AMALGAMATION AND INTERPOLATION IN ORDERED ALGEBRAS

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ABSTRACT. The first part of this paper provides a comprehensive and self-contained account of the inter-relationships between algebraic properties of varieties and properties of their free algebras and equational consequence relations. In particular, proofs are given of known equivalences between the amalgamation property and the Robinson property, the congruence extension property and the extension property, and the flat amalgamation property and the deductive interpolation property, as well as various dependencies between these properties. These relationships are then exploited in the second part of the paper in order to provide new proofs of amalgamation and deductive interpolation for the varieties of lattice-ordered abelian groups and MV-algebras, and to determine important subvarieties of residuated lattices where these properties hold or fail. In particular, a full description is provided of all subvarieties of commutative GMV-algebras possessing the amalgamation property.

1. INTRODUCTION

In Universal Algebra, a crucial and often extremely fruitful role is played by the fact that certain properties of a variety are "mirrored" in corresponding properties of their free algebras. In some cases, properties of free algebras may themselves be expressed as properties of associated equational consequence relations for the variety. The synthesis of these characterizations then typically provides an illuminating and potentially very useful "bridge" between the realms of algebra and logic.

A fundamental example of such a bridge is the relationship between the algebraic (or model-theoretic) property of amalgamation and the logical (or syntactic) property of interpolation. In this case, the amalgamation property for a variety is equivalent to its consequence relations possessing the Robinson property, which is equivalent in turn to a property of free algebras. These properties each imply the deductive interpolation property,

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which itself corresponds to an important property of free products: the flat amalgamation property. Conversely, the deductive interpolation property implies the amalgamation property in the presence of the congruence extension property or its syntactic equivalent, the extension property.

Relationships between these and other amalgamation, extension, and interpolation properties have already received considerable attention in the literature. Publications of particular relevance to our discussion include Bacsish [4], Czelakowski and Pigozzi [16], Galatos and Ono [24], Kihara and Ono [43, 44], Madarasz [45], Maksimova [46–48], Montagna [52], Pierce [58], Pigozzi [59], Powell and Tsinakis [60–63], and Wroński [70, 71]. However, we will defer more precise historical and bibliographical details to the appropriate points in the text.

Our goal in the first part of this paper is to provide a comprehensive and self-contained presentation in a universal algebra setting of the most important inter-relationships existing between amalgamation, interpolation, and extension properties. In contrast to the many other authors to have tackled this topic – in particular, the more general abstract algebraic logic approach of [16] – we focus for clarity of exposition on varieties of algebras and make use only of quite basic concepts from universal algebra in developing our account. The result is a more direct and accessible (but of course more restricted in scope) presentation of these topics. A further particular novelty of our approach is that we emphasize the fundamental role played by the equational consequence relation of a variety on a fixed countably infinite set of generators, thereby obtaining equivalent formulations of algebraic properties restricted to countable algebras.

The broad goal of the second part of the paper is to make use of the aforementioned relationships in order to investigate amalgamation and interpolation properties for specific varieties of ordered algebras. We first provide new "syntactic proofs" of the amalgamation property for abelian lattice-ordered groups and MV-algebras. We then turn our attention to varieties of residuated lattices, a framework that provides algebraic semantics for substructural logics as well as covering other important classes of algebras such as lattice-ordered groups. We study in some depth the amalgamation property for subvarieties of GBL-algebras; in particular, we provide a full description of all subvarieties of commutative GMV-algebras that have the amalgamation property.

Let us now be more specific about the structure and results of this paper. First, in Section 2, we recall some necessary background from universal algebra and examine the relationship between free algebras and equational consequence relations. In Section 3, we investigate the relationship between amalgamation and the Robinson property. In particular, (i) a criterion is given for a variety to have the amalgamation property (Theorem 3.2); (ii) it is shown that a variety has the amalgamation property if and only if (henceforth, iff) it has the Robinson property (Theorem 3.6). Similarly, in Section 4, it is shown that a variety has the congruence extension property iff its equational consequence relations has the extension property (Theorem 4.6). Section 5 is then devoted to interpolation properties. We show that the amalgamation property both implies the deductive interpolation property, and is implied by the conjunction of this property and the extension property (Theorem 5.1). We then establish the equivalence of the deductive interpolation property with the flat amalgamation property, a property of free products (Theorem 5.2), and a stronger version of the deductive interpolation property with the weak amalgamation property (Theorem 5.4). Finally, we show that the conjunction of the amalgamation property and the congruence extension property corresponds both to the Maehara interpolation property and to the transferable injections property (Theorem 5.8).

In Section 6 we make use of the results of the previous sections to obtain new syntactic proofs of the generation of the class of lattice-ordered abelian groups as a quasivariety by the integers (Theorem 6.3) and the deductive interpolation and amalgamation properties for this class (Theorem 6.4). We obtain, similarly, a new syntactic proof of the deductive interpolation and amalgamation properties for the variety of MV-algebras (Theorem 6.8). In Section 7 we introduce the class of residuated lattices. We show that a variety of semilinear residuated lattices satisfying the congruence extension property has the amalgamation property iff the class of its totally ordered members has the amalgamation property (Theorem 7.9). We also investigate the connection between the amalgamation property for a class of bounded residuated lattices and the class of its residuated lattice reducts (Theorem 7.10), and amalgamation in the join of two independent varieties of residuated lattices (Theorem 7.12). Finally, in Section 8, we make an in-depth study of amalgamation in classes of GBL-algebras. We provide a complete characterization of varieties of commutative GMV-algebras with the amalgamation property (Theorem 8.11), and determine whether amalgamation holds or fails for various classes of commutative GBL-algebras and n-potent GBL-algebras (Theorems 8.14, 8.16, 8.17, 8.23, and 8.24).

2. Equational Consequence Relations and Free Algebras

Our main goal in this preliminary section will be to relate the equational consequence relations of a variety of algebras to properties of the free algebras of the variety. To this end, let us first fix some terminology from universal algebra, referring to [9], [29], or [51] for all undefined notions.

Throughout this paper, we will understand a *signature* of algebras to be a pair $\mathcal{L} = \langle L, \tau \rangle$ consisting of a non-empty countable set L of *operation* *symbols* and a map $\tau\colon L \to \mathbb{N}$ where the image of an operation symbol under τ is called its *arity*. Nullary operation symbols will most often be referred to as *constant symbols* or simply as *constants*. We also fix for the whole paper a countably infinite set $\mathbb{X}$ of *variables*, denoting variables in general, which may belong to any set, by x, y, z.

The *formula (term) algebra* $\mathbf{Fm}(Y)$ for $\mathcal{L}$ over a set of variables Y exists if either $Y \neq \emptyset$ or $\mathcal{L}$ has a constant, and we call its members, denoted by α, β, *formulas*. We also let $\mathrm{Eq}(Y)$ be the set of ordered pairs of formulas from $\mathrm{Fm(Y)}$, called *equations*, written either (α, β) or $\alpha \approx \beta$ and denoted by ε, δ. Sets of equations will be denoted by Σ, Π, Δ. The variables occurring in a formula, equation, or set of equations S, is denoted by $\mathrm{Var}(S)$.

Given a variety $\mathcal{V}$ of algebras, we denote the *free algebra of* $\mathcal{V}$ *on a set* Z *of free generators* by $\mathbf{F}_{\mathcal{V}}(Z)$ or simply $\mathbf{F}(Z)$. Considering the *natural map* $h^Z_{\mathcal{V}}\colon \mathbf{Fm}(Z) \to \mathbf{F}_{\mathcal{V}}(Z)$, we write $\bar{\alpha}$ for $h^Z_{\mathcal{V}}(\alpha)$ for each $\alpha \in \mathrm{Fm(Z)}$. Similarly, we write $\bar{\varepsilon}$ for $\varepsilon = (\bar{\alpha}, \bar{\beta})$ with $(\alpha, \beta) \in \mathrm{Eq}(Z)$ and $\bar{\Sigma} = \{\bar{\varepsilon} \mid \varepsilon \in \Sigma\}$ for $\Sigma \subseteq \mathrm{Eq}(Z)$. We also define $\Theta^Z_{\mathcal{V}} = \ker h^Z_{\mathcal{V}}$.

We denote the *congruence lattice* of an algebra $\mathbf{A}$ by $\mathrm{Con}(\mathbf{A})$ and for $R \subseteq A^2$, we write $\mathrm{Cg}_{\mathbf{A}}(R)$ to denote the congruence relation on $\mathbf{A}$ generated by R, abbreviating to $\mathrm{Cg}_{\mathbf{A}}(a, b)$ for the principal congruence on $\mathbf{A}$ generated by a pair $(a, b) \in A^2$. For $\Theta \in \mathrm{Con}(\mathbf{A})$ and $a \in A$, we denote the equivalence class of a relative to Θ by $[a]_\Theta$ or simply $[a]$.

Let us now begin by noting the following useful result:

Lemma 2.1. *For any surjective homomorphism* $\varphi\colon \mathbf{A} \to \mathbf{B}$ *and* $R \subseteq A^2$,

$$\varphi^{-1}[\mathrm{Cg}_{\mathbf{B}}(\varphi[R])] = \mathrm{Cg}_{\mathbf{A}}(R) \vee \ker(\varphi),$$

where the join on the right-hand side of the equality takes place in $\mathrm{Con}(\mathbf{A})$.

Proof. It is clear that $\varphi^{-1}[\mathrm{Cg}_{\mathbf{B}}(\varphi[R])] \supseteq \mathrm{Cg}_{\mathbf{A}}(R) \vee \ker(\varphi)$. To prove the reverse inclusion, set $\Theta = \varphi[\mathrm{Cg}_{\mathbf{A}}(R) \vee \ker(\varphi)]$. Note that $\varphi[R] \subseteq \Theta \in \mathrm{Con}(\mathbf{B})$. So also $\mathrm{Cg}_{\mathbf{B}}(\varphi[R]) \subseteq \Theta$ and it follows that $\varphi^{-1}[\mathrm{Cg}_{\mathbf{B}}(\varphi[R])] \subseteq \varphi^{-1}[\Theta] = \mathrm{Cg}_{\mathbf{A}}(R) \vee \ker(\varphi)$. ☐

A crucial role will be played in subsequent sections of this paper by the fact that properties of the free algebras of a variety may be reflected in properties of the corresponding equational consequence relations of the variety; in particular, we may focus on properties of the equational consequence relation for the countably infinite set $\mathbb{X}$.

Let $\mathcal{K}$ be a class of algebras of the same signature and Y an arbitrary set of variables. For any $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$, we fix

$$\Sigma \models^Y_{\mathcal{K}} \varepsilon \quad \Leftrightarrow \quad \text{for all } \mathbf{A} \in \mathcal{K} \text{ and } \varphi \in \hom(\mathbf{Fm}(Y), \mathbf{A}),$$
$$\Sigma \subseteq \ker(\varphi) \quad \text{implies} \quad \varepsilon \in \ker(\varphi).$$

For any $\Sigma \cup \Delta \subseteq \mathrm{Eq}(Y)$, we also write $\Sigma \models^Y_{\mathcal{K}} \Delta$ to denote that $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ for all $\varepsilon \in \Delta$. We also drop the brackets and write just $\Sigma \models^Y_{\mathbf{A}} \varepsilon$ when $\mathcal{K}$ consists of just one algebra $\mathbf{A}$.

It follows that $\models^Y_{\mathcal{K}}$ is a "substitution-invariant consequence relation" in the sense that it satisfies for all $\Sigma \cup \Pi \cup \{\varepsilon, \delta\} \subseteq \mathrm{Eq}(Y)$:

(i) $\{\varepsilon\} \models^Y_{\mathcal{K}} \varepsilon$ (reflexivity);
(ii) $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ implies $\Sigma \cup \Pi \models^Y_{\mathcal{K}} \varepsilon$ (monotonicity);
(iii) $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ and $\Sigma \cup \{\varepsilon\} \models^Y_{\mathcal{K}} \delta$ implies $\Sigma \models^Y_{\mathcal{K}} \delta$ (transitivity);
(iv) $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ implies $\sigma(\Sigma) \models^Y_{\mathcal{K}} \sigma(\varepsilon)$ for any homomorphism $\sigma\colon \mathbf{Fm}(Y) \to \mathbf{Fm}(Y)$ (substitution-invariance).

Moreover, if $\mathcal{K}$ is a variety, then (see Corollary 2.3) also

(v) $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ implies $\Sigma' \models^Y_{\mathcal{K}} \varepsilon$ for some finite $\Sigma' \subseteq \Sigma$ (finitarity).

A first characterization of equational consequence relations in terms of free algebras is obtained for varieties as follows:

Lemma 2.2. *Let $\mathcal{V}$ be a variety and $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$. Then the following conditions are equivalent:*

(1) $\Sigma \models^Y_{\mathcal{V}} \varepsilon$;
(2) $\varepsilon \in \mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}}$;
(3) $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma})$.

Proof. (1) $\Rightarrow$ (2) Clearly, $\mathbf{Fm}(Y)/(\mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}})$ is a member of $\mathcal{V}$. So let ψ be the natural map from $\mathbf{Fm}(Y)$ to $\mathbf{Fm}(Y)/(\mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}})$. Then $\ker(\psi) = \mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}}$. Since $\Sigma \subseteq \ker(\psi)$ and $\Sigma \models^Y_{\mathcal{V}} \varepsilon$, also $\varepsilon \in \ker(\psi)$ as required.

(2) $\Rightarrow$ (1) Consider $\mathbf{A} \in \mathcal{V}$ and $\varphi \in \hom(\mathbf{Fm}(Y), \mathbf{A})$ and suppose that $\Sigma \subseteq \ker(\varphi)$. Notice that $\mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \subseteq \ker(\varphi)$ and $\Theta^Y_{\mathcal{V}} \subseteq \ker(\varphi)$. So also $\mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}} \subseteq \ker(\varphi)$ and $\varepsilon \in \ker(\varphi)$.

(2) $\Leftrightarrow$ (3) We make use of the previous lemma, taking φ to be the mapping $h^Y_{\mathcal{V}}\colon \mathbf{Fm}(Y) \to \mathbf{F}_{\mathcal{V}}(Y)$ sending α to $\bar{\alpha}$. Notice that $\ker(\varphi) = \Theta^Y_{\mathcal{V}}$. It follows that $\varepsilon \in \mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \ker(\varphi)$ iff $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{Fm}(Y)}(\bar{\Sigma})$ as required. $\square$

Observe that in (3) above, $\mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma}) = \bigcup\{\mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma'}) \mid \Sigma' \subseteq \Sigma,\ \Sigma' \text{ finite}\}$, and hence we obtain immediately:

Corollary 2.3. *Let $\mathcal{V}$ be a variety and $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$. Then $\Sigma \models^Y_{\mathcal{K}} \varepsilon$ implies $\Sigma' \models^Y_{\mathcal{K}} \varepsilon$ for some finite $\Sigma' \subseteq \Sigma$.*

The characterization presented in Lemma 2.2 may be sharpened to relate $\Sigma \models^Y_{\mathcal{V}} \varepsilon$ to free algebras for the variety over any countable set including the variables in Σ and ε. Algebraically, this corresponds to the possibility of extending congruences on free algebras as explained below.

Let $\mathbf{B}$ be a subalgebra of an algebra $\mathbf{A}$. We say that a congruence $\Theta \in \mathrm{Con}(\mathbf{B})$ can be *extended*, or has an *extension*, to $\mathbf{A}$ provided there exists a congruence $\Phi \in \mathrm{Con}(\mathbf{A})$ such that $\Phi \cap B^2 = \Theta$; we then refer to Φ as an *extension* of Θ to $\mathbf{A}$.

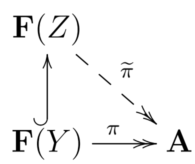
The proof of the following result is immediate.

Lemma 2.4. *Let $\mathbf{B}$ be a subalgebra of an algebra $\mathbf{A}$, and let $R \subseteq B^2$. Then $\mathrm{Cg}_{\mathbf{B}}(R)$ has an extension to $\mathbf{A}$ iff $\mathrm{Cg}_{\mathbf{A}}(R)$ is one such extension.*

The next result shows that free algebras have a restricted form of the congruence extension property considered in Section 4.

Lemma 2.5. *If $Y \subseteq Z$, then any congruence of $\mathbf{F}(Y)$ extends to $\mathbf{F}(Z)$.*

Proof. Consider $\mathrm{Cg}_{\mathbf{F}(Y)}(R) \in \mathrm{Con}(\mathbf{F}(Y))$. We prove $\mathrm{Cg}_{\mathbf{F}(Z)}(R) \cap F(Y)^2 \subseteq \mathrm{Cg}_{\mathbf{F}(Y)}(R)$, guided by the following diagram:



Let $\mathbf{A} = \mathbf{F}(Y)/\mathrm{Cg}_{\mathbf{F}(Y)}(R)$, and let $\pi\colon \mathbf{F}(Y) \to \mathbf{A}$ be the canonical epimorphism. Supposing first that $Y \neq \emptyset$, fix $y_0 \in Y$ and define a homomorphism $\widetilde{\pi}\colon \mathbf{F}(Z) \to \mathbf{A}$ such that $\widetilde{\pi} \restriction_Y = \pi \restriction_Y$, and $\widetilde{\pi} \restriction_{X-Y}$ is the constant map with value $\pi(y_0)$. We have for $(u, v) \in F(Y)^2, (u, v) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R)$ iff $\pi(u) = \pi(v)$ iff $\widetilde{\pi}(u) = \widetilde{\pi}(v)$ (since $\widetilde{\pi} \restriction_Y = \pi \restriction_Y$) iff $(u, v) \in \ker(\widetilde{\pi})$. Therefore $\mathrm{Cg}_{\mathbf{F}(Y)}(R) = \ker(\widetilde{\pi}) \cap F(Y)^2$. Since $\ker(\widetilde{\pi}) \supseteq \mathrm{Cg}_{\mathbf{F}(Z)}(R)$, we get $\mathrm{Cg}_{\mathbf{F}(Z)}(R) \cap F(Y)^2 \subseteq \mathrm{Cg}_{\mathbf{F}(Y)}(R)$ as required. If $Y = \emptyset$, then $\mathbf{F}(Y)$ exists only when the signature contains nullary operations, and hence in this case $\mathbf{F}(Y)$ is just the subalgebra of $\mathbf{F}(Z)$ generated by these operations. Assign the elements of Z to arbitrary elements of $F(Y)$, and let $\varphi\colon \mathbf{F}(Z) \to \mathbf{F}(Y)$ be the homomorphism extending this assignment. Clearly φ is onto and fixes the elements of $F(Y)$. Thus, if we let $\widetilde{\pi} = \pi\varphi$, then for $(u, v) \in F(Y)^2, (u, v) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R)$ iff $\pi(u) = \pi(v)$ iff $\widetilde{\pi}(u) = \widetilde{\pi}(v)$ iff $(u, v) \in \ker(\widetilde{\pi})$, and the remainder of the proof proceeds as above. □

Combining Lemmas 2.2 and 2.5, we obtain:

Corollary 2.6. *Let $\mathcal{V}$ be a variety, $Y \subseteq Z$, and $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$. Then the following conditions are equivalent:*

(1) $\Sigma \models^Z_{\mathcal{V}} \varepsilon$;
(2) $\Sigma \models^Y_{\mathcal{V}} \varepsilon$;
(3) $\varepsilon \in \mathrm{Cg}_{\mathbf{Fm}(Z)}(\Sigma) \vee \Theta^Z_{\mathcal{V}}$;

(4) $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(Z)}(\bar{\Sigma})$;
(5) $\varepsilon \in \mathrm{Cg}_{\mathbf{Fm}(Y)}(\Sigma) \vee \Theta^Y_{\mathcal{V}}$;
(6) $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma})$.

The result below, which is an immediate consequence of the preceding lemma, describes more precisely the relationship between equational consequence relations defined for different sets of variables.

Lemma 2.7. *Let $\mathcal{V}$ be a variety and suppose that $\psi\colon \mathbf{Fm}(Y) \to \mathbf{Fm}(Z)$ is an embedding such that $\psi[Y] = Z$. Then for all $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$:*

$$\Sigma \models^Y_{\mathcal{V}} \varepsilon \quad \text{iff} \quad \psi(\Sigma) \models^Z_{\mathcal{V}} \psi(\varepsilon).$$

Proof. In view of Corollary 2.6,

$$\Sigma \models^Y_{\mathcal{V}} \varepsilon \quad \text{iff} \quad \psi(\Sigma) \models^{\psi[Y]}_{\mathcal{V}} \psi(\varepsilon) \quad \text{iff} \quad \psi(\Sigma) \models^Z_{\mathcal{V}} \psi(\varepsilon). \qquad \square$$

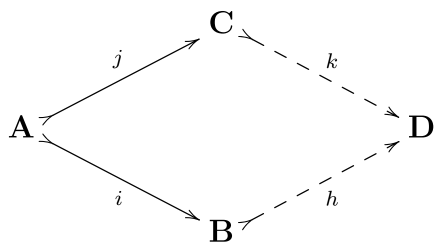
In light of this lemma, let us henceforth write $\Sigma \models_{\mathcal{V}} \varepsilon$ to denote that $\Sigma \models^Y_{\mathcal{V}} \varepsilon$ for *any* $Y \supseteq \mathrm{Var}(\Sigma \cup \{\varepsilon\})$. Note also that if $\mathrm{Var}(\Sigma \cup \{\varepsilon\})$ is *countable*, then it can be identified with a subset of $\mathbb{X}$ and we have $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Sigma \models^{\mathbb{X}}_{\mathcal{V}} \varepsilon$.

3. Amalgamation and the Robinson Property

This section has two main goals. First, we provide criteria for a variety $\mathcal{V}$ to admit the amalgamation property (Theorem 3.2). Secondly, we establish, in a self-contained exposition, a bridge between amalgamation for $\mathcal{V}$ and the Robinson property of the corresponding equational consequence relation $\models_{\mathcal{V}}$ (Theorem 3.6). A crucial intermediary role is played here by the Pigozzi property which characterizes amalgamation for a variety in terms of its free algebras.

The word "amalgamation" refers to the process of combining a pair of algebras in such a way as to preserve a common subalgebra. This is made precise in the following definitions.

Let $\mathcal{K}$ be a class of algebras of the same signature. A *V-formation in $\mathcal{K}$* is a 5-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}, \mathbf{C}$, respectively. Given two classes of algebras $\mathcal{K}$ and $\mathcal{K}'$ of the same signature and a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$, $(\mathbf{D}, h, k)$ is said to be an *amalgam* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}'$ if $\mathbf{D} \in \mathcal{K}'$ and h, k are embeddings of $\mathbf{B}, \mathbf{C}$, respectively, into $\mathbf{D}$ such that the compositions hi and kj coincide.



$\mathcal{K}$ is said to *have the amalgamation property with respect to* $\mathcal{K}'$ if each V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}'$. In particular, $\mathcal{K}$ has the *amalgamation property* AP if each V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}$.

Amalgamations were first considered for groups by Schreier [66] in the form of amalgamated free products. The general form of the amalgamation property was first formulated by Fraïsse [22] and the significance of this property to the study of algebraic systems was further demonstrated in Jónsson's pioneering work on the topic [36–40].

The following lemma, due to Grätzer [27], provides a useful necessary and sufficient condition for a variety $\mathcal{V}$ to have the amalgamation property.

Lemma 3.1. *The following are equivalent for any variety* $\mathcal{V}$*:*

(1) $\mathcal{V}$ *has the amalgamation property.*
(2) *For any V-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ *and* $x \neq y \in B$ *(respectively,* $x \neq y \in C$*), there exist* $\mathbf{D}_{xy} \in \mathcal{V}$ *and homomorphisms* $h_{xy} \colon \mathbf{B} \to \mathbf{D}_{xy}$ *and* $k_{xy} \colon \mathbf{C} \to \mathbf{D}_{xy}$ *such that* $h_{xy} i = k_{xy} j$ *and* $h_{xy}(x) \neq h_{xy}(y)$ *(respectively,* $k_{xy}(x) \neq k_{xy}(y)$*).*

Proof. We just need to show that (2) implies (1). Let $\mathbf{D}$ be the direct product of all algebras $\mathbf{D}_{xy}$, for all two-element sets $\{x, y\}$ as in the statement of (2). By the co-universality of $\mathbf{D}$, the homomorphisms $h_{xy} \colon \mathbf{B} \to \mathbf{D}_{xy}$ and $k_{xy} \colon \mathbf{C} \to \mathbf{D}_{xy}$ induce homomorphisms $h \colon \mathbf{B} \to \mathbf{D}$ and $k \colon \mathbf{C} \to \mathbf{D}$. The assumptions about the homomorphisms h_{xy} and k_{xy} guarantee that h and k are injective, and $hi = kj$. □

We now establish a set of sufficient conditions that allow us to deduce the amalgamation property for a variety $\mathcal{V}$ from the existence of a subclass $\mathcal{S}$ that contains all subdirectly irreducible members of $\mathcal{V}$, is closed under isomorphisms and subalgebras, satisfies a condition on congruences, and such that each V-formation in $\mathcal{S}$ has an amalgam in $\mathcal{V}$. We will make crucial use of this theorem in Section 7.

Theorem 3.2. *Let* $\mathcal{S}$ *be a subclass of a variety* $\mathcal{V}$ *that satisfies the following conditions:*

(i) *Every subdirectly irreducible member of* $\mathcal{V}$ *is in* $\mathcal{S}$*;*
(ii) $\mathcal{S}$ *is closed under isomorphisms and subalgebras;*

(iii) *For any algebra* $\mathbf{B} \in \mathcal{V}$ *and subalgebra* $\mathbf{A}$ *of* $\mathbf{B}$*, if* $\Theta \in \mathrm{Con}(\mathbf{A})$ *and* $\mathbf{A}/\Theta \in \mathcal{S}$*, then there exists* $\Phi \in \mathrm{Con}(\mathbf{B})$ *such that* $\Phi \cap A^2 = \Theta$ *and* $\mathbf{B}/\Phi \in \mathcal{S}$*;*

(iv) *Each V-formation consisting of algebras in* $\mathcal{S}$ *has an amalgam in* $\mathcal{V}$*.*

Then $\mathcal{V}$ *has the amalgamation property.*

Proof. We prove that $\mathcal{V}$ satisfies condition (2) of Lemma 3.1. Consider a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{V}$ and let $x \neq y \in \mathbf{B}$. Let Ψ be a congruence of $\mathbf{B}$ that is maximal with respect to $(x, y) \notin \Psi$. Set $\Theta = \Psi \cap A^2$, and note that the map i/Θ sending $[a]_\Theta \in \mathbf{A}/\Theta$ to $[a]_\Psi$ is an embedding of $\mathbf{A}/\Theta$ into $\mathbf{B}/\Psi$. (If $[a]_\Theta \neq [b]_\Theta$, then $(a, b) \notin \Theta$, therefore $(a, b) \notin \Psi$, as $\Theta = \Psi \cap A^2$.) Now $\mathbf{B}/\Psi$ is subdirectly irreducible, and hence it belongs to $\mathcal{S}$, by condition (i). It follows that $\mathbf{A}/\Theta \in \mathcal{S}$ by condition (ii). Hence by condition (iii), there is $\Phi \in \mathrm{Con}(\mathbf{C})$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{C}/\Phi \in \mathcal{S}$. Once again, the map j/Θ sending any $[a]_\Theta \in \mathbf{A}/\Theta$ to $[a]_\Phi$ is an embedding of $\mathbf{A}/\Theta$ into $\mathbf{C}/\Phi$.

It follows that $(\mathbf{A}/\Theta, \mathbf{B}/\Psi, \mathbf{C}/\Phi, i/\Theta, j/\Theta)$ is a V-formation in $\mathcal{S}$, and condition (iv) guarantees the existence of an amalgam $(\mathbf{D}, h, k)$ in $\mathcal{V}$. Consider the homomorphisms $h_\Psi \colon \mathbf{B} \to \mathbf{D}$ and $k_\Phi \colon \mathbf{C} \to \mathbf{D}$, defined by $h_\Psi(b) = h([b]_\Psi)$ and $k_\Phi(c) = k([c]_\Phi)$, for all $b \in B$ and $c \in C$. We have that $h_\Psi(x) \neq h_\Psi(y)$ (as h is injective and $[x]_\Psi \neq [y]_\Psi$) and for all $a \in \mathbf{A}$, $h_\Psi i(a) = h_\Psi(i(a)) = h([i(a)]_\Psi) = h((i/\Theta)([a]_\Theta)) = k((j/\Theta)([a]_\Theta)) = k([j(a)]_\Phi) = k_\Phi(j(a)) = k_\Phi j(a)$.

Thus the claim follows from Lemma 3.1. □

We now consider a characterization of the amalgamation property for a variety in terms of its free algebras, introduced by Pigozzi in [59].

A variety $\mathcal{V}$ is said to have the *Pigozzi property* PP if for any sets Y, Z, whenever

(i) $Y \cap Z \neq \emptyset$;

(ii) $\Theta_Y \in \mathrm{Con}(\mathbf{F}(Y))$ and $\Theta_Z \in \mathrm{Con}(\mathbf{F}(Z))$;

(iii) $\Theta_Y \cap F(Y \cap Z)^2 = \Theta_Z \cap F(Y \cap Z)^2$,

Θ_Y and Θ_Z have a common extension to $\mathbf{F}(Y \cup Z)$; explicitly, there exists $\Theta \in \mathrm{Con}(\mathbf{F}(Y \cup Z))$ with $\Theta_Y = \Theta \cap F(Y)^2$ and $\Theta_Z = \Theta \cap F(Z)^2$.

Lemma 3.3. *A variety* $\mathcal{V}$ *has the amalgamation property iff it has the Pigozzi property.*

Proof. ($\Rightarrow$) Suppose first that $\mathcal{V}$ has the AP. Let Y, Z, Θ_Y, and Θ_Z be as in the definition of the PP, and let $\Theta_0 = \Theta_Y \cap F(Y \cap Z)^2 = \Theta_Z \cap F(Y \cap Z)^2$. Let $\mathbf{A} = \mathbf{F}(Y \cap Z)/\Theta_0$, $\mathbf{B} = \mathbf{F}(Y)/\Theta_Y$, and $\mathbf{C} = \mathbf{F}(Z)/\Theta_Z$. The maps $i \colon \mathbf{A} \to \mathbf{B}$ and $j \colon \mathbf{A} \to \mathbf{C}$, defined, respectively, by $i([x]_{\Theta_0}) = [x]_{\Theta_Y}$ and $i([x]_{\Theta_0}) = [x]_{\Theta_Z}$ for all $x \in A$, are embeddings. Since $\mathcal{V}$ has the AP, the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ has an amalgam $(\mathbf{D}, h, k)$ in $\mathcal{V}$. We may assume, without loss of generality, that $\mathbf{D}$ is generated by $h(B) \cup k(C)$. Now let $g\colon Y \cup Z \to D$ be defined, for $y \in Y$ and $z \in Z$, by $g(y) = h([y]_{\Theta_Y})$ and $g(z) = k([z]_{\Theta_Z})$. Note that g is well defined. Indeed, if $u \in Y \cap Z$, then $h([u]_{\Theta_Y}) = h(i([u]_{\Theta_0})) = k(j([u]_{\Theta_0})) = [u]_{\Theta_Z}$. Let $\tilde{g}\colon \mathbf{F}(Y \cup Z) \to \mathbf{D}$ be the homomorphism extending g. Note that $\tilde{g}$ is surjective, since $\mathbf{D}$ is generated by $h(B) \cup k(C)$. Now let $\Theta = \ker(\tilde{g})$. We claim that $\Theta \cap F(Y)^2 = \Theta_Y$ and $\Theta \cap F(Z)^2 = \Theta_Z$. We just prove the non-trivial inclusion of the first equality. Suppose $(u, v) \in \Theta \cap F(Y)^2$. Then $\tilde{g}(u) = \tilde{g}(v)$, and hence $h([u]_{\Theta_Y}) = h([v]_{\Theta_Y})$. Since h is injective, we have $[u]_{\Theta_Y} = [v]_{\Theta_Y}$, and $(u, v) \in \Theta_Y$.

($\Leftarrow$) Suppose now that $\mathcal{V}$ has the PP. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in $\mathcal{V}$. Without loss of generality, we may assume that i and j are inclusion maps. Hence, A, B, and C will play the role of $Y \cap Z$, Y and Z, respectively, in the definition of the PP. Consider the surjective morphisms $\pi_A\colon \mathbf{F}(A) \to \mathbf{A}$, $\pi_B\colon \mathbf{F}(B) \to \mathbf{B}$, and $\pi_C\colon \mathbf{F}(C) \to \mathbf{C}$ extending the identity maps on A, B, and C, respectively. Let $\Theta_{\mathbf{A}} = \ker(\pi_{\mathbf{A}})$, $\Theta_{\mathbf{B}} = \ker(\pi_{\mathbf{B}})$, and $\Theta_{\mathbf{C}} = \ker(\pi_{\mathbf{C}})$. Since the restrictions of π_B and π_C on A is π_A, it follows that $\Theta_A = \Theta_B \cap F(A)^2 = \Theta_C \cap F(A)^2$. Therefore, by the PP, there exists $\Theta \in \mathrm{Con}(\mathbf{F}(B \cup C))$ such that $\Theta \cap F(B)^2 = \Theta_{\mathbf{B}}$ and $\Theta \cap F(C)^2 = \Theta_{\mathbf{C}}$. Let $\mathbf{D} = \mathbf{F}(B \cup C)/\Theta$, and let $\pi\colon \mathbf{F}(B \cup C) \to \mathbf{D}$ be the canonical homomorphism. Consider the inclusion homomorphisms $\alpha\colon \mathbf{F}(B) \to \mathbf{F}(B \cup C)$ and $\beta\colon \mathbf{F}(C) \to \mathbf{F}(B \cup C)$. Then $\ker(\pi\alpha) = \ker(\pi_{\mathbf{B}})$ and $\ker(\pi\beta) = \ker(\pi_{\mathbf{C}})$. So the general homomorphism theorem guarantees the existence of injective homomorphisms $h\colon \mathbf{B} \to \mathbf{D}$ and $k\colon \mathbf{C} \to \mathbf{D}$ such that $h\pi_{\mathbf{B}} = \pi\alpha$ and $h\pi_{\mathbf{C}} = \pi\beta$. Lastly, it is routine to verify that $hi = kj$, and so $(h, k, \mathbf{D})$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{V}$. □

The Pigozzi property may now be reformulated in terms of equational consequence relations, the result being a property introduced in the context of first-order logic by Robinson [65] (see also [13]).

A variety $\mathcal{V}$ has the *Robinson property* RP if for each set of variables Y, whenever

(i) $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$ and $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi) \neq \emptyset$;
(ii) $\Sigma \models_{\mathcal{V}} \delta$ iff $\Pi \models_{\mathcal{V}} \delta$, for all $\delta \in \mathrm{Eq}(Y)$ satisfying $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi)$;
(iii) $\mathrm{Var}(\varepsilon) \cap \mathrm{Var}(\Pi) \subseteq \mathrm{Var}(\Sigma)$;
(iv) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$,

also $\Sigma \models_{\mathcal{V}} \varepsilon$. We will say that $\mathcal{V}$ has the *countable Robinson property* (countable RP) if the above holds in particular for the set of variables $\mathbb{X}$.

Observe that the Robinson property is essentially the logical analogue of the Pigozzi property (see Lemma 3.4 below), if one takes into account that condition (iii) may be replaced by (iii)′ $\mathrm{Var}(\varepsilon) \subseteq \mathrm{Var}(\Sigma)$.

Let us verify that (iii) and (iii)′ are indeed equivalent in the presence of conditions (i), (ii), and (iv). It is clear that (iii)′ implies (iii). To prove the opposite implication, suppose that (i), (ii), (iii), and (iv) hold. Let $\Sigma' = \Sigma \cup \{x \approx x \mid x \in \mathrm{Var}(\varepsilon) - \mathrm{Var}(\Pi)\}$. Then conditions (i), (ii), (iii)′, and (iv) hold when Σ is replaced by Σ'; with regard to (ii), note that $\mathrm{Var}(\Sigma') \cap \mathrm{Var}(\Pi) = \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi)$. Hence (i), (ii), (iii)′, and (iv) imply that $\Sigma' \models_{\mathcal{V}} \varepsilon$, and hence, $\Sigma \models_{\mathcal{V}} \varepsilon$.

Lemma 3.4. *A variety $\mathcal{V}$ has the Pigozzi property iff it has the Robinson property.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{V}$ has the PP and that conditions (i), (ii), (iii)′, and (iv) above are satisfied for the RP. Let $Y = \mathrm{Var}(\Sigma)$ and $Z = \mathrm{Var}(\Pi)$. Let $\Theta_Y = \mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma})$ and $\Theta_Z = \mathrm{Cg}_{\mathbf{F}(Z)}(\bar{\Pi})$. Then by condition (ii), $\Theta_Y \cap F(Y \cap Z)^2 = \Theta_Z \cap F(Y \cap Z)^2$. So by the PP, there exists $\Theta \in \mathrm{Con}(\mathbf{F}(Y \cup Z))$ such that $\Theta_Y = \Theta \cap F(Y)^2$ and $\Theta_Z = \Theta \cap F(Z)^2$. We may assume that $\Theta = \mathrm{Cg}_{\mathbf{F}(Y\cup Z)}(\Theta_Y \cup \Theta_Z)$, in view of Lemma 2.4. Hence, by condition (iv), $\bar{\varepsilon} \in \Theta$. But since $\mathrm{Var}(\varepsilon) \subseteq Y$, we have $\bar{\varepsilon} \in \Theta_Y$ and $\Sigma \models_{\mathcal{V}} \varepsilon$.

($\Leftarrow$) Now suppose that $\mathcal{V}$ has the RP and that conditions (i), (ii), and (iii) are satisfied for the PP. We choose Σ and Π such that $\Theta_Y = \mathrm{Cg}_{\mathbf{F}(Y)}(\bar{\Sigma})$ and $\Theta_Z = \mathrm{Cg}_{\mathbf{F}(Z)}(\bar{\Pi})$. Clearly conditions (i) and (ii) of the RP hold. Let $\Theta = \mathrm{Cg}_{\mathbf{F}(Y\cup Z)}(\Theta_Y \cup \Theta_Z)$. Then if $\bar{\varepsilon} \in \Theta \cap F(Y)^2$, we have $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$ and $\mathrm{Var}(\varepsilon) \subseteq \mathrm{Var}(\Sigma)$. So by the RP, $\Sigma \models_{\mathcal{V}} \varepsilon$. Hence $\bar{\varepsilon} \in \Theta_Y$ and so $\Theta_Y = \Theta \cap F(Y)^2$. By the same reasoning, also $\Theta_Z = \Theta \cap F(Z)^2$. □

Lemma 3.5. *A variety $\mathcal{V}$ has the Robinson property iff it has the countable Robinson property.*

Proof. Suppose that $\mathcal{V}$ has the countable RP and consider an arbitrary set of variables Y. Exploiting compactness (Corollary 2.3), let us first fix for each $\Gamma \cup \{\delta\} \subseteq \mathrm{Eq}(Y)$ satisfying $\Gamma \models_{\mathcal{V}} \delta$, a finite set $\Gamma_\delta \subseteq \Gamma$ such that $\Gamma_\delta \models_{\mathcal{V}} \delta$.

Now suppose that conditions (i)-(iv) hold for the RP. Again by compactness (Corollary 2.3), there exist countable sets $\Sigma_0 \subseteq \Sigma$ and $\Pi_0 \subseteq \Pi$ such that $\Sigma_0 \cup \Pi_0 \models_{\mathcal{V}} \varepsilon$. Given $n \in \mathbb{N}$ and countable sets $\Sigma_n \subseteq \Sigma$ and $\Pi_n \subseteq \Pi$ satisfying $\Sigma_n \cup \Pi_n \models_{\mathcal{V}} \varepsilon$, let:

$$\Sigma_{n+1} = \Sigma_n \cup \bigcup\{\Sigma_\delta \subseteq \Sigma \mid \Sigma \models_{\mathcal{V}} \delta,\ \delta \in \mathrm{Eq}(\mathrm{Var}(\Sigma_n) \cap \mathrm{Var}(\Pi_n))\}$$
$$\Pi_{n+1} = \Pi_n \cup \bigcup\{\Pi_\delta \subseteq \Pi \mid \Pi \models_{\mathcal{V}} \delta,\ \delta \in \mathrm{Eq}(\mathrm{Var}(\Sigma_n) \cap \mathrm{Var}(\Pi_n))\}.$$

Clearly $\Sigma_{n+1} \cup \Pi_{n+1} \models_{\mathcal{V}} \varepsilon$. Moreover, since there are countably many equations with variables in Σ_n and Π_n and each Σ_δ and Π_δ is finite, Σ_{n+1} and Π_{n+1} are countable unions of countable sets, and hence themselves countable. Now define:

$$\Sigma' = \bigcup_{n \in \mathbb{N}} \Sigma_n \quad \text{and} \quad \Pi' = \bigcup_{n \in \mathbb{N}} \Pi_n.$$

Then $\Sigma' \cup \Pi' \models_{\mathcal{V}} \varepsilon$, and Σ' and Π' are countable. Moreover, by Lemma 2.7, we may assume that $\mathrm{Var}(\Sigma' \cup \Pi') \subseteq \mathbb{X}$.

Suppose that $\Sigma' \models_{\mathcal{V}} \delta$ for some $\delta \in \mathrm{Eq}(Y)$ satisfying $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma') \cap \mathrm{Var}(\Pi')$. Then by monotonicity, also $\Sigma \models_{\mathcal{V}} \delta$. So by condition (ii), $\Pi \models_{\mathcal{V}} \delta$. But $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma_n) \cap \mathrm{Var}(\Pi_n)$ for some $n \in \mathbb{N}$, so $\Pi_{n+1} \models_{\mathcal{V}} \delta$. That is, by monotonicity again, $\Pi' \models_{\mathcal{V}} \delta$. By the same reasoning, $\Sigma' \models_{\mathcal{V}} \delta$ iff $\Pi' \models_{\mathcal{V}} \delta$, for all $\delta \in \mathrm{Eq}(Y)$ satisfying $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma') \cap \mathrm{Var}(\Pi')$. That is, the countable sets Σ' and Π' satisfy conditions (i)-(iv) of the countable RP. So $\Sigma' \models_{\mathcal{V}} \varepsilon$ and, using monotonicity, $\Sigma \models_{\mathcal{V}} \varepsilon$ as required. □

Hence putting together Lemmas 3.3, 3.4, and 3.5, we obtain:

Theorem 3.6. *For a variety $\mathcal{V}$, the following are equivalent:*

(1) *$\mathcal{V}$ has the amalgamation property.*
(2) *$\mathcal{V}$ has the Pigozzi property.*
(3) *$\mathcal{V}$ has the Robinson property.*
(4) *$\mathcal{V}$ has the countable Robinson property.*

It follows from our considerations above that a variety $\mathcal{V}$ has the amalgamation property iff every V-formation of *countable* algebras from $\mathcal{V}$ has an amalgam in $\mathcal{V}$. We note, however, also the following stronger result of Grätzer and Lakser [31] (see also Grätzer [28, V.4: Corollaries 2,3]) which will not be needed in what follows:

Proposition 3.7 ([31]). *A variety $\mathcal{V}$ has the amalgamation property iff every V-formation of* finitely generated *algebras from $\mathcal{V}$ has an amalgam in $\mathcal{V}$.*

4. The Congruence Extension Property

An algebra $\mathbf{A}$ is said to have the *congruence extension property* CEP if any congruence of a subalgebra of $\mathbf{A}$ extends to $\mathbf{A}$. A variety $\mathcal{V}$ has the congruence extension property if all algebras in $\mathcal{V}$ have this property.

The aim of this section is to describe the congruence extension property for varieties in terms of their free algebras and equational consequence relations. This characterization will then play a key role in relating the amalgamation property to the interpolation properties investigated in Section 5. We first provide a necessary and sufficient condition for every homomorphic image of an algebra to have the congruence extension property (Proposition 4.2), which then leads naturally leads to Lemma 4.3, which states that the congruence extension property can be captured by a congruence condition of the free algebra in countably many generators. Theorem 4.6 shows that this condition is equivalent to the "extension property" for the equational consequence relations of the variety. The latter was studied in [55] under the name "limited GINT", and Theorem 4.6 may be understood as a refinement of Theorem 8 of that paper.

Lemma 4.1. *An algebra* $\mathbf{A}$ *has the congruence extension property if the compact congruences of every subalgebra of* $\mathbf{A}$ *extend to* $\mathbf{A}$.

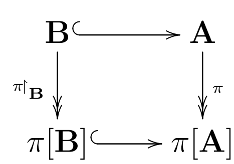
Proof. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. By assumption, every compact – that is, finitely generated – congruence of $\mathbf{B}$ has an extension to $\mathbf{A}$. In view of Lemma 2.4, this means that if R is a finite subset of B^2, then $\mathrm{Cg}_{\mathbf{B}}(R) = \mathrm{Cg}_{\mathbf{A}}(R) \cap B^2$. Consider now an arbitrary $\Theta \in \mathrm{Con}(\mathbf{B})$. We have, $\Theta = \bigcup\{\mathrm{Cg}_{\mathbf{B}}(R) \mid R \subseteq \Theta, R \text{ finite}\} = \bigcup\{\mathrm{Cg}_{\mathbf{A}}(R) \mid R \subseteq \Theta, R \text{ finite}\} \cap B^2$. Now $\{\mathrm{Cg}_{\mathbf{A}}(R) \mid R \subseteq \Theta, R \text{ finite}\}$ is an up-directed family of congruences of $\mathbf{A}$, and hence $\Phi = \bigcup\{\mathrm{Cg}_{\mathbf{A}}(R) \mid R \subseteq \Theta, R \text{ finite}\} \in \mathrm{Con}(\mathbf{A})$. So $\Theta = \Phi \cap B^2$, and Φ is an extension of Θ to $\mathbf{A}$. □

The next result provides a crucial step in relating the congruence extension property to properties of free algebras.

Proposition 4.2. *For an algebra* $\mathbf{A}$*, the following are equivalent:*

(1) *Every homomorphic image of* $\mathbf{A}$ *has the congruence extension property.*
(2) *Whenever* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$*,* $\Theta \in \mathrm{Con}(\mathbf{A})$*, and* $R \subseteq B^2$*, then*
$$[\Theta \vee \mathrm{Cg}_{\mathbf{A}}(R)] \cap B^2 = (\Theta \cap B^2) \vee' \mathrm{Cg}_{\mathbf{B}}(R)$$
where $\vee$ *denotes the join in* $\mathrm{Con}(\mathbf{A})$*, and* $\vee'$*, the join in* $\mathrm{Con}(\mathbf{B})$.

Proof. $(1) \Rightarrow (2)$ Suppose that every homomorphic image of $\mathbf{A}$ has the CEP. Consider a subalgebra $\mathbf{B}$ of the algebra of $\mathbf{A}$. Further, let $R \subseteq B^2$, $\Theta \in \mathrm{Con}(\mathbf{A})$, $\pi\colon \mathbf{A} \to \mathbf{A}/\Theta$ the canonical epimorphism, and $\pi\restriction_{\mathbf{B}}$ the restriction of π to $\mathbf{B}$. Then $\ker(\pi) = \Theta$ and $\ker(\pi\restriction_{\mathbf{B}}) = \Theta \cap B^2$. Let $\pi[\mathbf{A}] = A/\Theta$, and let $\pi[\mathbf{B}]$ be the image of $\mathbf{B}$ under π (refer to the commutative diagram below).



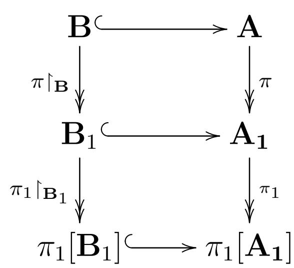
We obtain for $x, y \in B$:

$$\begin{array}{rll} (x,y) \in & [\Theta \vee \mathrm{Cg}_{\mathbf{A}}(R)] \cap B^2 & \text{iff} \\ (\pi(x),\pi(y)) \in & \mathrm{Cg}_{\pi[\mathbf{A}]}(\pi[R]) \cap \pi[B]^2 \ \ (\text{by Lemma 2.1}) & \text{iff} \\ (\pi(x),\pi(y)) \in & \mathrm{Cg}_{\pi[\mathbf{B}]}(\pi[R]) \ \ (\text{by the CEP for } \pi[\mathbf{A}]) & \text{iff} \\ (x,y) \in & (\Theta \cap B^2) \vee' \mathrm{Cg}_{\mathbf{B}}(R) \ \ (\text{by Lemma 2.1}). & \end{array}$$

(2) $\Rightarrow$ (1) Suppose now that $\mathbf{A}$ satisfies condition (2). Then $\mathbf{A}$ has the CEP: just let Θ be the identity congruence. Hence it suffices to show that also every homomorphic image of $\mathbf{A}$ satisfies (2). In other words, let $\mathbf{A_1}$ be a homomorphic image of $\mathbf{A}$, and let $\pi\colon \mathbf{A} \to \mathbf{A_1}$ be the corresponding surjective homomorphism. Further, let $\mathbf{B}_1$ be a subalgebra of $\mathbf{A_1}$, Θ_1 a congruence of $\mathbf{A_1}$, and $R_1 \subseteq B_1^2$. We claim that

$$(\Theta_1 \vee \mathrm{Cg}_{\mathbf{A_1}}(R_1)) \cap B_1^2 = (\Theta_1 \cap B_1^2) \vee' \ \mathrm{Cg}_{\mathbf{B}_1}(R_1).$$

As above, it is understood that $\vee$ is the join in $\mathrm{Con}(\mathbf{A_1})$, and $\vee'$ is the join in $\mathrm{Con}(\mathbf{B}_1)$.



Let $\Theta = \ker(\pi)$ and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ corresponding to the subuniverse $B = \pi^{-1}[B_1]$. If $\pi \restriction_{\mathbf{B}}$ denotes the restriction of π to $\mathbf{B}$, we have $\ker(\pi \restriction_{\mathbf{B}}) = \Theta \cap B^2$. Let $\pi_1\colon \mathbf{A_1} \to \mathbf{A_1}/\Theta_1$ be the canonical epimorphism, and let $\pi_1 \restriction_{\mathbf{B_1}}$ be the restriction of π_1 to $\mathbf{B}_1$. We have $\ker(\pi_1) = \Theta_1$ and $\ker(\pi_1 \restriction_{\mathbf{B_1}}) = \Theta_1 \cap B_1^2$. We denote the images of $\mathbf{B}_1$ and $\mathbf{A_1}$ under π_1 by $\pi_1[\mathbf{B}_1]$ and $\pi_1[\mathbf{A_1}]$, respectively (see the commutative diagram above). Set $\varphi = \pi_1\pi$, and let $\Phi = \ker(\varphi)$.

Let R be the inverse image of R_1 under $\pi \restriction_{\mathbf{B}}$. Consider an element $(x_1, y_1) \in (\Theta_1 \vee \mathrm{Cg}_{\mathbf{A_1}}(R_1)) \cap B_1^2$. There exist $x, y \in B$ such that $x_1 = \pi(x)$ and $y_1 = \pi(y)$. We have, $(\pi_1(x_1), \pi_1(y_1)) \in \pi_1[(\Theta_1 \vee \mathrm{Cg}_{\mathbf{A_1}}(R_1)) \cap B_1^2] =$ (by Lemma 2.1) $\mathrm{Cg}_{\pi_1[\mathbf{A_1}]}(\pi_1[R_1]) \cap \pi_1[\mathbf{B}_1]^2$. In other words, $(\varphi(x), \varphi(y)) \in \mathrm{Cg}_{\pi_1[\mathbf{A_1}]}(\varphi[R]) \cap \pi_1[\mathbf{B}_1]^2$. Invoking Lemma 2.1 and the fact that $\mathbf{A}$ satisfies condition (2), we obtain $(x, y) \in (\mathrm{Cg}_{\mathbf{A}}(R) \vee \Phi) \cap B^2 = \mathrm{Cg}_{\mathbf{B}}(R) \vee' (\Phi \cap B^2) = \mathrm{Cg}_{\mathbf{B}}(R) \vee' \ker(\pi \restriction_{\mathbf{B}}) \vee' (\Phi \cap B^2)$. Lastly, observe that $\ker(\pi \restriction_{\mathbf{B}}) = \Theta \cap B^2 \subseteq \Phi \cap B^2$, and $\pi[\Phi \cap B^2] \subseteq \Theta_1 \cap B_1^2$. Thus, invoking the correspondence theorem of universal algebra and Lemma 2.1, we get $(x_1, y_1) = (\pi(x), \pi(y)) \in \mathrm{Cg}_{\mathbf{B_1}}(R_1) \vee' \pi[\Phi \cap B^2] \subseteq \mathrm{Cg}_{\mathbf{B_1}}(R_1) \vee' (\Theta_1 \cap B_1^2)$. We have shown that $(\Theta_1 \vee \mathrm{Cg}_{\mathbf{A_1}}(R_1)) \cap B_1^2 \subseteq \ \mathrm{Cg}_{\mathbf{B_1}}(R_1) \vee' (\Theta_1 \cap B_1^2)$. Since the reverse inclusion is trivial, we have the desired equality. $\square$

Applying now Proposition 4.2 with $\mathbf{A} = \mathbf{F}(Y)$ for some set Y, we obtain a necessary and sufficient condition for a variety $\mathcal{V}$ to have the congruence extension property in terms of free algebras.

Lemma 4.3. *For a variety $\mathcal{V}$, the following are equivalent:*

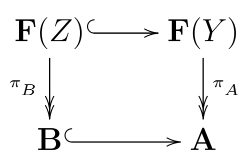
(1) *$\mathcal{V}$ has the congruence extension property.*
(2) *Whenever $Z \subseteq Y$, $\Theta \in \mathrm{Con}(\mathbf{F}(Y))$, and $R \subseteq F(Z)^2$, then*

$$(\Theta \vee \mathrm{Cg}_{\mathbf{F}(Y)}(R)) \cap F(Z)^2 = (\Theta \cap F(Z)^2) \vee' \ \mathrm{Cg}_{\mathbf{F}(Z)}(R),$$

where $\vee$ denotes the join in $\mathrm{Con}(\mathbf{F}(Y))$, and $\vee'$, the join in $\mathrm{Con}(\mathbf{F}(Z))$.

Proof. $(1) \Rightarrow (2)$ Follows directly from Lemma 4.2.

$(2) \Rightarrow (1)$ Suppose that condition (2) is satisfied and consider $\mathbf{A} \in \mathcal{V}$. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, and let $\mathrm{Cg}_{\mathbf{B}}(R') \in \mathrm{Con}(\mathbf{B})$. We need to prove that $\mathrm{Cg}_{\mathbf{B}}(R') = \mathrm{Cg}_{\mathbf{A}}(R') \cap B^2$. Choose $Z \subseteq Y$ and surjective maps $\pi_B \colon Z \to B$ and $\pi_A \colon Y \to A$ such that $\pi_A \restriction_Z = \pi_B$. We use the same symbols to denote the homomorphic extensions $\pi_B \colon \mathbf{F}(Z) \to B$ and $\pi_A \colon \mathbf{F}(Y) \to A$. It is clear that if $\Theta_A = \ker(\pi_A)$ and $\Theta_B = \ker(\pi_B)$, then $\Theta_A \cap F(Z)^2 = \Theta_B$.



Let $R \subseteq F(Z)^2$ be such that $\pi_B[R] = R'$. Also, let $x', y' \in B$ and $x, y \in F(Z)$ be such that $(x', y') \in \mathrm{Cg}_{\mathbf{A}}(R') \cap B^2$, $\pi_B(x) = x'$, and $\pi_B(y) = y'$. Now, $(x', y') \in \mathrm{Cg}_{\mathbf{A}}(R')$ implies, by Lemma 2.1, $(x, y) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \Theta_A$, and so $(x, y) \in [\mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \Theta_A] \cap F(Y_B)^2$. Invoking condition (2) and the relation $\Theta_A \cap F(Z)^2 = \Theta_B$, we get $(x, y) \in \mathrm{Cg}_{\mathbf{F}(Z)}(R) \vee' \Theta_B$. Another application of Lemma 2.1 gives $(x', y') \in \pi_B[\mathrm{Cg}_{F(Z)}(R) \vee' \Theta_B] = \mathrm{Cg}_{\mathbf{B}}(R')$. We have shown that $\mathrm{Cg}_{\mathbf{A}}(R') \cap B^2 \subseteq \mathrm{Cg}_{\mathbf{B}}(R')$. The reverse inclusion is trivial, and hence the proof of this implication is complete. □

We now consider a property of the equational consequence relation $\models_{\mathcal{V}}$ of a variety $\mathcal{V}$ that will turn out to correspond directly to the congruence extension property for $\mathcal{V}$.

A variety $\mathcal{V}$ is said to have the *extension property* EP if for any set of variables Y, whenever

(i) $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$;
(ii) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$,

there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that

(iii) $\Sigma \models_{\mathcal{V}} \Delta$;
(iv) $\Delta \cup \Pi \models_{\mathcal{V}} \varepsilon$;
(v) $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Pi \cup \{\varepsilon\})$.

We say also that $\mathcal{V}$ has the *countable extension property* (countable EP) if the above holds for the set of variables $\mathbb{X}$.

Lemma 4.4. *A variety $\mathcal{V}$ has the extension property iff it has the countable extension property.*

Proof. Suppose that $\mathcal{V}$ has the countable EP and consider an arbitrary set of variables Y satisfying (i) and (ii) above. By compactness, there exists finite $\Sigma' \subseteq \Sigma$ and $\Pi' \subseteq \Pi$ such that $\Sigma' \cup \Pi' \models_{\mathcal{V}} \varepsilon$. The result then follows using Lemma 2.7 and the countable EP. ☐

It is immediate using Corollary 2.6 that the extension property can be expressed in terms of congruences of free algebras as follows:

Lemma 4.5. *For a variety $\mathcal{V}$, the following statements are equivalent:*

(1) *$\mathcal{V}$ has the extension property.*
(2) *Whenever*
  (i) *$Z \subseteq Y$; $P \subseteq F(Y)^2$; $S \cup \{(u, v)\} \subseteq F(Z)^2$;*
  (ii) *$(u, v) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(P)$,*

  there exists $D \subseteq F(Z)^2$ such that
  (i) *$\mathrm{Cg}_{\mathbf{F}(Y)}(D) \subseteq \mathrm{Cg}_{\mathbf{F}(Y)}(P)$;*
  (ii) *$(u, v) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(D)$.*

We are now ready to establish the main result of this section.

Theorem 4.6. *For a variety $\mathcal{V}$, the following are equivalent:*

(1) *$\mathcal{V}$ has the congruence extension property.*
(2) *$\mathcal{V}$ has the extension property.*
(3) *$\mathcal{V}$ has the countable extension property.*

Proof. It suffices to prove that conditions (2) of Lemmas 4.3 and 4.5 are equivalent. Assume first that the variety $\mathcal{V}$ satisfies condition (2) of Lemma 4.5. Let $Z \subseteq Y, R \subseteq F(Z)^2, \Theta \in \mathrm{Con}(\mathbf{F}(Y))$, and $(u, v) \in (\Theta \vee \mathrm{Cg}_{\mathbf{F}(Y)}(R)) \cap F(Z)^2$, where $\vee$ is the join in $\mathrm{Con}(\mathbf{F}(Y))$. By assumption, there exists $D \subseteq \Theta \cap F(Z)^2$, such that $(u, v) \in (\mathrm{Cg}_{\mathbf{F}(Y)}(D) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(R)) \cap F(Z)^2 = (\mathrm{Cg}_{\mathbf{F}(Y)}(D \cup R)) \cap F(Z)^2 = \mathrm{Cg}_{\mathbf{F}(Z)}(D \cup R)$. The last equality is a consequence of Lemma 2.5. Thus, $(u, v) \in \mathrm{Cg}_{\mathbf{F}(Z)}(D) \vee' \mathrm{Cg}_{\mathbf{F}(Z)}(R) \subseteq (\Theta \cap F(Z)^2) \vee' \mathrm{Cg}_{\mathbf{F}(Z)}(R)$, where $\vee'$ denotes the join in $\mathrm{Con}(\mathbf{F}(Z))$. We have shown that $(\Theta \vee \mathrm{Cg}_{\mathbf{F}(Y)}(R)) \cap F(Z)^2 \subseteq (\Theta \cap F(Z)^2) \vee' \mathrm{Cg}_{\mathbf{F}(Z)}(R)$, which proves condition (2) of Lemma 4.3, since the reverse inclusion is trivial.

Conversely, suppose that condition (2) of Lemma 4.3 holds. Let $Z \subseteq Y$, $S \cup \{(u, v)\} \subseteq F(Z)^2$, $P \subseteq F(Y)^2$, and $(u, v) \in \mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(P)$. We have: $(u, v) \in [\mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(P)] \cap F(Z)^2 = \mathrm{Cg}_{\mathbf{F}(Z)}(R) \vee'$

$(\mathrm{Cg}_{\mathbf{F}(Y)}(P) \cap F(Z)^2) \subseteq \mathrm{Cg}_{\mathbf{F}(Y)}(R) \vee \mathrm{Cg}_{\mathbf{F}(Y)}(D)$, where $D = \mathrm{Cg}_{\mathbf{F}(Y)}(P) \cap F(Z)^2$. □

It follows from the proof of the previous theorem that a variety $\mathcal{V}$ has the congruence extension property iff all countable algebras in $\mathcal{V}$ have this property. However, as in the case of the amalgamation property, we may obtain the following stronger result:

Proposition 4.7. *A variety $\mathcal{V}$ has the congruence extension property iff all finitely generated algebras in $\mathcal{V}$ have this property.*

Proof. Let us assume that all finitely generated algebras in $\mathcal{V}$ have the CEP. Let $\mathbf{A} \in \mathcal{V}$, and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. In view of Lemma 2.5, we need to prove that every compact congruence of $\mathbf{B}$ extends to $\mathbf{A}$. Let Θ be such a congruence. There is a finite set $R \subseteq B^2$ such that $\Theta = \mathrm{Cg}_{\mathbf{B}}(R)$. Observe that if Θ cannot be extended to $\mathbf{A}$, then there exists a finitely generated subalgebra $\mathbf{C}$ of $\mathbf{B}$ such that $\mathrm{Cg}_{\mathbf{C}}(R)$ cannot be extended to $\mathbf{A}$. Indeed, suppose that $(x, y) \in \mathrm{Cg}_{\mathbf{A}}(R) \cap B^2$, but $(x, y) \notin \Theta$. Let Y be a finite subset of B such that $\{(x, y)\} \cup R \subseteq Y^2$, and let $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ generated by Y. It is now clear that $\mathrm{Cg}_{\mathbf{C}}(R)$ does not extend to $\mathbf{A}$, since $(x, y) \in \mathrm{Cg}_{\mathbf{A}}(R) \cap C^2$, but $(x, y) \notin \mathrm{Cg}_{\mathbf{C}}(R) \subseteq \mathrm{Cg}_{\mathbf{B}}(R)$.

Hence, we may assume that $\mathbf{B}$ is finitely generated. Consider the set $\mathcal{S}$ of all finitely generated subuniverses of $\mathbf{A}$ that contain B. Clearly, $\mathcal{S}$ is an up-directed family under set-inclusion, and $\bigcup \mathcal{S} = A$. By assumption, we have $\Theta = \mathrm{Cg}_{\mathbf{C}}(R) \cap B^2$, for every $C \in \mathcal{S}$. It is routine to verify that $\Phi = \bigcup\{\mathrm{Cg}_{\mathbf{C}}(R) \mid C \in \mathcal{S}\} \in \mathrm{Con}(\mathbf{A})$, and $\Phi \cap B^2 = \Theta$. Thus, Φ extends Θ to $\mathbf{A}$. □

5. Interpolation Properties

As shown in Section 3, the amalgamation property for a variety may be described using either the Pigozzi property for the free algebras of the variety or the Robinson property for the corresponding equational consequence relations (Theorem 3.6). Similarly, in Section 4, we have seen that the congruence extension property for a variety corresponds both to a property of its free algebras (Lemma 4.3) and to the extension property for its equational consequence relations (Theorem 4.6). In both cases, these properties may be restricted to, respectively, countable algebras and the equational consequence relation over countably infinitely many variables. In this section we consider other closely related "interpolation properties" of equational consequence relations and their algebraic equivalents: in particular, the *deductive interpolation property*, *contextualized deductive interpolation property*, and *Maehara interpolation property*.

$$
\begin{array}{ccccccc}
\mathsf{CEP} + \mathsf{FAP} & & & & & & \\
\Updownarrow & & & & & & \\
\mathsf{TIP} & \Longrightarrow & \mathsf{AP} & \Longrightarrow & \mathsf{WAP} & \Longrightarrow & \mathsf{FAP} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\mathsf{MIP} & \Longrightarrow & \mathsf{RP} & \Longrightarrow & \mathsf{CDIP} & \Longrightarrow & \mathsf{DIP} \\
\Updownarrow & & & & & & \\
\mathsf{DIP} + \mathsf{EP} & & & & & &
\end{array}
$$

FIGURE 1. Relationships between algebraic and syntactic properties

We first give simple, direct syntactic proofs that the Robinson property (equivalently, the amalgamation property) both implies the deductive interpolation property, and is implied by the conjunction of the deductive interpolation property and the extension property (Theorem 5.1). The first proof of this useful fact appeared in [40], and is credited there to unpublished work of H.J. Keisler. As observed in [59], the essential ideas underlying the proof may be traced back to Magnus' work in group theory. Moreover, it was already noted in [38] (see also [31]) that a seemingly stronger algebraic condition, referred to below as the *weak amalgamation property* WAP, implies the amalgamation property in the presence of the congruence extension property. Our second objective here is to show that the weak amalgamation property corresponds to the contextualized deductive interpolation property, a strengthening of the deductive interpolation property (Theorem 5.4), while the deductive interpolation property itself corresponds algebraically to an important property of free products, called the *flat amalgamation property* FAP (Theorem 5.2). Finally, we show that the Maehara interpolation property, a strengthening of both the deductive interpolation property and the contextualized deductive interpolation property – studied, for example, in [70], [55], and [18] – corresponds both to the conjunction of the amalgamation property and the congruence extension property, and to the *transferable injections property* TIP. These relationships are summarized for the reader's convenience in Figure 1.

A variety $\mathcal{V}$ is said to have the *deductive interpolation property* DIP if for any set of variables Y, whenever

(i) $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$ and $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$;
(ii) $\Sigma \models_{\mathcal{V}} \varepsilon$,

there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that

(iii) $\Sigma \models_{\mathcal{V}} \Delta$;
(iv) $\Delta \models_{\mathcal{V}} \varepsilon$;
(v) $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$.

We say also that $\mathcal{V}$ has the *countable deductive interpolation property* (countable DIP) if the above holds for the set of variables $\mathbb{X}$.

Theorem 5.1. *Let $\mathcal{V}$ be a variety.*

(a) *$\mathcal{V}$ has the deductive interpolation property iff $\mathcal{V}$ has the countable deductive interpolation property.*
(b) *If $\mathcal{V}$ has the Robinson property, then $\mathcal{V}$ has the deductive interpolation property.*
(c) *If $\mathcal{V}$ has the deductive interpolation property and the extension property, then it has the Robinson property.*

Proof. (a) Follows exactly the same pattern as the proof of Lemma 4.4, using Lemma 2.7 and compactness (Corollary 2.3).

(b) Let $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$ with $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$, and suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$. We define

$$\Delta = \{\delta \in \mathrm{Eq}(Y) \mid \mathrm{Var}(\delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \text{ and } \Sigma \models_{\mathcal{V}} \delta\}.$$

Clearly, $\Sigma \models_{\mathcal{V}} \Delta$ and $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$. Moreover, (i)-(iv) in the definition of the RP are satisfied (with Δ taking the role of Σ and Σ taking the role of Π), and hence $\Delta \models_{\mathcal{V}} \varepsilon$.

(c) Suppose that conditions (i)-(iv) of the RP are satisfied. Then by the EP, there exists $\Delta' \subseteq \mathrm{Eq}(Y)$ such that

(i) $\Sigma \cup \Delta' \models_{\mathcal{V}} \varepsilon$;
(ii) $\mathrm{Var}(\Delta') \subseteq \mathrm{Var}(\Sigma) \cup \mathrm{Var}(\varepsilon)$;
(iii) $\Pi \models_{\mathcal{V}} \Delta'$.

Then by the DIP applied to $\Pi \models_{\mathcal{V}} \delta$ for each $\delta \in \Delta'$, there exists $\Delta_\delta \subseteq \mathrm{Eq}(Y)$ such that

(iv) $\Pi \models_{\mathcal{V}} \Delta_\delta$;
(v) $\Delta_\delta \models_{\mathcal{V}} \delta$;
(vi) $\mathrm{Var}(\Delta_\delta) \subseteq \mathrm{Var}(\Pi) \cap [\mathrm{Var}(\Sigma) \cup \mathrm{Var}(\varepsilon)] \subseteq \mathrm{Var}(\Pi) \cap \mathrm{Var}(\Sigma)$.

Letting $\Delta = \bigcup\{\Delta_\delta \mid \delta \in \Delta'\}$, it follows that:

(iv') $\Pi \models_{\mathcal{V}} \Delta$;
(v)' $\Delta \models_{\mathcal{V}} \Delta'$;
(vi)' $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Pi) \cap [\mathrm{Var}(\Sigma) \cup \mathrm{Var}(\varepsilon)] \subseteq \mathrm{Var}(\Pi) \cap \mathrm{Var}(\Sigma)$.

Hence by (i) and (v)', $\Sigma \cup \Delta \models_{\mathcal{V}} \varepsilon$. But then using (iv)' and the conditions for the RP, we obtain $\Sigma \models_{\mathcal{V}} \Delta$. So $\Sigma \models_{\mathcal{V}} \varepsilon$ as required. □

The deductive interpolation property corresponds to an important property of free products: the flat amalgamation property, presented in [38].

Before introducing this property, we recall the concept of a free product. Let $\mathcal{K}$ be a class of algebras of the same signature, and let $(\mathbf{A_i} \mid i \in I)$ be a family of algebras in $\mathcal{K}$. The $\mathcal{K}$-free product of this family is an algebra $\mathbf{A}$ together with a family of injective homomorphisms $(\varphi_i \colon \mathbf{A_i} \to \mathbf{A} \mid i \in I)$ such that

(1) $\bigcup_{i \in I} \varphi_i[A_i]$ generates $\mathbf{A}$;
(2) for any algebra $\mathbf{B}$ in $\mathcal{K}$ and any family of homomorphisms $(\psi_i \colon \mathbf{A_i} \to \mathbf{B} \mid i \in I)$, there exists a (necessarily unique) homomorphism $\psi \colon \mathbf{A} \to \mathbf{B}$ satisfying $\psi\varphi_i = \psi_i$, for all $i \in I$.

The algebra $\mathbf{A}$ in the preceding definition is denoted by $*^{\mathcal{K}}_{i \in I}\mathbf{A_i}$ or simply $*_{i \in I}\mathbf{A_i}$. Following usual practice, we speak of $*_{i \in I}\mathbf{A_i}$ as the $\mathcal{K}$-free product of the family $(\mathbf{A_i} \mid i \in I)$. To further simplify the notation, we use the "internal" definition of a free product, that is, we identify each free factor $\mathbf{A_i}$ with its isomorphic image $\varphi_i[A_i]$ in $*_{i \in I}\mathbf{A_i}$; thus we think of each $\mathbf{A_i}$ as a subalgebra of $*_{i \in I}\mathbf{A_i}$, and each φ_i as the inclusion homomorphism.

The reader has undoubtedly noted that free products are just coproducts satisfying the additional assumption that the associated homomorphisms are injective. In what follows, we only consider $\mathcal{V}$-free products when $\mathcal{V}$ is a variety. In this setting, $\mathcal{V}$-coproducts always exist (see, for example, [29], p. 186); this is not the case with $\mathcal{V}$-free products. Their existence is a consequence of the *embedding property* for $\mathcal{V}$: given any two algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{V}$, there exists an algebra $\mathbf{C}$ in $\mathcal{V}$ into which both $\mathbf{A}$ and $\mathbf{B}$ can be embedded. Most varieties of algebras admit free products. For example, this is the case with all varieties of residuated lattices (see Section 7), groups, etc. To see this, note that each residuated lattice has a one-element subalgebra $\{e\}$, and so $\mathbf{C}$ can be taken to be the direct product $\mathbf{A} \times \mathbf{B}$. Moreover, it also holds in other varieties of bounded residuated lattices, such as Boolean algebras and MV algebras, even though its verification is less obvious. In what follows, we use the expression "the variety $\mathcal{V}$ admits free products" to indicate that the free product of any family of algebras in $\mathcal{V}$ exists.

Suppose that $\mathcal{V}$ admits free products. $\mathcal{V}$ is said to have the *flat amalgamation property* FAP if whenever $\mathbf{A}$ and $\mathbf{B}$ are algebras in $\mathcal{V}$, with $\mathbf{A}$ a subalgebra of $\mathbf{B}$, and Y any non-empty set, the induced homomorphism $\mathbf{A} * \mathbf{F}(Y) \to \mathbf{B} * \mathbf{F}(Y)$ is injective.

Theorem 5.2. *Let $\mathcal{V}$ be a variety admitting free products. The following are equivalent:*

(1) *$\mathcal{V}$ has the countable deductive interpolation property.*
(2) *$\mathcal{V}$ has the deductive interpolation property.*
(3) *$\mathcal{V}$ has the flat amalgamation property.*

Proof. The proof of this result is a simpler version of the proof of Theorem 5.4, and is therefore omitted. □

Corollary 5.3. *If a variety has the flat amalgamation property and the congruence extension property, then it has the amalgamation property.*

As already remarked at the beginning of this section, there is an algebraic condition, an important property of free products introduced in [38], apparently stronger than the flat amalgamation property, that also implies the amalgamation property in the presence of the congruence extension property. Let $\mathcal{V}$ be a variety admitting free products. $\mathcal{V}$ is said to have the *weak amalgamation property* WAP if whenever $\mathbf{A}$ and $\mathbf{B}$ are algebras in $\mathcal{V}$, and $\mathbf{A_1}$ is a subalgebra of $\mathbf{A}$, the subalgebra of the free product $\mathbf{A} * \mathbf{B}$ generated by $A_1 \cup B$ is isomorphic to the free product $\mathbf{A_1} * \mathbf{B}$. It is clear that $\mathcal{V}$ has the weak amalgamation property iff whenever $\mathbf{A}$ and $\mathbf{B}$ are algebras in $\mathcal{V}$, and $\mathbf{A_1}$ and $\mathbf{B_1}$ are subalgebras of $\mathbf{A}$ and $\mathbf{B}$, respectively, the subalgebra of the free product $\mathbf{A} * \mathbf{B}$ generated by $A_1 \cup B_1$ is isomorphic to the free product $\mathbf{A_1} * \mathbf{B}_1$.

The weak amalgamation property corresponds to a variant of the deductive interpolation property. A variety $\mathcal{V}$ is said to have the *contextualized deductive interpolation property* CDIP if, for any set of variables Y, whenever

(i) $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$, $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi) = \emptyset$, and $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$;
(ii) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$,

there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that

(iii) $\Sigma \models_{\mathcal{V}} \Delta$;
(iv) $\Delta \cup \Pi \models_{\mathcal{V}} \varepsilon$;
(v) $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$.

$\mathcal{V}$ is said to have the *countable contextualized deductive interpolation property* (countable CDIP) if the above holds for the set of variables $\mathbb{X}$.

Note that $\mathcal{V}$ has the contextualized deductive interpolation property iff, for any set of variables Y, whenever

(i) $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$, $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi) = \emptyset$, $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$, and $\mathrm{Var}(\Pi) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$;
(ii) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$;

then there exist $\Delta \cup \Gamma \subseteq \mathrm{Eq}(Y)$ such that:

(iv) $\Sigma \models_{\mathcal{V}} \Delta, \Pi \models_{\mathcal{V}} \Gamma$;
(v) $\Delta \cup \Gamma \models_{\mathcal{V}} \varepsilon$; and
(vi) $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$, $\mathrm{Var}(\Gamma) \subseteq \mathrm{Var}(\Pi) \cap \mathrm{Var}(\varepsilon)$.

Theorem 5.4. *Let $\mathcal{V}$ be a variety admitting free products. The following are equivalent:*

(1) *$\mathcal{V}$ has the countable contextualized deductive interpolation property.*
(2) *$\mathcal{V}$ has the contextualized deductive interpolation property.*
(3) *$\mathcal{V}$ has the weak amalgamation property.*

Proof.
(1) $\Leftrightarrow$ (2) Again, as in the proof of Lemma 4.4, using Lemma 2.7 and compactness (Corollary 2.3).
(2) $\Rightarrow$ (3) Suppose that $\mathcal{V}$ has the CDIP. Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and let $\mathbf{A_1}$ be a subalgebras of $\mathbf{A}$. We need to prove that $\mathbf{A_1} * \mathbf{B}$ is isomorphic to the subalgebra $\langle A_1 \cup B \rangle$ of $\mathbf{A} * \mathbf{B}$ generated by $A_1 \cup B$. We may assume that $A \cup B \subseteq Y$ and $A \cap B = \emptyset$.

Let $\pi_{\mathbf{A}} \colon \mathbf{F}(A) \to \mathbf{A}$ be the homomorphism that extends the identity on A, and likewise for $\pi_{\mathbf{B}} \colon \mathbf{F}(B) \to \mathbf{B}$. Let $\Theta_{\mathbf{A}} = \ker(\pi_{\mathbf{A}})$ and $\Theta_{\mathbf{B}} = \ker(\pi_{\mathbf{B}})$. Then the homomorphism $\pi \colon \mathbf{F}(A \cup B) \to \mathbf{A} * \mathbf{B}$ that extends the identity on $A \cup B$ has as kernel the congruence $\mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}} \cup \Theta_{\mathbf{B}})$. In the diagram below

$$\begin{array}{ccc} \mathbf{F}(A_1 \cup B) & \hookrightarrow & \mathbf{F}(A \cup B) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \langle A_1 \cup B \rangle & \hookrightarrow & \mathbf{A} * \mathbf{B} \end{array}$$

$\pi' \colon \mathbf{F}(A_1 \cup B) \to \langle A_1 \cup B \rangle$ is the restriction of π to $\mathbf{F}(A_1 \cup B)$. Note that $\ker(\pi') = \mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}} \cup \Theta_{\mathbf{B}}) \cap F(A_1 \cup B)^2$.

Consider the homomorphisms $\pi_{\mathbf{A_1}} \colon \mathbf{F}(A_1) \to \mathbf{A_1}$ extending the identity map on A_1, and let $\Theta_{\mathbf{A_1}}$ be its kernel. To prove that $\langle A_1 \cup B \rangle \cong \mathbf{A_1} * \mathbf{B}$, it will suffice to verify that $\mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}} \cup \Theta_{\mathbf{B}}) \cap F(A_1 \cup B)^2 = \mathrm{Cg}_{\mathbf{F}(A_1\cup B)}(\Theta_{\mathbf{A_1}} \cup \Theta_{\mathbf{B}})$. We verify the inclusion $\mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}} \cup \Theta_{\mathbf{B}}) \cap F(A_1 \cup B)^2 \subseteq \mathrm{Cg}_{\mathbf{F}(A_1\cup B)}(\Theta_{\mathbf{A_1}} \cup \Theta_{\mathbf{B}})$, since the reverse inclusion is trivial. So let $\bar{\varepsilon}$ be an element of the left-hand side. Applying the CDIP to $\bar{\varepsilon}$ and $\mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}} \cup \Theta_{\mathbf{B}})$, we conclude that there exists subset $S \subseteq F(A_1)^2$ such that $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(A\cup B)}(S \cup \Theta_{\mathbf{B}})$ and $\mathrm{Cg}_{\mathbf{F}(A\cup B)}(S) \subseteq \mathrm{Cg}_{\mathbf{F}(A\cup B)}(\Theta_{\mathbf{A}})$. Now $S \cup \Theta_{\mathbf{B}} \subseteq F(A_1 \cup B)^2$, and so by Lemma 2.4, $\bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(A\cup B)}(S \cup \Theta_{\mathbf{B}}) \cap F(A_1 \cup B)^2 = \mathrm{Cg}_{\mathbf{F}(A_1\cup B)}(S \cup \Theta_{\mathbf{B}}) \subseteq \mathrm{Cg}_{\mathbf{F}(A_1\cup B)}(\Theta_{\mathbf{A_1}} \cup \Theta_{\mathbf{B}})$. The last inclusion follows from the relation $S \subseteq \Theta_{\mathbf{A_1}}$. The proof of the implication is now complete.
(3) $\Rightarrow$ (2) Suppose that $\mathcal{V}$ has the WAP. Let Σ, Π and ε satisfy (i) and (ii) of the CDIP. Without loss of generality we may assume that $\mathrm{Var}(\varepsilon) \subseteq$

$\mathrm{Var}(\Sigma) \cup \mathrm{Var}(\Pi)$. We need to verify conditions (iii)-(v). Define $A = \mathrm{Var}(\Sigma), B = \mathrm{Var}(\Pi)$, and $A_1 = \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon)$.

$$\begin{array}{ccc} \mathbf{F}(A_1 \cup B) & \hookrightarrow & \mathbf{F}(A \cup B) \\ \big\downarrow{\scriptstyle \bar{\pi}'} & & \big\downarrow{\scriptstyle \bar{\pi}} \\ \bar{\mathbf{A}}_\mathbf{1} * \bar{\mathbf{B}} & \hookrightarrow & \bar{\mathbf{A}} * \bar{\mathbf{B}} \end{array}$$

Let $\Phi_A = \mathrm{Cg}_{\mathbf{F}(\mathbb{X})}(\bar{\Sigma}) \cap F(A)^2$, $\Phi_B = \mathrm{Cg}_{\mathbf{F}(\mathbb{X})}(\bar{\Pi}) \cap F(B)^2$, $\bar{\mathbf{A}} = \mathbf{F}(A)/\Phi_A$, and $\bar{\mathbf{B}} = \mathbf{F}(B)/\Phi_B$. Further, let $\bar{\pi}_A \colon \mathbf{F}(A) \to \bar{\mathbf{A}}$ and $\bar{\pi}_B \colon \mathbf{F}(B) \to \bar{\mathbf{B}}$ be the canonical epimorphisms, and $\bar{\mathbf{A}}_1$ the subalgebra of $\bar{\mathbf{A}}$ with subuniverse $\bar{A}_1 = \bar{\pi}_A[A_1]$. Let $\bar{\pi} \colon \mathbf{F}(A \cup B) \to \bar{\mathbf{A}} * \bar{\mathbf{B}}$ be the epimorphism extending $\bar{\pi}_A$ and $\bar{\pi}_B$, and let $\bar{\pi}'$ be the restriction of $\bar{\pi}$ on $F(A_1 \cup B)$. Note that $\ker(\bar{\pi}) = \mathrm{Cg}_{\mathbf{F}(\mathbb{X})}(\Phi_A \cup \Phi_B) \cap F(A \cup B)^2$, and $\ker(\bar{\pi}') = \mathrm{Cg}_{\mathbf{F}(\mathbb{X})}(\Phi_A \cup \Phi_B) \cap F(A_1 \cup B)^2$. In particular, $\bar{\varepsilon} \in \ker(\bar{\pi}')$. Now the image of $\bar{\pi}'$ is the subalgebra $\langle A_1 \cup B \rangle$ of $\bar{\mathbf{A}} * \bar{\mathbf{B}}$ generated by $A_1 \cup B$. Since $\mathcal{V}$ has the WAP, it follows that $\langle A_1 \cup B \rangle \cong \bar{\mathbf{A}}_1 * \bar{\mathbf{B}}$. But this implies that $\ker(\bar{\pi}') = \mathrm{Cg}_{\mathbf{F}(A_1 \cup B_1)}([\Phi_A \cap F(A_1 \cup B)^2] \cup [\Phi_B \cap F(A_1 \cup B)^2])$. Let $\Delta, \Gamma \subseteq \mathrm{Eq}(\mathbb{X})$ such that $\bar{\Delta} = \Phi_A \cap F(A_1 \cup B_1)^2$ and $\bar{\Delta} = \Phi_B \cap F(A_1 \cup B_1)^2$. Then it is clear that $\Sigma, \Pi, \Delta, \Gamma$ and ε satisfy (iii)-(v) of the CDIP.

□

We conclude this section with a discussion on another important interpolation property: the Maehara interpolation property. This property implies all the interpolation properties discussed until now, and is equivalent to the conjunction of the deductive interpolation property and the extension property. Its algebraic counterpart is a strengthening of the amalgamation property called the transferable injections property.

A variety $\mathcal{V}$ has the *Maehara interpolation property* MIP if for any set of variables Y, whenever

(i) $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$ and $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi \cup \{\varepsilon\}) \neq \emptyset$;
(ii) $\Sigma \cup \Pi \models_\mathcal{V} \varepsilon$,

there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that

(iii) $\Sigma \models_\mathcal{V} \Delta$;
(iv) $\Delta \cup \Pi \models_\mathcal{V} \varepsilon$;
(v) $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi \cup \{\varepsilon\})$.

We say also that $\mathcal{V}$ has the *countable Maehara interpolation property* (countable MIP) if the above holds for the set of variables $\mathbb{X}$.

Lemma 5.5. *A variety $\mathcal{V}$ has the Maehara interpolation property iff it has both the extension property and the deductive interpolation property.*

Proof. ($\Rightarrow$) It is immediate that the EP follows from the MIP; also, the DIP is the special case of the MIP where $\Pi = \emptyset$.

($\Leftarrow$) Conversely, suppose that both the EP and the DIP hold, and that $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(Y)$, $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi \cup \{\varepsilon\}) \neq \emptyset$, and $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$. By the EP, there exists $\Delta' \subseteq \mathrm{Eq}(Y)$ such that $\Sigma \models_{\mathcal{V}} \Delta'$, $\Delta' \cup \Pi \models_{\mathcal{V}} \varepsilon$, and $\mathrm{Var}(\Delta') \subseteq \mathrm{Var}(\Pi \cup \{\varepsilon\})$. Then by the DIP, there exists $\Delta \subseteq \mathrm{Eq}(Y)$ such that $\Sigma \models_{\mathcal{V}} \Delta$, $\Delta \models_{\mathcal{V}} \Delta'$, and $\mathrm{Var}(\Delta) \subseteq \mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi \cup \{\varepsilon\})$. But then also $\Delta \cup \Pi \models_{\mathcal{V}} \varepsilon$ as required. □

Corollary 5.6. *The Maehara interpolation property implies the Robinson property and the contextualized deductive interpolation property.*

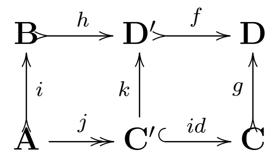
Proof. The MIP implies the DIP and the EP, which together imply the RP; also, the CDIP is the special case of the MIP where $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\Pi) = \emptyset$, and $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$. □

A variety $\mathcal{V}$ has the *transferable injections property* TIP if whenever $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$, i is an embedding of $\mathbf{A}$ into $\mathbf{B}$, and j is a homomorphism from $\mathbf{A}$ into $\mathbf{C}$, there exist an algebra $\mathbf{D} \in \mathcal{V}$, a homomorphism h from $\mathbf{B}$ into $\mathbf{D}$, and an embedding k from $\mathbf{C}$ into $\mathbf{D}$ such that $hi = kj$.

Lemma 5.7. *A variety $\mathcal{V}$ has the transferable injections property iff it has both the amalgamation property and the congruence extension property.*

Proof. If, in the definition of TIP, we assume that both i and j are embeddings, TIP provides homomorphisms $h_{x,y}$ and $k_{x,y}$ as in Lemma 3.1(2), for all $x \neq y$ in B, Hence, the AP holds. Moreover if in the definition of the TIP we assume j to be surjective, we obtain CEP. Indeed, given a subalgebra $\mathbf{A}$ of $\mathbf{B}$ in $\mathcal{V}$ and $\Theta \in \mathrm{Con}(\mathbf{A})$, let $j_\Theta \colon \mathbf{A} \to \mathbf{A}/\Theta$ be the epimorphism induced by Θ, and let $i \colon \mathbf{A} \to \mathbf{B}$ be the inclusion homomorphism. Then by the TIP, there exists $\mathbf{D} \in \mathcal{V}$ and homomorphisms $h \colon \mathbf{B} \to \mathbf{D}$ and $k \colon \mathbf{A}/\Theta \to \mathbf{D}$ such that k is an embedding, and $kj_\Theta = hi$. Since $k[\mathbf{A}/\Theta] = (hi)[\mathbf{A}] \subseteq [\mathbf{B}]$, we may assume, without loss of generality, that $\mathbf{D} = h[\mathbf{B}]$. Now let $\Phi = \ker(h)$. Since $hi = kj_\Theta$ and k is injective, it follows that Φ is an extension of Θ to $\mathbf{B}$. Hence, $\mathcal{V}$ has the CEP.

Conversely, suppose that $\mathcal{V}$ has the AP and CEP. Given $\mathbf{A}, \mathbf{B}, \mathbf{C}$, i and j as in the definition of the TIP, let $\mathbf{C}' = j[\mathbf{A}]$. Then $j \colon \mathbf{A} \to \mathbf{C}'$ is surjective, and by the CEP there are $\mathbf{D}' \in \mathcal{V}$, an embedding $h \colon \mathbf{B} \to \mathbf{D}'$ and a homomorphism $k \colon \mathbf{C}' \to \mathbf{D}'$ such that $hi = kj$. Now by AP the V-formation $\langle \mathbf{C}', \mathbf{D}', \mathbf{C}, h, id \rangle$, where $id = id_{\mathbf{C}'} \colon \mathbf{C}' \to \mathbf{C}$ is the inclusion homomorphism, has an amalgam $\langle \mathbf{D}, f, g \rangle$. But then $\mathbf{D}$ and the homomorphisms fh and g witness the validity of the TIP.



□

Theorem 5.8. *For a variety $\mathcal{V}$, the following are equivalent:*

(1) *$\mathcal{V}$ has the countable Maehara interpolation property.*
(2) *$\mathcal{V}$ has the Maehara interpolation property.*
(3) *$\mathcal{V}$ has the extension property and deductive interpolation property.*
(4) *$\mathcal{V}$ has the amalgamation property and congruence extension property.*
(5) *$\mathcal{V}$ has the transferable injections property.*

Proof. (1) $\Leftrightarrow$ (2) This equivalence follows as in the proof of Lemma 4.4, using Lemma 2.7 and compactness (Corollary 2.3):

(2) $\Leftrightarrow$ (3) Lemma 5.5.

(3) $\Leftrightarrow$ (4) The CEP and EP are equivalent by Theorem 4.6. But also, the AP implies the DIP, and the DIP and EP together imply the AP.

(4) $\Leftrightarrow$ (5) Lemma 5.7. □

The equivalence between the Maehara interpolation property and the transferable injection property is shown in [70]. That the Maehara interpolation property is equivalent to the deductive interpolation property plus the congruence extension property is shown in [17], Theorem 2.2, where the congruence extension property is called the *filter extension property*, and the equivalence of the Maehara interpolation property with the extension property plus the Robinson property is proved in [55], Theorem 4, where the Maehara interpolation property, Robinson property, and extension property are called the GINT, ROB*, and LimitedGINT, respectively.

6. Lattice-ordered Abelian Groups and MV-algebras

In this section we obtain new proofs of the amalgamation property for the varieties $\mathcal{A}$ of lattice-ordered abelian groups (abelian ℓ-groups) and $\mathcal{MV}$ of MV-algebras, by establishing the deductive interpolation property for these varieties and then applying the results of the preceding sections. We also obtain a new proof of Weinberg's theorem [67] that $\mathcal{A}$ is generated as a quasivariety by the abelian ℓ-group $\mathbf{Z}$ of the integers. Amalgamation was first established for $\mathcal{A}$ by Pierce in [58]; other algebraic proofs are given by Powell and Tsinakis in [60, 61]. The self-contained proof given below is closest to the model-theoretic approach of Weispfenning [68] based on quantifier elimination for totally and densely ordered abelian ℓ-groups. Here, however, we prove the deductive interpolation property via an elimination step that is sound for all abelian ℓ-groups and invertible for $\mathbf{Z}$, thereby obtaining also a direct proof that $\mathbf{Z}$ generates $\mathcal{A}$ as a quasivariety.

An *abelian ℓ-group* is an algebra $(A, \wedge, \vee, +, -, 0)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, +, -, 0)$ is an abelian group, and $+$ is order preserving; namely, $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in A$. The pivotal example of an abelian ℓ-group is provided by the integers $\mathbf{Z} = (\mathbb{Z}, \min, \max, +, -, 0)$.

Given elements $a, b, a_1, \ldots, a_n$ of any algebra of this signature (including $\mathbf{Fm}(\mathbb{X})$), we write as usual $a - b$ for $a + (-b)$ and $a_1 + a_2 + \ldots + a_n$ for $a_1 + (a_2 + (\ldots + a_n) \ldots)$. We also define $0a = 0$ and $(n + 1)a = a + na$, $-(n+1)a = (-n)a - a$ for each $n \in \mathbb{N}$. We call terms in $\mathrm{Fm}(\mathbb{X})$ containing only variables, $+$, $-$, and 0, *group terms*, and make frequent use of the (easily proved) fact that for any abelian ℓ-group equation $\varepsilon \in \mathrm{Eq}(\mathbb{X})$, there exist conjunctive and disjunctive forms

$$\varepsilon_1 = (0 \leq \bigwedge_{i \in I} \bigvee_{j \in J} \alpha_{ij}) \quad \text{and} \quad \varepsilon_2 = (0 \leq \bigvee_{i \in I'} \bigwedge_{j \in J'} \beta_{ij})$$

such that α_{ij} and $\beta_{i'j'}$ are group terms for all $i \in I, j \in J, i' \in I', j' \in J'$, and $\{\varepsilon_i\} \models_{\mathcal{A}} \varepsilon$ and $\{\varepsilon\} \models_{\mathcal{A}} \varepsilon_i$ for $i = 1, 2$.

The variety $\mathcal{A}$ is term-equivalent to a variety of commutative residuated lattices. These structures are defined and investigated in some detail in Section 7; here, however, we need just Lemma 7.3, which tells us in particular that $\mathcal{A}$ has the extension property and congruence extension property. Hence in order to establish the amalgamation property for this variety, it will be enough to establish the deductive interpolation property.

Let us first recall some useful facts about $\mathcal{A}$.

Lemma 6.1.

(a) $\{0 \leq (x + y) \vee z\} \models_{\mathcal{A}} 0 \leq x \vee y \vee z$;
(b) $\{0 \leq x \wedge y \wedge z\} \models_{\mathcal{A}} 0 \leq (x + y) \wedge z$;
(c) $\{0 \leq nx \vee y\} \models_{\mathcal{A}} 0 \leq x \vee y$ *for all* $n \in \mathbb{Z}^+$;
(d) *Suppose that* $\Sigma \subseteq \mathrm{Eq}(\mathbb{X})$ *and* $\alpha, \beta \in \mathrm{Fm}(\mathbb{X})$. *If* $\Sigma \cup \{0 \leq \alpha\} \models_{\mathcal{A}} \varepsilon$ *and* $\Sigma \cup \{0 \leq \beta\} \models_{\mathcal{A}} \varepsilon$, *then* $\Sigma \cup \{0 \leq \alpha \vee \beta\} \models_{\mathcal{A}} \varepsilon$.

Proof. It follows easily from the definition of an abelian ℓ-group that $\models_{\mathcal{A}} 0 \leq -x \vee x$ and therefore $\models_{\mathcal{A}} 0 \leq (-(x + y) + y) \vee (0 + x)$. Hence also $\models_{\mathcal{A}} 0 \leq (-(x + y) + (x \vee y)) \vee (0 + (x \vee y))$. So using distributivity properties, $\models_{\mathcal{A}} 0 \leq -((x + y) \wedge 0) + (x \vee y)$; i.e., $\models_{\mathcal{A}} (x + y) \wedge 0 \leq x \vee y$. Hence also $\models_{\mathcal{A}} ((x + y) \wedge 0) \vee z \leq x \vee y \vee z$. So if $0 \leq (x + y) \vee z$ holds in some abelian ℓ-group $\mathbf{A}$, then $0 \leq ((x + y) \wedge 0) \vee z$ in $\mathbf{A}$ and hence, as required, $0 \leq x \vee y \vee z$ in $\mathbf{A}$.

(b) Very similar to (a).

(c) Follows immediately from (a).

(d) Suppose that $\Sigma \cup \{0 \leq \alpha\} \models_{\mathcal{A}} \varepsilon$ and $\Sigma \cup \{0 \leq \beta\} \models_{\mathcal{A}} \varepsilon$. We may assume also that $\varepsilon = (0 \leq \gamma)$. By Lemma 7.3 twice, $\Sigma \models_{\mathcal{A}} (\alpha \wedge 0)^m \leq \gamma$ and $\Sigma \models_{\mathcal{A}} (\beta \wedge 0)^n \leq \gamma$ for some $m, n \in \mathbb{N}$. Hence $\Sigma \models_{\mathcal{A}} (\alpha \wedge 0)^{\max(m,n)} \vee (\beta \wedge 0)^{\max(m,n)} \leq \gamma$. Since (by a simple induction) $(a \vee b)^n \leq a^n \vee b^n$ in all abelian ℓ-groups, we obtain $\Sigma \models_{\mathcal{A}} ((\alpha \wedge 0) \vee (\beta \wedge 0))^{\max(m,n)} \leq \gamma$. By distributivity, $\Sigma \models_{\mathcal{A}} ((\alpha \vee \beta) \wedge 0)^{\max(m,n)} \leq \gamma$. So finally by Lemma 7.3 again, $\Sigma \cup \{0 \leq \alpha \vee \beta\} \models_{\mathcal{A}} \varepsilon$. ☐

We now establish the crucial lemma (essentially a quantifier elimination step) that allows us to reduce the number of different variables in consequences between equations. Since it will be helpful to have an equation that fails in all non-trivial abelian ℓ-groups, we reserve a variable $x_0 \in \mathbb{X}$ and fix $\mathbb{X}_0 = \mathbb{X} - \{x_0\}$.

Lemma 6.2. *If $\delta \in \mathrm{Eq}(\mathbb{X}_0)$, $\varepsilon \in \mathrm{Eq}(\mathbb{X}_0) \cup \{0 \leq x_0\}$, and $x \in (\mathrm{Var}(\varepsilon) \cup \mathrm{Var}(\delta)) - \{x_0\}$, then there exist $\delta' \in \mathrm{Eq}(\mathbb{X}_0 - \{x\})$ and $\varepsilon' \in \mathrm{Eq}(\mathbb{X}_0 - \{x\}) \cup \{0 \leq x_0\}$ satisfying:*

(i) $\{\delta\} \models_{\mathcal{A}} \delta'$;
(ii) *If $\{\delta'\} \models_{\mathcal{A}} \varepsilon'$, then $\{\delta\} \models_{\mathcal{A}} \varepsilon$;*
(iii) *If $x \notin \mathrm{Var}(\delta)$, then $\delta' = \delta$ and $\{\varepsilon'\} \models_{\mathcal{A}} \varepsilon$;*
(iv) *If $x \notin \mathrm{Var}(\varepsilon)$, then $\varepsilon' = \varepsilon$;*
(v) *If $\{\delta\} \models_{\mathbf{Z}} \varepsilon$, then $\{\delta'\} \models_{\mathbf{Z}} \varepsilon'$.*

Proof. Using basic equivalences in abelian ℓ-groups and Lemma 6.1, we may assume $\delta = (0 \leq \bigwedge D)$ and $\varepsilon = (0 \leq \bigvee E)$ where for some $n \in \mathbb{Z}^+$:

$$\begin{array}{l} \emptyset \neq D \subseteq \{\gamma_\delta, (\alpha' - nx), (nx + \beta')\} \\ \emptyset \neq E \subseteq \{\gamma_\varepsilon, (\alpha - nx), (nx + \beta)\} \\ x \notin \mathrm{Var}(\{\gamma_\delta, \gamma_\varepsilon, \alpha, \beta, \alpha', \beta'\}). \end{array}$$

Let D' and E' be the smallest sets satisfying

$$\begin{array}{rcrcl} & & \gamma_\delta \in D & \Rightarrow & \gamma_\delta \in D' \\ & & \gamma_\varepsilon \in E & \Rightarrow & \gamma_\varepsilon \in E' \\ (\alpha' - nx) \in D & \text{and} & (nx + \beta') \in D & \Rightarrow & (\alpha' + \beta') \in D' \\ (\alpha - nx) \in E & \text{and} & (nx + \beta) \in E & \Rightarrow & (\alpha + \beta) \in E' \\ (\alpha - nx) \in D & \text{and} & (\alpha' - nx) \in E & \Rightarrow & (\alpha - \alpha') \in E' \\ (nx + \beta) \in D & \text{and} & (nx + \beta') \in E & \Rightarrow & (\beta - \beta') \in E'. \end{array}$$

We define

$$\delta' = \begin{cases} 0 \leq 0 & \text{if } D' = \emptyset \\ 0 \leq \bigwedge D' & \text{otherwise} \end{cases} \quad \text{and} \quad \varepsilon' = \begin{cases} 0 \leq x_0 & \text{if } E' = \emptyset \\ 0 \leq \bigvee E' & \text{otherwise.} \end{cases}$$

Notice first that (i), i.e., $\{\delta\} \models_{\mathcal{A}} \delta'$, follows directly from Lemma 6.1 (b). Also, by Lemma 6.1 (a):

$$\begin{array}{rcl} \{0 \leq \alpha + \beta\} & \models_{\mathcal{A}} & 0 \leq (\alpha - nx) \vee (nx + \beta) \\ \{0 \leq \alpha - \alpha', 0 \leq \alpha' - nx\} & \models_{\mathcal{A}} & 0 \leq \alpha - nx \\ \{0 \leq \beta - \beta', 0 \leq nx + \beta'\} & \models_{\mathcal{A}} & 0 \leq nx + \beta. \end{array}$$

Hence, by repeated applications of Lemma 6.1 (d), $\{\varepsilon', \delta\} \models_{\mathcal{A}} \varepsilon$. Consider, e.g., the most complicated case:

$$\begin{array}{l} \{0 \leq \gamma_\varepsilon \vee (\alpha + \beta) \vee (\alpha - \alpha') \vee (\beta - \beta'), \\ \qquad 0 \leq \gamma_\delta \wedge (\alpha' - nx) \wedge (nx + \beta')\} \quad \models_{\mathcal{A}} \quad \gamma_\varepsilon \vee (\alpha - nx) \vee (nx + \beta). \end{array}$$

So if $\{\delta'\} \models_{\mathcal{A}} \varepsilon'$, then, since also $\{\delta\} \models_{\mathcal{A}} \delta'$, we obtain $\{\delta\} \models_{\mathcal{A}} \varepsilon$ as required for (ii).

Suppose $x \notin \mathrm{Var}(\delta)$. Then $\delta' = \delta = (0 \leq \gamma_\delta)$. Also, either $\varepsilon' = \varepsilon = (0 \leq \gamma_\varepsilon)$, or $\varepsilon = (0 \leq \gamma_\varepsilon \vee (\alpha - nx) \vee (nx + \beta))$ and $\varepsilon' = (0 \leq \gamma_\varepsilon \vee (\alpha + \beta))$. Hence, using Lemma 6.1 (a) for the latter case, $\{\varepsilon'\} \models_{\mathcal{A}} \varepsilon$. I.e., (iii) is satisfied. Similarly, if $x \notin \mathrm{Var}(\varepsilon)$, then $\varepsilon' = \varepsilon = (0 \leq \gamma_\varepsilon)$, so (iv) is satisfied.

Finally, for (v), let us assume, since other cases are very similar (and in fact easier), that

$$\begin{array}{rcl} \delta & = & (0 \leq \gamma_\delta \wedge (\alpha' - nx) \wedge (nx + \beta')) \\ \varepsilon & = & (0 \leq \gamma_\varepsilon \vee (\alpha - nx) \vee (nx + \beta)) \\ \delta' & = & (0 \leq \gamma_\delta \wedge (\alpha' + \beta')) \\ \varepsilon' & = & (0 \leq \gamma_\varepsilon \vee (\alpha + \beta) \vee (\alpha - \alpha') \vee (\beta - \beta')). \end{array}$$

Suppose contrapositively that $\{\delta'\} \not\models_{\mathbf{Z}} \varepsilon'$. Then for some $\varphi\colon \mathbf{Fm}(\mathbb{X} - \{x\}) \to \mathbf{Z}$, we have $0 \leq \varphi(\gamma_\delta)$, $-\varphi(\alpha') \leq \varphi(\beta')$, $\varphi(\gamma_\varepsilon) < 0$, $\varphi(\alpha) < -\varphi(\beta)$, $\varphi(\alpha) < \varphi(\alpha')$, and $\varphi(\beta) < \varphi(\beta')$. These same inequalities hold for any $m \in \mathbb{Z}^+$ and $\varphi_m\colon \mathbf{Fm}(\mathbb{X} - \{x\}) \to \mathbf{Z}$ defined by $\varphi_m(y) = m\varphi(y)$ for all $y \in \mathbb{X} - \{x\}$. Hence we may assume that $\varphi(\alpha)$ is divisible by $2n$ for any $\alpha \in \mathbf{Fm}(\mathbb{X} - \{x\})$.

Our aim now is to extend φ to $\varphi\colon \mathbf{Fm}(\mathbb{X}) \to \mathbf{Z}$ by choosing an appropriate value of x such that $0 \leq \varphi(\delta)$ and $0 > \varphi(\varepsilon)$. We define

$$\varphi(x) = \frac{\min(\varphi(\alpha'), -\varphi(\beta)) + \max(\varphi(\alpha), -\varphi(\beta'))}{2n}.$$

By calculation in $\mathbf{Z}$, it follows that $\varphi(\alpha') \geq \varphi(nx)$, $-\varphi(nx) \geq \varphi(\beta')$, $\varphi(\alpha) < \varphi(nx)$, and $-\varphi(nx) < \varphi(\beta)$. So $\{\delta\} \not\models_{\mathbf{Z}} \varepsilon$ as required. □

This reduction lemma provides the key ingredient for a new proof of Weinberg's generation theorem [67] for abelian ℓ-groups.

Theorem 6.3. *For any* $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\mathbb{X})$*:*

$$\Sigma \models_{\mathcal{A}} \varepsilon \qquad \Leftrightarrow \qquad \Sigma \models_{\mathbf{Z}} \varepsilon.$$

I.e., $\mathcal{A}$ is generated by $\mathbf{Z}$ as a quasivariety.

Proof. The left-to-right direction is immediate. For the converse direction, by Corollary 2.3 and renaming of variables, we can consider a finite $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\mathbb{X}_0)$. Then, by taking a suitable equation in place of Σ, it suffices to show that for any $\delta \in \mathrm{Eq}(\mathbb{X}_0)$ and $\varepsilon \in \mathrm{Eq}(\mathbb{X}_0) \cup \{0 \leq x_0\}$:

$$\{\delta\} \models_{\mathbf{Z}} \varepsilon \qquad \Rightarrow \qquad \{\delta\} \models_{\mathcal{A}} \varepsilon.$$

We prove this claim by induction on $|(\mathrm{Var}(\varepsilon) \cup \mathrm{Var}(\delta)) - \{x_0\}|$. If $\mathrm{Var}(\varepsilon) \cup \mathrm{Var}(\delta) = \emptyset$, then ε and δ contain no variables and both hold in all members of $\mathcal{A}$. I.e., $\{\delta\} \models_{\mathcal{A}} \varepsilon$. If $\mathrm{Var}(\varepsilon) = \{x_0\}$ and $\mathrm{Var}(\delta) = \emptyset$, then $\varepsilon = (0 \leq x_0)$. Hence, evaluating x_0 as -1, we obtain $\{\delta\} \not\models_{\mathbf{Z}} \varepsilon$.

For the induction step, pick $x \in (\mathrm{Var}(\varepsilon) \cup \mathrm{Var}(\delta)) - \{x_0\}$. By Lemma 6.2, we obtain $\delta' \in \mathrm{Eq}(\mathbb{X}_0 - \{x\})$, $\varepsilon' \in \mathrm{Eq}(\mathbb{X}_0 - \{x\}) \cup \{0 \leq x_0\}$ such that

$$\{\delta\} \models_{\mathbf{Z}} \varepsilon \;\;\Rightarrow\;\; \{\delta'\} \models_{\mathbf{Z}} \varepsilon' \qquad \text{and} \qquad \{\delta'\} \models_{\mathcal{A}} \varepsilon' \;\;\Rightarrow\;\; \{\delta\} \models_{\mathcal{A}} \varepsilon.$$

If $\{\delta\} \models_{\mathbf{Z}} \varepsilon$, then by the first implication, $\{\delta'\} \models_{\mathbf{Z}} \varepsilon'$. So, by the induction hypothesis, $\{\delta'\} \models_{\mathcal{A}} \varepsilon'$, and, by the second implication, $\{\delta\} \models_{\mathcal{A}} \varepsilon$. □

Note, moreover, that the proof of Lemma 6.2 describes explicitly how to check $\Sigma \models_{\mathcal{A}} \varepsilon$ when Σ is finite by constructing Σ' and ε' containing one fewer variable than Σ and ε, such that $\Sigma \models_{\mathcal{A}} \varepsilon$ iff $\Sigma' \models_{\mathcal{A}} \varepsilon'$. Repeating the process with Σ' and ε', the algorithm terminates after finitely many steps. That is, the quasiequational theory of $\mathcal{A}$ is shown here to be decidable.

We now tackle the main result of this section for abelian ℓ-groups.

Theorem 6.4. *The variety of abelian ℓ-groups has the deductive interpolation property, amalgamation property, and Robinson property.*

Proof. By the results of previous sections and the fact that the variety $\mathcal{A}$ admits the CEP, it suffices to show that $\mathcal{A}$ has the DIP. Suppose then that:

(i) $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\mathbb{X})$ with $\mathrm{Var}(\Sigma) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$;
(ii) $\Sigma \models_{\mathcal{A}} \varepsilon$.

Then there exists a finite $\Sigma' \subseteq \Sigma$ with $\Sigma' \models_{\mathcal{A}} \varepsilon$. Moreover, we may assume (using variable renaming if necessary) that $\Sigma \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\mathbb{X}_0)$. Hence, by rewriting using equivalences in abelian ℓ-groups, there also exists $\delta \in \mathrm{Eq}(\mathbb{X}_0)$ with $\mathrm{Var}(\delta) \cap \mathrm{Var}(\varepsilon) \neq \emptyset$ such that $\{\delta\} \models_{\mathcal{A}} \varepsilon$ and $\Sigma \models_{\mathcal{A}} \delta$. It therefore suffices to show that there exists $\gamma \in \mathrm{Eq}(\mathbb{X}_0)$ such that

(iii) $\{\delta\} \models_{\mathcal{A}} \gamma$;
(iv) $\{\gamma\} \models_{\mathcal{A}} \varepsilon$;
(v) $\mathrm{Var}(\gamma) \subseteq \mathrm{Var}(\delta) \cap \mathrm{Var}(\varepsilon)$.

We proceed by induction on $|\mathrm{Var}(\delta) - \mathrm{Var}(\varepsilon)|$. If $\mathrm{Var}(\delta) \subseteq \mathrm{Var}(\varepsilon)$, then we can take γ to be δ. Otherwise, choose $x \in \mathrm{Var}(\delta) - \mathrm{Var}(\varepsilon)$. Using Lemma 6.2 and Theorem 6.3, we obtain $\delta' \in \mathrm{Eq}(\mathbb{X}_0)$ with $\mathrm{Var}(\delta') \subseteq$

$\mathrm{Var}(\delta) - \{x\}$ (noting that in Lemma 6.2, the ε' is ε since $x \notin \mathrm{Var}(\varepsilon)$) such that $\{\delta\} \models_{\mathcal{A}} \delta'$ and $\{\delta'\} \models_{\mathcal{A}} \varepsilon$. By the induction hypothesis, there exists $\gamma \in \mathrm{Eq}(\mathbb{X}_0)$ such that $\{\delta'\} \models_{\mathcal{A}} \gamma$, $\{\gamma\} \models_{\mathcal{A}} \varepsilon$, and $\mathrm{Var}(\gamma) \subseteq \mathrm{Var}(\delta') \cap \mathrm{Var}(\varepsilon)$. But then also $\{\delta\} \models_{\mathcal{A}} \gamma$ and $\mathrm{Var}(\gamma) \subseteq \mathrm{Var}(\delta) \cap \mathrm{Var}(\varepsilon)$, so we are done. □

We turn our attention now to the variety $\mathcal{MV}$ of MV-algebras, the algebraic semantics of Łukasiewicz logic. Amalgamation for $\mathcal{MV}$ follows from the categorical equivalence between MV-algebras and abelian ℓ-groups with strong unit established in [53] (see also [12, 14] for further details regarding MV-algebras and Łukasiewicz logic). A direct geometrical proof established via the Robinson property for $\mathcal{MV}$ is given by Busaniche and Mundici in [10]. Here we provide a shorter proof that makes use of the fact that consequences in the standard MV-algebra $[\mathbf{0}, \mathbf{1}]$ on the real unit interval can be translated back-and-forth between consequences in an abelian ℓ-group with strong unit based on the real numbers. The categorical equivalence is not required for this step; however, in concluding that the variety of MV-algebras has the deductive interpolation property, we make use of Di Nola and Lettieri's result that the standard MV-algebra generates $\mathcal{MV}$ as a quasivariety. We note also that a related proof of the deductive interpolation property for $\mathcal{MV}$, obtained independently by Mundici in a geometric setting (subsequently to the proof given here) has appeared in [11].

Let $\mathcal{L}_{\mathcal{MV}}$ be the signature with a binary operation $\oplus$, a unary operation $\neg$, and a constant 0. An *MV-algebra* is an algebraic structure $\mathbf{A} = (A, \oplus, \neg, 0)$ for $\mathcal{L}_{\mathcal{MV}}$ such that $(A, \oplus, 0)$ is a commutative monoid and $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ for all $x, y \in A$. Further operations are defined as $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \vee y = \neg(\neg x \oplus y) \oplus y$, and $x \wedge y = \neg(\neg x \vee \neg y)$. It is easily verified that with these definitions $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.

The fundamental example of an MV-algebra is $[\mathbf{0}, \mathbf{1}] = ([0, 1], \oplus, \neg, 0)$ where $x \oplus y = \min(x + y, 1)$ and $\neg x = 1 - x$. We will make use in what follows of the result that $\mathcal{MV}$ is generated as a quasivariety by this algebra (see [14] for a proof and references). It is shown that $\mathcal{MV}$ has the deductive interpolation property and hence also the amalgamation property, by exploiting a relationship between the algebra $[\mathbf{0}, \mathbf{1}]$ and the abelian ℓ-group with strong unit $\mathbf{R} = (\mathbb{R}, \min, \max, +, -, 0, 1)$ in the signature of abelian ℓ-groups with an additional constant 1, which we denote $\mathcal{L}_{\mathcal{A}}$.

The following lemma is then proved in almost exactly the same way as Lemma 6.2. The main differences, which result in no essential changes in the proof are, firstly, that conditions (i) and (ii) refer to the single structure $\mathbf{R}$ rather than a class of structures, and, secondly, that since there is an additional constant 1, the equation that fails in $\mathbf{R}$ may be taken to be $1 \leq 0$ and there is no need for a reserved variable.

Lemma 6.5. *Suppose that* $\delta, \varepsilon \in \mathrm{Eq}(\mathbb{X})$ *and* $x \in \mathrm{Var}(\varepsilon) \cup \mathrm{Var}(\delta)$. *Then there exist* $\delta', \varepsilon' \in \mathrm{Eq}(\mathbb{X} - \{x\})$ *such that*

(i) $\{\delta\} \models_{\mathbf{R}} \delta'$;
(ii) *If* $\{\delta'\} \models_{\mathbf{R}} \varepsilon'$, *then* $\{\delta\} \models_{\mathbf{R}} \varepsilon$;
(iii) *If* $x \notin \mathrm{Var}(\delta)$, *then* $\delta' = \delta$ *and* $\{\varepsilon'\} \models_{\mathbf{R}} \varepsilon$;
(iv) *If* $x \notin \mathrm{Var}(\varepsilon)$, *then* $\varepsilon' = \varepsilon$;
(v) *If* $\{\delta\} \models_{\mathbf{R}} \varepsilon$, *then* $\{\delta'\} \models_{\mathbf{R}} \varepsilon'$.

Now for a formula α of $\mathcal{L}_{\mathcal{A}}$, let us define (following [14, 64]):

$$\alpha^{\#} = (\alpha \wedge 0) \vee 1.$$

The next lemma then follows by an induction on the number of symbols in a formula of $\mathcal{L}_{\mathcal{MV}}$:

Lemma 6.6. *For any formula* β *of* $\mathcal{L}_{\mathcal{MV}}$, *there exists a formula* α *of* $\mathcal{L}_{\mathcal{A}}$ *such that* $\mathrm{Var}(\alpha) = \mathrm{Var}(\beta)$ *and* $\beta^{[\mathbf{0},\mathbf{1}]} = (\alpha^{\#})^{\mathbf{R}}$.

Now consider any group formula α of $\mathcal{L}_{\mathcal{A}}$ built using $+$, $-$, 0, and 1. Such an α is equivalent in $\mathbf{R}$ to (assuming an order on the variables) a formula of the form $k + \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \in \mathbb{Z}$ for $i = 1 \ldots n$ and k stands for $k(1)$ with $k \in \mathbb{Z}$. Following again [14, 64], we define the formula β_α of $\mathcal{L}_{\mathcal{MV}}$ by induction on $\sigma(\alpha) = \sum_{i=1}^{n} |\lambda_i|$:

(1) If $\sigma(\alpha) = 0$, then α is equivalent to k and define:

$$\beta_\alpha = \begin{cases} 1 & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0. \end{cases}$$

(2) For $\sigma(\alpha) > 0$, let $j = \min\{i \mid \lambda_i \neq 0\}$ and define:

$$\beta_\alpha = \begin{cases} (\beta_{\alpha - x_j} \oplus x_j) \odot \beta_{\alpha - x_j + 1} & \text{if } \lambda_j > 0 \\ \neg((\beta_{1-\alpha-x_j} \oplus x_j) \odot \beta_{2-\alpha-x_j}) & \text{otherwise.} \end{cases}$$

This definition is extended to all formulas α of $\mathcal{L}_{\mathcal{A}}$ by observing that as for abelian ℓ-groups, α is equivalent in $\mathbf{R}$ to a formula $\bigwedge_{i \in I} \bigvee_{j \in J_i} \alpha_{ij}$ where each α_{ij} is a group term, and we can set $\beta_\alpha = \bigwedge_{i \in I} \bigvee_{j \in J_i} \beta_{\alpha_{ij}}$.

Lemma 6.7. *For each formula* α *of* $\mathcal{L}_{\mathcal{A}}$: $(\alpha^{\#})^{\mathbf{R}} = \beta_\alpha^{[\mathbf{0},\mathbf{1}]}$.

Proof. Let us consider the case where $\alpha = k + \sum_{i=1}^{n} \lambda_i x_i$ is a group formula of $\mathcal{L}_{\mathcal{A}}$ since the more general case follows easily. We proceed by induction on $\sigma(\alpha)$. The base case is immediate. For $\sigma(\alpha) > 0$, suppose without loss of generality that $\min\{i \mid \lambda_i \neq 0\} = 1$. For $\lambda_1 > 0$, let $\alpha' = k + (\lambda_1 - 1)x_1 + \sum_{i=2}^{n} \lambda_i x_i$; i.e., α' is equivalent to $\alpha - x_1$. Then $\beta_\alpha = (\alpha' \oplus x_1) \odot \beta_{\alpha'+1}$.

By the induction hypothesis $(\alpha'^{\#})^{\mathbf{R}} = \beta_{\alpha'}^{[\mathbf{0},\mathbf{1}]}$ and $((\alpha'+1)^{\#})^{\mathbf{R}} = \beta_{\alpha'+1}^{[\mathbf{0},\mathbf{1}]}$. We may then easily check in $\mathbf{R}$ that $(\alpha^{\#})^{\mathbf{R}} = ((\alpha'+x_1)^{\#})^{\mathbf{R}} = \beta_{\alpha}^{[\mathbf{0},\mathbf{1}]}$. The case where $\lambda_1 < 0$ is very similar. □

Theorem 6.8. *The variety of MV-algebras has the deductive interpolation property, amalgamation property, and Robinson property.*

Proof. It suffices to show that for any formulas β_1, β_2 of $\mathcal{L}_{\mathcal{MV}}$ such that $\{1 \leq \beta_1\} \models_{[\mathbf{0},\mathbf{1}]} 1 \leq \beta_2$, there exists a formula β_3 of $\mathcal{L}_{\mathcal{MV}}$ with $\mathrm{Var}(\beta_3) \subseteq \mathrm{Var}(\beta_1) \cap \mathrm{Var}(\beta_2)$ satisfying $\{1 \leq \beta_1\} \models_{[\mathbf{0},\mathbf{1}]} 1 \leq \beta_3$ and $\{1 \leq \beta_3\} \models_{[\mathbf{0},\mathbf{1}]} 1 \leq \beta_2$.

By Lemma 6.6, there exist formulas α_1, α_2 of $\mathcal{L}_{\mathcal{A}}$ such that $\mathrm{Var}(\alpha_1) = \mathrm{Var}(\beta_1)$, $\mathrm{Var}(\alpha_2) = \mathrm{Var}(\beta_2)$, $\beta_1^{[\mathbf{0},\mathbf{1}]} = (\alpha_1^{\#})^{\mathbf{R}}$, and $\beta_2^{[\mathbf{0},\mathbf{1}]} = (\alpha_2^{\#})^{\mathbf{R}}$. Hence $\{1 \leq \alpha_1^{\#}\} \models_{\mathbf{R}} 1 \leq \alpha_2^{\#}$, and by Lemma 6.5, there exists α_3 with $\mathrm{Var}(\alpha_3) \subseteq \mathrm{Var}(\alpha_1) \cap \mathrm{Var}(\alpha_2)$ satisfying $\{1 \leq \alpha_1^{\#}\} \models_{\mathbf{R}} 1 \leq \alpha_3$ and $\{1 \leq \alpha_3\} \models_{\mathbf{R}} 1 \leq \alpha_2^{\#}$. But then also $\{1 \leq \alpha_1^{\#}\} \models_{\mathbf{R}} 1 \leq \alpha_3^{\#}$ and $\{1 \leq \alpha_3^{\#}\} \models_{\mathbf{R}} 1 \leq \alpha_2^{\#}$. By Lemma 6.7, there exists $\beta_3 = \beta_{\alpha_3}$ with $\mathrm{Var}(\beta_3) \subseteq \mathrm{Var}(\beta_1) \cap \mathrm{Var}(\beta_2)$ such that $(\alpha_3^{\#})^{\mathbf{R}} = \beta_3^{[\mathbf{0},\mathbf{1}]}$ and we have $\{1 \leq \beta_1\} \models_{[\mathbf{0},\mathbf{1}]} 1 \leq \beta_3$ and $\{1 \leq \beta_3\} \models_{[\mathbf{0},\mathbf{1}]} 1 \leq \beta_2$ as required. □

7. Residuated Lattices

In this section we establish some general conditions for the amalgamation property (and therefore, by the results of Sections 3 through 5, also the deductive interpolation property) in varieties of residuated lattices and pointed residuated lattices. These varieties provide algebraic semantics for substructural logics as well as encompassing other important classes of algebras such as lattice-ordered groups (see [23] for a comprehensive overview). We focus here in particular on semilinear (also referred to in the literature as "representable") varieties: that is, varieties generated by their totally ordered members (see, e.g., [8]).

A *residuated lattice* is an algebra $\mathbf{L} = (L, \cdot, \backslash, /, \vee, \wedge, e)$ satisfying:

(a) $(L, \cdot, e)$ is a monoid;
(b) $(L, \vee, \wedge)$ is a lattice with order $\leq$;
(c) $\backslash$ and $/$ are binary operations satisfying the residuation property:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y.$$

The symbol $\cdot$ is often omitted when writing elements of these algebras.

A *pointed residuated lattice* is an algebra $\mathbf{L} = (L, \cdot, \backslash, /, \vee, \wedge, e, f)$ whose residuated lattice reduct $(L, \cdot, \backslash, /, \vee, \wedge, e)$ is a residuated lattice. Since residuated lattices may be identified with pointed residuated lattices satisfying $e = f$, general theorems applying to classes of pointed residuated lattices apply also to classes of residuated lattices, although not necessarily vice versa. We will therefore write pointed residuated lattice when we mean an algebra of either of these classes.

We also define here a *bounded residuated lattice* to be a pointed residuated lattice with bottom element f (and therefore also, top element $f\backslash f$), emphasizing that "bounded" implies that the constant f representing the bottom element is in the signature. A pointed residuated lattice is said to be *integral* if its top element is e. Note, however, that in a bounded residuated lattice, e may not be the top element, and, conversely, an integral pointed residuated lattice may not be bounded.

A pointed residuated lattice is *commutative* if it satisfies $xy = yx$, in which case, $x\backslash y$ and y/x coincide and are denoted by $x \to y$. Let us also define $x^0 = e$ and $x^{n+1} = x(x^n)$ for $n \in \mathbb{N}$. Then a pointed residuated lattice is said to be *idempotent* if it satisfies $x^2 = x$ and, more generally, n-*potent* for $2 \leq n \in \mathbb{N}$, if it satisfies $x^{n+1} = x^n$. A pointed residuated lattice is said to be *divisible* if $x \leq y$ implies $y(y\backslash x) = (x/y)y = x$, *cancellative* if $xyu = xzu$ implies $y = z$, and *semilinear* if it is isomorphic to a subdirect product of totally ordered pointed residuated lattices.

It is easily shown (see [8], [5]) that the class of pointed residuated lattices forms a congruence distributive variety and that all the preceding conditions may be expressed equationally. This variety provides algebraic semantics for the Full Lambek calculus (pointed residuated lattices are therefore often referred to also as FL-algebras) and its subvarieties correspond to substructural logics. Moreover, lattice-ordered groups (or ℓ-*groups*) (see [3], [26]) can be presented as residuated lattices satisfying $x(x\backslash e) = e$. It suffices to let $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$.

Let us now briefly recall some structure theory for pointed residuated lattices, referring to [8], [35], and [23] for further details. We fix a pointed residuated lattice $\mathbf{L}$. If $F \subseteq L$, we write F^- for the set of "negative" elements of F; i.e., $F^- = \{x \in F \mid x \leq e\}$. The *negative cone* of a residuated lattice $\mathbf{L}$ is the algebra $\mathbf{L}^-$ with domain L^- and lattice operations and the monoid operation of $\mathbf{L}^-$, the restrictions to $\mathbf{L}^-$ of the corresponding operations in $\mathbf{L}$. The residuals $\backslash^-$ and $/^-$ are defined by

$$x\backslash^- y = (x\backslash y) \wedge e \qquad \text{and} \qquad y/^- x = (y/x) \wedge e$$

where $\backslash$ and $/$ denote the residuals of $\mathbf{L}$.

Given $a \in L$, define $\rho_a(x) = (ax/a) \wedge e$ and $\lambda_a(x) = (a\backslash xa) \wedge e$, for all $x \in L$. We refer to ρ_a and λ_a respectively as *right conjugation* and *left conjugation* by a. An *iterated conjugation* map is a composition $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$, where each γ_i is a right conjugation or a left conjugation by an element $a_i \in L$. The set of all iterated conjugation maps will be denoted by Γ.

A lattice filter F of a pointed residuated lattice $\mathbf{L}$ is said to be *normal* if (i) it contains e; (ii) it is *multiplicative*, that is, it is closed under multiplication, and (iii) for all $x \in F$ and $y \in L$, $y\backslash xy \in F$ and $yx/y \in F$. Note that condition (iii) is equivalent to the following condition: (iii)$'$ F is closed under all iterated conjugation maps.

Given a normal filter F of $\mathbf{L}$, $\Theta_F = \{(x,y) \in L^2 \mid (x\backslash y) \wedge (y\backslash x) \in F\}$ is a congruence of $\mathbf{L}$. Conversely, given a congruence Θ, the upper set $F_\Theta = \uparrow[e]_\Theta$ of the equivalence class $[e]_\Theta$ is a normal filter. Moreover:

Lemma 7.1 ([8], [35], [7]; see also [69] or [23])**.** *The lattice $\mathcal{N}F(\mathbf{L})$ of normal filters of a pointed residuated lattice $\mathbf{L}$ is isomorphic to its congruence lattice $\mathrm{Con}(\mathbf{L})$. The isomorphism is given by the mutually inverse maps $F \mapsto \Theta_F$ and $\Theta \mapsto \uparrow[e]_\Theta$.*

In what follows, if F is a normal filter of $\mathbf{L}$, we write $\mathbf{L}/F$ for the quotient algebra $\mathbf{L}/\Theta_F$. We mention the trivial fact that in a commutative pointed residuated lattice, a lattice filter is normal iff it satisfies conditions (i) and (ii) above.

Lemma 7.2 ([8], [35])**.** *If F is a normal filter of a pointed residuated lattice $\mathbf{L}$, then $[e]_{\Theta_F} = \{x \mid x \wedge (x\backslash e) \wedge e \in F\} = \{x \mid \exists\, a \in F^-, a \leq x \leq a\backslash e\}$.*

Amalgamation and interpolation properties for varieties of pointed residuated lattices (and their associated substructural logics) have been investigated by a number of authors (see, e.g., [24], [52], [44], [49], [50]), very often making use of various relationships between these properties. Let us begin here by recalling some useful facts for *commutative* varieties. In particular, such varieties possess the following local deduction property, a more refined version of the extension property discussed in Section 4.

The following is a reformulation of Lemma 2.7 and Corollary 2.8 in [32].

Lemma 7.3 ([32])**.** *Let $\mathcal{V}$ be a variety of pointed commutative residuated lattices. The following are equivalent for $\Sigma \subseteq \mathrm{Eq}(\mathbb{X})$ and $\alpha, \beta \in \mathbf{Fm}(\mathbb{X})$:*

(1) $\Sigma \cup \{e \leq \alpha\} \models_{\mathcal{V}} e \leq \beta$.
(2) $\Sigma \models_{\mathcal{V}} (\alpha \wedge e)^n \leq \beta$ *for some* $n \in \mathbb{N}$.

In particular, $\mathcal{V}$ has the extension property.

It then follows by the results of previous sections that:

Corollary 7.4. *For any variety $\mathcal{V}$ of pointed commutative residuated lattices:*

(a) *$\mathcal{V}$ has the congruence extension property (equivalently, the extension property).*

(b) *$\mathcal{V}$ has the amalgamation property (equivalently, the Robinson property or Pigozzi property) iff $\mathcal{V}$ has the deductive interpolation property iff $\mathcal{V}$ has the Maehara interpolation property (equivalently, the transferable injections property).*

The deductive interpolation property and hence also the amalgamation property, has been established for many varieties of pointed commutative residuated lattices using a proof-theoretic strategy. In fact a stronger (in the presence of commutativity) property, the Craig interpolation property, is established. Namely, it is shown that if $\models_{\mathcal{V}} \alpha \leq \beta$, then there exists γ with $\mathrm{Var}(\gamma) \subseteq \mathrm{Var}(\alpha) \cap \mathrm{Var}(\beta)$ such that $\models_{\mathcal{V}} \alpha \leq \gamma$ and $\models_{\mathcal{V}} \gamma \leq \beta$. This approach requires a suitable cut-free Gentzen-style calculus for the variety and typically constructs interpolants by induction on the height of derivations in the calculus. It is successful for varieties of (integral, idempotent, n-potent, bounded) commutative residuated lattices [56,57] but usually fails in the presence of divisibility and cancellativity where suitable calculi are not available. If commutativity is lacking, then Craig interpolation may be established, but the deductive interpolation property and amalgamation property may not follow. Moreover, in the presence of semilinearity, Craig interpolation may be known to fail (see, e.g., [50]) but this provides no information on whether or not the deductive interpolation property and amalgamation property hold for the variety. The deductive interpolation property and amalgamation property have also been established for certain varieties of pointed commutative residuated lattices, including MV-algebras ([54]) and BL-algebras ([52]). More recent proofs using a model-theoretic strategy based on quantifier-elimination can be found in [15], [49], and [50].[1]

In the remainder of this section, we describe some general conditions for the amalgamation property in varieties of pointed residuated lattices. In particular, we show that a variety $\mathcal{V}$ of semilinear residuated lattices satisfying the congruence extension property has the amalgamation property iff the class $\mathcal{V}_{lin}$ of totally ordered members of $\mathcal{V}$ has the amalgamation property (Theorem 7.9). We also investigate the connection between the amalgamation property for a class of bounded residuated lattices and the class of its residuated lattice reducts (Theorem 7.10), and the amalgamation property in the join of two independent varieties of residuated lattices (Theorem 7.12). These conditions will be employed in subsequent sections to obtain new results for some specific varieties.

Our proof of Theorem 7.9 below makes use of Theorem 3.2 and four auxiliary lemmas. Recall first that a normal filter F in a pointed residuated lattice $\mathbf{L}$ is called *prime* if it is prime in the usual lattice-theoretic sense; that is, whenever $x, y \in L$ satisfy $x \vee y \in F$, then $x \in F$ or $y \in F$.

[1]Indeed, certain results of [49] and [50] rely on the previously announced Theorem 7.9.

Lemma 7.5. *Let $\mathbf{L}$ be a semilinear pointed residuated lattice, and let F be a normal filter of $\mathbf{L}$. The following statements are equivalent:*

(1) *F is prime.*
(2) *For all $x, y \in L$, whenever $x \vee y = e$, then $x \in F$ or $y \in F$.*
(3) *$\mathbf{L}/F$ is totally ordered.*

Proof. (1) $\Rightarrow$ (2) By specialization.

(2) $\Rightarrow$ (3) Assume (2) holds, and let $[x], [y] \in L/F$. We need to show that $[x] \leq [y]$ or $[y] \leq [x]$. Now in $\mathbf{L}$, $e \leq (x\backslash y) \vee (y\backslash x)$, since $\mathbf{L}$ is semilinear, and so $((x\backslash y)\wedge e)\vee((y\backslash x)\wedge e) = e$. By (2), either $(x\backslash y)\wedge e \in F$ or $(y\backslash x) \wedge e \in F$. Without loss of generality, we may assume that $(x\backslash y) \wedge e \in F$. Hence, by Lemma 7.2, $[(x\backslash y) \wedge e] = [e]$, so $([x]\backslash[y]) \wedge [e] = [e]$. Thus, $[e] \leq [x]\backslash[y]$ in $\mathbf{L}/F$, which implies that $[x] \leq [y]$. This establishes condition (3).

(3) $\Rightarrow$ (1) Suppose that condition (3) is satisfied, and let $x, y \in L$ such that $x \vee y \in F$. We need to prove that $x \in F$ or $y \in F$. Now, $x \vee y \in F$ implies that $(x \vee y) \wedge e = (x \wedge e) \vee (y \wedge e) \in F$, and so by Lemma 7.2, $[(x \wedge e) \vee (y \wedge e)] = [e]$. Thus, $[x \wedge e] \vee [y \wedge e] = [e]$ in $\mathbf{L}/F$. Since $\mathbf{L}/F$ is totally ordered, we may assume that $[x \wedge e] \leq [y \wedge e]$, and so, $[y \wedge e] = [e]$. It follows that $y \in {\uparrow}[e] = F$. ☐

Lemma 7.6 ([8])**.** *Let $\mathbf{L}$ be a pointed residuated lattice, and let $S \subseteq L^-$. Denote by Γ the set of all iterated conjugate maps on $\mathbf{L}$. Then the normal filter of $\mathbf{L}$ generated by S is the upper set $\uparrow \hat{S}$, where $\hat{S}$ is the submonoid of $\mathbf{L}$ generated by $\{\gamma(s) \mid s \in S, \gamma \in \Gamma\}$.*

Lemma 7.7 ([8])**.** *Let $\mathbf{L}$ be a pointed residuated lattice and $\{a_i \mid 1 \leq i \leq n\}, \{b_j \mid 1 \leq j \leq m\} \subseteq L^-$ finite subsets of the negative cone of $\mathbf{L}$ with the property that $a_i \vee b_j = e$, for any i and j. Then $(\prod_{i=1}^n a_i) \vee (\prod_{j=1}^m b_j) = e$.*

Lemma 7.8 ([8])**.** *Suppose that $\mathbf{L}$ is a semilinear residuated lattice. Then for all $a, b \in L^-$ and for any iterated conjugation maps γ_1, γ_2, if $a \vee b = e$ then $\gamma_1(a) \vee \gamma_2(b) = e$.*

Theorem 7.9. *Let $\mathcal{V}$ be a variety of semilinear pointed residuated lattices with the congruence extension property, and let $\mathcal{V}_{lin}$ be the class of totally ordered members of $\mathcal{V}$. If every V-formation in $\mathcal{V}_{lin}$ has an amalgam in $\mathcal{V}$, then $\mathcal{V}$ has the amalgamation property.*

Proof. It suffices by Theorem 3.2 to show that:

(a) All subdirectly irreducible members of $\mathcal{V}$ are in $\mathcal{V}_{lin}$.
(b) $\mathcal{V}_{lin}$ is closed under (isomorphic images and) subalgebras.
(c) For any $\mathbf{B} \in \mathcal{V}$, any subalgebra $\mathbf{A}$ of $\mathbf{B}$, and $P \in \mathcal{NF}(\mathbf{A})$ such that $\mathbf{A}/P \in \mathcal{V}_{lin}$, there is $Q \in \mathcal{NF}(\mathbf{B})$ such that $Q \cap A = P$ and $\mathbf{B}/Q \in \mathcal{V}_{lin}$.

It is clear that (a) and (b) are satisfied. We prove (c). Let $\mathbf{A}, \mathbf{B}$ and P be as in the statement of (c). Since $\mathcal{V}$ has the CEP, there is a normal filter F of $\mathbf{B}$, such that $P = F \cap A$. Let $\mathcal{X}$ denote the poset, under set-inclusion, of all normal filters of $\mathbf{B}$ whose intersection with A is P. $\mathcal{X} \neq \emptyset$, since $F \in \mathcal{X}$. By Zorn's Lemma, $\mathcal{X}$ has a maximal element Q.

Given elements $x \in A$ and $y \in B$, we write $[x]_P$ for the equivalence class of x in $\mathbf{A}/P$, and $[y]_Q$ for the equivalence class of y in $\mathbf{B}/Q$. Since $P = Q \cap A$, the map $\varphi\colon \mathbf{A}/P \to \mathbf{B}/Q$ is an embedding.

We complete the proof of (c) by showing that Q is a prime normal filter of $\mathbf{B}$, and hence, by Lemma 7.5, $\mathbf{B}/Q \in \mathcal{V}_{lin}$. Suppose otherwise, and let $a, b \in B$ such that $a \vee b = e$, but $a \notin Q$ and $b \notin Q$. Let Q_a and Q_b be the normal filters of $\mathbf{B}$ generated by $Q \cup \{a\}$ and $Q \cup \{b\}$, respectively. Then, by the maximality of Q, P is a proper subset of the normal filters $Q_a \cap A$ and $Q_b \cap A$ of $\mathbf{A}$. By Lemma 7.6, there exist elements $c \in A - P, d \in B - P, q_1, \ldots, q_k, r_1, \ldots, r_l \in Q^-, \gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_l \in \Gamma$ (the set of iterated conjugation maps), and $n_1, \ldots, n_k, m_1, \ldots, m_l \in \mathbb{Z}^+$, such that $\prod_{i=1}^k q_i\gamma_i(a^{n_i}) \le c \le e$, and $\prod_{j=1}^l r_j\delta_j(a^{m_j}) \le d \le e$. Since $a \vee b = e$, by Lemmas 7.7 and 7.8, $(\prod_{i=1}^k \gamma_i(a^{n_i})) \vee (\prod_{j=1}^l \delta_j(a^{m_j})) = e$. Note further that, in view of Lemma 7.2, for any i, j, $[q_i]_Q = [r_j]_Q = [e]_Q$. Thus, $[\prod_{i=1}^k q_i\gamma_i(a^{n_i})]_Q \vee [\prod_{j=1}^l r_j\delta_j(a^{m_j})]_Q = [\prod_{i=1}^k \gamma_i(a^{n_i})]_Q \vee [\prod_{j=1}^l \delta_j(a^{m_j})]_Q = [(\prod_{i=1}^k \gamma_i(a^{n_i})) \vee (\prod_{j=1}^l \delta_j(a^{m_j}))]_Q = [e]_Q$. On the other hand, since $\mathbf{A}/P$ is totally ordered, we may assume that $[c]_P \le [d]_P$ in $\mathbf{A}/P$, and so $[c]_Q \le [d]_Q$ in $\mathbf{B}/Q$. But then,

$$[e]_Q = [\prod_{i=1}^k q_i\gamma_i(a^{n_i})]_Q \vee [\prod_{j=1}^l r_j\delta_j(a^{m_j})]_Q \le [c]_Q \vee [d]_Q = [d]_Q.$$

Since $[d]_Q \le [e]_Q$, we get that $[d]_Q = [e]_Q$. But then $d \in Q \cap A = P$, which is a contradiction. Thus, Q is a prime normal filter of $\mathbf{B}$, and the proof of the theorem is complete. □

Given a class $\mathcal{K}$ of integral bounded residuated lattices, let us fix $\mathcal{K}_*$ to be the class of all residuated lattice subreducts of algebras in $\mathcal{K}$ (i.e., subalgebras of reducts of algebras in $\mathcal{K}$ in the language of residuated lattices). The next result relates the amalgamation property for a variety $\mathcal{V}$ of integral bounded residuated lattices to the amalgamation property for the class $\mathcal{V}_*$.

Theorem 7.10. *Let $\mathcal{V}$ be a variety of integral bounded residuated lattices that has the congruence extension property.*

(a) *$\mathcal{V}_*$ is a variety.*
(b) *If $\mathcal{K} \subseteq \mathcal{V}$ and $\mathcal{V} = \mathbb{V}(\mathcal{K})$, then $\mathcal{V}_* = \mathbb{V}(\mathcal{K}_*)$.*

(c) *If $\mathcal{V}_*$ has the amalgamation property, then $\mathcal{V}$ has the amalgamation property.*

Proof. (a) $\mathcal{V}_*$ is clearly closed under subalgebras and direct products. To prove that $\mathcal{V}_*$ is closed under quotients, consider $\mathbf{A} \in \mathcal{V}$, and let $\mathbf{B}$ be a residuated lattice subreduct of $\mathbf{A}$ and F a normal filter of $\mathbf{B}$. Since $\mathcal{V}$ has the CEP, there exists a normal filter F' of $\mathbf{A}$ such that $F' \cap B = F$. It follows that $\mathbf{B}/F$ embeds into the residuated lattice reduct of $\mathbf{A}/F'$, and that $\mathbf{A}/F' \in \mathcal{V}$. So $\mathbf{B}/F \in \mathcal{V}_*$.

(b) The residuated lattice reducts of homomorphic images (subalgebras, direct products, respectively) of algebras in $\mathcal{K}$ are clearly homomorphic images (subalgebras, direct products) of the residuated lattice reducts of the same algebras considered as elements of $\mathcal{K}_*$. Hence $\mathbb{V}(\mathcal{K}_*)$ contains all the residuated lattice reducts of algebras of $\mathcal{V}$. Since $\mathbb{V}(\mathcal{K}_*)$ is closed under subalgebras, it contains all residuated lattice subreducts of algebras in $\mathcal{V}$. That is, $\mathcal{V}_* \subseteq \mathbb{V}(\mathcal{K}_*)$, and since $\mathcal{V}_*$ is a variety by (a) that contains $\mathcal{K}_*$, $\mathcal{V}_* = \mathbb{V}(\mathcal{K}_*)$.

(c) Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in $\mathcal{V}$, and let $\mathbf{A}_*, \mathbf{B}_*$, and $\mathbf{C}_*$ denote the residuated lattice reducts of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$. Since $\mathcal{V}_*$ has the AP, $(\mathbf{A}_*, \mathbf{B}_*, \mathbf{C}_*, i, j)$ has an amalgam $(\mathbf{D}_*, h, k)$ in $\mathcal{V}_*$. Let f be the minimum of $\mathbf{A}$. Then $i(f)$ is the minimum of $\mathbf{B}$ and $j(f)$ is the minimum of $\mathbf{C}$. Let $m = h(i(f)) = k(j(f))$. Then m is an idempotent element of $\mathbf{D}_*$. Now let D_m be the set of all $d \in D_*$ such that $m \leq d \leq e$. Then, making use of integrality, D_m is closed under all operations of residuated lattices and hence is the domain of a subalgebra $\mathbf{D}_m$ of $\mathbf{D}_*$. Now let $\mathbf{D}_{m,f}$ be the algebra obtained from $\mathbf{D}_m$ by adding the interpretation of f as m. Then $(\mathbf{D}_{m,f}, h, k)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{V}$. □

We close this section by investigating the amalgamation property for joins of independent varieties of residuated lattices. Recall that two varieties $\mathcal{U}$, $\mathcal{V}$ of the same signature are said to be *independent* ([30]) provided there exists a binary term $t(x, y)$ such that

$$\mathcal{U} \models t(x, y) \approx x \quad \text{and} \quad \mathcal{V} \models t(x, y) \approx y.$$

It is shown in [30] that if $\mathcal{U}$ and $\mathcal{V}$ are independent varieties, then they are disjoint, meaning that their intersection consists of trivial algebras, and $\mathcal{U} \vee \mathcal{V} = \mathcal{U} \times \mathcal{V}$. The last equation simply means that every algebra in $\mathcal{U} \times \mathcal{V}$ decomposes as a direct product of an algebra in $\mathcal{U}$ and an algebra in $\mathcal{V}$. The following partial converse, established in [42], also holds: If $\mathcal{U}$ and $\mathcal{V}$ are disjoint subvarieties of a congruence permutable variety, then $\mathcal{U}$ and $\mathcal{V}$ are independent and $\mathcal{U} \vee \mathcal{V} = \mathcal{U} \times \mathcal{V}$. In particular, two varieties of residuated lattices are independent iff they are disjoint.

Examples of independent varieties of residuated lattices are:

(1) Any variety of ℓ-groups and any variety of integral residuated lattices. In this case, let $t(x,y) = ((x\backslash e)\backslash e)y(y\backslash e)$.
(2) Any variety of n-potent integral residuated lattices and any variety of negative cones of ℓ-groups. In this example, we can let $t(x,y) = (x^n\backslash x^{n+1})((y^n\backslash y^{n+1})\backslash y)$.

Lemma 7.11. *For two independent varieties of residuated lattices $\mathcal{U},\mathcal{V}$:*

(a) *For every algebra $\mathbf{A} \in \mathcal{U} \vee \mathcal{V}$ there are subalgebras $\mathbf{A}_1 \in \mathcal{U}$ and $\mathbf{A}_2 \in \mathcal{V}$ of $\mathbf{A}$ such that $\mathbf{A}$ is the direct sum $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$; that is, $A_1 \cap A_2 = \{e\}$, and for every element $a \in A$ there are uniquely determined elements $x \in A_1$ and $y \in A_2$ such that $a = xy$. Moreover, $\mathbf{A}$ is isomorphic to $\mathbf{A}_1 \times \mathbf{A}_2$.*
(b) *With the same notation as in (a), let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2, \mathbf{B} = \mathbf{B}_1 \oplus \mathbf{B}_2 \in \mathcal{U} \vee \mathcal{V}$, and let $\varphi\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then there are homomorphisms $\varphi_i\colon \mathbf{A}_i \to \mathbf{B}_i$ $(i = 1, 2)$ such that for all $x \in A_1$ and $y \in A_2$, $\varphi(xy) = \varphi_1(x)\varphi_2(y)$. (We express this fact by writing $\varphi = \varphi_1 \oplus \varphi_2$.) Moreover, φ is an embedding iff both φ_1 and φ_2 are embeddings.*

Proof. (a) We know that $\mathbf{A} = \mathbf{A}'_1 \times \mathbf{A}'_2$ for some $\mathbf{A}'_1 \in \mathcal{U}$ and $\mathbf{A}'_2 \in \mathcal{V}$. Let $A_1 = \{(u, e_{\mathbf{A}'_2}) \mid u \in A'_1\}$ and $A_2 = \{(e_{\mathbf{A}'_1}, v) \mid v \in A'_2\}$. It is easy to see that A_1 and A_2 are the domains of subalgebras of $\mathbf{A}$ with the desired properties.

(b) Let φ_i be the restriction of φ to A_i $(i = 0, 1)$. Then, for $x \in A_1$ and $y \in A_2$, $\varphi(xy) = \varphi(x)\varphi(y) = \varphi_1(x)\varphi_2(y)$. Moreover $\varphi(x) \in \varphi[A_1]$ and $\varphi(y) \in \varphi[A_2]$. Since $\varphi(A_i)$ is the domain of an algebra in $\mathcal{V}_i$, and $\mathcal{V}_1, \mathcal{V}_2$ have in common only trivial algebras, $\varphi[A_i] \subseteq B_i$ $(i = 0, 1)$, $\varphi(x) \in B_1$ and $\varphi(y) \in B_2$.

If φ is an embedding, then also φ_1 and φ_2 are embeddings, because they are restrictions of φ. Moreover, by the uniqueness of the decomposition, if φ_1 and φ_2 are embeddings, then φ is also an embedding. □

Theorem 7.12. *Let $\mathcal{U}$, $\mathcal{V}$ be two independent varieties of residuated lattices. The following are equivalent:*

(1) *Both $\mathcal{U}$ and $\mathcal{V}$ have the amalgamation property.*
(2) *The join $\mathcal{U} \vee \mathcal{V}$ (in the lattice of subvarieties of residuated lattices) has the amalgamation property.*

Proof. Suppose that $\mathcal{U} \vee \mathcal{V}$ has the AP. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in $\mathcal{U}$. Then there is an amalgam $(\mathbf{D}, h, k)$ of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{U} \vee \mathcal{V}$. Moreover, by Lemma 7.11, $\mathbf{D}$ has the form $\mathbf{D}_1 \oplus \mathbf{D}_2$ with $\mathbf{D}_1 \in \mathcal{U}$ and $\mathbf{D}_2 \in \mathcal{V}$, and h and k are embeddings of $\mathbf{B}$ and $\mathbf{C}$, respectively, into $\mathbf{D}_1 \oplus \mathbf{D}_2$. Since $h[\mathbf{B}]$, $k[\mathbf{C}] \in \mathcal{U}$, $h[B] \cap D_2 = k[C] \cap D_2 = \{e\}$. Hence, $h[\mathbf{B}]$ and $k[\mathbf{C}]$

are subalgebras of $\mathbf{D}_1$, and $(\mathbf{D}_1, h, k)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{U}$. The same argument shows that $\mathcal{V}$ has the AP.

Conversely, suppose that $\mathcal{U}$ and $\mathcal{V}$ have the AP. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in $\mathcal{U} \vee \mathcal{V}$. By Lemma 7.11, $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, $\mathbf{B} = \mathbf{B}_1 \oplus \mathbf{B}_2$, and $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2$ with $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1 \in \mathcal{U}$ and $\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2 \in \mathcal{V}$. Moreover, i and j decompose as $i = i_1 \oplus i_2$ and $j = j_1 \oplus j_2$, and so $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, i_1, j_1)$ is a V-formation in $\mathcal{U}$, and $(\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2, i_2, j_2)$ is a V-formation in $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ have the AP, there is an amalgam $(\mathbf{D}_1, h_1, k_1)$ of $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, i_1, j_1)$ in $\mathcal{U}$, and an amalgam $(\mathbf{D}_2, h_2, k_2)$ of $(\mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2, i_2, j_2)$ in $\mathcal{V}$. It follows that $(\mathbf{D}_1 \oplus \mathbf{D}_2, h_1 \oplus h_2, k_1 \oplus k_2)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{U} \vee \mathcal{V}$. □

8. GBL-ALGEBRAS AND GMV-ALGEBRAS

In this section, we investigate the amalgamation property and deductive interpolation property for some varieties of residuated lattices that both enjoy a close relationship with lattice-ordered groups and encompass important classes of algebras from logic such as MV-algebras and BL-algebras. In particular, a *GBL-algebra* (see [25]) is a residuated lattice satisfying the divisibility condition

$$x \leq y \quad \text{implies} \quad y(y\backslash x) = (x/y)y = x,$$

and a *GMV-algebra* is a residuated lattice satisfying

$$y/((x\backslash y) \wedge e) = ((y/x) \wedge e)\backslash y = x \vee y.$$

A GMV-algebra is a GBL-algebra, but the converse need not be true. An *MV-algebra* may be defined as a commutative integral bounded residuated lattice whose residuated lattice reduct is a GMV-algebra.[2] A *BL-algebra* is a commutative integral semilinear bounded residuated lattice whose residuated lattice reduct is a GBL-algebra. Moreover, a *Heyting algebra* may be defined as (or is term-equivalent to) a commutative integral idempotent bounded residuated lattice whose residuated lattice reduct is a GBL-algebra; a *Gödel algebra* is a semilinear Heyting algebra.

Let us recall some principal results from the literature on GBL-algebras and GMV-algebras. First, note that in [5] it is shown that the class of negative cones of ℓ-groups, the class of cancellative and integral GMV-algebras, and the class of cancellative and integral GBL-algebras all coincide. Moreover, each GBL-algebra decomposes as follows:

[2]This definition is equivalent to that given in Section 6: for an MV-algebra as defined there, we obtain an MV-algebra in the new sense by letting $x \cdot y = \neg(\neg x \oplus \neg y)$ and $x \to y = \neg x \oplus y$. Conversely, for an MV-algebra according to the present definition, we obtain an MV-algebra as defined in Section 6 by letting $\neg x = x \to f$ and $x \oplus y = (x \to f) \to y$.

Proposition 8.1 ([25])**.** *Every GBL-algebra (GMV-algebra, respectively) is a direct product of an ℓ-group and an integral GBL-algebra (GMV-algebra, respectively).*

Corollary 8.2 ([33])**.** *Any totally ordered GMV-algebra is either an ℓ-group, the GMV-reduct of a bounded and integral GMV-algebra, or the negative cone of an ℓ-group.*

Thus we obtain a general negative result:

Corollary 8.3. *Any variety $\mathcal{V}$ of GBL-algebras containing the class of all ℓ-groups does not have the amalgamation property. In particular, the variety of GBL-algebras and the variety of GMV-algebras do not have the amalgamation property.*

Proof. If $\mathcal{V}$ is the class of all ℓ-groups, then it does not have the AP by a result of Pierce [58]. Otherwise, by Proposition 8.1, $\mathcal{V}$ is the join of the variety of ℓ-groups and a variety of integral GBL-algebras. Such varieties are independent. Since the variety of ℓ-groups does not have the AP, the claim follows from Theorem 7.12. □

Since, by this corollary, the varieties of GMV-algebras and GBL-algebras do not themselves have the amalgamation property, we focus our attention in this section on smaller classes where we are able to obtain various characterizations of algebras admitting the amalgamation property. We first provide a complete account of the varieties of commutative GMV-algebras with the amalgamation property (Theorem 8.11), then investigate whether amalgamation holds or fails for various classes of commutative GBL-algebras and n-potent GBL-algebras (Theorems 8.14, 8.16, 8.17, 8.23, and 8.24).

8.1. **Commutative GMV-algebras.** A neat characterization of the amalgamable varieties of *MV-algebras* is provided by Di Nola and Lettieri in [20]: a variety of MV-algebras has the amalgamation property iff it is generated by a single chain. Observe, however, that for commutative GMV-algebras, the absence of the constant f makes a significant difference with respect to embeddings. In particular, any two GMV-algebras $\mathbf{A}$ and $\mathbf{B}$ embed into their direct product $\mathbf{A} \times \mathbf{B}$ via the embeddings h and k defined, for $a \in A$ and $b \in B$, by $h(a) = (a, e_{\mathbf{B}})$ and $k(b) = (e_{\mathbf{A}}, b)$. But if $\mathbf{A}$ and $\mathbf{B}$ are MV-algebras, then the maps h and k defined above do not preserve f and are therefore not MV-homomorphisms (they are, of course, homomorphisms of their GMV-reducts). More generally, if $\mathbf{A}$ and $\mathbf{B}$ are finite MV-chains such that neither embeds into the other, then their GMV-reducts embed into the direct product, but the whole algebras do not.

In order to determine which varieties of commutative GMV-algebras admit the amalgamation property, we first provide a general description of these varieties. Varieties of MV-algebras are fully described in [19], while a description of the varieties of commutative integral GMV-algebras may be found in [2]. This latter characterization refers, however, to *Wajsberg hoops* (see [6]): subreducts of MV-algebras with respect to the signature $\cdot, \to, e$. These algebras are lattice-ordered by $x \leq y$ iff $x \to y = e$ with lattice operations defined by $x \wedge y = x \cdot (x \to y)$ and $x \vee y = (x \to y) \to y$, and $\cdot$ and $\to$ form a residuated pair with respect to this order; i.e., $x \cdot y \leq z$ iff $x \leq y \to z$. Hence, Wajsberg hoops can be extended to commutative integral residuated lattices satisfying $(x \to y) \to y = x \vee y = (y \to x) \to x$, and are therefore term equivalent to commutative integral GMV-algebras.

Since every subdirectly irreducible Wajsberg hoop is either the negative cone of a totally ordered abelian group or the reduct of an MV-chain [2], Wajsberg hoops are semilinear, so are also commutative integral GMV-algebras. Moreover, by Proposition 8.1, every commutative GMV-algebra is the direct product of an abelian ℓ-group and an integral commutative GMV-algebra, so commutative GMV-algebras are semilinear.

In order to describe the varieties $\mathcal{MV}$ of MV-algebras and $\mathcal{CIGMV}$ of commutative integral GMV-algebras (equivalently, Wajsberg hoops), we begin by recalling some fundamental facts about these algebras. Note first that each MV-algebra can be identified with the interval $[e, u]$ of an abelian ℓ-group $\mathbf{G}$ with strong unit u (i.e., for all $a \in G$, there exists $n \in \mathbb{N}$ such that $a \leq u^n$), written $(\mathbf{G}, u)$, with operations $\cdot$ and $\to$ defined by $x \cdot y = (xyu^{-1}) \vee e$ and $x \to y = (ux^{-1}y) \wedge u$. The MV-algebra obtained in this way is denoted $\Gamma(\mathbf{G}, u)$. With reference to the definition given in Section 6, an MV-algebra is characterized as the interval $[e, u]$ of $(\mathbf{G}, u)$, with operations $\oplus$ and $\neg$ defined by $x \oplus y = (xy) \wedge u$ and $\neg x = ux^{-1}$.

The connection between MV-algebras and abelian ℓ-groups with strong unit may be carried further. Let $(\mathbf{G}, u)$, $(\mathbf{H}, w)$ be abelian ℓ-groups with strong unit. A morphism from $(\mathbf{G}, u)$ into $(\mathbf{H}, w)$ is a homomorphism h from $\mathbf{G}$ into $\mathbf{H}$ such that $h(u) = w$. For any such morphism h we denote by $\Gamma(h)$ its restriction to $\Gamma(\mathbf{G}, u)$. Then Γ becomes a functor from the category of abelian ℓ-groups with strong unit into the category of MV-algebras, with homomorphisms as morphisms. Moreover Γ has an adjoint Γ^{-1} such that the pair (Γ, Γ^{-1}) is an equivalence of categories (see [53] for details).

Here, we will refer in fact to the isomorphic copy $\Gamma(\mathbf{G}, u^{-1})$ of $\Gamma(\mathbf{G}, u)$ defined as follows: the domain of $\Gamma(\mathbf{G}, u^{-1})$ is the interval $[u^{-1}, e]$ with top element e and bottom element u^{-1}, $x \cdot y = xy \vee u^{-1}$, and $x \to y = x^{-1}y \wedge e$. We make use in particular of the abelian ℓ-groups $\mathbf{R}$ of reals and $\mathbf{Z}$ of integers, denoting the group operation in these cases by $+$, the neutral element by 0, and the inverse of x by $-x$.

The lattice of subvarieties of MV-algebras and the lattice of subvarieties of commutative GMV-algebras can now be described as follows (see [19] and [2]):

(1) The variety $\mathcal{MV}$ of MV-algebras is generated by $\Gamma(\mathbf{R}, -1)$ and the variety $\mathcal{CIGMV}$ of commutative integral GMV-algebras is generated by its GMV-reduct $\Gamma(\mathbf{R}, -1)_*$ (see Theorem 7.10).

(2) Every proper subvariety of $\mathcal{MV}$ is generated by a finite number of chains having one of the forms: $\mathbf{L}_n = \Gamma(\mathbf{Z}, -n)$ or $\mathbf{K}_n = \Gamma(\mathbf{Z} \times_{lex} \mathbf{Z}, (-n, 0))$, where $n \in \mathbb{N}$ and $\mathbf{Z} \times_{lex} \mathbf{Z}$ is the product of two copies of $\mathbf{Z}$ with the lexicographic order. The varieties generated by $\mathbf{L}_n$ and $\mathbf{K}_n$ will be denoted by $\mathcal{MV}_n$ and $\mathcal{MV}_n^\omega$, respectively.

(3) Every proper subvariety of $\mathcal{CIGMV}$ is generated by a finite number of chains of the form $\mathbf{L}_{n*}$, $\mathbf{K}_{n*}$ (the GMV-reducts of $\mathbf{L}_n$ and $\mathbf{K}_n$), and $\mathbf{Z}^-$ (the negative cone of $\mathbf{Z}$). The varieties of GMV-algebras generated by $\mathbf{L}_{n*}$, $\mathbf{K}_{n*}$, and $\mathbf{Z}^-$ will be denoted by $\mathcal{MV}_{n*}$, $\mathcal{MV}_{n*}^\omega$ and $\mathcal{A}^-$, respectively.

(4) The following inclusions hold (see [19], [2]): $\mathcal{MV}_n \subseteq \mathcal{MV}_m$ iff $\mathcal{MV}_n^\omega \subseteq \mathcal{MV}_m^\omega$ iff $\mathcal{MV}_n \subseteq \mathcal{MV}_m^\omega$ iff $\mathcal{MV}_{n*} \subseteq \mathcal{MV}_{m*}$ iff $\mathcal{MV}_{n*}^\omega \subseteq \mathcal{MV}_{m*}^\omega$ iff $\mathcal{MV}_{n*} \subseteq \mathcal{MV}_{m*}^\omega$ iff n divides m. Moreover, each $\mathcal{MV}_n$ and $\mathcal{MV}_n^\omega$ is properly included in $\mathcal{MV}$, and each $\mathcal{MV}_{n*}$ and $\mathcal{MV}_{n*}^\omega$ is properly included in $\mathcal{CIGMV}$. Finally, for each n, $\mathcal{A}^- \subset \mathcal{MV}_{n*}^\omega$. No other inclusions between varieties of the form $\mathcal{MV}$, $\mathcal{MV}_n$, and $\mathcal{MV}_n^\omega$ or of the form $\mathcal{CIGMV}$, $\mathcal{A}^-$, $\mathcal{MV}_{n*}$, and $\mathcal{MV}_{n*}^\omega$ hold.

(5) By Proposition 8.1, the variety $\mathcal{CGMV}$ of commutative GMV-algebras is the join of the independent varieties $\mathcal{CIGMV}$ and the variety of abelian ℓ-groups $\mathcal{A}$, and is therefore generated by $\Gamma(\mathbf{R}, -1)_*$ and $\mathbf{Z}$. Moreover, since $\mathcal{A}$ is an atom in the lattice of varieties of residuated lattices, any proper subvariety of $\mathcal{CGMV}$ which is not contained in $\mathcal{CIGMV}$ is generated by $\mathbf{Z}$ and a finite number of algebras of the form $\mathbf{L}_{n*}$, $\mathbf{K}_{n*}$, or $\mathbf{Z}^-$, and hence is the join of $\mathcal{A}$ and a finite number of varieties of the form $\mathcal{MV}_{n*}$, $\mathcal{MV}_{n*}^\omega$, or $\mathcal{A}^-$.

Since any variety of residuated lattices is congruence distributive, we can make use of Jónsson's result [41] that each subdirectly irreducible member of a congruence distributive join of two varieties is in one of the two varieties, to obtain:

Lemma 8.4.

(a) *Let $\mathbf{A}$ be any commutative integral GMV-algebra that does not generate the whole of $\mathcal{CIGMV}$. Then $\mathbf{A}$ is a subdirect product of a finite number of algebras from $\mathcal{A}^-$, $\mathcal{MV}_n$, and $\mathcal{MV}_n^\omega$.*

(b) *Let* $\mathbf{A}$ *be any commutative GMV-algebra that generates a variety not containing* $\mathcal{CIGMV}$. *Then* $\mathbf{A}$ *is a subdirect product of a finite number of algebras from* $\mathcal{A}$, $\mathcal{A}^-$, $\mathcal{MV}_{n*}$, *and* $\mathcal{MV}^{\omega}_{n*}$.

Now let $\mathbf{A} = \Gamma(\mathbf{G}, u^{-1})$ be an MV-algebra and $n \in \mathbb{Z}^+$. We say that u^{-1} *has an* n^{th} *root in* $\mathbf{A}$ if there is a (necessarily unique) element $x \in A$ (denoted by $u^{-\frac{1}{n}}$) such that $x^n = u^{-1}$ in $\mathbf{G}$. In what follows we write $u^{-\frac{k}{n}}$ instead of $(u^{-\frac{1}{n}})^k$. We say that $\nu \leq e$ is a *co-infinitesimal* of $\mathbf{A}$ if $\nu^n \geq u^{-1}$ holds in $\mathbf{G}$ for each $n \in \mathbb{N}$. It is easy to prove that ν is co-infinitesimal iff the equation $\nu = \nu^n \to \nu^{n+1}$ holds in $\Gamma(\mathbf{G}, u^{-1})$ for all $n \in \mathbb{N}$. Finally, the set of all co-infinitesimals of an MV-algebra is the intersection of all its maximal filters (co-radical). The following lemma is almost immediate:

Lemma 8.5. *For any MV-algebra* $\mathbf{A} = \Gamma(\mathbf{G}, u^{-1})$ *and* $n \in \mathbb{Z}^+$*:*

(a) $\mathbf{L}_n$ *embeds into* $\mathbf{A}$ *iff* u^{-1} *has an* n^{th} *root in* $\mathbf{A}$. *Moreover, the isomorphic image of* $\mathbf{L}_n$ *consists of* $e, u^{-\frac{1}{n}}, u^{-\frac{2}{n}}, \ldots, u^{-\frac{(n-1)}{n}}, u^{-1}$.

(b) *If* $\mathbf{L}_n$ *embeds into* $\mathbf{A}$ *and* ν *is a co-infinitesimal of* $\mathbf{A}$, *then* $\nu > u^{-\frac{1}{n}}$.

As a consequence, we obtain a slightly simpler proof and a generalization of two results from [19].

Lemma 8.6. *For any MV-algebra* $\mathbf{A} = \Gamma(\mathbf{G}, u^{-1})$ *and* $m, n \in \mathbb{Z}^+$ *with* $q = lcm(m, n)$*:*

(a) *If* $\mathbf{A}$ *has subalgebras isomorphic to* $\mathbf{L}_n$ *and* $\mathbf{L}_m$, *then it has a subalgebra isomorphic to* $\mathbf{L}_q$.

(b) *If* $\mathbf{A}$ *has subalgebras isomorphic to* $\mathbf{L}_n$ *and* $\mathbf{K}_m$, *then it has a subalgebra isomorphic to* $\mathbf{K}_q$.

Proof. (a) By Lemma 8.5, u^{-1} has both an n^{th} root and an m^{th} root in $\mathbf{A}$. Hence, $\mathbf{A}$ contains isomorphic copies of $\mathbf{L}_n$ and $\mathbf{L}_m$ consisting of the elements $e, u^{-\frac{1}{n}}, u^{-\frac{2}{n}}, \ldots, u^{-\frac{n-1}{n}}, u^{-1}$ and $e, u^{-\frac{1}{m}}, u^{-\frac{2}{m}}, \ldots, u^{-\frac{m-1}{m}}, u^{-1}$, respectively. We claim that u^{-1} has a q^{th} root in $\mathbf{A}$. Let $d = gcd(n, m)$. Then $mn = qd$, and there are integers a, b such that $an + bm = d$, and hence $a\frac{1}{m} + b\frac{1}{n} = \frac{1}{q}$. Consider now the divisible hull, $\mathbf{G}_d$, of $\mathbf{G}$. In $\mathbf{G}_d$, $u^{-\frac{1}{q}} = u^{-\frac{a}{m} - \frac{b}{n}}$ and since $u^{-\frac{1}{m}}$ and $u^{-\frac{1}{n}}$ are in $\mathbf{G}$, $u^{-\frac{1}{q}} = (u^{-\frac{1}{m}})^a (u^{-\frac{1}{n}})^b \in \mathbf{G}$. Hence, u^{-1} has a q^{th} root in $\mathbf{A}$ and $\mathbf{L}_q$ embeds into $\mathbf{A}$.

(b) Let ν be the element in $\mathbf{A}$ corresponding to $(0, -1) \in \mathbf{K}_m = \Gamma(\mathbf{Z} \times_{lex} \mathbf{Z}, (m, 0))$. Then ν is a co-infinitesimal of $\mathbf{A}$. Since $\mathbf{L}_m$ is a subalgebra of $\mathbf{K}_m$, $\mathbf{L}_m$ and $\mathbf{L}_n$ are isomorphic to subalgebras of $\mathbf{A}$, and, by (a), $\mathbf{A}$ contains the q^{th} root $u^{-\frac{1}{q}}$ of u^{-1}. Now let for every $a, b \in \mathbf{Z}$, $\phi(a, b) = u^{\frac{a}{q}} \nu^{-b}$. Then ϕ is an embedding of $\mathbf{Z} \times_{lex} \mathbf{Z}$ into $\mathbf{G}$ such that $\phi(-q, 0) = u^{-1}$. Hence, the restriction of ϕ to $[(-q, 0), (0, 0)]$ is an embedding of $\mathbf{K}_q$ into $\mathbf{A}$. $\square$

We make use of the following result from [14] and a useful corollary:

Proposition 8.7 ([14])**.**

(a) *An MV-chain belongs to* $\mathcal{MV}_n$ *iff it is a subalgebra of* $\mathbf{L}_n$*.*
(b) *An MV-chain belongs to* $\mathcal{MV}_n^\omega$ *iff its quotient modulo its co-radical belongs to* $\mathcal{MV}_n$*.*

Corollary 8.8.

(a) *A commutative integral GMV-chain* $\mathbf{A}$ *belongs to* $\mathcal{MV}_{n*}$ *iff it is a subalgebra of* $\mathbf{L}_{n*}$*.*
(b) *A commutative integral GMV-chain* $\mathbf{A}$ *belongs to* $\mathcal{MV}_{n*}^\omega$ *iff either it is the negative cone of an abelian* ℓ*-group or its quotient modulo its co-radical belongs to* $\mathcal{MV}_{n*}$*.*

Proof. (a) By Theorem 7.10, $\mathbf{A}$ is in $\mathcal{MV}_{n*}$ iff it is a subreduct of a chain in $\mathcal{MV}_n$, and the claim follows from Proposition 8.7.

(b) By Theorem 7.10, $\mathbf{A}$ is in $\mathcal{MV}_{n*}^\omega$ iff it is a subreduct of a chain in $\mathcal{MV}_n^\omega$. This is the case if either $\mathbf{A}$ is a negative cone of an abelian ℓ-group or the reduct of a chain in $\mathcal{MV}_n^\omega$. The claim follows from Proposition 8.7. □

Proposition 8.9 ([20])**.** *For each* $n \in \mathbb{Z}^+$*:*

(a) *For every MV-chain* $\mathbf{A}$*, there is a maximum subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\mathbf{B} \in \mathcal{MV}_n$*.*
(b) *For every MV-chain* $\mathbf{A}$*, there is a maximum subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\mathbf{B} \in \mathcal{MV}_n^\omega$*.*
(c) *For every commutative integral GMV-chain* $\mathbf{A}$*, there is a maximum subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\mathbf{B} \in \mathcal{MV}_{n*}$*.*
(d) *For every commutative integral GMV-chain* $\mathbf{A}$*, there is a maximum subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\mathbf{B} \in \mathcal{MV}_{n*}^\omega$*.*

Proof. (a) and (c). There are only finitely many chains in $\mathcal{MV}_n$, ($\mathcal{MV}_{n*}$, respectively): namely, all subalgebras of $\mathbf{L}_n$, ($\mathbf{L}_{n*}$, respectively) (see Proposition 8.7 and Corollary 8.8). Moreover, if $\mathbf{L}_h$ and $\mathbf{L}_k$ ($\mathbf{L}_{h*}$ and $\mathbf{L}_{k*}$, respectively) are subalgebras of $\mathbf{A}$, then, by Lemma 8.6, also $\mathbf{L}_q$ ($\mathbf{L}_{q*}$, respectively) with $q = lcm(h,k)$ is a subalgebra of $\mathbf{A}$. Hence, the maximum subalgebra of $\mathbf{A}$ in $\mathcal{MV}_n$, ($\mathcal{MV}_{n*}$, respectively) is $\mathbf{L}_h$, ($\mathbf{L}_{h*}$, respectively), where h is the greatest natural number which divides n such that u^{-1} has a h^{th} root in $\mathbf{A}$.

(b) and (d). Suppose first that $\mathbf{A}$ is an MV-chain or the reduct of an MV-chain. Let R be its co-radical and $\mathbf{A}/R$ its associated quotient, and for all $a \in A$, let $[a]_R$ denote the congruence class of a modulo R. Let h be the maximum natural number such that h divides n and u^{-1} has a h^{th} root in $\mathbf{A}/R$. Then, by (a), $\mathbf{L}_h$ ($\mathbf{L}_{h*}$, respectively) is the maximum subalgebra of $\mathbf{A}/R$ which is in $\mathcal{MV}_n$, ($\mathcal{MV}_{n*}$ respectively) and by Proposition 8.7 and Corollary 8.8, $\{a \in A \mid [a]_R \in L_h\}$ is the domain of the maximum subalgebra of $\mathbf{A}$ in $\mathcal{MV}_n$, ($\mathcal{MV}_{n*}$, respectively).

Finally, if $\mathbf{A}$ is the negative cone of an abelian ℓ-group, then $\mathbf{A}$ is itself in $\mathcal{MV}_n^\omega$. □

We are ready now to provide a characterization of amalgamable varieties of commutative GMV-algebras. As a bonus, we obtain a slightly simplified proof of Di Nola's and Lettieri's characterization of amalgamable varieties of MV-algebras. Our starting point is the following auxiliary result:

Proposition 8.10. *The classes of MV-chains and commutative integral GMV-chains have the amalgamation property.*

Proof. We begin by proving that the class of MV-chains has the AP. Note first that Mundici's categorical equivalence between MV-algebras and abelian ℓ-groups with strong unit [53] specializes to a categorical equivalence between ordered abelian groups with strong unit and MV-chains. Hence, any V-formation of MV-chains (or of reducts of MV-chains) has the form

$$(\Gamma(\mathbf{G}, u_{\mathbf{G}}^{-1}), \Gamma(\mathbf{F}, u_{\mathbf{F}}^{-1}), \Gamma(\mathbf{H}, u_{\mathbf{H}}^{-1}), \Gamma(i), \Gamma(j)),$$

where $(\mathbf{G}, u_{\mathbf{G}})$, $(\mathbf{F}, u_{\mathbf{F}})$, $(\mathbf{H}, u_{\mathbf{H}})$ are abelian ℓ-groups with strong unit and i, j are homomorphisms from $\mathbf{G}$ into $\mathbf{F}$ and $\mathbf{H}$, respectively, which preserve the strong unit. But totally ordered abelian ℓ-groups have the AP, and hence there is an amalgam, $(\mathbf{K}, h, k)$, of $(\mathbf{G}, \mathbf{F}, \mathbf{H}, i, j)$ such that $\mathbf{K}$ is a totally ordered abelian ℓ-group. Clearly, $h(u_{\mathbf{F}}) = h(i(u_{\mathbf{G}})) = k(i(u_{\mathbf{G}})) = k(u_{\mathbf{H}})$. Moreover, after replacing $\mathbf{K}$ by its convex subgroup generated by $h(u_{\mathbf{F}})$, we can assume without loss of generality that $h(u_{\mathbf{F}})$ is a strong unit of $\mathbf{K}$. Hence, $(\Gamma(\mathbf{K}, h(u_{\mathbf{K}})^{-1}), \Gamma(h), \Gamma(k))$ is an amalgam of $(\Gamma(\mathbf{G}, u_{\mathbf{G}}^{-1}), \Gamma(\mathbf{F}, u_{\mathbf{F}}^{-1}), \Gamma(\mathbf{H}, u_{\mathbf{H}}^{-1}), \Gamma(i), \Gamma(j))$ in the class of MV-chains.

Now consider a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ of commutative integral GMV-chains. We distinguish several cases. First, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are reducts of MV-chains, then, since MV-chains have only two idempotent elements, the minimum and the maximum, i and j must preserve the minimum element. Hence, they are also MV-homomorphisms and the proof proceeds as in the previous case. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are negative cones of abelian ℓ-groups, the claim can be reduced to the AP for abelian ℓ-groups, observing that the functor associating to every abelian ℓ-group its negative cone and to each morphism of abelian ℓ-groups its restriction to the negative cone induces an equivalence between the category of abelian ℓ-groups and the category of their negative cones.

It remains to consider V-formations of commutative integral GMV-chains $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}$ is a negative cone of an abelian ℓ-group and either $\mathbf{B}$ or $\mathbf{C}$ (or both) is the reduct of an MV-chain. Assuming for instance that $\mathbf{B}$ is the negative cone of an abelian ℓ-group and $\mathbf{C}$ is the reduct of an MV-chain, we reason as follows: j, being an embedding, maps $\mathbf{A}$ into the co-radical, $\mathbf{R}$, of $\mathbf{C}$, (considered as a subalgebra of $\mathbf{C}$) which is the negative cone, $\mathbf{F}^-$, of a totally ordered abelian ℓ-group $\mathbf{F}$. Hence, $\Gamma(\mathbf{Z} \otimes_{lex} \mathbf{F}, (-1, 0_\mathbf{F}))$ embeds into $\mathbf{C}$ by the embedding l defined, for all $f \in R = F^-$, by $l(-1, f^{-1}) = \neg f$ and $l(0, f) = f$. Moreover, i and j extend to embeddings i' and j' from $\mathbf{A}' = \Gamma(\mathbf{Z} \otimes_{lex} \mathbf{G}, (-1, 0_\mathbf{G}))$ into $\mathbf{B}' = \Gamma(\mathbf{Z} \otimes_{lex} \mathbf{H}, (-1, 0_\mathbf{H}))$ and $\mathbf{C}$, respectively, defined for all $g \in A$, by $i'(-1, g^{-1}) = (-1, i(g)^{-1})$ and $i'(0, g) = (0, i(g))$, and by $j'(-1, g^{-1}) = \neg j(g)$ and $j'(0, g) = j(g)$.

But then $(\mathbf{A}', \mathbf{B}', \mathbf{C}, i', j')$ is a V-formation in the class of reducts of MV-chains, and there is an amalgam $(\mathbf{D}, h, k)$ of $(\mathbf{A}', \mathbf{B}', \mathbf{C}, i,', j')$. Taking the restriction h' of h to $\mathbf{B}$, we obtain an amalgam $(\mathbf{D}, h', k)$ of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in the class of all commutative integral GMV-chains.

The case where $\mathbf{A}$ is the negative cone of an abelian ℓ-group and $\mathbf{B}$ and $\mathbf{C}$ are both reducts of MV-algebras is very similar. □

Theorem 8.11.

(a) *A variety $\mathcal{V}$ of MV-algebras has the amalgamation property iff either it is the trivial variety, $\mathcal{V} = \mathcal{MV}$, $\mathcal{V} = \mathcal{MV}_n$ for some n, or $\mathcal{V} = \mathcal{MV}_n^\omega$ for some n. In other words, $\mathcal{V}$ has the amalgamation property iff it is generated by a single chain (see* [20]*).*

(b) *A variety $\mathcal{V}$ of commutative GMV-algebras has the amalgamation property iff either it is the trivial variety, $\mathcal{V} = \mathcal{CIGMV}$, $\mathcal{V} = \mathcal{MV}_{n*}$, $\mathcal{V} = \mathcal{MV}_{n*}^\omega$, $\mathcal{V} = \mathcal{A}^- \vee \mathcal{MV}_{n*}$ for some n, or $\mathcal{V}$ is the join of one of the above varieties with $\mathcal{A}$, that is, if $\mathcal{V}$ is one of $\mathcal{A}$ (the join of $\mathcal{A}$ with the trivial variety) or $\mathcal{CGMV} = \mathcal{A} \vee \mathcal{CIGMV}$, $\mathcal{A} \vee \mathcal{MV}_{n*}$, $\mathcal{A} \vee \mathcal{MV}_{n*}^\omega$, or $\mathcal{A} \vee \mathcal{A}^- \vee \mathcal{MV}_{n*}$, for some n.*

Proof. (a) That $\mathcal{MV}$ has the AP is well-known [54] and has been proved directly in Section 6. It also follows immediately from Corollary 7.9 and Proposition 8.10. That the trivial variety has the AP is obvious. So, let us prove that for every n, $\mathcal{MV}_n$ and $\mathcal{MV}_n^\omega$ have the AP. By Corollary 7.9, it is sufficient to prove that every V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{MV}_n$ ($\mathcal{MV}_n^\omega$) consisting of chains has an amalgam in $\mathcal{MV}_n$ ($\mathcal{MV}_n^\omega$). By Proposition 8.10, $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ has an amalgam $(\mathbf{D}, h, k)$ in $\mathcal{MV}$ such that $\mathbf{D}$ is a chain. Now by Proposition 8.9, there is a maximum subalgebra $\mathbf{D}_0$ of $\mathbf{D}$ such that $\mathbf{D}_0 \in \mathcal{MV}_n$ ($\mathbf{D}_0 \in \mathcal{MV}_n^\omega$, respectively). Moreover $h(\mathbf{B})$ and $k(\mathbf{C})$ are in $\mathcal{MV}_n$ ($\mathcal{MV}_n^\omega$, respectively), because varieties are closed under homomorphic images. Hence, $h(\mathbf{B})$ and $k(\mathbf{C})$ are subalgebras of $\mathbf{D}_0$ and $(\mathbf{D}_0, h, k)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{MV}_n$ ($\mathcal{MV}_n^\omega$, respectively).

Now let $\mathcal{V}$ be another variety of MV-algebras. Then $\mathcal{V}$ is not generated by a single chain, but it is generated by a finite set of chains of the form $\mathbf{L}_n$ or $\mathbf{K}_n$. Let X be a generating set with minimum cardinality. Suppose that $\mathbf{L}_n$ and $\mathbf{L}_m$ are distinct elements in X, and let $q = lcm(n, m)$. Then $\mathbf{L}_q \notin \mathcal{V}$, otherwise we might reduce the cardinality of X replacing $\mathbf{L}_n$ and

$\mathbf{L}_m$ by $\mathbf{L}_q$. Now consider the V-formation $(\mathbf{L}_1, \mathbf{L}_m, \mathbf{L}_n, i, j)$, where $\mathbf{L}_1$ is the two element MV-algebra and i and j are the unique embeddings of $\mathbf{L}_1$ into $\mathbf{L}_m$ and $\mathbf{L}_n$, respectively. If $(\mathbf{A}, h, k)$ is an amalgam of $(\mathbf{L}_1, \mathbf{L}_m, \mathbf{L}_n, i, j)$, then $\mathbf{A}$ must contain both $\mathbf{L}_m$ and $\mathbf{L}_n$, and hence it must contain $\mathbf{L}_q$. Since $\mathbf{L}_q \notin \mathcal{V}$, the V-formation $(\mathbf{L}_1, \mathbf{L}_m, \mathbf{L}_n, i, j)$ cannot have an amalgam in $\mathcal{V}$.

If we assume that $\mathbf{K}_m, \mathbf{K}_n \in X$, or that $\mathbf{K}_m, \mathbf{L}_n \in X$, then by the same argument we see that, if $q = lcm(m, n)$, then $\mathbf{K}_q \notin \mathcal{V}$, but if $(\mathbf{A}, h, k)$ is an amalgam of $(\mathbf{L}_1, \mathbf{K}_m, \mathbf{K}_n, i, j)$ ($(\mathbf{L}_1, \mathbf{K}_m, \mathbf{L}_n, i, j)$, respectively), then $\mathbf{A}$ must contain $\mathbf{K}_q$. It follows that the V-formation $(\mathbf{L}_1, \mathbf{K}_m, \mathbf{K}_n, i, j)$ ($(\mathbf{L}_1, \mathbf{K}_m, \mathbf{L}_n, i, j)$, respectively) cannot have an amalgam in $\mathcal{V}$.

(b) That $\mathcal{CIGMV}$ has the AP follows from Proposition 8.10 and Corollary 7.9. That $\mathcal{A}^-$ and $\mathcal{A}$ have the AP follows from the AP for abelian ℓ-groups, see [58] and Section 6, and that the trivial variety has the AP is obvious. The proof that $\mathcal{MV}_{n*}$ and $\mathcal{MV}_{n*}^{\omega}$ have the AP is similar to the proof that $\mathcal{MV}_n$ and $\mathcal{MV}_n^{\omega}$ have the AP. If a variety $\mathcal{V}$ of GMV-algebras is generated by a finite set X of algebras of the form $\mathbf{K}_{m*}$ or $\mathbf{L}_{n*}$, but not by a single chain, then letting $X_0 = \{\mathbf{K}_m \mid \mathbf{K}_{m*} \in \mathcal{V}\} \cup \{\mathbf{L}_m \mid \mathbf{L}_{m*} \in \mathcal{V}\}$, and $\mathcal{V}_0$ the variety generated by X_0, we have that $\mathcal{V}_0$ does not have the AP by (a). Moreover $\mathcal{V}$ is the class of residuated lattice subreducts of algebras in $\mathcal{V}_0$, and by Theorem 7.10, $\mathcal{V}$ does not have the AP. Finally, by Theorem 7.12, if either $\mathcal{V} = \mathcal{V}' \vee \mathcal{A}^-$, where $\mathcal{V}' \subseteq \mathcal{CIGMV}$ and $\mathcal{A}^- \not\subseteq \mathcal{V}'$, or $\mathcal{V} = \mathcal{A} \vee \mathcal{V}'$, with $\mathcal{V}' \subseteq \mathcal{CIGMV}$, then $\mathcal{V}$ has the AP iff $\mathcal{V}'$ has the AP. □

8.2. **Commutative GBL-algebras.** We turn our attention now to the amalgamation problem for varieties of commutative GBL-algebras, recalling that, by Corollary 8.3, the class of all GBL-algebras does not have the amalgamation property. Although we have not been able to solve the amalgamation problem for the whole class of commutative GBL-algebras (equivalently, by Theorem 7.12, the amalgamation problem for integral commutative GBL-algebras), we are nevertheless able to distinguish some important classes of commutative GBL-algebras having the amalgamation property. In particular, we focus here on varieties of semilinear commutative GBL-algebras, beginning with some auxiliary concepts and results.

Let $(I, \leq)$ be a totally ordered set and $(\mathbf{H}_i \mid i \in I)$ an ordered family of integral GBL-algebras such that for $i \neq j$, $H_i \cap H_j = \{e\}$. Suppose also that for each $i \in I$, either e is join irreducible in $\mathbf{H}_i$, $i = \max(I)$, or i has an immediate successor, denoted by $s(i)$ (i.e., $i < s(i)$ and there is no $j \in I$ with $i < j < s(i)$) and $\mathbf{H}_{s(i)}$ is bounded. The *ordinal sum* $\bigoplus_{i \in I} \mathbf{H}_i$ of the family $(\mathbf{H}_i \mid i \in I)$ consists of $\bigcup_{i \in I} H_i$ with the following operations (where the subscript $_i$ denotes the realization of the operation in $\mathbf{H}_i$):

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in H_i \quad (i \in I) \\ x & \text{if } x \in H_i - \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if } y \in H_i - \{e\}, x \in H_j \text{ with } i < j \end{cases}$$

$$x \backslash y = \begin{cases} x \backslash_i y & \text{if } x, y \in H_i \quad (i \in I) \\ e & \text{if } x \in H_i - \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if } y \in H_i - \{e\}, x \in H_j \text{ with } i < j \end{cases}$$

$$y / x = \begin{cases} y /_i x & \text{if } x, y \in H_i \quad (i \in I) \\ e & \text{if } x \in H_i - \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if } y \in H_i - \{e\}, x \in H_j \text{ with } i < j \end{cases}$$

$$x \wedge y = \begin{cases} x \wedge_i y & \text{if } x, y \in H_i \quad (i \in I) \\ x & \text{if } x \in H_i - \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if } y \in H_i - \{e\}, x \in H_j \text{ with } i < j \end{cases}$$

$$x \vee y = \begin{cases} e & \text{if } e \in \{x, y\} \\ x \vee_i y & \text{if } x, y \in H_i \text{ and either } i = \max(I) \text{ or } x \vee_i y < e \\ \min(\mathbf{H}_{s(i)}) & \text{if } i \neq \max(I),\ x, y \in H_i - \{e\},\ \text{and } x \vee_i y = e \\ x & \text{if } y \in H_i - \{e\}, x \in H_j \text{ with } i < j \\ y & \text{if } x \in H_i - \{e\}, y \in H_j \text{ with } i < j. \end{cases}$$

Note that if e is join irreducible in every $\mathbf{H}_i$, the third condition in the definition of $x \vee y$ never occurs, and we do not need to assume that $s(i)$ and $\min(\mathbf{H}_{s(i)})$ exist.

It is easy to verify that the ordinal sum of any ordered family of integral GBL-algebras is an integral GBL-algebra. Moreover, it is shown in [1] (using the terminology of hoops), that:

Theorem 8.12 ([1])**.** *Every commutative integral GBL-chain* $\mathbf{A}$ *is the ordinal sum of an ordered family* $(\mathbf{U}_i \mid i \in I)$ *of commutative integral GMV-algebras.*

We will refer to the algebras $\mathbf{U}_i$ in Theorem 8.12 as the *GMV-components* of $\mathbf{A}$. We note also that Theorem 8.12 was extended by Dvurečenskij [21] to the non-commutative case.

We now introduce a definition of poset product of residuated lattices, taken from [33] (referred to there as *poset sums*). Let $(P, \leq)$ be a poset and $(\mathbf{A}_p \mid p \in P)$ a collection of residuated lattices with a common neutral element e, such that if p is not minimal, then $\mathbf{A}_p$ is integral and if p is not maximal, then $\mathbf{A}_p$ has a minimum element 0 (common to all $\mathbf{A}_p$ with p not minimal). The *poset product* $\bigotimes_{p\in P} \mathbf{A}_p$ is the algebra consisting of all maps $h \in \prod_{p\in P} A_p$ such that for any $p \in P$ if $h(p) \neq e$, then for all $q < p$, $h(q) = 0$, with monoid operation and lattice operations defined pointwise, and residual operations defined as follows:

$$(h\backslash g)(p) \quad = \quad \begin{cases} h(p)\backslash_p g(p) & \text{if for all } q > p,\ h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$

$$(g/h)(p) \quad = \quad \begin{cases} g(p)/_p h(p) & \text{if for all } q > p,\ h(q) \leq_p (q) \\ 0 & \text{otherwise} \end{cases}$$

where the subscript $_p$ denotes realization of operations and order in $\mathbf{A}_p$.

In [33] and [34], the following is shown:

Proposition 8.13.

(a) *The poset product of a collection of integral bounded GBL-algebras is an integral bounded GBL-algebra, which is commutative when all its factors are commutative.*
(b) *Every finite GBL-algebra can be represented as the poset product of finite MV-chains.*
(c) *Every n-potent GBL-algebra embeds into a poset product of finite n-potent MV-chains.*
(d) *Every commutative integral GBL-algebra embeds into a poset product of MV-chains.*

Let us outline the construction in the proof of (c). First of all, any n-potent GBL-algebra $\mathbf{A}$ is commutative and integral [33]. Let $\Delta(\mathbf{A})$ be the collection of all completely meet irreducible filters (values) of $\mathbf{A}$, ordered by reverse inclusion. Using a result from [33], for each $F \in \Delta(\mathbf{A})$, the quotient $\mathbf{A}/F$ decomposes as $\mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is an n-potent GBL-algebra and $\mathbf{W}_F$ is a finite non-trivial n-potent MV-chain. Also, for every $a \in A$, if F is maximal among all filters G such that $a \notin G$, then $[a]_F \in W_F - \{e\}$. Now let

$$h_a(F) = \begin{cases} [a]_F & \text{if } [a]_F \in \mathbf{W}_F \\ 0 & \text{otherwise.} \end{cases}$$

Then in [34] it is proved that the map $\Phi\colon a \mapsto h_a$ is the desired embedding of $\mathbf{A}$ into $\bigotimes_{F\in\Delta(\mathbf{A})} \mathbf{W}_F$.

The proof essentially shows that any n-potent GBL-algebra embeds into a poset product $\bigotimes_{F\in\Delta(A)} \mathbf{W}_F$ in such a way that for every $F \in \Delta(\mathbf{A})$, the projection π_F of $\mathbf{A}$ into $\mathbf{W}_F$ – defined, for all $h \in \bigotimes_{F\in\Delta(A)} \mathbf{W}_F$, by $\pi_F(h) = h(F)$ – is surjective. We express this property by saying that $\mathbf{A}$ is a *subdirect poset product* of the family $\{\mathbf{W}_F \mid F \in \Delta(\mathbf{A})\}$ with respect to the poset $(\Delta(\mathbf{A}), \subseteq)$, and we write $\mathbf{A} \subseteq_s \bigotimes_{F\in\Delta(\mathbf{A})} \mathbf{W}_F$.

Amalgamation for commutative semilinear GBL-algebras is essentially proved in [52], but here we present a slightly more general result (i.e., we do not assume integrality), and a slightly more elegant proof.

Theorem 8.14. *The variety of commutative semilinear GBL-algebras has the amalgamation property.*

Proof. The variety of commutative semilinear GBL-algebras is the join of the independent varieties of abelian ℓ-groups and commutative integral semilinear GBL-algebras. Since the variety of abelian ℓ-groups has the AP, by Theorem 7.12, it suffices to prove that the variety of commutative integral semilinear GBL-algebras has the AP. Moreover by Corollary 7.9, it is sufficient to prove that any V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ consisting of commutative integral GBL-chains has an amalgam.

Using Theorem 8.12, we can assume that $\mathbf{A} = \bigoplus_{r\in R} \mathbf{U}_r$, $\mathbf{B} = \bigoplus_{s\in S} \mathbf{V}_s$, and $\mathbf{C} = \bigoplus_{t\in T} \mathbf{W}_t$, where $\mathbf{U}_r$, $\mathbf{V}_s$, and $\mathbf{W}_t$ are commutative integral GMV-chains and $\mathbf{R} = (R, \leq_R)$, $\mathbf{S} = (S, \leq_S)$, and $\mathbf{T} = (T, \leq_T)$ are totally ordered sets.

It follows from the definition of ordinal sum that two elements x, y of a commutative integral GBL-chain are in the same GMV-component iff $(x \to y) \to y = (y \to x) \to x$. Moreover, if $x, y \neq e$ belong to different components $\mathbf{U}_r$ and $\mathbf{U}_{r'}$, then $r < r'$ iff $(y \to x) \to x = e$. Hence, i and j map elements from the same component into elements of the same component. Moreover letting for all $r \in R$, $i^*(r) = s$ iff for all $x \in U_r$, $i(x) \in V_S$ and $j^*(r) = t$ iff for all $x \in U_r$, $j(x) \in W_t$, the maps i^* and j^* are embeddings of $\mathbf{R}$ into $\mathbf{S}$ and $\mathbf{T}$, respectively.

It is easily seen that the V-formation $(\mathbf{R}, \mathbf{S}, \mathbf{T}, i^*, j^*)$ has an amalgam $(\mathbf{M}, h^*, k^*)$ in the class of totally ordered sets such that $M = h^*(S) \cup k^*(T)$ and $h^*(S) \cap k^*(T) = h^*(i^*(R)) = k^*(j^*(R))$.

Since the class of commutative integral GMV-chains has the AP, for each V-formation $(\mathbf{U}_r, \mathbf{V}_{i^*(r)}, \mathbf{W}_{j^*(r)}, i_{\restriction \mathbf{U}_r}, j_{\restriction \mathbf{U}_r})$, we can obtain an amalgam $(\mathbf{Z}_{h^*(i^*(r))}, h_{i^*(r)}, k_{j^*(r)})$ such that each $\mathbf{Z}_{h^*(i^*(r))}$ is a commutative integral GMV-chain.

We are now ready to construct the desired amalgam $(\mathbf{D}, h, k)$ of the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$. We have already defined $\mathbf{Z}_m$ for $m \in h^*(i^*(R))$. If $m \in M \backslash h^*(i^*(R))$, then either $m \in h^*(S) \backslash k^*(T)$ or $m \in k^*(T) \backslash h^*(S)$. In the former case, define $\mathbf{Z}_m = \mathbf{V}_s$ where s is the unique $s \in S$ such that $h^*(s) = m$. In the latter case, define $\mathbf{Z}_m = \mathbf{W}_t$ where t is the unique $t \in T$ such that $k^*(t) = m$. Up to isomorphism, we may assume that if $m \neq m'$, then $Z_m \cap Z_{m'} = \{e\}$. Now let $\mathbf{D} = \bigoplus_{m \in M} \mathbf{Z}_m$, and define, for $x \in B$ and $y \in C$:

$$h(x) = \begin{cases} h_s(x) & \text{if } x \in V_s \text{ and } s \in i^*(R) \\ x & \text{otherwise} \end{cases}$$

$$k(y) = \begin{cases} k_t(x) & \text{if } x \in W_t \text{ and } t \in i^*(R) \\ x & \text{otherwise.} \end{cases}$$

It is easily seen that $(\mathbf{D}, h, k)$ is an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$. □

The proof of Theorem 8.14 also shows that the variety of commutative integral semilinear GBL-algebras has the amalgamation property. Since this variety is the class of subreducts of BL-algebras, by Theorem 7.10 we obtain (see also [52]):

Theorem 8.15. *The variety of BL-algebras has the amalgamation property.*

For every positive integer n, the class of all commutative integral GBL-algebras that are subdirect products of ordinal sums of at most n GMV-chains, is a variety, denoted here by $n\mathcal{CIGBL}$. Indeed, as shown in [1], $n\mathcal{CIGBL}$ is axiomatized in the signature of BL-algebras and hoops, by the equation

(nCIGBL) $$\bigwedge_{i=1}^{n}((x_{i+1} \to x_i) \to x_i) \leq \bigvee_{i=1}^{n+1} x_i.$$

Somewhat surprisingly, we can prove:

Theorem 8.16. *$n\mathcal{CIGBL}$ has the amalgamation property iff $n = 1$.*

Proof. $1\mathcal{CIGBL}$ is just the variety of commutative integral GMV-algebras, which is known to have the AP. We present a proof that $2\mathcal{CIGBL}$ does not have the AP, which then generalizes to any $n > 1$. Let $\mathbf{A}$ be any non-trivial GMV-chain, and let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two isomorphic copies of $\mathbf{A}$ such that $A_1 \cap A_2 = \{e\}$. Let for every $a \in A$, a_1 and a_2 denote the copies of a in $\mathbf{A}_1$ and $\mathbf{A}_2$. Let $\mathbf{B} = \mathbf{C} = \mathbf{A}_1 \oplus \mathbf{A}_2$ and let for $a \in A$, $i(a) = a_1$ and $j(a) = a_2$. Suppose that the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ has an amalgam $(\mathbf{D}, h, k)$ in $2\mathcal{CIGBL}$. Let $x \ll y$ mean that $x < y$ and x and y are not in the same GMV component. Note that if $x, y \neq e$, then $x \ll y$ iff $(y \to x) \to x = e$.

Now for all $a \in A \backslash \{e\}$, we have $i(a) \ll j(a)$ and hence $(j(a) \to i(a)) \to i(a) = e$. It follows that $(k(j(a)) \to k(i(a))) \to k(i(a)) = e$ and $(h(j(a)) \to h(i(a))) \to h(i(a)) = e$. But $k(j(a)) = h(i(a))$, and hence, letting $b_1 = k(i(a))$, $b_2 = h(i(a)) = k(j(a))$, and $b_3 = h(j(a))$, we have $(b_{i+1} \to b_i) \to b_i = e$ for $i = 1, 2$; i.e., $b_1 \ll b_2 \ll b_3$. It follows that $\mathbf{D} \notin 2\mathcal{CIGBL}$, a contradiction. □

8.3. **Varieties of n-potent GBL-algebras.** Finally, we investigate the amalgamation property for varieties of n-potent GBL-algebras. We will prove that the variety $\mathcal{GBL}np$ of all n-potent GBL-algebras has the amalgamation property iff $n \leq 2$. Nevertheless, for every n there is a variety of n-potent GBL-algebras with the amalgamation property, namely, the variety $\mathcal{GBL}_n$ consisting of all GBL-algebras which embed into the poset product of subalgebras of $\mathbf{L}_{n*}$. As a particular case, for $n = 1$ we obtain that the variety of Heyting algebras has the amalgamation property.

We prove the negative result first.

Theorem 8.17. *If $n > 2$, then the variety $\mathcal{GBL}np$ of n-potent GBL-algebras does not have the amalgamation property.*

Proof. By Theorem 7.10, it suffices to prove that the variety $\mathcal{GBL}np_0$ of n-potent *bounded* GBL-algebras does not have the AP. If $n > 2$, then there is an $m < n$ such that m does not divide n. Now let i and j be the unique embeddings of $\mathbf{L}_1$ into $\mathbf{L}_n$ and into $\mathbf{L}_m$, respectively. Then the V-formation $(\mathbf{L}_1, \mathbf{L}_n, \mathbf{L}_m, i, j)$ is in $\mathcal{GBL}np_0$, and it suffices to prove that it does not have an amalgam in $\mathcal{GBL}np_0$. Suppose, by way of contradiction, that $(\mathbf{D}, h, k)$ is an amalgam of $(\mathbf{L}_1, \mathbf{L}_n, \mathbf{L}_m, i, j)$ in $\mathcal{GBL}np_0$. Then $\mathbf{D}$ is a subdirect poset product of the form $\mathbf{D} \subseteq_s \bigotimes_{p\in\mathbf{P}} \mathbf{A}_p$, where for every p, $\mathbf{A}_p$ is an MV-chain of cardinality at most $n + 1$.

Let a and b denote the coatoms of $\mathbf{L}_n$ and of $\mathbf{L}_m$, respectively, and let $c = h(a)$ and $d = k(b)$. Note that $c^n = d^m = 0$, $c^{n-1} = \neg c$, and $d^{m-1} = \neg d$. It follows that for every $p \in P$, $c(p) < e$, otherwise $c^n(p) = e \neq 0$. Likewise, $d(p) < e$ for all $p \in P$. Hence, if p is not maximal in $\mathbf{P}$, then $c(p) = 0$, since otherwise, for all $q > p$ we would have $c(q) = e$. Similarly, $d(p) = 0$ for all non-maximal $p \in P$. Since $c^{n-1} = \neg c$, there must be a (necessarily maximal) $p \in P$ such that $c(p) > 0$. But the maximality of p, together with the definition of poset product, entails $c^{n-1}(p) = \neg c(p) = c(p) \rightarrow_p 0$. Hence, the equation $\neg x = x^{n-1}$ has a solution in $\mathbf{A}_p$, and $\mathbf{A}_p$ contains an isomorphic copy of $\mathbf{L}_n$. By the same token, $\mathbf{A}_p$ contains an isomorphic copy of $\mathbf{L}_m$.

It follows that $\mathbf{A}_p$ is an MV-algebra extending both $\mathbf{L}_n$ and $\mathbf{L}_m$, and by Lemma 8.6, letting $q = lcm(n, m)$, we have that $\mathbf{L}_q$ is a subalgebra of $\mathbf{A}_p$. But since $q > n$, this implies that $\mathbf{A}_p$, and hence $\mathbf{D}$, is not n-potent, and the V-formation $(\mathbf{L}_1, \mathbf{L}_n, \mathbf{L}_m, i, j)$ does not have an amalgam in $\mathcal{GBL}np_0$. □

We now investigate the amalgamation property for the class $\mathcal{GBL}_n$ of all subdirect poset products of subalgebras of $\mathbf{L}_{n*}$. We will prove that $\mathcal{GBL}_n$ is a variety which enjoys the amalgamation property. As a corollary we will obtain that $\mathcal{GBL}1p$, $\mathcal{GBL}2p$, as well as the variety $\mathcal{H}$ of Heyting algebras, have the amalgamation property.

First we prove that $\mathcal{GBL}_n$ is a variety by providing an explicit equational axiomatization. Clearly, every member of $\mathcal{GBL}_n$ is an n-potent GBL-algebra, and hence satisfies the equation $x^{n+1} = x^n$. Moreover:

Lemma 8.18. *If $m < n$ and m does not divide n, then every algebra in $\mathcal{GBL}_n$ satisfies the equation*

$$(\mathrm{GBL}_{n,m}) \qquad ((x \to x^n) \leftrightarrow x^{m-1})^n \leq x.$$

Proof. Let $\mathbf{A} \in \mathcal{GBL}_n$. Then $\mathbf{A}$ is a subdirect poset product of the form $\mathbf{A} \subseteq_s \bigotimes_{p \in \mathbf{P}} \mathbf{W}_p$ where $\mathbf{W}_p$ is a subalgebra of $\mathbf{L}_{n*}$ for every p. Now let $f_m(x) = ((x \to x^n) \leftrightarrow x^{m-1})$ and $g_m(x) = f_m(x)^n$. We prove that for all $p \in P$, $g_m(x)(p) \leq x(p)$. The claim is trivial if $x(p) = e$, so let us assume $x(p) < e$. We distinguish the following cases:

(a) $0 < x(p) < e$. Then for all $q > p$, $x(q) = (x(q))^{m-1} = x(q)^n = e$, and hence, by the definition of implication in a poset product, $((x \to x^n) \leftrightarrow x^{m-1})(p) = ((x(p) \to_p x(p)^n) \leftrightarrow_p x(p)^{m-1})$, where the subscript $_p$ denotes the realization of operations in $\mathbf{W}_p$. Now $x(p)^n = 0$, $x(p) \to_p x(p)^n = \neg_p x(p)$, and since m does not divide n, we have $\neg_p x(p) \neq x(p)^{m-1}$ (since otherwise, $\mathbf{L}_{m*}$ would embed into $\mathbf{W}_p$ and hence into $\mathbf{L}_{n*}$). It follows that $f_m(x)(p) < e$ and $g_m(x)(p) = f_m(p)^n = 0$. Hence $g_m(p) \leq x(p)$.

(b) $x(p) = 0$ and there is $q > p$ such that $0 < x(q) < 1$. Then by (a), $g_m(x)(q) = 0$, and by the definition of poset product, $g_m(x)(p) = 0 \leq x(p)$.

(c) $x(p) = 0$ and for all $q > p$, either $x(q) = 0$ or $x(q) = e$. Hence for all $q > p$, $x(q) = x^n(q)$. So $(x \to x^n)(p) = x(p) \to_p x^n(p) = e$. Since $x^{m-1}(p) = 0$, $((x \to x^n) \leftrightarrow x^{m-1})(p) = 0$, and $g_m(x)(p) = 0 \leq x(p)$.

In each case, $g_m(x)(p) \leq x(p)$, and ($\mathrm{GBL}_{n,m}$) holds as required. □

Theorem 8.19. *$\mathcal{GBL}_n$ is axiomatized by the equations $x^{n+1} = x^n$ and all equations ($\mathrm{GBL}_{n,m}$) such that $m < n$ and m does not divide n (thus $\mathcal{GBL}_n$ is a finitely based variety).*

Proof. By Lemma 8.18, all equations ($\mathrm{GBL}_{n,m}$) such that $m < n$ and m does not divide n are valid in $\mathcal{GBL}_n$. Clearly, $\mathcal{GBL}_n$ satisfies the equation $x^{n+1} = x^n$.

For the other direction, suppose that a GBL-algebra $\mathbf{A}$ satisfies $x^{n+1} = x^n$ and all equations ($\mathrm{GBL}_{n,m}$) such that $m < n$ and m does not divide n. Then $\mathbf{A}$ is n-potent. Hence, $\mathbf{A}$ is a subdirect poset product $\bigotimes_{p \in \mathbf{P}} \mathbf{W}_p$, where for every $p \in P$, $\mathbf{W}_p$ is an n-potent MV-chain, hence, an MV-chain with cardinality $\leq n + 1$. Let $m_p = |W_p| - 1$. If for every $p \in P$, m_p divides n, then for every p, $\mathbf{W}_p$ is a subalgebra of $\mathbf{L}_{n*}$ and $\mathbf{A} \in \mathcal{GBL}_n$. Now suppose, by way of contradiction, that for some $p \in P$, m_p does not divide n. Let c be the coatom of $\mathbf{W}_p$, and let $x \in A$ be such that $x(p) = c$. Then $x^n(p) = 0$, and $x(q) = x^n(q) = e$ for all $q > p$. It follows that $(x \to$

$x^n)(p) = x(p) \to_p 0 = \neg_p x(p) = x(p)^{m_p - 1}$ and $g_{m_p}(x) = e > c = x(p)$. Hence, ($\mathrm{GBL}_{n,m_p}$) is not valid in $\mathbf{A}$ and $\mathbf{A} \notin \mathcal{GBL}_n$. □

We prove now that every variety $\mathcal{GBL}_n$ has the amalgamation property. Consider a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ of algebras in $\mathcal{GBL}_n$. Without loss of generality, we will assume that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$ and that i and j are the identity embeddings. Then $\mathbf{B} \subseteq_s \bigotimes_{F \in \Delta(\mathbf{B})} \mathbf{W}_F$ and $\mathbf{C} \subseteq_s \bigotimes_{G \in \Delta(\mathbf{C})} \mathbf{W}_G$, where for each $F \in \Delta(\mathbf{B})$ and for each $G \in \Delta(\mathbf{C})$, $\mathbf{W}_F$ and $\mathbf{W}_G$ are subalgebras of $\mathbf{L}_{n*}$. Now for every $F \in \Delta(\mathbf{B})$ (every $G \in \Delta(\mathbf{C})$, respectively), let F^* (G^*, respectively) denote the set of all filters H of $\mathbf{C}$ (of $\mathbf{B}$, respectively) which are maximal with respect to the property $H \cap A = F \cap A$ ($H \cap A = G \cap A$, respectively). By Zorn's lemma, F^* and G^* are non-empty. We define a poset $\Delta(\mathbf{B}, \mathbf{C})$ as described below.

The domain of $\Delta(\mathbf{B}, \mathbf{C})$ is the set of all pairs (F, G) such that either $F \in \Delta(\mathbf{B})$ and $G \in F^*$, or $G \in \Delta(\mathbf{C})$ and $F \in G^*$ (note that these possibilities are not mutually exclusive). The order $\leq$ is as follows: $(F, G) \leq (F', G')$ iff $F \subseteq F'$ and $G \subseteq G'$. Now let $\mathbf{D} = \bigotimes_{(F,G) \in \Delta(\mathbf{B},\mathbf{C})} \mathbf{W}_{(F,G)}$, where for all $(F, G) \in \Delta(\mathbf{B}, \mathbf{C})$, $\mathbf{W}_{(F,G)} = \mathbf{L}_{n*}$. Clearly, $\mathbf{D} \in \mathcal{GBL}_n$. Note that for every $F \in \Delta(\mathbf{B})$ and for every $G \in \Delta(\mathbf{C})$, there is a unique embedding i_F (i_G, respectively) of $\mathbf{W}_F$ ($\mathbf{W}_G$, respectively) into $\mathbf{L}_{n*}$. We define maps h and k from B into D and from C into D, respectively, as follows:

$$(h(b))(F, G) = \begin{cases} e & \text{if } b \in F \\ i_F([b]_F) & \text{if } F \in \Delta(\mathbf{B}),\ G \in F^*,\ \text{and } [b]_F \in \mathbf{W}_F \\ 0 & \text{otherwise} \end{cases}$$

$$(k(c))(F, G) = \begin{cases} e & \text{if } c \in G \\ i_G([c]_G) & \text{if } G \in \Delta(\mathbf{C}),\ F \in G^*,\ \text{and } [c]_G \in \mathbf{W}_G \\ 0 & \text{otherwise.} \end{cases}$$

Warning. It is possible that $F \in \Delta(\mathbf{B})$, $G \in \Delta(\mathbf{C})$, $F \in G^*$, and $G \in F^*$; however, it is also possible that $F \in \Delta(\mathbf{B})$, $G \in \Delta(\mathbf{C})$, and either $F \notin G^*$ and $G \in F^*$, or $F \in G^*$ and $G \notin F^*$. If for instance $F \in \Delta(\mathbf{B}) \cap G^*$, $G \in \Delta(\mathbf{C})$, $G \notin F^*$, and $[b]_F \in \mathbf{W}_F$, but $b \notin F$, then, according to our definition, $h(b)(F) = 0$.

Theorem 8.20. $(\mathbf{D}, h, k)$ *is an amalgam of* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$*, and hence* $\mathcal{GBL}_n$ *has the amalgamation property.*

Proof. Let for every $b \in B$ (for every $c \in C$, respectively), $\Delta(\mathbf{B}, b)$ ($\Delta(\mathbf{C}, c)$, respectively) denote the set of filters H of $\mathbf{B}$ (of $\mathbf{C}$, respectively) which are maximal with respect to the property that $b \notin H$ ($c \notin H$, respectively). We require the following:

Lemma 8.21. *The following conditions hold:*

(a) $F \in \Delta(\mathbf{B}, b)$ *iff* $[b]_F \in W_F - \{e\}$ *and* $G \in \Delta(\mathbf{C}, c)$ *iff* $[c]_G \in W_G - \{e\}$.

(b) *If* $F \in \Delta(\mathbf{B})$, *if* $G \in F^*$ *and* $(F, G) < (F', G')$, *then* $F \subset F'$. *Moreover, if* $b \in B$ *and* $[b]_F \in W_F$, *then* $b \in F'$. *Likewise, if* $G \in \Delta(\mathbf{C})$, *if* $F \in G^*$ *and* $(F, G) < (F', G')$, *then* $G \subset G'$. *Moreover, if* $c \in C$ *and* $[c]_G \in W_G$, *then* $c \in G'$.

(c) *Let* $a \in A$ *and* $(F, G) \in \Delta(\mathbf{B}, \mathbf{C})$. *Then* $a \in F$ *iff* $a \in G$. *Moreover,* $F \in \Delta(\mathbf{B}, a)$ *iff* $G \in \Delta(\mathbf{C}, a)$, *and in this case,* $F \in G^*$ *and* $G \in F^*$. *Finally, the map from* $\mathbf{W}_F \cap \mathbf{A}/F$ *into* $\mathbf{W}_G \cap \mathbf{A}/G$ *sending* $[x]_F$ *into* $[x]_G$ *is a well defined isomorphism.*

Proof. (a) It is readily seen that W_F is the minimum filter of $\mathbf{B}/F$ (it coincides with the filter generated by the unique coatom of $\mathbf{W}_F$, which is also the unique coatom of $\mathbf{B}/F$). Hence, $F \in \Delta(\mathbf{B}, b)$ iff $[b]_F \neq e$ is in the minimum filter of $\mathbf{B}/F$ iff $[b]_F \in W_F - \{e\}$. The proof that $G \in \Delta(\mathbf{C}, c)$ iff $[c]_G \in W_G - \{e\}$ is similar.

(b) If $F = F'$, then from $(F, G) < (F', G')$, we deduce that $G \subset G'$. On the other hand, we must have $F \cap A = G \cap A = F' \cap A = G' \cap A$, which contradicts the maximality of G among all filters H of C such that $H \cap A = F \cap A$. Now suppose that $[b]_F \in W_F$. By the second homomorphism theorem, $\mathbf{B}/F' = (\mathbf{B}/F)/(F'/F)$, where F'/F denotes the set of all equivalence classes, modulo the congruence associated to F, of all elements of F'. Since W_F is the minimum filter of $\mathbf{B}/F$, F'/F contains W_F. Hence, $[b]_F \in F'/F$, and $b \in F'$. The proof of the second half of (b) is quite similar.

(c) Since $F \cap A = G \cap A$, we have that $a \in F$ iff $a \in G$. Now suppose that $F \in \Delta(\mathbf{B}, a)$ and $G \in F^*$. We prove that $F \cap A$ is maximal among the filters of $\mathbf{A}$ which do not contain a. Indeed, for $x \in A - F$ we have that a belongs to the filter of $\mathbf{B}$, generated by $F \cup \{x\}$. This implies that, for some n, $x^n \to a \in F$, and since $x^n \to a \in A$, we conclude that $x^n \to a \in F \cap A$. Therefore, a belongs to the filter of $\mathbf{A}$ generated by $(F \cap A) \cup \{x\}$. Hence, if $K \supset G$, then $K \cap A \supset G \cap A = F \cap A$, and $a \in K$. It follows that G is maximal among all filters of $\mathbf{C}$ which do not contain a, that is, $G \in \Delta(\mathbf{C}, a)$. The same argument also shows that G is maximal with respect to the property $G \cap A = F \cap A$. Hence, $G \in \Delta(\mathbf{C}, a)$, and $F \in G^*$.

Finally, for $x, y \in A$, we have $[x]_F = [y]_F$ iff $x \leftrightarrow y \in A \cap F$ iff $x \leftrightarrow y \in A \cap G$ iff $[x]_G = [y]_G$. Hence, the map $[x]_F \mapsto [x]_G$ is well defined and is a bijection between $\mathbf{W}_F \cap \mathbf{A}/F$ and $\mathbf{W}_G \cap \mathbf{A}/G$, because it has an inverse, namely, the map $[x]_G \mapsto [x]_F$. Clearly, the above defined map is a homomorphism, and hence it is an isomorphism. This settles claim (c). □

Continuing now the proof of Theorem 8.20, we prove the following claims:

Claim 1. h and k map B and C, respectively, into D.

Proof of claim 1. Let $b \in B$ and $(F, G) \in \Delta(\mathbf{B}, \mathbf{C})$ be given. Suppose $(h(b))(F', G') < e$ and $(F, G) < (F', G')$. Then $b \notin F'$, and hence $b \notin F$. Hence, if either $F \notin \Delta(\mathbf{B})$ or $G \notin F^*$, then $h(b)(F, G) = 0$. Suppose now $F \in \Delta(\mathbf{B})$ and $G \in F^*$. Then by Lemma 8.21, $F \subset F'$. Moreover, $h(b)(F, G) > 0$ would imply $[b]_F \in W_F$ and, again by Lemma 8.21, $b \in F'$, contradicting our assumption. Thus, in any case $h(b)(F, G) = 0$. By the definition of poset product, this shows that $h(b) \in D$ for all $b \in B$. The proof for k is similar.

Claim 2. h and k are homomorphisms of lattice-ordered monoids.

Proof of claim 2. The claim follows from the fact that lattice operations and the monoid operation in a poset product are defined pointwise.

Claim 3. h and k preserve implication.

Proof of claim 3. We prove the claim for h, the proof for k being similar. Let $b, b' \in B$ and $(F, G) \in \Delta(\mathbf{B}, \mathbf{C})$ be given. We distinguish several cases:

(3.1) If $b \to b' \in F$, then $h_{b \to b'}(F, G) = e$. Moreover, $[b]_F \leq [b']_F$. Therefore, according to the definition of h, $(h(b))(F, G) \leq (h(b'))(F, G)$, and for every $(F', G') > (F, G)$, we have $b \to b' \in F'$ and hence $(h(b))(F', G') \leq (h(b'))(F', G')$. By the definition of implication in a poset product, we have:

$$(h(b) \to h(b'))(F, G) = (h(b))(F, G) \to_{\mathbf{W}_F} (h(b'))(F, G) =$$
$$= [b]_F \to_{\mathbf{W}_F} [b']_F = e,$$

and the claim follows.

(3.2) If $b \to b' \notin F$, but $h_{b \to b'}(F, G) > 0$, then, according to the definition of h, we must have $F \in \Delta(\mathbf{B})$, $G \in F^*$, $h_{b \to b'}(F, G) = i_F([b \to b']_F) = i_F([b]_F \to_{\mathbf{B}/F} [b']_F) \in W_F - \{e\}$. Moreover, recalling that $\mathbf{B}/F$ has the form $\mathbf{H}_F \oplus \mathbf{W}_F$, by the definition of ordinal sum, if $[b]_F \to_{\mathbf{B}/F} [b']_F \in W_F - \{e\}$, then $[b']_F \in W_F$, $[b]_F \in W_F$ and $[b']_F < [b]_F$. Hence,

$$h_{b \to b'}(F, G) = i_F([b]_F \to_{\mathbf{B}/F} [b']_F) = i_F([b]_F \to_{\mathbf{W}/F} [b']_F) =$$
$$= i_F([b]_F) \to_{\mathbf{L}_{n*}} i_F([b']_F).$$

On the other hand, by Lemma 8.21, (b), if $(F', G') > (F, G)$, then $b \in F'$, $b' \in F'$ and $(h(b))(F', G') \leq (h(b'))(F', G') = e$. By the definition of implication in a poset product, it follows

$$(h(b \to b'))(F, G) = (h(b))(F, G) \to_{\mathbf{L}_{n*}} (h(b'))(F, G) = h_{b \to b'}(F, G).$$

(3.3) If $h_{b \to b'}(F, G) = 0$, then $b \to b' \notin F$, and we have to distinguish the following subcases:

(3.3.a). If $F \notin \Delta(\mathbf{B}, b \to b')$, then there is $F' \in \Delta(\mathbf{B}, b \to b')$ such that $F \subset F'$. Take $G' \supseteq G$ maximal among all filters H of $\mathbf{C}$ such that $H \cap A = F' \cap A$. Then $G' \in F'^*$, $(F', G') \in \Delta(\mathbf{B}, \mathbf{C})$, $(F', G') > (F, G)$, and, by Lemma 8.21, (a), $[(b \to b')]_{F'} \in W_{F'} - \{e\}$. As in case (3.2), we see that $[b]_{F'} \in W_{F'}$, $[b']_{F'} \in W_{F'}$, and $[b]_{F'} > [b']_{F'}$. Thus, $h(b)(F', G') > h(b')(F', G')$ for some $(F', G') > (F, G)$, and by the definition of implication in a poset product, $(h(b) \to h(b'))(F, G) = 0$.

(3.3.b). If $F \in \Delta(\mathbf{B}, b \to b')$ and $G \notin F^*$, then there is $G' \in F^*$ such that $G \subset G'$. Then $(F, G') > (F, G)$ and by Lemma 8.21, (a), $(h(b \to b'))(F, G') \in W_{F'} - \{e\}$. Hence, as in case (3.2), we obtain that $[b]_{F'} \in W_{F'}$, that $[b']_{F'} \in W_{F'}$, and that $[b]_{F'} > [b']_{F'}$. It follows that $(h(b))(F, G') > (h(b'))(F, G')$ for some $(F, G') > (F, G)$, and by the definition of implication in a poset product, we obtain $(h(b) \to h(b'))(F, G) = 0$.

(3.3.c). If $F \in \Delta(\mathbf{B}, b \to b')$ and $G' \in F^*$, then by Lemma 8.21, (a), $[(b \to b')]_F \in W_F - \{e\}$ and $0 = (h(b \to b'))(F, G) = i_F([(b \to b')]_F) \in W_F - \{e\}$. By the usual argument, $[b]_F \in W_F$ and $[b']_F \in W_F$. Since $0 = i_F(([b]_F) \to_{\mathbf{W}_F} ([b']_F))$, the only possibility is that $[b]_F$ is the maximum of $\mathbf{W}_F$ and $[b']_F$ is the minimum of $\mathbf{W}_F$. Thus, $(h(b))(F, G) = e$, $(h(b'))(F, G) = 0$ and $(h(b) \to h(b'))(F, G) = 0$. This settles claim (3.3).

Claim 3.4. h and k are injective.

Proof of claim 3.4. As usual, we only prove the claim for h. It suffices to prove that for all $b \in B$, if $h(b) = e$, then $b = e$. If $b < e$, then there is a filter $F \in \Delta(\mathbf{B}, b)$ and a filter $G \in F^*$. Then $(h(b))(F, G) = i_F([b]_F) < e$, and the claim is proved.

Claim 3.5. If $a \in A$, then $h(a) = k(a)$.

Proof of claim 3.5. Let $(F, G) \in \Delta(\mathbf{B}, \mathbf{C})$. By Lemma 8.21, we have that $a \in F$ iff $a \in G$. Hence, $h(a)(F, G) = e$ iff $a \in F$ iff $a \in G$ iff $k(a)(F, G) = e$. Now suppose $a \notin F$ and $a \notin G$. Then by Lemma 8.21 (c), $F \in \Delta(a, \mathbf{B})$ iff $G \in \Delta(a, \mathbf{C})$. Thus if $F \notin \Delta(a, \mathbf{B})$, then $G \notin \Delta(a, \mathbf{C})$, $[a]_F \notin W_F$, $[a]_G \notin W_G$ and $h(a)(F, G) = k(a)(F, G) = 0$. Finally, if $F \in \Delta(a, \mathbf{B})$ and $G \in \Delta(a, \mathbf{C})$, then, again by Lemma 8.21, (c), $F \in G^*$ and $G \in F^*$. Moreover $[a]_F \in W_F$, $[a]_G \in W_G$, $h(a)(F, G) = i_F([a]_F)$ and $k(a)(F, G) = i_G([a]_G)$. Since the map $[x]_F \mapsto [x]_G$ is an isomorphism from $\mathbf{W}_F \cap \mathbf{A}/F$ onto $\mathbf{W}_G \cap \mathbf{A}/G$, the isomorphic copies of $[a]_F$ and $[a]_G$ in $\mathbf{L}_{n*}$ must coincide, i.e., $h(a)(F, G) = i_F([a]_F) = i_G([a]_G) = k(a)(F, G)$. This concludes the proof. □

Corollary 8.22. *$\mathcal{GBL}1p$, $\mathcal{GBL}2p$, and the variety $\mathcal{H}$ of Heyting algebras have the amalgamation property.*

Proof. Every 1-potent GBL-algebra is a subdirect poset product of algebras isomorphic to $\mathbf{L}_{1*}$ and hence $\mathcal{GBL}1p = \mathcal{GBL}_1$. Moreover, every 2-potent GBL-algebra is a subdirect poset product of algebras isomorphic either to $\mathbf{L}_{1*}$ or to $\mathbf{L}_{2*}$. Hence, every 2-potent GBL-algebra is a subdirect poset product of subalgebras of $\mathbf{L}_{2*}$, and hence $\mathcal{GBL}2p = \mathcal{GBL}_2$. So $\mathcal{GBL}1p$ and $\mathcal{GBL}2p$ have the amalgamation property by Theorem 8.20.

The elements of $\mathcal{GBL}_1$ are precisely the subreducts of Heyting algebras in the signature of commutative GBL-algebras, and the claim then follows from Theorem 8.20 and Theorem 7.10. □

Finally, we remark that notable *non-equational* classes of commutative GBL-algebras satisfying the amalgamation property include the class of finite GBL-algebras and the class of *finitely-potent GBL-algebras*: the GBL-algebras that are n-potent for some n.

Note that each n-potent GBL-algebra is a subdirect poset product of algebras of the form $\mathbf{L}_{m*}$ for some $m \leq n$. These algebras are subalgebras of $\mathbf{L}_{n!*}$, and hence, $\mathcal{GBL}np \subseteq \mathcal{GBL}_{n!}$.

Now a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ of finitely-potent GBL-algebras is a V-formation in $\mathcal{GBL}np$ for some n, and hence a V-formation in $\mathcal{GBL}_{n!}$. By Theorem 8.20, such a V-formation has an amalgam in $\mathcal{GBL}_{n!}$, and hence in the class of finitely-potent GBL-algebras. It follows:

Theorem 8.23. *The class of finitely-potent GBL-algebras has the amalgamation property.*

Now let us consider finite GBL-algebras. Clearly, a finite GBL-algebra is n-potent for a suitable n, and hence it is an element of $\mathcal{GBL}_{n!}$. Moreover, if in Theorem 8.20 the algebras $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are supposed to be finite, then $\Delta(\mathbf{B}, \mathbf{C})$ is finite and the amalgam $\mathbf{D}$ in $\mathcal{GBL}_{n!}$ provided by Theorem 8.20 is finite, because it is a poset product, with respect to a finite poset, of a finite family of finite algebras. It follows:

Theorem 8.24. *The class of finite GBL-algebras has the amalgamation property.*

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