Constantine Tsinakis
Han Zhang

Order Algebras as Models of Linear Logic

Abstract. The starting point of the present study is the interpretation of intuitionistic linear logic in Petri nets proposed by U. Engberg and G. Winskel. We show that several categories of order algebras provide equivalent interpretations of this logic, and identify the category of the so called strongly coherent quantales arising in these interpretations. The equivalence of the interpretations is intimately related to the categorical facts that the aforementioned categories are connected with each other via adjunctions, and the compositions of the connecting functors with co-domain the category of strongly coherent quantales are dense. In particular, each quantale canonically induces a Petri net, and this association gives rise to an adjunction between the category of quantales and a category whose objects are all Petri nets.

Keywords: Petri nets, linear logic, quantales, net semantics, partially ordered monoids

1. Introduction

Throughout this paper, we shall exclusively consider commutative algebraic structures, and omit the adjective "commutative" when referring to them. For example, we shall use the terms "monoid" for "commutative monoid," "quantale" for "commutative quantale," "free quantale" for "free commutative quantale," etc.

Linear logic, introduced by J. -Y. Girard [5], is a refinement of classical logic. It has been described as a "resource conscious" logic because resources are consumed, or discarded under explicit rules. The connections between variants of linear logic, Petri nets, and quantales are the subject of many studies, including [1], [3], [4], [6], [8], [14], and [7]. In what follows, we shall use the term *linear logic* to refer to the fragment of *commutative, intuitionistic, propositional linear logic* that does not include the modality ? and the connective ⅋ (see Section 6.1 for pertinent definitions).

The starting point of this work is the interpretation of this logic in Petri nets, proposed by U. Engberg and G. Winskel in [3]. Their order theoretic approach, henceforth referred to as *net semantics*, is a variant of Girard's phase semantics in [5], as well as Yetter's quantale semantics in [14], and is based on the fact that every Petri net induces a preorder $\preceq$ (reflexive and

Special Issue Algebra and Substructural Logics: Take Two — Part 1
Edited by **Hiroakira Ono and Tomasz Kowalski**

Studia Logica **76**: 195–219, 2004.

transitive relation) on its set of markings, called the reachability relation. The markings $S^{\otimes}$ form a free monoid over the set S of places of the net, and the reachability relation is compatible with the binary operation, that is, if $a \preceq b$, then $(a + c) \preceq (b + c)$, for all $a, b, c \in S^{\otimes}$. Thus, $(S^{\otimes}, \preceq)$ is a preordered monoid (see Section 3), and the quantale associated with the Petri net is the set of down sets of $(S^{\otimes}, \preceq)$. Recall that a subset I of preorder set $(P, \preceq)$ is a *down set* of P if $I = \downarrow I$, where, for each $A \subseteq P$, $\downarrow A = \{p \in P \mid \exists\, a \in A, p \preceq a\}$. *It should be noted that, unlike quantale semantics, values of the formulas in net semantics are restricted to those that extend the assignments of atomic propositions to principal down sets of the corresponding quantale.* The choice of the quantales and valuation maps in net semantics reflects the view that each place of the net is an atomic proposition, and its interpretation is the set of markings that are prerequisites for it.

Evidently, the quantales associated with Petri nets are distributive lattices (in fact frames), since joins and meets are simply unions and intersections, respectively. Thus, Petri nets are potentially useful for modeling the extension of linear logic that includes the lattice-distributive law as an axiom, but they are inadequate to interpret the full linear logic. It is worth contrasting the Engberg-Winskel approach with that in [8], where the models of the logic are certain linear categories. Linear categories are essentially the categorical analogues of residuated lattices. Categorical models share the same mathematical ideas with the interpretations by residuated lattices or quantales (see [13]). For example, completeness of the models can be tested by a "free" linear category, essentially the "Tarski–Lindenbaum Category" of the theory.

Category theory provides the framework for exploring further the nature of the correspondence between Petri nets and quantales as models of linear logic. The pertinent categories are the category *Quant* of quantales, and the category *Petri* consisting of Petri nets and morphisms preserving the static structure and behavior of the net (see Section 2.2). An important fact underlying many of the considerations of the present paper is that Petri nets are essentially relational systems of the form $(S^{\otimes}, R)$, where $S^{\otimes}$ is the free monoid over the set S of places of the net and R is a binary relation on $S^{\otimes}$. Thus, all preordered monoids of the form $(S^{\otimes}, \preceq)$ may be viewed as Petri nets, and a category *PreoPetri*, with objects this algebraic class of nets, suffices for net semantics.

The nature of the correspondence between Petri nets and quantales has hitherto been opaque. The first question to consider is whether the quantales arising in net semantics can be characterized abstractly within the class of all quantales.

- *Characterize, up to isomorphism, the (necessarily distributive) quantales associated with net semantics.*

It is shown in Section 5 that these are the so called *strongly coherent quantales*, that is, those quantales whose completely join-prime elements form a submonoid that order generates the quantale. It is evident that the quantale of down sets of a preordered monoid of the form $(S^{\otimes}, \preceq)$ is strongly coherent. The nontrivial direction of the result is to establish that every strongly coherent quantale is isomorphic to the quantale of down sets of the reachability relation of some Petri net.

It is also important to inquire whether the aforementioned correspondence is categorical, and whether every quantale canonically induces a Petri net.

- *Is the correspondence from Petri to Quant functorial? Moreover, is there a functorial correspondence in the opposite direction?*

The first part of the question has been answered by J. Lilius in [7], where it is shown that the reachability relation induces a functor from *Petri* to *Quant*, but the category *SCohQuant* of strongly coherent quantales is not identified there. We extend Lilius' result by showing that the functor in question has a right adjoint, thereby establishing that the categories *Quant* and *Petri* are closely connected. A crucial step in establishing this result is to show that the categories *Petri* and *PoMon* – of partially ordered monoids and monotone monoid homomorphisms – are connected by an adjunction, and are *equivalent* models of the logic. Here, equivalence means that for every sequent that is valid in a member of *Petri*, there exists a member of *PoMon* in which the sequent is valid, and conversely. In this sense, it appears that the algebraic category *PoMon* is a convenient category for abstracting the Engberg-Winskel correspondence. Given a partially ordered monoid, one passes from the logic to quantales by mapping the atomic propositions to the elements of the monoid and then taking the down sets of this monoid (see Section 6.3).

It should be noted that the interpretations of linear logic in *Petri* and *PreoPetri* are "external" to the categories, in the sense that they require the use of the category *Quant*, or more precisely *SCohQuant*. This leads to the following question.

- *Is there a category of order algebras isomorphic to Petri or PreoPetri in which linear logic can be interpreted internally?*

In Section 6, we define the category *FrQuantOp*, of free quantales with a single operator, which is isomorphic to *PreoPetri*, and in which linear logic can be interpreted internally. Its definition involves quantic nuclei on free quantales and provides an alternative way for viewing the correspondence between Petri nets and strongly coherent quantales. Given a monoid M, there is a canonical way to construct a quantale $\wp(M)$ with lattice structure the Boolean lattice of all subsets of M, and multiplication $\circ$ defined as $A \circ B = \{ab \,|\, a \in A, b \in B\}$, for all $A, B \in \wp(M)$. As was noted earlier, the set $S^{\otimes}$ of markings of a Petri net is a free monoid over the set S of places of the net, and hence, for any set S, $\wp(S^{\otimes})$ is the free quantale generated by S. Now every quantale is "canonically" isomorphic to the image of a quantic nucleus on a quantale of the form $\wp(S^{\otimes})$ and, as was shown in [7], the reachability relation on $S^{\otimes}$ gives rise to a quantic nucleus whose image is the quantale of down sets of S. Such a quantic nucleus is evidently *linear*, that is, it preserves arbitrary joins. We show that there exists a bijective correspondence between linear quantic nuclei on $\wp(S^{\otimes})$ and preorders on $S^{\otimes}$ that are compatible with the semigroup operation. This correspondence actually extends to an isomorphism of the categories *PreoPetri* and *FrQuantOp*. The objects of the category *PreoPetri* are preordered monoids of the form $(S^{\otimes}, \preceq)$, which may be viewed as special types of Petri nets, and those of *FrQuantOp* are pairs in the form $(\wp(S^{\otimes}), \alpha)$, with α a linear quantic nucleus on $\wp(S^{\otimes})$.

2. Preliminaries

In this section we introduce the categories *Petri* of Petri nets and *RelPetri* of relational Petri nets. The category *Petri* has the same objects as the category with the same name in [7], but its morphisms are more restrictive. The category *RelPetri* casts Petri nets in a familiar algebraic form and provides the first step in a hierarchy of categories that provide equivalent models of linear logic and are connected with each other via adjunctions.

2.1. Multisets

A **multiset** is a set in which the multiplicity of an element is taken into account. In what follows, we will restrict our attention to **finite multisets**, that is, multisets in which at most finitely many elements have nonzero multiplicities. More formally, we have the following definition.

Definition 2.1. *A* **multiset** *p over a set S is a map $p : S \longrightarrow \mathbb{N}$ from S to the natural numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$. A multiset p over S is* **finite** *if $p(a) = 0$ for all but finitely many $a \in S$. The set of all finite multisets over S is denoted by $S^{\otimes}$.*

There is a natural operation $+$ and a natural partial order $\leq$ on $S^{\otimes}$, defined respectively as

$$(p+q)(a) = p(a) + q(a), \quad \forall p, q \in S^{\otimes},\ \forall a \in S;\ \text{and}$$

$$p \leq q \text{ iff } p(a) \leq q(a), \quad \forall p, q \in S^{\otimes},\ \forall a \in S.$$

For $a \in S$, we write $\underline{a}$ for the characteristic function of $\{a\}$, that is, the multiset given by $\underline{a}(a) = 1$ and $\underline{a}(s) = 0$, for all $s \neq a$. The constant function $\mathbf{0}$, which maps all elements of S to 0, is the unit for the operation $+$ and the bottom element for the partial order. We may also define a partial operation $-$ on S given by $(q-p)(a) = q(a) - p(a), \forall a \in S$, whenever $p \leq q$.

We state the following simple result for future reference.

PROPOSITION 2.2. *Let S be a set and let $S^{\otimes}$ be the set of finite multisets over S. The following properties hold.*

1. $(S^{\otimes}, +, \mathbf{0})$ *is isomorphic to the free monoid over S. Moreover, $\underline{S} = \{\underline{a} \mid a \in S\}$ is the set of free generators for it.*
2. $(S^{\otimes}, +, \mathbf{0})$ *satisfies the cancellation law for $+$, that is, if $p + t = q + t$, then $p = q$, for all $p, q, t \in S^{\otimes}$.*

2.2. Petri Nets

We start with the standard definition of the concept of a Petri net. The interested reader may consult [11] for an introduction to Petri nets and their connections with the theory of concurrency.

A Petri net is a triple $\boldsymbol{N} = (S, T, F)$ consisting of a set S of places, a disjoint T of transitions, and a finite multiset F – called the causal dependency relation – over $(S \times T) \cup (T \times S)$.

Several categories of Petri nets have been proposed in the literature, see for example [9], [10], and [7]. The category *Petri* below has the same objects as the category with the same name in [7], but its morphisms are more restrictive. The definition of its objects is motivated by the following observation. In the preceding definition, each $t \in T$ gives rise to the multisets $i(t), o(t) \in S^{\otimes}$ defined by $i(t)(s) = F(s,t)$ and $o(t)(s) = F(t,s)$, for all $s \in S$. Note that F uniquely determines the maps $i, o : T \longrightarrow S^{\otimes}$, with values $i(t)$ and $o(t)$, and conversely. This leads to the following definition for the objects of *Petri*.

DEFINITION 2.3. *A* **Petri net** *is a quadruple $\boldsymbol{N} = (S, T, i, o)$, where S is a set whose elements are called* **places**, *T is a disjoint set whose elements are called* **transitions**, *and $i, o : T \longrightarrow S^{\otimes}$ are arbitrary maps. For each $t \in T$, $i(t)$ is referred to as the* **preset** *of t and $o(t)$ as the* **postset** *of t.*

The morphisms must preserve the static structure and the behavior of the net. That is, a morphism from (S, T, i, o) to (S', T', i', o') must take places to places and transitions to transitions, and hence it must be a pair $\langle f, g\rangle$ of maps $f : S \longrightarrow S'$ and $g : T \longrightarrow T'$. In addition, it must also preserve the structure of the presets and postsets of the transitions of the nets. This is achieved by demanding that the following diagrams commute.

$$
\begin{array}{ccc}
T & \xrightarrow{\;i\;} & S^{\otimes} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
T' & \xrightarrow{\;i'\;} & S'^{\otimes}
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & \xrightarrow{\;o\;} & S^{\otimes} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
T' & \xrightarrow{\;o'\;} & S'^{\otimes}
\end{array}
$$

Here $\bar{f}$ is the free monoid homomorphic extension of the map $\underline{a} \mapsto \underline{f(a)}$, for all $a \in S$. We want to point out that the composition is just the componentwise compositions, that is, $\langle f, g\rangle \circ \langle f', g'\rangle = \langle f \circ f', g \circ g'\rangle$.

We next define the category *RelPetri* of relational Petri nets, which gives a simple algebraic description of the traditionally defined Petri nets.

DEFINITION 2.4. *A* **relational Petri net** *is a pair* $\boldsymbol{N} = (S^{\otimes}, R)$*, where* $S^{\otimes}$ *is the set of all finite multisets over a set* S *and* R *is a binary relation on* $S^{\otimes}$*.*

The objects in the category *RelPetri* are Petri nets as defined in Definition 2.4. A morphism $f : (S^{\otimes}, R) \longrightarrow (S'^{\otimes}, R')$ is a monoid homomorphism $f : S^{\otimes} \longrightarrow S'^{\otimes}$ such that (1) f maps $\underline{S} = \{\underline{a} \,|\, a \in S\}$ to $\underline{S'} = \{\underline{a'} \,|\, a' \in S'\}$, and (2) f preserves the relation R, that is, if mRn in $S^{\otimes}$, then $f(m)R'f(n)$ in $S'^{\otimes}$.
We remark the only difference between the categories *Petri* and *RelPetri* is that there are no distinct transitions in *RelPetri* with the same preset and postset. It is also worth mentioning that the categories *Petri* and *RelPetri* are connected by an adjunction. We omit the proof of this fact, but show in the next section that *Petri* and a useful subcategory of *RelPetri* are linked by an adjunction.

3. Preordered Petri Nets

In this section, we consider the full subcategory *PreoPetri* of *RelPetri* whose objects are defined in Definition 3.1. The main result of this section is Theorem 3.3 below which asserts that the categories *Petri* and *PreoPetri* are connected via an adjunction. This result is crucial in linking net and quantale semantics of linear logic, and is used to characterize the quantales arising in net semantics.

DEFINITION 3.1. *A* **preordered Petri net** *is a pair* $\boldsymbol{N} = (S^{\otimes}, \preceq)$*, where* $S^{\otimes}$ *is the set of all finite multisets over a set of places* S *and* $\preceq$ *is a preorder on* $S^{\otimes}$ *that is* compatible *with the + operation on* $S^{\otimes}$*, that is, if* $a \preceq b$*, then* $(a + c) \preceq (b + c)$*, for all* $a, b, c \in S^{\otimes}$*.*

We proceed with the definition of two functors $\mathcal{R} : \underline{Petri} \longrightarrow \underline{PreoPetri}$ and $\Pi : \underline{PreoPetri} \longrightarrow \underline{Petri}$.

The object part of the functor $\mathcal{R}$ is defined in three steps:

1. Given (S, T, i, o) in $\underline{Petri}$, let R be the binary relation on $S^{\otimes}$ defined by $R = \{(i(t), o(t)) \,|\, t \in T\}$.
2. Define a second binary relation W on $S^{\otimes}$ as follows:
$$(m, m') \in W \text{ iff } \exists\, (a, b) \in R \text{ such that } a \leq m \text{ and } a + m' = b + m$$
3. Let $\preceq$ be the reflexive and transitive closure of W, and define $\mathcal{R}(S, T, i, o) = (S^{\otimes}, \preceq)$.

Obviously $R \subseteq W \subseteq \preceq$, and it is easy to verify that the preorder $\preceq$ on $S^{\otimes}$ coincides with the reachability relation in [3].

If $\langle f, g\rangle : (S, T, i, o) \longrightarrow (S', T', i', o')$ is a morphism in $\underline{Petri}$, then define $\mathcal{R}(\langle f, g\rangle) = \bar{f}$ as the free monoid homomorphic extension of the map $\underline{a} \mapsto \underline{f(a)}$, for all $a \in S$.

Conversely, given $(S^{\otimes}, \preceq)$ in $\underline{PreoPetri}$, let $\Pi(S^{\otimes}, \preceq) = (S, \preceq, \pi_1, \pi_2)$, with $\pi_1, \pi_2 : \preceq \longrightarrow S^{\otimes}$ being the projection maps. Clearly Π has the correct co-domain at the object level.

Given a morphism $f : (S^{\otimes}, \preceq) \longrightarrow (S'^{\otimes}, \preceq')$ in $\underline{PreoPetri}$, let $\Pi(f) = \langle h, g\rangle$, where $h : S \longrightarrow S'$ is the map defined by $h(a) = b$ if and only if $f(\underline{a}) = \underline{b}$, and $g : \preceq \longrightarrow \preceq'$ is defined as $g(m, n) = (f(m), f(n))$. Since f is monotone, $\Pi(f)$ is a map from $\Pi(S^{\otimes}, \preceq)$ to $\Pi(S'^{\otimes}, \preceq')$. The proof of Theorem 3.3 will be preceded by a useful lemma.

LEMMA 3.2. *Let* $(S^{\otimes}, \preceq)$ *be in* $\underline{PreoPetri}$*. Then* $\mathcal{R}(\Pi(S^{\otimes}, \preceq)) = (S^{\otimes}, \preceq)$*.*

PROOF. Let $(S^{\otimes}, \preceq) \in \underline{PreoPetri}$, and let $\Pi(S^{\otimes}, \preceq) = (S, \preceq, \pi_1, \pi_2)$. We construct $\mathcal{R}(S, \preceq, \pi_1, \pi_2) = (S^{\otimes}, \preceq')$ using the three–step definition above. The first step yields $(S^{\otimes}, R)$, where $R = \{(\pi_1(t), \pi_2(t)) \,|\, t \in \preceq\} = \preceq$. So, all we have to show is that the last two steps will not change $\preceq$. First, we check the second relation W.

$$\forall m, m' \in S^{\otimes},\ m\ W\ m' \text{ iff}$$
$$\exists a, b \in S^{\otimes} \text{ such that } a \preceq b,\ a \leq m \text{ and } a + m' = b + m$$

We wish to show that $\preceq\ =\ W$. Clearly $\preceq\ \subseteq W$, since whenever $a \preceq b$, the relation $a \leq a$ implies $a + b = b + a$. Conversely, if $m\,W\,m'$, then there exist $a, b \in S^{\otimes}$ such that $a \preceq b, a \leq m$ and $a + m' = b + m$. But $a \leq m$ implies that there exists $n \in S^{\otimes}$ such that $m = a + n$ (just let $n = m - a$), and hence $m' = b + m - a = b + n$. But because $\preceq$ is compatible with $+$, $(a + n) \preceq (b + n)$ for any n, and hence, $m \preceq m'$. Finally, since $\preceq$ is already reflexive and transitive, we have $\preceq' = \preceq$, and $\mathcal{R}(\Pi(S^{\otimes}, \preceq)) = (S^{\otimes}, \preceq)$. ■

THEOREM 3.3. *The assignments* $\mathcal{R}$: <u>*Petri*</u> $\longrightarrow$ <u>*PreoPetri*</u> *and* Π : <u>*PreoPetri*</u> $\longrightarrow$ <u>*Petri*</u> *form an adjunction.*

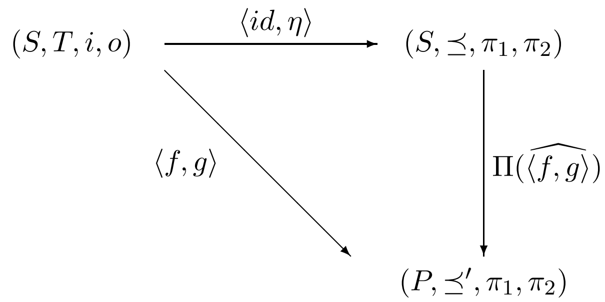
PROOF. We start the proof by showing that Π and $\mathcal{R}$ are functors. To begin with, we already mentioned that Π is well-defined at the object level. As for the morphisms, let $f : (P^{\otimes}, \preceq) \longrightarrow (P'^{\otimes}, \preceq')$ be a morphism in <u>*PreoPetri*</u>. Then $\Pi(f) = \langle h, g\rangle$ (see definition above) is a morphism in <u>*Petri*</u>, in view of the equalities $f(\pi_i(m, n)) = \pi_i(f(m), f(n)) = \pi_i(g(m, n))$, for all $(m, n) \in \preceq$ and $i = 1, 2$.

Next, we consider $\mathcal{R}$. First we need to show that given (S, T, i, o) in <u>*Petri*</u>, $\mathcal{R}(S, T, i, o) = (S^{\otimes}, \preceq)$ is in <u>*PreoPetri*</u>. Here $\preceq$ is constructed in three steps as defined above and is already a preorder. It remains to show that $\preceq$ is compatible with $+$. Let R and W be the binary relations defined as above. We only need show that W is compatible with $+$, since then clearly its reflexive and transitive closure will also be compatible with the operation $+$. Suppose $m\,W\,m'$ for some $m, m' \in S^{\otimes}$. By the definition of W, there exists $(a, b) \in R$, such that $a \leq m$ and $a + m' = b + m$. If $n \in S^{\otimes}$, then clearly $a \leq m + n$ and $a + (m' + n) = b + (m + n)$. Thus, $(m + n)W\,(m' + n)$ for all $n \in S^{\otimes}$, showing that W is compatible with $+$.

For the morphisms, let $\langle f, g\rangle : (S, T, i, o) \longrightarrow (S', T', i', o')$ be a morphism in <u>*Petri*</u>, $\mathcal{R}(S, T, i, o) = (S^{\otimes}, \preceq)$, $\mathcal{R}(S', T', i', o') = (S'^{\otimes}, \preceq')$, and R, R', W, W' be the corresponding binary relations on $S^{\otimes}$ and $S'^{\otimes}$ respectively. Since $\bar{f} = \mathcal{R}(\langle f, g\rangle)$ is already a monoid homomorphism by definition, we only have to show that it is monotone, equivalently, that it preserves the relations W and W'. We first show that $\bar{f}$ preserves the relations R and R'. Let $m\,R\,n$. There exists $t \in T$ such that $m = i(t)$ and $n = o(t)$. By the commutativity of the diagram above, $\bar{f}(m) = i'(g(t))$ and $\bar{f}(n) = o'(g(t))$; hence $\bar{f}(m)\,R'\,\bar{f}(n)$. We next show that $\bar{f}$ preserves the relations W and W'. Suppose $m, m' \in S^{\otimes}$ and mWm'. Then there exist $a, b \in S^{\otimes}$ such that $a\,R\,b$, $a \leq m$ and $a + m' = b + m$. It follows that $\bar{f}(a)\,R'\,\bar{f}(b)$ and

$\bar{f}(a) + \bar{f}(m') = \bar{f}(b) + \bar{f}(m)$. Note that $\bar{f}(a) \leq \bar{f}(m)$. Indeed $a \leq m$ implies that there exists $n \in S^{\otimes}$ such that $m = a + n$. Thus $\bar{f}(m) = \bar{f}(a) + \bar{f}(n)$ and $\bar{f}(a) \leq \bar{f}(m)$. We have shown that $\bar{f}(m)\, W'\, \bar{f}(m')$. The remaining categorical properties are easily checked.

We now prove that the pair $(\mathcal{R}, \Pi)$ forms an adjunction. Let (S, T, i, o) be in *Petri*, $\mathcal{R}(S, T, i, o) = (S^{\otimes}, \preceq)$, and $(S, \preceq, \pi_1, \pi_2) = \Pi(S^{\otimes}, \preceq)$. If $\eta : T \longrightarrow \preceq$ is the map defined by $\eta(t) = (i(t), o(t))$, and id is the identity map on S, then it is easily seen that $\langle id, \eta \rangle : (S, T, i, o) \longrightarrow (S, \preceq, \pi_1, \pi_2)$ is a morphism in *Petri*. Now let $(P^{\otimes}, \preceq')$ be in *PreoPetri*, $(P, \preceq', \pi_1, \pi_2) = \Pi(P^{\otimes}, \preceq')$, and $\langle f, g \rangle : (S, T, i, o) \longrightarrow (P, \preceq', \pi_1, \pi_2)$ be a morphism in *Petri*. We need to show that there exists a unique morphism $\widehat{\langle f, g \rangle} : (S^{\otimes}, \preceq) \longrightarrow (P^{\otimes}, \preceq')$ such that the following diagram commutes.



First, let $\widehat{\langle f, g \rangle} = \mathcal{R}(\langle f, g \rangle) = \bar{f}$. In view of Lemma 3.2, $\widehat{\langle f, g \rangle}$ is a morphism from $\mathcal{R}(S, T, i, o) = (S^{\otimes}, \preceq)$ to $\mathcal{R}(P, \preceq', \pi_1, \pi_2) = (P^{\otimes}, \preceq')$.

Now clearly $\Pi(\widehat{\langle f, g \rangle}) = \Pi(\bar{f}) = \langle f, \hat{g} \rangle$, where $\hat{g} : \preceq \longrightarrow \preceq'$ is defined by $\hat{g}(m, n) = (\bar{f}(m), \bar{f}(n))$. We have to show that $\langle f, \hat{g} \rangle \circ \langle id, \eta \rangle = \langle f \circ id, \hat{g} \circ \eta \rangle = \langle f, g \rangle$, or $\hat{g} \circ \eta = g$. For $t \in T$, $\hat{g} \circ \eta(t) = \hat{g}(i(t), o(t)) = (\bar{f}(i(t)), \bar{f}(o(t)))$. On the other hand, since $\langle f, g \rangle$ is a morphism in *Petri*, $\pi_1(g(t)) = \bar{f}(i(t))$, and $\pi_2(g(t)) = \bar{f}(o(t))$. It follows that $\hat{g} \circ \eta(t) = g(t)$, and hence $\hat{g} \circ \eta = g$, as was to be shown. The uniqueness of $\widehat{\langle f, g \rangle}$ is clear because if there exists a morphism $h : (S^{\otimes}, \preceq) \longrightarrow (P^{\otimes}, \preceq')$ such that $\Pi(h) \circ \langle id, \eta \rangle = \langle f, g \rangle$, then h has to agree with $\bar{f}$ on the generators of $S^{\otimes}$, hence they must be equal. ∎

4. Partially Ordered Monoids

This auxiliary section connects the category *PreoPetri* with the familiar category *PoMon* of partially ordered monoids and monotone monoid homomorphisms. The category *PoMon* provides the last link for the chain of adjunctions between *Petri* and *SCohQuant* and serves as an intermediate step for canonically associating Petri nets with quantales. In light of the results of the last section, $\underline{PoMon}$ is the most convenient category for abstracting the Engberg-Winskel correspondence in a purely algebraic setting.

A *partially ordered monoid* is a system $(M, \cdot, \mathbf{1}, \leq)$ consisting of a (commutative) monoid $(M, \cdot, \mathbf{1})$ and a partial order $\leq$ on M which is compatible with the monoid operation. We shall denote a partially ordered monoid by $(M, \leq)$, or even by M if there is no danger of confusion.

It is clear that $\underline{PreoPetri}$ is a subcategory of the category of preordered sets with monotone maps. There exists a well-known functor from the category of preordered sets with monotone maps to the category of partially ordered sets with monotone maps. We show that this functor induces a functor $\mathcal{M} : \underline{PreoPetri} \longrightarrow \underline{PoMon}$. For any $(S^{\otimes}, \preceq)$ in $\underline{PreoPetri}$, let $\equiv_{S^{\otimes}}$ be the equivalence relation associated with $\preceq$: $a \equiv_{S^{\otimes}} b$ iff $a \preceq b$ and $b \preceq a$. The relation $\equiv_{S^{\otimes}}$ is a monoid congruence, since $\preceq$ is compatible with the monoid operation $\cdot$. Therefore $S^{\otimes}/\equiv_{S^{\otimes}}$ is a partially ordered monoid with the binary operation defined by $[a] \cdot [b] = [a \cdot b]$, and the partial order by $[a] \leq [b] \iff a \preceq b$. Given $(S^{\otimes}, \preceq)$ in $\underline{PreoPetri}$, let $\mathcal{M}(S^{\otimes}, \preceq) = (S^{\otimes}/\equiv_{S^{\otimes}}, \leq)$. Given $f : (S^{\otimes}, \preceq) \longrightarrow (S'^{\otimes}, \preceq)$, a morphism in $\underline{PreoPetri}$, let $\mathcal{M}(f) : (S^{\otimes}/\equiv_{S^{\otimes}}, \leq) \longrightarrow (S'^{\otimes}/\equiv_{S'^{\otimes}}, \leq)$ be defined by $\mathcal{M}(f)([a]) = [f(a)]$, for all $a \in S^{\otimes}$. It is easy to see that the preceding assignments define a functor $\mathcal{M} : \underline{PreoPetri} \longrightarrow \underline{PoMon}$.

Conversely, we define a functor $\mathcal{N} : \underline{PoMon} \longrightarrow \underline{PreoPetri}$ as follows. First, recall that given any set M, $M^{\otimes}$ is the free monoid over M, so if in addition M is already a monoid, there exists a natural monoid epimorphism $\gamma_M : M^{\otimes} \longrightarrow M$ extending the map $\underline{a} \mapsto a (a \in M)$. Given $(M, \leq)$ in the category $\underline{PoMon}$, let $\mathcal{N}(M, \leq) = (M^{\otimes}, \preceq)$, with $\preceq$ being defined by

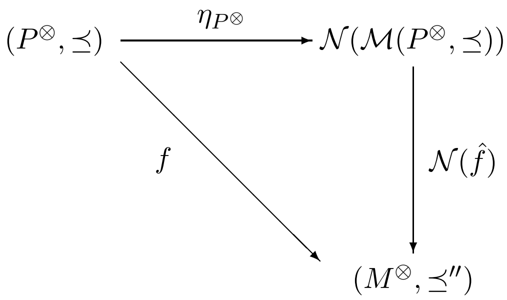
$$m_1 \preceq m_2 \iff \gamma_M(m_1) \leq \gamma_M(m_2), \ \forall m_1, m_2 \in M^{\otimes}.$$

The relation $\preceq$ is clearly a preorder and compatible with $+$ on $M^{\otimes}$, since γ_M is a monoid homomorphism. As for the morphisms, if $f : M \longrightarrow N$ is a morphism in $\underline{PoMon}$, let $\mathcal{N}(f) : (M^{\otimes}, \preceq) \longrightarrow (N^{\otimes}, \preceq')$ be the monoid homomorphic extension of the map $\underline{a} \mapsto \underline{f(a)}$, for all $a \in M$. It is easy to see that $\mathcal{N}(f)$ is a morphism in $\underline{PreoPetri}$. Indeed, note that $f \circ \gamma_M = \gamma_N \circ \mathcal{N}(f)$, and hence $m \preceq n$ in $M^{\otimes} \iff \gamma_M(m) \leq \gamma_M(n) \Longrightarrow f(\gamma_M(m)) \leq f(\gamma_M(n)) \iff \gamma_N(\mathcal{N}(f)(m)) \leq \gamma_N(\mathcal{N}(f)(n)) \iff \mathcal{N}(f)(m)) \preceq' \mathcal{N}(f)(n)$ in $N^{\otimes}$.

THEOREM 4.1. *The functor $\mathcal{M} : \underline{PreoPetri} \longrightarrow \underline{PoMon}$ is a left adjoint to the functor $\mathcal{N} : \underline{PoMon} \longrightarrow \underline{PreoPetri}$.*

PROOF. Let $(P^{\otimes}, \preceq) \in \underline{PreoPetri}$, $(Q, \leq) = \mathcal{M}(P^{\otimes}, \preceq) = (P^{\otimes}/\equiv_{P^{\otimes}}, \leq)$, $(M, \leq) \in \underline{PoMon}$, $\mathcal{N}(M, \leq) = (M^{\otimes}, \preceq'')$. Consider a morphism

$f : (P^{\otimes}, \preceq) \longrightarrow (M^{\otimes}, \preceq'')$ in *PreoPetri*. We need to find a morphism $\eta_{P^{\otimes}}$: $(P^{\otimes}, \preceq) \longrightarrow \mathcal{N}(\mathcal{M}(P^{\otimes}, \preceq))$ and a unique morphism $\hat{f} : (Q, \leq) \longrightarrow (M, \leq)$ such that the following diagram commutes.



To this end, let $\mathcal{N}(\mathcal{M}(P^{\otimes}, \preceq)) = \mathcal{N}(Q, \leq) = (Q^{\otimes}, \preceq')$, and let $\eta_{P^{\otimes}}$: $(P^{\otimes}, \preceq) \longrightarrow (Q^{\otimes}, \preceq')$ be the monoid homomorphic extension of the map $\underline{p} \longmapsto \underline{[p]}$ $(p \in P)$. By definition, $\eta_{P^{\otimes}}$ is a monoid homomorphism. To show that it is monotone, let $m, n \in P^{\otimes}$ such that $m \preceq n$. Note that $\gamma_Q \circ \eta_{P^{\otimes}} = \pi$, where $\pi : (P^{\otimes}, \preceq) \longrightarrow (Q, \leq)$ is the canonical projection. It follows that $\gamma_Q(\eta_{P^{\otimes}}(m)) \leq \gamma_Q(\eta_{P^{\otimes}}(n))$, and hence, $\eta_{P^{\otimes}}(m) \preceq' \eta_{P^{\otimes}}(n)$, by the definition of $\preceq'$. We have shown that $\eta_{P^{\otimes}}$ is a morphism in *PreoPetri*. Next, let $\hat{f} : (Q, \leq) \longrightarrow (M, \leq)$ be defined by $\hat{f}([r]) = \gamma_M \circ f(r)$, for all $r \in P^{\otimes}$. We show that $\hat{f}$ is well-defined. Let $r, s \in P^{\otimes}$ such that $[r] = [s]$. Then $r \preceq s$ and $s \preceq r$. It follows that $\gamma_M(f(r)) \leq \gamma_M(f(s))$ and $\gamma_M(f(s)) \leq \gamma_M(f(r))$, since f and γ_M are monotone. Thus, $\gamma_M(f(s)) = \gamma_M(f(r))$, by antisymmetry.

Next, note that if $p \in P$, then $\underline{\hat{f}([\underline{p}])} = f(\underline{p})$. Indeed, since f maps $\{\underline{p} \,|\, p \in P\}$ to $\{\underline{q} \,|\, q \in M\}$, there exists $q \in M$ such that $f(\underline{p}) = \underline{q}$. It follows that $\underline{\hat{f}([\underline{p}])} = \underline{\gamma_M(f(\underline{p}))} = \underline{\gamma_M(\underline{q})} = \underline{q} = f(\underline{p})$. Now recall that $\mathcal{N}(\hat{f}) : (Q^{\otimes}, \preceq') \longrightarrow (M^{\otimes}, \preceq'')$ is the monoid homomorphism extending the map $\underline{[m]} \mapsto \underline{\hat{f}([m])}$. Hence for each generator $\underline{p}$ of $P^{\otimes}$, $\mathcal{N}(\hat{f})(\eta_{P^{\otimes}}(\underline{p})) = \mathcal{N}(\hat{f})(\underline{[p]}) = \underline{\hat{f}([p])} = f(\underline{p})$. It follows that $\mathcal{N}(\hat{f}) \circ \eta_{P^{\otimes}} = f$, establishing the commutativity of the diagram. The uniqueness of $\hat{f}$ is obvious since if there exists $g : (Q, \leq) \longrightarrow (M, \leq)$ such that $\mathcal{N}(g) \circ \eta_{P^{\otimes}} = f$, then g and $\hat{f}$ agree on the generators of $P^{\otimes}$, so they must be equal. ■

5. Strongly Coherent Quantales

In this section, we settle the first two questions raised in Section 1. We start by showing in Theorem 5.1 that the categories *PoMon* and *Quant* are linked by an adjunction. A complete answer to the second question raised in

Section 1 is provided by Theorem 5.2, which asserts that the categories *Petri* and *Quant* are linked by an adjunction. This result is a direct consequence of Theorem 5.1 and the adjunctions of Sections 3 and 4. The adjunction between *Petri* and *Quant* restricted at the object level, provides a canonical way for associating a net to a given quantale, and demonstrates the deep connection between Petri nets and quantales. The answer to the first question is provided by Theorem 5.7, which describes abstractly the quantales arising in net semantics as the *strongly coherent quantales*, that is, those quantales whose join-prime elements form a submonoid that order generates the quantale. The proof of this result is based on the fact (see Lemma 5.6) that the left adjoint of the adjunction between *Petri* and *SCohQuant* is dense. Lemma 5.6 will also be important for establishing in Section 6.3 the equivalence of the categories *Petri*, *PreoPetri*, and *PoMon* as models of linear logic.

A quantale $\mathcal{Q}$ is a system $\mathcal{Q} = (Q, \vee, \perp, \circ, \mathbf{1})$ satisfying the following properties.

1. $(Q, \vee, \perp)$ is a complete join-semilattice with least element $\perp$;
2. $(Q, \circ, \mathbf{1})$ is a monoid; and
3. The following distributive law holds:

$$a \circ \bigvee D = \bigvee \{a \circ b \mid b \in D\}, \forall\, a \in Q, \forall\, D \subseteq Q.$$

In what follows, we shall denote a quantale $\mathcal{Q}$ by its carrier Q. A quantale is evidently a complete lattice, with the meet operation and the greatest element denoted, respectively, by $\wedge$ and $\top$. We remark that the distributive law above is equivalent to the existence of a binary operation $\multimap$ on Q satisfying $a \circ b \leq c \iff b \leq a \multimap c$, for all $a, b, c \in Q$. We shall refer to the operation $\multimap$ as the *residual* of $\circ$.

A map between two complete join-semilattices is said to be *linear* if it preserves all joins. A *quantale homomorphism* is a linear monoid homomorphism. A *closure (coclosure) operator* on a quantale Q is a monotone map r on Q satisfying $a \leq r(a)$ $(r(a) \leq a)$ and $r(r(a)) = r(a)$, for all $a \in Q$. A *quantic nucleus* on Q is a closure operator r on Q such that $r(a) \circ r(b) \leq r(a \circ b)$, for all $a, b \in Q$.

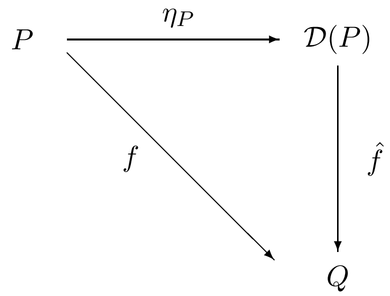
The category of quantales and quantale homomorphisms will be denoted by *Quant*.

It is simple to verify that the forgetful functor U : *Quant* $\longrightarrow$ *PoMon* has a left adjoint $\mathcal{D}$: *PoMon* $\longrightarrow$ *Quant*. Given P in *PoMon* , we let $\mathcal{D}(P)$ be the complete join-semilattice of down sets of P. That is, $\mathcal{D}(P) =$

$\{A \in \wp(P) \,|\, A = \downarrow A\}$. Arbitrary joins and meets in $\mathcal{D}(P)$ are unions and intersections, respectively. It was noted in [7] that the map α defined by $\alpha(A) = \downarrow A$, for all $A \in \wp(P)$, is a quantic nucleus. Hence, $\mathcal{D}(P)$, being the image of α, can be made into a quantale with multiplication defined by $A \circ' B = \downarrow (A \circ B)$, for all $A, B \in \mathcal{D}(P)$ (see Theorem 3.1.1 of [12]). Also, if $f : D \longrightarrow E$ is a morphism in <u>*PoMon*</u>, then it is easily checked that $\mathcal{D}(f) : \mathcal{D}(P) \longrightarrow \mathcal{D}(E)$, defined by $\mathcal{D}(f)(A) = \downarrow f(A)$, is a quantale homomorphism.

THEOREM 5.1. *The functor $\mathcal{D}$:* <u>*PoMon*</u> $\longrightarrow$ <u>*Quant*</u> *is the left adjoint of the forgetful functor U :* <u>*Quant*</u> $\longrightarrow$ <u>*PoMon*</u>.

PROOF. Throughout the proof, we will suppress the use of the symbol U. Let P be a partially ordered monoid, Q be a quantale, and $f : P \longrightarrow Q$ be a morphism in <u>*PoMon*</u>. We need to find a morphism $\eta_P : P \longrightarrow \mathcal{D}(P)$ in <u>*PoMon*</u> and a unique quantale homomorphism $\hat{f} : \mathcal{D}(P) \longrightarrow Q$ such that the following diagram commutes.



Here η_P is defined by $\eta_P(a) = \downarrow a$, for all $a \in P$. It is clear that η_P is a morphism in <u>*PoMon*</u>. Indeed, η_P is obviously monotone. Furthermore, $\eta_P(ab) = \downarrow (ab) = \downarrow (\downarrow a \circ \downarrow b) = \downarrow a \circ' \downarrow b$, for all $a, b \in P$. Next, let $\hat{f}$ be defined by $\hat{f}(A) = \bigvee f(A) = \bigvee \{f(a) \,|\, a \in A\}$. We need to show that $\hat{f}$ is a quantale homomorphism from $\mathcal{D}(P)$ to Q. First, it is easily checked that $\hat{f}$ is well-defined, linear and preserves the unit ($\hat{f}(\downarrow \mathbf{1}) = \mathbf{1}$). It also preserves multiplication, since for all $A, B \in \mathcal{D}(P)$,

$$\begin{aligned}
\hat{f}(A \circ' B) &= \bigvee f(\downarrow (A \circ B)) \\
&= \bigvee f(A \circ B) \\
&= \bigvee \{f(a \circ b) \,|\, a \in A, b \in B\} \\
&= \bigvee \{f(a) \circ f(b) \,|\, a \in A, b \in B\} \\
&= \bigvee_{a \in A} \{ \bigvee_{b \in B} f(a) \circ f(b)\}
\end{aligned}$$

$$= \bigvee_{a\in A} \{f(a) \circ \bigvee f(B)\}$$
$$= \bigvee f(A) \circ \bigvee f(B)$$
$$= \hat{f}(A) \circ \hat{f}(B).$$

Observe now that for all $a \in P$, $\hat{f}(\eta_P(a)) = \hat{f}(\downarrow a) = \bigvee f(\downarrow a) = f(a)$. So $\hat{f} \circ \eta_P = f$. For the uniqueness of $\hat{f}$, suppose there exists a quantale homomorphism $g : \mathcal{D}(P) \longrightarrow Q$ such that $g \circ \eta_P = f$, then $g(\downarrow a) = f(a) = \hat{f}(\downarrow a)$, for all $a \in P$. Thus $g(A) = g(\bigcup\{\downarrow a \,|\, a \in A\} = \bigvee\{g(\downarrow a) \,|\, a \in A\} = \bigvee\{\hat{f}(\downarrow a) \,|\, a \in A\} = \hat{f}(\bigcup\{\downarrow a \,|\, a \in A\} = \hat{f}(A)$, for all $A \in \mathcal{D}(P)$. Hence, $g = \hat{f}$. ■

The diagram below describes the main sequence of the adjunctions studied in this paper.

$$\underline{Petri} \underset{\Pi}{\overset{\mathcal{R}}{\rightleftarrows}} \underline{PreoPetri} \underset{\mathcal{N}}{\overset{\mathcal{M}}{\rightleftarrows}} \underline{PoMon} \underset{U}{\overset{\mathcal{D}}{\rightleftarrows}} \underline{Quant}$$

Combining Theorems 3.3, 4.1, and 5.1, we obtain the following result relating Petri nets and quantales.

THEOREM 5.2. *The functor* $\mathcal{D} \circ \mathcal{M} \circ \mathcal{R} : \underline{Petri} \longrightarrow \underline{Quant}$ *is the left adjoint of the functor* $\Pi \circ \mathcal{N} \circ \mathcal{U} : \underline{Quant} \longrightarrow \underline{Petri}$.

The remainder of this section is occupied with the proof of Theorem 5.7.

Let us recall that in a complete lattice L, an element p is a *completely join prime* if $p \neq \bot$ and for all $A \subseteq L$,the relation $p \leq \bigvee A$ implies the existence of $a \in A$ such that $p \leq a$. The set of all completely join primes of L is denoted by $\mathcal{CJP}(L)$. We say that $\mathcal{CJP}(L)$ *order generates* L if for all $a \in L$, $a = \bigvee(\downarrow a \bigcap \mathcal{CJP}(L))$. Such a lattice is an algebraic and dually algebraic distributive lattice (see, for example, p. 83 of [2]). In particular, it is a *frame*, that is, it satisfies the join-infinite distributive law $a \wedge \bigvee D = \bigvee\{a \wedge b \,|\, b \in D\}$.

DEFINITION 5.3. *A quantale Q is said to be* **strongly coherent** *if $\mathcal{CJP}(Q)$ is a submonoid of Q that order generates Q.*

Let *SCohQuant* be the full subcategory of *Quant* whose objects are strongly coherent quantales. It is clear that for every partially ordered monoid P, the quantale $\mathcal{D}(P)$ is strongly coherent, with $\mathcal{CJP}(\mathcal{D}(P)) = \{\downarrow a \,|\, a \in P\}$. Thus, the adjunction in Theorem 5.1 may be viewed as an adjunction between the categories *PoMon* and *SCohQuant*, and the main sequence of adjunctions takes the following form.

$$\underline{Petri} \underset{\Pi}{\overset{\mathcal{R}}{\rightleftarrows}} \underline{PreoPetri} \underset{\mathcal{N}}{\overset{\mathcal{M}}{\rightleftarrows}} \underline{PoMon} \underset{U}{\overset{\mathcal{D}}{\rightleftarrows}} \underline{SCohQuant}$$

Lemma 5.6 below asserts that all compositions of left adjoints, in the picture above, with co-domain $\underline{SCohQuant}$ are dense. We recall the concept of a dense functor, and note that compositions of dense functors are again dense.

DEFINITION 5.4. *A functor $\mathcal{G} : \underline{D} \longrightarrow \underline{E}$ is said to be* **dense** *if for every object B in the category $\underline{E}$, there exists an object A in the category $\underline{D}$ such that $\mathcal{G}(A)$ is isomorphic to B.*

In what follows, we extend the definition of $\mathcal{D}$ so that it applies on preordered monoids as well. That is, if P is a preordered monoid, then $\mathcal{D}(P)$ is the set of down sets of D. As in the case of a partial order, $\mathcal{D}(P)$ is a complete join-semilattice whose joins and meets are unions and intersections, respectively. Moreover, the multiplication $A \circ' B = \downarrow(A \circ B)$, for all $A, B \in \mathcal{D}(P)$, makes it into a strongly coherent quantale. A significant observation for our purposes is the following result.

LEMMA 5.5. *If $(P, \preceq)$ be a preordered monoid, then the strongly coherent quantales $\mathcal{D}(P)$ and $\mathcal{D}(P/\equiv_P)$ are isomorphic. Here $\equiv_P$ is the standard congruence associated with $\preceq$. It is defined by $a \equiv_P b$ iff $a \preceq b$ and $b \preceq a$, for all $a, b \in P$.*

PROOF. We leave to the reader to verify that the map $\theta : \mathcal{D}(P) \longrightarrow \mathcal{D}(P/\equiv_P)$, defined by $\theta(A) = \{[a] \mid a \in A\}$, is a quantale isomorphism from $\mathcal{D}(P)$ to $\mathcal{D}(P/\equiv_P)$. ■

LEMMA 5.6. *The functors $\mathcal{D}$, $\mathcal{D} \circ \mathcal{M}$, and $\mathcal{D} \circ \mathcal{M} \circ \mathcal{R}$ are dense.*

PROOF. We begin by showing that the functors $\mathcal{R}$ and $\mathcal{D}$ are dense. The functor $\mathcal{R}$ is dense by Lemma 3.2. The density of $\mathcal{D}$ follows from the fact that for each object $Q \in \underline{SCohQuant}$, the map $\lambda_Q : \mathcal{D}(\mathcal{CJP}(Q)) \longrightarrow Q$, defined by $\lambda_Q(A) = \bigvee A$, is a quantale isomorphism. Indeed, since $\mathcal{CJP}(Q)$ order generates Q, λ_Q is an order isomorphism, and hence linear. It is also clear that λ_Q preserves the $\mathbf{1}$. Finally, for $A, B \in \mathcal{D}(\mathcal{CJP}(Q))$,

$$\begin{aligned} \lambda_Q(A \circ B) &= \lambda_Q(\bigcup\{\downarrow a \circ \downarrow b \mid a \in A, b \in B\}) \\ &= \bigvee\{\lambda_Q(\downarrow a \circ \downarrow b) \mid a \in A, b \in B\} \\ &= \bigvee\{\bigvee(\downarrow a \circ \downarrow b) \mid a \in A, b \in B\} \end{aligned}$$

$$= \bigvee\{a \circ b) \mid a \in A, b \in B\}$$
$$= \bigvee A \circ \bigvee B$$
$$= \lambda_Q(A) \circ \lambda_Q(B).$$

This completes the proof that λ_Q is an isomorphism.

Next, since $\mathcal{D}(\mathcal{CJP}(Q)) \cong Q$, for every $Q \in$ $\underline{\mathit{SCohQuant}}$, to show that $\mathcal{D} \circ \mathcal{M}$ is dense, it will suffice to show that, if $(M, \leq)$ is a partially ordered monoid, then $\mathcal{D}(M) \cong \mathcal{D}(\mathcal{M}(\mathcal{N}(M))) = \mathcal{D}(M^{\otimes}/ \equiv_{M^{\otimes}})$. In light of Lemma 5.5, we already have that $\mathcal{D}(M^{\otimes}) \cong \mathcal{D}(M^{\otimes}/ \equiv_{M^{\otimes}})$, and hence we need to establish the isomorphism $\mathcal{D}(M) \cong \mathcal{D}(M^{\otimes})$. Let $\gamma_M : M^{\otimes} \longrightarrow M$ be the monoid epimorphism satisfying $\gamma_M(\underline{a}) = a$, for all $a \in M$. Recall that the order $\preceq$ on $M^{\otimes}$ is defined by $m_1 \preceq m_2 \iff \gamma_M(m_1) \leq \gamma_M(m_2)$, for all $m_1, m_2 \in M^{\otimes}$. Now, define the map $\theta_M : \mathcal{D}(M^{\otimes}) \longrightarrow \mathcal{D}(M)$ by $\theta_M(A) = \gamma_M(A) = \{\gamma_M(a) \mid a \in A\}$ for any $A \in \mathcal{D}(M^{\otimes})$. First, θ_M is well-defined, meaning that $\gamma_M(A)$ is in $\mathcal{D}(M^{\otimes})$. Indeed, if $x \leq \gamma_M(a)$ for some $a \in A$, then $x = \gamma_M(\underline{x}) \leq \gamma_M(a)$. Hence $\underline{x} \preceq a$, and $x \in \gamma_M(A)$. A direct check shows that θ_M is a quantale epimorphism. We show θ_M is injective, and hence a quantale isomorphism between $\mathcal{D}(M)$ and $idl(M^{\otimes})$. Let $A, B \in \mathcal{D}(M^{\otimes})$ with $\theta_M(A) = \theta_M(B)$. For every $a \in A$, there exists $b \in B$ such that $\gamma_M(a) = \gamma_M(b)$, and hence $a \preceq b$. This yields the relations $a \in B$ and $A \subseteq B$. By symmetry, $B \subseteq A$, and hence $A = B$. We have established the injectivity of θ_M and the density of the functor $\mathcal{D} \circ \mathcal{M}$.

Lastly, $\mathcal{D} \circ \mathcal{M} \circ \mathcal{R}$ is dense, since $\mathcal{D} \circ \mathcal{M}$ and $\mathcal{R}$ are dense. ■

Lemma 5.6 directly implies the final and main result of this section.

THEOREM 5.7. *The quantales arising in net semantics are the strongly coherent quantales. More specifically, given a strongly coherent quantale Q, there exists a Petri net $\boldsymbol{N} = (S, T, F)$ such that Q is isomorphic to the quantale $\mathcal{D}(S^{\otimes})$ of down sets of $(S^{\otimes}, \preceq)$, where $\preceq$ is the reachability relation on the markings $S^{\otimes}$ of $\boldsymbol{N}$. $\boldsymbol{N}$ may be taken to be the net $\Pi \circ \mathcal{N} \circ U(Q)$.*

6. Free Quantales with Operators

The objective of this section is to provide an affirmative answer to the third question raised in Section 1, by introducing the category $\underline{\mathit{FrQuantOp}}$ that permits an internal interpretation of linear logic in $\underline{\mathit{Petri}}$. In particular, the use of linear quantic nuclei on free quantales provides an alternative way for viewing the correspondence between Petri nets and strongly coherent quantales.

Let *Mon* denote the category of monoids and monoid homomorphisms. Obviously, there is a forgetful functor U from *Quant* to *Mon* that forgets the lattice structure and join preserving property of the morphisms. On the other hand, given a monoid M, there is a canonical way to construct a quantale $\wp(M)$ with lattice structure the power set of M and multiplication $\circ$ defined as $A \circ B = \{ab \,|\, a \in A, b \in B\}$, for all $A, B \in \wp(M)$. For each monoid homomorphism $f : M \longrightarrow N$, let $\wp(f) : \wp(M) \longrightarrow \wp(N)$ be defined by $\wp(f)(A) = f(A)$, for all $A \subseteq M$. It is a well-known fact, and easy to verify, that the preceding assignments define a functor $\wp$: *Mon* $\longrightarrow$ *Quant* that is a left adjoint to the forgetful functor U : *Quant* $\longrightarrow$ *Mon*. We also note that for any set P, $\wp(P^{\otimes})$ is the free quantale generated by P.

In Section 3, we defined the category *PreoPetri* of preordered Petri nets, whose objects are preordered free monoids, and the morphisms are monotone monoid homomorphisms. We wish to relate *PreoPetri* to some category of free quantales with a single operator. Besides being free quantales, the objects must reflect the preorder on the side of the Petri nets. We define *FrQuantOp* to be the category, whose objects are pairs in the form (Q, α), with Q a free commutative quantale and α a linear quantic nucleus on Q. Every free quantale is isomorphic to $\wp(P^{\otimes})$, for some set P, and hence we shall always write an object in *FrQuantOp* in the form $(\wp(P^{\otimes}), \alpha)$. The definition of the morphism in *FrQuantOp* is more involved. Let $(\wp(P^{\otimes}), \alpha)$ and $(\wp(S^{\otimes}), \beta)$ be objects in *FrQuantOp*. A map $f : (\wp(P^{\otimes}), \alpha) \longrightarrow (\wp(S^{\otimes}), \beta)$ is a morphism in *FrQuantOp* if:

1. f is a quantale homomorphism from $\wp(P^{\otimes})$ to $\wp(S^{\otimes})$;
2. f maps singletons to singletons; and
3. $f \circ \alpha \leq \beta \circ f$.

Let $\wp$ be the assignment from *PreoPetri* to *FrQuantOp* defined as follows. If $(P^{\otimes}, \preceq)$ is in *PreoPetri*, let $\wp(P^{\otimes}, \preceq) = (\wp(P^{\otimes}), \alpha)$, with $\alpha : \wp(P^{\otimes}) \longrightarrow \wp(P^{\otimes})$ defined by $\alpha(A) = \downarrow A = \{p \in P^{\otimes} \,|\, p \preceq a, \exists\, a \in A\}$, for all $A \in \wp(P^{\otimes})$. If $f : (P^{\otimes}, \preceq_p) \longrightarrow (S^{\otimes}, \preceq_s)$ is a morphism in *PreoPetri*, let $\wp(f) : (\wp(P^{\otimes}), \alpha) \longrightarrow (\wp(S^{\otimes}), \beta)$ be defined by $\wp(f)(A) = f(A)$, for all $A \in \wp(P^{\otimes})$.

Next, let $\mathcal{N}_0$ be the assignment from *FrQuantOp* to *PreoPetri* defined as follows. If $(\wp(P^{\otimes}), \alpha)$ is in *FrQuantOp*, let $\mathcal{N}_0(\wp(P^{\otimes}), \alpha) = (P^{\otimes}, \preceq_\alpha)$, with $m \preceq_\alpha n \iff \alpha(\{m\}) \subseteq \alpha(\{n\})$, for all $m, n \in P^{\otimes}$. Moreover, if $g : (\wp(P^{\otimes}), \alpha) \longrightarrow (\wp(Q^{\otimes}), \beta)$ is a morphism in *FrQuantOp*, let $\mathcal{N}_0(g) : (P^{\otimes}, \preceq_\alpha) \longrightarrow (Q^{\otimes}, \preceq_\beta)$ be the map from $P^{\otimes}$ to $Q^{\otimes}$, defined by $\mathcal{N}_0(g)(m) = n$, if $g(\{m\}) = \{n\}$, for all $m \in P^{\otimes}$ and $n \in Q^{\otimes}$. Clearly $\mathcal{N}_0(g)(m) = \mathcal{N}_0(g)(n) \iff g(\{m\}) = g(\{n\})$, for all $m \in P^{\otimes}$, $n \in Q^{\otimes}$.

THEOREM 6.1. *The assignment $\wp$: PreoPetri $\longrightarrow$ FrQuantOp is an isomorphism with inverse $\mathcal{N}_0$: FrQuantOp $\longrightarrow$ PreoPetri.*

PROOF. We first show that $\wp$ is a functor. If $(P^{\otimes}, \preceq)$ is in *PreoPetri*, let $\wp(P^{\otimes}, \preceq) = (\wp(P^{\otimes}), \alpha)$, where α is defined as above. We need to prove that α is a quantic nucleus. It is clear that α is a linear closure operator. Hence, it remains to show that $\alpha(A) \circ \alpha(B) \subseteq \alpha(A \circ B)$, or $\downarrow A \circ \downarrow B \subseteq \downarrow (A \circ B)$. Now $m \in \downarrow A \circ \downarrow B$ implies the existence of $a \in A,\ b \in B,\ c, d \in P^{\otimes}$, such that $m = c + d,\ c \preceq a$, and $d \preceq b$. It follows, since $\preceq$ is compatible with $+$, that $m = c + d \preceq a + b$, showing that $m \in \downarrow (A \circ B)$.

If $f : (P^{\otimes}, \preceq_p) \longrightarrow (S^{\otimes}, \preceq_s)$ is a morphism in *PreoPetri*, then $\wp(f)$ is clearly a quantale homomorphism that maps singletons to singletons. We need to check the inequality $f \circ \alpha \leq \beta \circ f$, or $f(\downarrow A) \subseteq \downarrow f(A)$. Let $x \in f(\downarrow A)$. There exist $a \in A$ and $m \in P^{\otimes}$, such that $m \preceq_p a$, and $x = f(m)$. Since f preserves the preorders, $x \preceq_s f(a)$, and $x \in \downarrow f(A)$. We have shown that $f \circ \alpha \leq \beta \circ f$. The remaining categorical properties of $\wp$ are easily checked.

We next show that $\mathcal{N}_0$ is a functor. For $(\wp(P^{\otimes}), \alpha)$ in *FrQuantOp*, let $\mathcal{N}_0(\wp(P^{\otimes}), \alpha) = (P^{\otimes}, \preceq_\alpha)$, with $\preceq_\alpha$ defined as above. Clearly $\preceq_\alpha$ is a preorder. If $m, n, t \in P^{\otimes}$ and $m \preceq_\alpha n$, then $m + t \in \{m + t\} = \{m\} \circ \{t\} \subseteq \alpha(\{m\}) \circ \alpha(\{t\}) \subseteq \alpha(\{n\}) \circ \alpha(\{t\}) \subseteq \alpha(\{n\} \circ \{t\}) = \alpha(\{n + t\})$, since $\circ$ is monotone and α is a quantic nucleus. Thus, $\alpha(\{m + t\}) \subseteq \alpha(\{n + t\})$, using the fact that α is a closure operator. Thus, $m + t \preceq_\alpha n + t$, showing that $\preceq_\alpha$ is compatible with $+$ in $P^{\otimes}$.

If $g : (\wp(P^{\otimes}), \alpha) \longrightarrow (\wp(Q^{\otimes}), \beta)$ is a morphism in *FrQuantOp*, then $\mathcal{N}_0(g)$ is clearly a monoid homomorphism. We only need to show that $\mathcal{N}_0(g)$ preserves the preorder. If $m \preceq_\alpha n$ in $(P^{\otimes}, \preceq_\alpha)$, then by the definition of $\preceq_\alpha$, $\alpha(\{m\}) \subseteq \alpha(\{n\})$, and hence $\{m\} \subseteq \alpha(\{n\})$, since α is a closure operator. But then, $g(\{m\}) \subseteq g(\alpha(\{n\})) \subseteq \beta(g(\{n\}))$, which yields $\beta(g(\{m\})) \subseteq \beta(g(\{n\}))$, or $\mathcal{N}_0(g)(m) \preceq_\beta \mathcal{N}_0(g)(n)$. Again the remaining categorical properties of $\mathcal{N}_0$ are easily verified. Thus $\mathcal{N}_0$: *FrQuantOp* $\longrightarrow$ *PreoPetri* is a functor.

We next show that the functors $\wp$ and $\mathcal{N}_0$ are mutually inverse isomorphisms, by showing that $\mathcal{N}_0 \circ \wp = id_{\underline{PreoPetri}}$ and $\wp \circ \mathcal{N}_0 = id_{\underline{FrQuantOp}}$.

First, let $(P^{\otimes}, \preceq)$ be in *PreoPetri*. Then $\mathcal{N}_0(\wp(P^{\otimes}, \preceq)) = (P^{\otimes}, \preceq_\alpha)$, where $\preceq_\alpha$ is defined by $m \preceq_\alpha n$ iff $\downarrow m \subseteq \downarrow n$. This is evidently equivalent to $m \preceq n$, and hence, $(P^{\otimes}, \preceq) = \mathcal{N}_0(\wp(P^{\otimes}, \preceq))$.

Next, let $(\wp(P^{\otimes}), \alpha)$ be an object in the category *FrQuantOp*. Then $\wp(\mathcal{N}_0(\wp(P^{\otimes}), \alpha)) = (\wp(P^{\otimes}), \alpha')$, where α' is defined as $\alpha'(A) = \{m \in P^{\otimes} \mid m \preceq_\alpha a, \exists\ a \in A\}$, for all $A \in \wp(P^{\otimes})$. We have to show that $\alpha'(A) = \alpha(A)$. Let $m \in \alpha(A)$. Now $\alpha(A) = \alpha(\bigcup_{a \in A} \{\{a\}\}) = \bigcup_{a \in A} \alpha(\{a\})$, since α preserves arbitrary joins, and hence, $m \in \alpha(\{a\})$, for some $a \in A$. It follows that $m \preceq_\alpha a$, or $m \in \alpha'(A)$. Conversely, if $m \in \alpha'(A)$, then $m \in \alpha(\{a\})$ for some $a \in A$. So $m \in \alpha(\{a\}) \subseteq \alpha(A)$, as α is monotone. Therefore $\alpha' = \alpha$, and hence, $(\wp(P^{\otimes}), \alpha) = \wp(\mathcal{N}_0(\wp(P^{\otimes}), \alpha)$. ■

Order-Semantics of Linear Logic

We start this section with a brief review of intuitionistic linear logic. Then we show that the categories *Petri*, *PreoPetri*, and *PoMon* provide interpretations of linear logic equivalent to net semantics. The equivalence of the interpretations is established with the use of Lemma 5.6 which asserts that the compositions of left adjoints with co-domain *SCohQuant* are dense. We also outline in Section 6.2 how the objects in *FrQuantOp* may be viewed as models of the logic, and leave to the reader to verify the equivalence of this interpretation and net semantics.

It follows from the considerations of the present section that net semantics is equivalent to a restricted form of *SCohQuant*-semantics. More specifically, unlike quantale semantics, values of the formulas in restricted *SCohQuant*-semantics are limited to those that extend the assignments of atomic propositions to completely join-prime elements of the strongly coherent quantale. It is of interest to inquire whether there is a category $\mathcal{C}$ of quantales for which $\mathcal{C}$-semantics is equivalent to net semantics. The category *SCohQuant* is, of course, a natural candidate for such a category.

6.1. Linear Logic

Linear logic, introduced by Girard in [5], has attracted considerable attention because of its potential applications in parallel and distributive computing. Intuitionistic linear logic is one of the variants of linear logic. We refer the reader to [13] for additional information concerning this and other variants of linear logic. The major difference between intuitionistic linear logic and traditional intuitionistic logic is the controlled use of the two structural rules contraction and weakening, and the introduction of the connectives $\otimes$ (with unit $\mathbf{1}$) and !. The traditional conjunction and disjunction are denoted by & (with unit $\top$) and $\oplus$ (with unit $\perp$). The implication $\multimap$ is now called *linear implication.* A *formula* is either an atomic proposition, a logic constant, or a compound formula constructed using the logical connectives.

Linear logic is usually introduced by Gentzen's sequent calculus. A *sequent* $A_1, A_2, \cdots, A_n \vdash A$ of the logic consists of a collection of assumption formulas $A_1, A_2, \cdots, A_n$, a single formula A, and the symbol $\vdash$, which is read as "deduce." A sequence of assumption formulas is usually denoted by upper case Greek letters Γ or Δ. The axioms and inference rules of intuitionistic linear logic are presented below (see also, for example, [3] or [7]). Reverting to our practice initiated earlier, we shall refer to "intuitionistic linear logic" as "linear logic."

Axiom:

$$\frac{}{A \vdash A}\ \text{(Identity)}$$

Structural Rules:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \text{(Exchange)} \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \text{(Cut)}$$

Logical Rules:

$$\frac{}{\vdash \mathbf{1}}\ (\mathbf{1}\text{-R}) \qquad \frac{\Gamma \vdash A}{\Gamma, \mathbf{1} \vdash A}\ (\mathbf{1}\text{-L})$$

$$\frac{}{\Gamma \vdash \top}\ (\top\text{-R}) \qquad \frac{}{\Gamma, \bot \vdash A}\ (\bot\text{-L})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes\text{-R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes\text{-L})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&\text{-R}) \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}\ (\&\text{-L})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus\text{-R}) \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ (\oplus\text{-L})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap\text{-R}) \qquad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\ (\multimap\text{-L})$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\text{-R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \text{(Dereliction)}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{(Restricted Contraction)} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{(Restricted Weakening)}$$

The so called "of course" operator ! is also referred to as an "exponential" or "storage" operator. One can think of $!A$ as a storage of A which can provide unlimited copies of A.

6.2. Models of Linear Logic

In this subsection we describe the interpretations of linear logic in the categories Petri, PreoPetri, PoMon , and FrQuantOp.

DEFINITION 6.2. *Let Q be in Quant. A coclosure operator $! : Q \longrightarrow Q$ is said to be a* **modality** *over Q (see p.78 of [13]) provided*

1. $!x \leq \mathbf{1}$*, for all* $x \in Q$*;*
2. $!\mathbf{1} = \mathbf{1}$*; and*
3. $!(x \wedge y) = !x \circ !y$*, for all* $x, y \in Q$*.*

Note that clause (3) implies that $(!x)^2 = !x$ and $!x \circ !y = !(\,!x \circ !y)$, for all $x, y \in Q$. One can easily check (see p.79 of [13]) that, for a given quantale Q, the assignment $x \mapsto !x = \bigvee\{y \in Q \mid y \leq \mathbf{1} \wedge x,\ y^2 = y\}$ defines a modality over Q.

Let L be the set of all formulas of the logic. L is the underlying set for the absolutely free algebra $\mathcal{L}$, corresponding to the connectives of the logic as function symbols, over the set C of atomic propositions.

$$\mathcal{L} = (L, \otimes, \multimap, \&, \oplus, !, \mathbf{1}, \bot, \top)$$

The type of any quantale Q with a modality $!$ may be expanded in an obvious way to the type of $\mathcal{L}$, if one disregards the infinite nature of joins and meets. Thus, every assignment $f : C \longrightarrow Q$ can be extended to an $\mathcal{L}$-homomorphism $[\![\text{-}]\!] : \mathcal{L} \longrightarrow Q$ satisfying the following properties, for all $a \in C$, and $A, B \in \mathcal{L}$.

$$\begin{array}{lcl}
[\![\top]\!] & = & \top \\
[\![\bot]\!] & = & \bot \\
[\![\mathbf{1}]\!] & = & \mathbf{1} \\
[\![a]\!] & = & f(a) \\
[\![A \otimes B]\!] & = & [\![A]\!] \circ [\![B]\!] \\
[\![A \& B]\!] & = & [\![A]\!] \wedge [\![B]\!] \\
[\![A \oplus B]\!] & = & [\![A]\!] \vee [\![B]\!] \\
[\![A \multimap B]\!] & = & [\![A]\!] \multimap [\![B]\!] \\
[\![!A]\!] & = & ![\![A]\!]
\end{array}$$

We refer to the homomorphism $[\![\text{-}]\!]$ as a *valuation* of the logic on $(Q, !)$. Further, we say that a sequent $A_1, A_2, \cdots, A_n \vdash A$ is *valid* in $(Q, !)$ if and only if $[\![A_1]\!] \circ \cdots \circ [\![A_n]\!] \leq [\![A]\!]$ (or $\mathbf{1} \leq [\![A]\!]$ if $n = 0$), for every valuation $[\![\text{-}]\!] : \mathcal{L} \longrightarrow Q$. The proof of soundness and completeness of linear logic with respect to the preceding semantics can be found in [13].

The discussion in the preceding sections describes how to associate an object in any one of the categories *Petri*, *PreoPetri*, *PoMon* , or *FrQuantOp* to a quantale. Thus, in order to describe the interpretation of linear logic in each category, we only need to describe the assignment from the set C of atomic propositions.

Let (P,T,i,o) (or $(P^{\otimes},\preceq)$) be in *Petri* (or *PreoPetri*) with corresponding quantale $\mathcal{D}(P^{\otimes})$ (the quantale of down sets of $(P^{\otimes},\preceq)$). Let $g : C \longrightarrow P$ be any assignment. Then the assignment $f : C \longrightarrow \mathcal{D}(P^{\otimes})$ is defined by the formula $f(a) = \downarrow(\underline{g(a)})$.

Similarly if M is in *PoMon* and $g : C \longrightarrow M$ is any assignment, then the assignment $f : C \longrightarrow \mathcal{D}(M)$ is defined by $f(a) = \downarrow g(a)$, for all $a \in C$.

If $(\wp(P^{\otimes}),\alpha)$ is in *FrQuantOp*, then the interpretation takes place within the category. For any assignment $g : C \longrightarrow P$, the valuation function $[\![\text{-}]\!] : \mathcal{L} \longrightarrow \wp(P^{\otimes})$ is the homomorphism corresponding to the assignment $a \mapsto \alpha(\{\underline{g(a)}\})$. Hence, we have the following, for all $a \in C$, and $A, B \in \mathcal{L}$.

$$
\begin{array}{lll}
[\![\top]\!] & = & P^{\otimes} \\
[\![\bot]\!] & = & \emptyset \\
[\![\mathbf{1}]\!] & = & \alpha(\{\mathbf{0}\}), \qquad \text{where } \mathbf{0} \text{ is the empty multiset over } P \\
[\![a]\!] & = & \alpha(\{\underline{g(a)}\}) \\
[\![A \otimes B]\!] & = & \alpha([\![A]\!] \circ [\![B]\!]) \\
[\![A \& B]\!] & = & [\![A]\!] \cap [\![B]\!] \\
[\![A \oplus B]\!] & = & [\![A]\!] \cup [\![B]\!] \\
[\![A \multimap B]\!] & = & [\![A]\!] \multimap [\![B]\!] \\
[\![!A]\!] & = & \bigcup\{[\![B]\!] \mid [\![B]\!] \subseteq [\![\mathbf{1}]\!] \cap [\![A]\!],\ \alpha([\![B]\!]^2) = [\![B]\!]\}
\end{array}
$$

6.3. The Equivalence of the Interpretations of Linear Logic

We formally define the equivalence of classes of models of linear logic as follows.

DEFINITION 6.3. *Let $\mathcal{L}$ be the set of formulas of linear logic, and $\mathcal{M}_1$, $\mathcal{M}_2$ be two classes of models of the logic. We say that $\mathcal{M}_2$ is* **finer** *than $\mathcal{M}_1$ if for any model M_1 in $\mathcal{M}_1$ and any valuation function $[\![\text{-}]\!]_1 : \mathcal{L} \longrightarrow M_1$, there exist a model M_2 in $\mathcal{M}_2$ and a valuation function $[\![\text{-}]\!]_2 : \mathcal{L} \longrightarrow M_2$ such that a sequent is valid in $\mathcal{M}_1$ if and only if it is valid in $\mathcal{M}_2$. We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are* **equivalent** *as models of the logic if $\mathcal{M}_1$ is finer than $\mathcal{M}_2$ and $\mathcal{M}_2$ is finer than $\mathcal{M}_1$.*

An object H in any one of the categories *Petri*, *PreoPetri*, or *PoMon*, gives rise to a quantale Q_1 in *SCohQuant*. In light of Lemma 5.6, there exists an object K in each of the other categories inducing a quantale Q_2 in *SCohQuant* isomorphic to Q_2. Since the interpretations actually takes place outside the categories, we need to apply the equivalence definition above with a slight modification. Namely, we need to check that for every valuation function $[\![\text{-}]\!]_1 : \mathcal{L} \longrightarrow Q_1$ constructed through H, there exists a valuation function $[\![\text{-}]\!]_2 : \mathcal{L} \longrightarrow Q_2$ constructed through K such that a sequent is valid in Q_1 if and only if it is valid in Q_2.

THEOREM 6.4. *The categories* *Petri*, *PreoPetri*, *and* *PoMon* *provide equivalent interpretations of linear logic.*

PROOF. It is clear that *Petri* and *PreoPetri* are equivalent, since they induce the same quantale and the same valuation function.

We next show that *PoMon* is finer than *PreoPetri*. Let $(M^{\otimes}, \preceq)$ be in *PreoPetri*, and let $g : C \longrightarrow M^{\otimes}$ be an assignment from the atomic propositions C to $M^{\otimes}$. Now note that the corresponding valuation function $[\![\text{-}]\!]_1 : \mathcal{L} \longrightarrow \mathcal{D}(M^{\otimes})$ extends the map $a \mapsto \downarrow g(a)$ from C to $\mathcal{D}(M^{\otimes})$. Consider the partially ordered monoid $\mathcal{M}(M^{\otimes}, \preceq) = (M^{\otimes}/\equiv_{M^{\otimes}}, \leq)$ (see Section 4), and the map $g' : C \longrightarrow M^{\otimes}/\equiv_{M^{\otimes}}$, defined by $g'(a) = [g(a)]$, for all $a, b \in C$. The corresponding valuation function $[\![\text{-}]\!]_2 : \mathcal{L} \longrightarrow \mathcal{D}(M^{\otimes}/\equiv_{M^{\otimes}})$ extends the map $a \mapsto \downarrow [g(a)]$ from C to $\mathcal{D}(M^{\otimes}/\equiv_{M^{\otimes}})$. By Lemma 5.5, the quantales $\mathcal{D}(M^{\otimes}/\equiv_{M^{\otimes}})$ and $\mathcal{D}(M^{\otimes})$ are isomorphic. The isomorphism is implemented by the map $\chi_{M^{\otimes}} : \mathcal{D}(M^{\otimes}) \longrightarrow \mathcal{D}(M^{\otimes}/\equiv_{M^{\otimes}})$, defined by $\chi_{M^{\otimes}}(A) = \{[a] \mid a \in A\}$. Note that the quantale isomorphism $\chi_{M^{\otimes}}$ preserves the operations $\wedge$, $\multimap$, and $\top$, and satisfies the equality $[\![\text{-}]\!]_2 = \chi_M \circ [\![\text{-}]\!]_1$. It follows that $[\![\text{-}]\!]_1$ and $[\![\text{-}]\!]_2$ are equivalent, that is, a sequent is valid in $\mathcal{D}(M^{\otimes})$ with respect to $[\![\text{-}]\!]_1$ if and only if it is valid in $\mathcal{D}(M^{\otimes}/\equiv_{M^{\otimes}})$ with respect to $[\![\text{-}]\!]_2$.

We lastly show that *PreoPetri* is finer than *PoMon*. Let $(M, \leq)$ be in *PoMon* and let $\mathcal{N}(M, \leq) = (M^{\otimes}, \preceq)$ (see Section 4). Recall that if $\gamma_M : M^{\otimes} \longrightarrow M$ is the monoid homomorphism satisfying $\gamma_M(\underline{a}) = a$, for all $a \in M$, then the order $\preceq$ on $M^{\otimes}$ is defined by $m_1 \preceq m_2 \iff \gamma_M(m_1) \leq \gamma_M(m_2)$, for all $m_1, m_2 \in M^{\otimes}$. Let $g : C \longrightarrow M$ be an assignment. The corresponding valuation function $[\![\text{-}]\!]_1 : \mathcal{L} \longrightarrow \mathcal{D}(M)$ extends the mapping $a \mapsto \downarrow g(a)$ from C to $\mathcal{D}(M)$. Let $g' : C \longrightarrow M^{\otimes}$ be defined by $g'(a) = \underline{g(a)}$. The corresponding valuation function $[\![\text{-}]\!]_2 : \mathcal{L} \longrightarrow \mathcal{D}(M^{\otimes})$ extends the map $a \mapsto \downarrow g'(a) = \downarrow \underline{g(a)}$ from C to $\mathcal{D}(M^{\otimes})$. It was established in the proof of Lemma 5.6 that the map $\theta_M : \mathcal{D}(M^{\otimes}) \longrightarrow \mathcal{D}(M)$ defined by $\theta_M(A) = \gamma_M(A) = \{\gamma_M(a) \mid a \in A\}$, is a quantale isomorphism. Furthermore, an easy check establishes that $[\![\text{-}]\!]_1 = \eta_M \circ [\![\text{-}]\!]_2$. Hence, the two valuation functions are equivalent, since η_M is a quantale isomorphism. We have shown that $\underline{\mathit{PreoPetri}}$ is finer than $\underline{\mathit{PoMon}}$, as was to be shown. ■

It was remarked at the beginning of this section that net semantics is equivalent to a restricted form of $\underline{\mathit{SCohQuant}}$-semantics. The values of the formulas in restricted $\underline{\mathit{SCohQuant}}$-semantics are limited to those that extend the assignments of atomic propositions to completely join-prime elements of the strongly coherent quantale. It is clear that this semantics is equivalent to $\underline{\mathit{PoMon}}$ -semantics, and hence its equivalence with net semantics follows from Theorem 6.4.

References

[1] Brown, C., *Petri nets as quantales,* Technical Report ECS-LFCS-89-96, Laboratory for Foundations of Computer Science, University of Edinburgh, Scotland, 1989.

[2] Crawley, P., and R. P. Dilworth, *Algebraic Theory of Lattices,* Prentice-Hall, 1973.

[3] Engberg, U., and G. Winskel, 'Petri Nets as Models of Linear Logic', in *CAAP' 90, Coll. on Trees in Algebra and Programming (Copenhagen)*, LNCS No. 431, Springer-Verlag, 1990, pp. 147–161.

[4] Engberg, U., and G. Winskel, 'Completeness Results for Linear Logic on Petri Nets', in *Mathematical Foundations of Computer Science* 1993, LNCS No. 711, Springer-Verlag, 1993, pp. 442–452.

[5] Girard, J. -Y. 'Linear Logic', *Theoretical Computer Science* 50 (1987), 1–102.

[6] Gunter, C., and V. Gehlot, 'Nets as Tensor Theories', *10th International Conference on Applications and Theory of Petri Nets. Bonn*, 1989, pp. 174–191.

[7] Lilius, J., *On the Compositionality and Analysis of Algebraic High-level Nets*, Research Reports, Series **A**, No. 16, Digital Systems Laboratory, Helsinki University of Technology, March 1991.

[8] Martí-Oliet, N., and J. Meseguer, 'From Petri Nets to Linear Logic', in *Category Theory and Computer Science*, Manchester, UK, LNCS No. 389, Springer-Verlag, 1989.

[9] Meseguer, J., and U. Montanari, 'Petri Nets Are Monoids: A New Algebraic Foundation for the Net Theory', *Third Annual IEEE Symposium on Logic in Computer Science*, Edinburgh, Scotland, July 1988, pp. 155–164.

[10] Meseguer, J., and U. Montanari, 'Petri Nets Are Monoids', *Information and Computation* 88 (1990), 105–155.

[11] Reising, W., *Petri Nets: An Introduction,* Springer-Verlag, 1985.

[12] ROSENTHAL, K., *Quantales and their applications,* Longman Scientific & Technical, 1990.

[13] TROELSTRA, A. S., *Lectures on Linear Logic,* CSLI Lecture Notes, No. 29, Stanford University, 1992.

[14] YETTER, D., 'Quantales and (Non-commutative) Linear Logic', *The Journal of Symbolic Logic* 55 (1990), 41–64.

CONSTANTINE TSINAKIS
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240
constantine.tsinakis@vanderbilt.edu

HAN ZHANG
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240
han.zhang@math.vanderbilt.edu