kappa_{x\approx y}\}$ for $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$, both of which are finite, hence compact. □

Lemma 6.9. *If* u *is compact in* $\mathbf{A}$*, then the compact elements of* $\mathbf{A}u$ *are exactly of the form* au*, where* a *is compact in* $\mathbf{A}$*; in symbols* $K_{\mathbf{A}u} = K_{\mathbf{A}}u$*.*

Proof. Clearly, if a is compact in $\mathbf{A}$, then au is also compact in $\mathbf{A}$, and hence also compact in $\mathbf{A}u$. Conversely, let au be compact in $\mathbf{A}u$. Since $\mathbf{A}$ is finitary, $a = \bigvee^{\mathbf{A}} C$, where C is the set of compact elements of $\mathbf{A}$ below a. Thus, $au = \bigvee^{\mathbf{A}}\{cu : c \in C\} = \bigvee^{\mathbf{A}u}\{cu : c \in C\}$. Note that $\{cu : c \in C\}$ is a directed set of compact elements of $\mathbf{A}u$, as u is compact in $\mathbf{A}u$. Since au is compact in $\mathbf{A}u$, there exists $c \in C$ such that $au = cu$. □

Note that for the inclusion $K_{\mathbf{A}u} \subseteq K_{\mathbf{A}}u$ it suffices to assume that u is compact in $\mathbf{A}u$.

Corollary 6.10. *Every regular module is finitary.*

Proof. Every regular module is isomorphic to $\mathbf{A}u$, for u idempotent and compact in $\mathbf{A}$. An arbitrary element of Au is of the form au, where $a \in A$.

So, $au = (\bigvee K_{\mathbf{A}}(\downarrow a))u = (\bigvee K_{\mathbf{A}}(\downarrow a)u)$. Since $K_{\mathbf{A}}(\downarrow a) = K_{\mathbf{A}} \cap \downarrow a$, by Lemma 6.9 $K_{\mathbf{A}}(\downarrow a)u$ are compact elements in $\mathbf{A}u$. So every element of $\mathbf{A}u$ is a join of compact elements of $\mathbf{A}u$. $\square$

The following lemma shows that for cyclic objects $\mathbf{A}u$ in ${}_{\mathbf{A}}\mathcal{M}$ such that u is compact in $\mathbf{A}$, projectivity in ${}_{\mathbf{A}}\mathcal{M}$ implies projectivity in ${}_{\mathbf{A}}\mathcal{FM}$.

Lemma 6.11. *Regular* $\mathbf{A}$*-modules are projective in the category* ${}_{\mathbf{A}}\mathcal{FM}$.

Proof. We will show that if the $\mathbf{A}$-modules $\mathbf{P}, \mathbf{Q}, \mathbf{R}$, the module morphism $k : \mathbf{P} \to \mathbf{R}$, and the surjective module morphism $g : \mathbf{Q} \to \mathbf{R}$ are all finitary and if, further, $\mathbf{P}$ is regular, then there exists a finitary module morphism $h : \mathbf{P} \to \mathbf{Q}$ such that $gh = k$.

$$\begin{tikzcd} \mathbf{P} \arrow[r, dashed, "h"] \arrow[dr, "k"'] & \mathbf{Q} \arrow[d, two heads, "g"] \\ & \mathbf{R} \end{tikzcd}$$

In view of Theorem 5.7 and the definition of a regular module, we may assume that $\mathbf{P} = \mathbf{A}u$, where u is an idempotent element of $\mathbf{A}$ that is compact in $\mathbf{A}$, and hence in $\mathbf{A}u$. Consider the element $y = k(u)$ of $\mathbf{R}$. It is clear that y is a compact element of $\mathbf{R}$. We claim that there exists compact w in $\mathbf{Q}$ such that $y = g(w)$. Indeed there exists x in $\mathbf{Q}$ such that $y = g(x)$. Now, $x = \bigvee^{\mathbf{Q}} X$, for some set X of compact elements of $\mathbf{Q}$, and so $g(x) = \bigvee^{\mathbf{R}} g[X]$. By the compactness of y in $\mathbf{R}$, there exists a finite subset Y of X such that $\delta(x) = \bigvee^{\mathbf{R}} g[Y]$. But then, if w denotes the compact element $\bigvee^{\mathbf{Q}} Y$ in $\mathbf{Q}$, we get $y = g(w)$, as was to be shown. Let $z = u \star_{\mathbf{Q}} w$. Then z is a compact element of $\mathbf{Q}$. We claim that the map $\tau_z : P \to Q$, defined by $au \mapsto a \star_{\mathbf{Q}} z$, is a finitary module morphism from $\mathbf{P}$ to $\mathbf{Q}$ such that $g\tau_z = k$.

We first note that τ_z is a well-defined map. Indeed, suppose that $au = bu$, for $a, b \in A$. Then $a \star_{\mathbf{Q}} z = a \star_{\mathbf{Q}} (u \star_{\mathbf{Q}} w) = (au \star_{\mathbf{Q}} w) = (bu \star_{\mathbf{Q}} w) = \ldots = b \star_{\mathbf{Q}} z$. We next show that τ_z is residuated. We have for all $a \in A$ and $q \in Q$, $\tau_z(au) \leq q \Rightarrow a \star_{\mathbf{Q}} z \leq q \Rightarrow a \leq q/_{\mathbf{Q}} z \Rightarrow au \leq (q/_{\mathbf{Q}} z)u \Rightarrow (au) \star_{\mathbf{Q}} z \leq ((q/_{\mathbf{Q}} z)u) \star_{\mathbf{Q}} z \Rightarrow a \star_{\mathbf{Q}} z \leq (q/_{\mathbf{Q}} z) \star_{\mathbf{Q}} z \Rightarrow a \star_{\mathbf{Q}} z \leq q$. We have shown that $\tau_z(au) \leq q$ iff $au \leq (q/_{\mathbf{Q}} z)u$. Thus, τ_z is residuated and its residual is the map $(\tau_z)_* : \mathbf{Q} \to \mathbf{P}$, defined by $(\tau_z)_*(q) = (q/_{\mathbf{Q}} z)u$. To prove that τ_z is a module morphism, consider $a, b \in A$. We have, $a\tau_z(bu) = a \star_{\mathbf{Q}} (b \star_{\mathbf{Q}} z) = (ab) \star_{\mathbf{Q}} z = \tau_z(a(bu))$.

It remains to verify that τ_z is finitary. In view of Lemma 6.9, for every compact element cu of $\mathbf{A}u$ we can take c to be compact in $\mathbf{A}$. Then $\tau_z(cu) = c \star_{\mathbf{Q}} z$, which is a compact element of $\mathbf{Q}$, since $\mathbf{Q}$ is finitary, c is a compact element of $\mathbf{A}$ and z is a compact element of $\mathbf{Q}$. $\square$

In view of Remark 3.10, the following corollary implies Theorem 2.4.

Corollary 6.12. *Every finitary representation (hence also every isomorphism) between finitary consequence relations on the sets* $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$ *and* $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ *is induced by a finitary translator.*

We will not need the following result, but we state it since it is interesting and relevant to our discussion, as it provides an insight to the nature of module morphisms. Moreover, it builds on notions developed in [3]. Its proof follows ideas similar to the ones in the proof of the Lemma 6.11. Let $\mathbf{Q}$ be an $\mathbf{A}$-module and $a \in A$. An element y of Q is called *a-invariant*, if $a \star y = y$.

Theorem 6.13. *Assume that the* $\mathbf{A}$*-module* $\mathbf{P}$ *is cyclic projective with respect to the elements* v *and* u*, and that* $\mathbf{Q}$ *is also an* $\mathbf{A}$*-module. Then, there is a bijection between module morphisms* τ *from* $\mathbf{P}$ *to* $\mathbf{Q}$ *and* u*-invariant elements* y *of* Q*, given by* $\tau \mapsto \tau(v)$ *and* $y \mapsto \tau_y$*, where* $\tau_y(x) = (x /_{\star} v) \star y$.

Lemma 6.14. *Let* $\mathbf{P}_i$ *be finitary lattices, for all* $i \in I$*. An element of* $\prod_{i \in I} \mathbf{P}_i$ *is compact iff it has finitely many non-zero coordinates and those are occupied by compact elements of the corresponding factors.*

Proof. Let x_i be a compact element of $\mathbf{P}_i$, for some $i \in I$, and let $\bar{x_i}$ be the element of $\mathbf{P}$ with i-th coordinate equal to x_i and all other coordinates equal to $\perp$. Clearly, $\bar{x_i}$ is compact in $\mathbf{P}$, as any directed join exceeding it contains elements with all but the i-th coordinate equal to $\perp$. The directed join in $\mathbf{P}_i$ of the elements in the i-th coordinate exceed x_i, so one of them exceeds x_i. The corresponding element of $\mathbf{P}$ exceeds $\bar{x_i}$. Since the finite join of compact elements is also compact, we have one direction of the lemma.

Conversely, assume that $x = (x_i)_{i \in I}$ is a compact element of $\mathbf{P}$. Clearly, $x = \bigvee_{i \in I} \bar{x_i}$, so there is a finite subset I_0 of I such that $x = \bigvee_{i \in I_0} \bar{x_i}$. □

We are now ready to prove the main result of this section.

Theorem 6.15. *The coproduct in* ${}_{\mathbf{A}}\mathcal{M}$ *of a family of regular* $\mathbf{A}$*-modules is projective in* ${}_{\mathbf{A}}\mathcal{FM}$.

Proof. We will show that if $\mathbf{P}$ is the coproduct of a family of regular $\mathbf{A}$-modules, $\mathbf{Q}$ an $\mathbf{A}$-module, γ a structural closure operator on $\mathbf{P}$, δ a finitary structural closure operator on $\mathbf{Q}$ and f a finitary representation of γ in δ, then f is induced by a finitary module morphism $\tau : \mathbf{P} \to \mathbf{Q}$.

For each $i \in I$, let $\sigma_i : \mathbf{P}_i \to \mathbf{P}$ be the injective module morphism associated with the coproduct.

$$
\begin{tikzcd}
\mathbf{P}_i \arrow[d, tail, "\sigma_i"'] \arrow[dr, dashed, "\tau_i"] & \\
\mathbf{P} \arrow[r, dashed, "\tau"] \arrow[d, "\gamma"'] & \mathbf{Q} \arrow[d, "\delta"] \\
\mathbf{P}_{\gamma} \arrow[r, tail, "f"] & \mathbf{Q}_{\delta}
\end{tikzcd}
$$

Note that the map $f\gamma\sigma_i$ is a finitary module morphism, and hence Lemma 6.11 implies that there exists a finitary module morphism $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ such that $f\gamma\sigma_i = \delta\tau_i$. Now by the universal property of the coproduct, there exists $\tau : \mathbf{P} \to \mathbf{Q}$ such that $\tau\sigma_i = \tau_i$, for all $i \in I$.

To show that τ is finitary, let $x = (x_i)_{i\in I}$ be a compact element of $\mathbf{P}$. By Lemma 6.14, there is a finite subset I_0 of I such that $x_j = \bot$ for all $j \notin I_0$, and x_i is compact in $\mathbf{P}_i$, for all $i \in I_0$. Since τ_i is finitary, $\tau_i(x_i)$ is compact in $\mathbf{Q}$, for $i \in I_0$. Also, $\tau_j(x_j) = \tau_j(\bot) = \bot$, for $j \notin I_0$. Therefore, by Lemma 5.11 we have, $\tau((x_i)_{i\in I}) = \bigvee_{i\in I} \tau_i(x_i) = \bigvee_{i\in I_0} \tau_i(x_i)$, which is compact, being a finite join of compact elements. □

Corollary 6.16. *Let $\mathbf{P}$, $\mathbf{Q}$ be each a coproduct in ${}_{\mathbf{A}}\mathcal{M}$ of regular $\mathbf{A}$-modules, and let γ, δ be finitary structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$, respectively. Then every equivalence between γ and δ is induced by finitary module morphism.*

In view of Theorem 5.13, Corollary 6.8 and Corollary 6.16, we have the following result.

Corollary 6.17. *Every finitary representation (hence also every isomorphism) between consequence relations on the $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}\mathcal{P}_1})$ and $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}\mathcal{P}_2})$ is induced by a finitary translator, where $\mathcal{L}$ is any algebraic language and $\mathcal{P}_1$ and $\mathcal{P}_2$ are any predicate-only languages.*

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