C. Tsinakis
A. M. Wille

Minimal Varieties of Involutive Residuated Lattices

Dedicated to the memory of Willem Johannes Blok

Abstract. We establish the existence uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. The proof utilizes a construction used in the proof of the corresponding result for residuated lattices and is based on the fact that every residuated lattice with greatest element can be associated in a canonical way with an involutive residuated lattice.

Keywords: residuated lattice, involutive residuated lattice, module over a residuated lattice, minimal variety

1. Introduction

A binary operation $\cdot$ on a partially ordered set $\mathbf{P} = \langle P, \leq \rangle$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

$$x \cdot y \leq z \ \text{ iff } \ x \leq z/y \ \text{ iff } \ y \leq x\backslash z.$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$.

An *involutive residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, {}', e \rangle$ such that

(i) $\langle L, \wedge, \vee \rangle$ is a lattice;

(ii) $\langle L, \cdot, e \rangle$ is a monoid;

(iii) the unary operation $'$ is an involution of the lattice $\langle L, \wedge, \vee \rangle$, that is, a dual automorphism such that $x'' = x$, for all $x \in L$; and

(iv) $xy \leq z \iff y \leq (z'x)' \iff x \leq (yz')'$, for all $x, y, z \in L$.

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The term "involutive residuated lattice" reflects the fact that multiplication is residuated in any such algebra. In fact, it is immediate, from condition (iv) above, that for all elements $x, y \in L$, $x\backslash y = (y'x)'$ and $y/x = (xy')'$.

Throughout this paper, the class of involutive residuated lattices will be denoted by $\mathcal{I}n\mathcal{RL}$. It is routine to verify that the equivalences of condition (iv) can be described by finitely many equations and thus $\mathcal{I}n\mathcal{RL}$ is a finitely based variety.

Involutive residuated lattices have received considerable attention both from the logic and algebra communities. From a logical perspective, they are the algebraic counterparts of the propositional non-commutative linear logic without exponentials. From an algebraic perspective, they provide a common framework within which a host of disparate structures – including Boolean algebras, MV-algebras, lattice-ordered groups and relation algebras – can be studied. The defining properties that describe the class $\mathcal{I}n\mathcal{RL}$ are few and easy to grasp and the theory is sufficiently robust to yield significant results. Recent publications focussing on residuated structures include [2], [3], [6], [7], [11], [12], [13] and [15].

The primary purpose of this article is to establish the following result.

Theorem (see Theorem 3.12) *The subvariety lattice,* $\mathbf{L}(\mathcal{I}n\mathcal{RL})$, *of* $\mathcal{I}n\mathcal{RL}$ *contains uncountably many atoms.* ∎

2. Background

2.1. Residuated Lattices

We refer the reader to [4] and [9] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

A *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that

(i) $\langle L, \wedge, \vee\rangle$ is a lattice;

(ii) $\langle L, \cdot, e\rangle$ is a monoid; and

(iii) the operation "$\cdot$" is a residuated map on $\langle L, \wedge, \vee\rangle$ with residuals $\backslash$ and $/$.

Throughout this paper, the class of residuated lattices will be denoted by $\mathcal{RL}$. It is easy to see that the equivalences defining residuation can be captured by finitely many equations and thus $\mathcal{RL}$ is a finitely based variety.

2.2. Involutive Residuated Lattices

In what follows, we will mostly use a term-equivalent description of involutive residuated lattices. An algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is said to be a *dualizing residuated lattice* provided it satisfies the following conditions:

(i) $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /\rangle$ is a residuated lattice; and

(ii) d is a *cyclic dualizing* element. That is, for all $x \in L$, $d/x = x\backslash d$ (d is *cyclic*) and $d/(x\backslash d) = (d/x)\backslash d = x$ (d is *dualizing*).

Note that if $\mathbf{L}_d = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is an dualizing residuated lattice and we define $x' = d/x$, for all $x \in L$, then $\mathbf{L}' = \langle L, \wedge, \vee, \cdot, {}', e\rangle$ becomes an involutive residuated lattice. On the other hand, if $\mathbf{L}' = \langle L, \wedge, \vee, \cdot, {}', e\rangle$ is an involutive residuated lattice, then the algebra $\mathbf{L}_d = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ – defined by (i) $d = e'$; and (ii) $x\backslash z = (z'x)'$, $z/x = (xz')'$, for all $x, z \in L$ – is a dualizing residuated lattice.

2.3. Congruence Relations

$\mathcal{RL}$ is a congruence permutable and congruence distributive variety. Another key property is that each residuated lattice $\mathbf{L}$ is *e-regular*, that is, each congruence relation on $\mathbf{L}$ is determined by its identity block.

We discuss the latter property in more detail. Proofs of this and the aforementioned facts are presented in [4] and [9]. For $a \in L$, we define the notion of right and left *conjugation* by a as follows: $\lambda_a(x) = [a\backslash(xa)] \wedge e$ and $\rho_a(x) = [(ax)/a] \wedge e$, respectively. These are unary operations on the universe of $\mathbf{L}$ that correspond to the analogous concepts from group theory. A subalgebra $\mathbf{H}$ of $\mathbf{L}$ is called *normal* if $\lambda_a(x), \rho_a(x) \in H$ for all $a \in L$ and all $x \in H$. The algebraic closure families of congruence relations and (order) convex normal subalgebras of a residuated lattice $\mathbf{L}$ will be denoted by $\mathrm{Con}(\mathbf{L})$ and $\mathrm{CN}(\mathbf{L})$, respectively.

Proposition 2.1. ([4]) If θ is a congruence relation of a residuated lattice $\mathbf{L}$, then $[e]_\theta$ – the θ-block of e – is a convex normal subalgebra (more precisely, subuniverse) of $\mathbf{L}$. Conversely, if H is a convex normal subalgebra of $\mathbf{L}$, then $\theta_H = \{(x, y) : x\backslash y \wedge y\backslash x \wedge e \in H\}$ is a congruence relation of $\mathbf{L}$. Moreover, the maps $H \mapsto \theta_H$ and $\theta \mapsto [e]_\theta$ are mutually inverse isomorphisms between $\mathrm{Con}(\mathbf{L})$ and $\mathrm{CN}(\mathbf{L})$.

It is clear that the congruence lattice of a dualizing residuated lattice, and hence that of a involutive residuated lattice, is the congruence lattice of its residuated lattice reduct. It follows that $\mathcal{I}n\mathcal{RL}$ is a congruence permutable, congruence distributive and e-regular variety of algebras.

Involutive residuated lattices are referred to in [14] as $*$-autonomous lattices. The latter term is reserved in F. Paoli's monograph, [12], for involutive dually residuated lattices. That is, a *$*$-autonomous lattice* is an algebra $\mathbf{L} = \langle L, \sqcap, \sqcup, +, \ ', 0\rangle$ such that

(i) $\langle L, +, 0\rangle$ is an arbitrary monoid;

(ii) the unary operation $'$ is an involution on the lattice $\langle L, \sqcap, \sqcup\rangle$; and

(iii) $z \leq x + y \iff y(z' + x)' \leq y \iff (y + z')' \leq x$, for all $x, y, z \in L$.

It is clear that $\langle L, \wedge, \vee, \cdot, e,' \rangle$ is an involutive residuated lattice if and only if $\langle L, \sqcap, \sqcup, +, 0,' \rangle$ is a $*$-autonomous lattice, where $\sqcap = \vee$, $\sqcup = \wedge$, $+ = \cdot$ and $0 = e$. In other words, each of these two structures is obtained from the other by simply reversing the lattice order. Moreover, they are term equivalent to the corresponding dualizing residuated lattice. These observations imply that a number of results in [12] and [13] regarding congruences and ideals of $*$-autonomous lattices can be obtained directly from the corresponding results for residuated lattices in [8] and [4] .

3. Proof of the Main Result

3.1. The atomic structure of $\mathbf{L}(\mathcal{RL})$

The starting point in the proof of the main result is a construction in [9], which produces uncountably many strictly simple residuated chains $\mathbf{C}_S$ ($S \subseteq \omega$) that generate distinct atoms of $\mathbf{L}(\mathcal{RL})$. Recall that a non-trivial algebra $\mathbf{L}$ is called *strictly simple* if it is simple and any proper subalgebras of it are trivial.

For any subset S of ω, the universe of the algebra $\mathbf{C}_S$ is the set

$$\{\bot, a, b, e, \top\} \cup \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}.$$

The linear order is given by

$$\bot < a < b < c_0 < c_1 < c_2 < \cdots < \cdots < d_2 < d_1 < d_0 < e < \top.$$

Multiplication is defined by $ex = x = xe$; $\bot x = \bot = x\bot$; $ax = \bot = xa$, $\top x = x = x\top$, if $x \neq e$; and $bx = \bot = xb$, if $x \neq \{e, \top\}$. Furthermore, for all $i, j \in \omega$, $c_i c_j = \bot$, $d_i d_j = b$,

$$c_i d_j = \begin{cases} \bot, & \text{if } i < j \\ a, & \text{if } i = j \text{ or } (i = j+1 \text{ and } j \in S) \\ b, & \text{otherwise.} \end{cases}$$

$$d_i c_j = \begin{cases} \bot, & \text{if } i \geq j \\ b, & \text{otherwise.} \end{cases}$$

The multiplication is illustrated by the table below. Depending on the subset S, the elements s_i in the table are either equal to a (if $i \in S$) or b (if $i \notin S$).

$\cdot$	$\top$	e	d_0	d_1	d_2	$\ldots$	$\ldots$	c_2	c_1	c_0	b	a	$\bot$
$\top$	$\top$	$\top$	d_0	d_1	d_2	$\ldots$	$\ldots$	c_2	c_1	c_0	b	a	$\bot$
e	$\top$	e	d_0	d_1	d_2	$\ldots$	$\ldots$	c_2	c_1	c_0	b	a	$\bot$
d_0	d_0	d_0	b	b	b	$\ldots$	$\ldots$	b	b	$\bot$	$\bot$	$\bot$	$\bot$
d_1	d_1	d_1	b	b	b	$\ldots$	$\ldots$	b	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
d_2	d_2	d_2	b	b	b	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_2	c_2	c_2	b	s_1	a	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c_1	c_1	c_1	s_0	a	$\bot$	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c_0	c_0	c_0	a	$\bot$	$\bot$	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	b	b	$\bot$	$\bot$	$\bot$	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	a	$\bot$	$\bot$	$\bot$	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\ldots$	$\ldots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Note that $xyz = \bot$ whenever $e, \top \notin \{x, y, z\}$, and hence multiplication is associative. Further, the algebra is generated by $\bot$ since $\top = \bot\backslash\bot$, $d_0 = \top\backslash e$, $c_i = d_i\backslash\bot$ and $d_{i+1} = c_i\backslash\bot$. This easily implies that $\mathbf{C}_S$ is strictly simple. The reader may verify directly, or refer to the proof of Theorem 3.12 below for guidance, that each $\mathbf{C}_S$ generates an atom in the subvariety lattice and that for distinct subsets S_1 and S_2 of ω, one can find an equation that holds in $\mathbf{C}_{S_1}$ but not in $\mathbf{C}_{S_2}$, and vice versa. Thus, these algebras generate distinct varieties.

3.2. From modules to dualizing residuated lattices

In this subsection, we show that every residuated lattice with greatest element can be associated in a canonical way with a dualizing residuated lattice. Our development owes considerable debt to a related result of P. H. Chu regarding the embedding of an arbitrary quantale to a Girard quantale (see the appendix of [1] and [14]). Our approach illuminates and extends Chu's construction by relating it to the concept of a module over a residuated lattice. While we restrict our attention to the framework of residuated lattices, the observant reader will undoubtedly note that the constructions of this section have suitable extensions in the setting of semigroups.

Let **L** be a residuated lattice and let $\mathbf{M} = \langle M, \wedge, \vee, \bot \rangle$ be a lower bounded lattice with least element $\bot$. A *right module action* of **L** into **M** is a map $* : M \times L \longrightarrow M$ satisfying the following conditions for all $x \in M$ and $a, b \in L$.

1. $x * e = x$;
2. $x * (ab) = (x * a) * b$;
3. $*$ is a residuated map.

If the preceding conditions are satisfied, we will refer to **M** as a *right* **L**-*module*. Left **L**-*modules* are defined analogously with the module action on the left. In what follows, we will use the term **L**-*bimodule* for a left and right **L**-module that satisfies the following associative law, for all $x \in M$ and $a, b \in L$.

(4) $(a * x) * b = a * (x * b)$.

Note: For the sake of notational simplicity, the two actions and their residuals will be denoted by the symbols $*$, $\backslash_*$ and $/_*$, respectively. This choices will not create any confusion if the reader keeps in mind that, throughout this section, the letters a, b, c will denote elements of L, while x, y, z will denote elements of M. For example, $a\backslash_* x$ refers to the left action, as seen by the equivalence, $y \leq a\backslash_* x \iff a * y \leq x$. Likewise, $x\backslash_* a$ refers to the right action.

LEMMA 3.2. Let **M** be a right **L**-module. Then, for all $a \in L$ and $x, y \in M$, the following properties hold:

$$\begin{aligned} \bot * a &= \bot \\ (x \vee y) * a &= x * a \vee y * a. \end{aligned}$$

The corresponding properties hold for left **L**-modules.

PROOF. For each $a \in L$, the assignment $x \mapsto x * a$ $(x \in M)$ is a residuated map on **M**. Thus, it preserves all existing joins, including binary joins and the empty join, which is equal to $\bot$. ■

THEOREM 3.3. Every **L**-bimodule **M** gives rise to a residuated lattice $\mathbf{L} \diamond \mathbf{M} = \langle L \times M, \wedge, \vee, \cdot, \backslash, /, (e, \bot)\rangle$ defined as follows:

$$\begin{aligned}
(a,x) \wedge (b,y) &= (a \wedge b, x \wedge y)\\
(a,x) \vee (b,y) &= (a \vee b, x \vee y)\\
(a,x)(b,y) &= (ab, a * y \vee x * b)\\
(a,x)\backslash(b,y) &= (a\backslash b \wedge x\backslash_* y, a\backslash_* y)\\
(a,x)/(b,y) &= (a/b \wedge x/_* y, x/_* b)
\end{aligned}$$

PROOF. We need to show now that $\mathbf{L} \diamond \mathbf{M}$ is a monoid and that the multiplication is residuated with respect to the lattice structure. Invoking Lemma 3.2, we have for all $a \in L$ and $x \in M$,

$$(a,x)(e,\bot) = (ae, a * \bot \vee x * e) = (a, \bot \vee x) = (a,x),$$

and similarly, $(e,\bot)(a,x) = (a,x)$. Hence, $(e,\bot)$ is a neutral element. To show associativity, we use the second property of Lemma 3.2, the associativity of the monoid multiplication of **L**, and property (4) in the definition of $*$. For all $a,b,c \in L$ and $x,y,z \in M$,

$$\begin{aligned}
[(a,x)(b,y)]\,(c,z) &= ((ab)c, (ab) * z \vee (a * y \vee x * b) * c)\\
&= ((ab)c, a * (b * z) \vee (a * y) * c \vee (x * b) * c)\\
&= (a(bc), a * (b * z) \vee a * (y * c) \vee x * (bc))\\
&= (a(bc), a * (b * z \vee y * c) \vee x * (bc))\\
&= (a,x)\,[(b,y)(c,z)].
\end{aligned}$$

It remains to prove that the multiplication on $\mathbf{L} \diamond \mathbf{M}$ is residuated; equivalently, we must prove that for all $a,b,c \in L$ and $x,y,z \in M$,

$$(a,x)(b,y) \leq (c,z) \text{ iff } (a,x) \leq (c,z)/(b,y) \text{ iff } (b,y) \leq (c,z)\backslash(a,x).$$

The first equivalence, which can also be expressed as

$$(ab, a * y \vee x * b) \leq (c,z) \text{ iff } (a,x) \leq (c/b \wedge z/_* y, z/_* b),$$

follows from the following computation:

$$\begin{array}{rl} & (a,x)(b,y) \leq (c,z) \\ \Leftrightarrow & ab \leq c \text{ and } a * y \vee x * b \leq z \\ \Leftrightarrow & ab \leq c \text{ and } a * y \leq z \text{ and } x * b \leq z \\ \Leftrightarrow & a \leq c/b \text{ and } a \leq z/_{*}y \text{ and } x \leq z/_{*}b \\ \Leftrightarrow & a \leq c/b \wedge z/_{*}y \text{ and } x \leq z/_{*}b. \end{array}$$

The second equivalence,

$$(a,x)(b,y) \leq (c,z) \text{ iff } (b,y) \leq (c,z)\backslash(a,x),$$

is obtained in a similar fashion. ■

Let $\mathbf{L}$ be a residuated lattice with greatest element $\top$ and let $\mathbf{M}$ be the order dual $\langle L, \wedge, \vee, \top\rangle^{op}$ of $\langle L, \wedge, \vee, \top\rangle$. $\mathbf{M}$ can be endowed with an $\mathbf{L}$-bimodule structure by defining the scalar multiplications by $a * x = x/a$ and $x * a = a\backslash x$, for all $a, x \in L$. Note that, with respect to these scalar multiplications, $x\backslash_{*}y = x/y$, $x/_{*}y = x\backslash y$, $a\backslash_{*}y = ya$ and $x/_{*}b = bx$. Thus the next result follows directly from Theorem 3.3.

Corollary 3.4. Let $\mathbf{L}$ be a residuated lattice with greatest element $\top$. Then $\widehat{\mathbf{L}} = \langle L \times L, \wedge, \vee, \cdot, \backslash, /, (e, \top)\rangle$ is a residuated lattice with respect to the following operations:

$$\begin{array}{rcl} (a,x) \wedge (b,y) & = & (a \wedge b, x \vee y) \\ (a,x) \vee (b,y) & = & (a \vee b, x \wedge y) \\ (a,x)(b,y) & = & (ab, y/a \wedge b\backslash x) \\ (a,x)\backslash(b,y) & = & (a\backslash b \wedge x/y, ya) \\ (a,x)/(b,y) & = & (a/b \wedge x\backslash y, bx) \end{array}$$

■

Conditions (1) and (2) of the following result were originally established by P. H. Chu (see the appendix of [1] and [14]).

Corollary 3.5. Maintaining the notation established in Corollary 3.4, we have the following:

1. The element $D = (\top, e)$ is a cyclic dualizing element of $\widehat{\mathbf{L}}$. More specifically, for all $a, x \in L$,
$$(a,x)\backslash(\top,e) = (x,a) = (\top,e)/(a,x).$$

2. $\tilde{\mathbf{L}} = \langle L \times L, \wedge, \vee, \cdot, \backslash, /, E, D\rangle$ is a dualizing residuated lattice, where $E = (e, \top), D = (\top, e)$ and the other operations are defined as in Corollary 3.4.

3. Let $\widehat{\mathbf{L}}^\star = \langle \widehat{L}^\star, \vee, \wedge, \cdot, \backslash^\star, /^\star, E\rangle$, where

$$\begin{aligned} \widehat{L}^\star &= L \times \{\top\} \\ B/^\star A &= B/A \wedge (\top, \top), \\ A\backslash^\star B &= A\backslash B \wedge (\top, \top). \end{aligned}$$

Then the map $\varepsilon : \mathbf{L} \to \widehat{\mathbf{L}}^\star$, defined by $\varepsilon(a) = (a, \top)$ for all $a \in L$, is a residuated lattice isomorphism. Furthermore, it restricts to a residuated lattice isomorphism from $\mathbf{L}^-$ to $\widehat{\mathbf{L}}^-$.

Proof. An easy calculation establishes condition (1); thus, condition (2) follows from condition (1) and Corollary 3.4. To establish condition (3), note that, for all $a, b \in L$,

$$\varepsilon(a)\varepsilon(b) \ = \ (a, \top)(b, \top) \ = \ (ab, \top/a \wedge b\backslash\top) \ = \ (ab, \top),$$

and

$$\varepsilon(a)/^\star\varepsilon(b) = (a, \top)/^\star(b, \top) = (a/b \wedge \top\backslash\top, b\top) \wedge (\top, \top) = (a/b, \top).$$

Furthermore,

$$\begin{aligned} \varepsilon(a)\backslash^\star\varepsilon(b) &= (a, \top)\backslash^\star(b, \top) &&= (a\backslash b, \top), \\ \varepsilon(a) \wedge \varepsilon(b) &= (a, \top) \wedge (b, \top) &&= (a \wedge b, \top), \\ \varepsilon(a) \vee \varepsilon(b) &= (a, \top) \vee (b, \top) &&= (a \vee b, \top), \quad \text{and} \\ \varepsilon(e) &= (e, \top) &&= E. \end{aligned}$$

Hence, since ε is clearly a bijection, $\widehat{\mathbf{L}}^\star$ is a residuated lattice and ε is a residuated lattice isomorphism. Lastly, it is also clear that ε restricts to a residuated lattice isomorphism from $\mathbf{L}^-$ to $\widehat{\mathbf{L}}^-$. ■

Corollary 3.6. Every integral residuated lattice is isomorphic to the negative cone of a dualizing (equivalently, an involutive) residuated lattice. More specifically, if $\mathbf{L}$ be an integral residuated lattice then the map $\varepsilon : \mathbf{L} \to \tilde{\mathbf{L}}^-$, defined by $\varepsilon(a) = (a, e)$ for all $a \in L$, is a residuated lattice isomorphism.

■

The lattice of subvarieties of $\mathcal{RL}$ will be denoted by $\mathbf{L}(\mathcal{R}L)$ and that of $\mathcal{I}n\mathcal{RL}$ by $\mathbf{L}(\mathcal{I}n\mathcal{RL})$.

3.3. An uncountable family of involutive residuated lattices

The third step in the proof of the main result is to construct uncountably many strictly simple involutive residuated lattices that will generate distinct atoms in $\mathbf{L}(\mathcal{I}n\mathcal{RL})$. In what follows, it will be convenient to use the language of dualizing residuated lattices.

Let S be a subset of ω, let $\mathbf{C}_S$ be the strictly simple residuated chain constructed in 3.1 and let $\tilde{\mathbf{C}}_S = \mathbf{C}_S \times \mathbf{C}_S$ be defined as in 3.2. Then, by Corollary 3.5, the algebra $\tilde{\mathbf{C}}_S$ is a dualizing residuated lattice, where, for $(a,x),(b,y) \in \tilde{\mathbf{C}}_S$,

$$\begin{aligned}
(a,x)\wedge(b,y) &= (a\wedge b, x\vee y)\\
(a,x)\vee(b,y) &= (a\vee b, x\wedge y)\\
(a,x)(b,y) &= (ab, y/a\wedge b\backslash x)\\
(a,x)\backslash(b,y) &= (a\backslash b\wedge x/y, ya)\\
(a,x)/(b,y) &= (a/b\wedge x\backslash y, bx)\\
E &= (e,\top)\\
D &= (\top, e).
\end{aligned}$$

(See Figure 1.)

Let $\mathbf{L}_S$ be the subalgebra of $\tilde{\mathbf{C}}_S$ generated by E and D. Note, that $\mathbf{L}_S$ contains elements other than E and D, since $D\cdot D = (\top, e/\top\wedge\top\backslash e) = (\top, d_0)\in L_S$.

Lemma 3.7.

$$\{(x,\top) : x\in C_S\backslash\{\top\}\} \subseteq L_S \quad \text{and} \quad \{(\top,x) : x\in C_S\backslash\{\top\}\} \subseteq L_S.$$

Furthermore, $\{(x,\top) : x\in C_S\backslash\{\top\}\}$ is closed under multiplication.

Proof. First, let $x,y\in C_S\backslash\{\top\}$. Then, $xy\in C_S\backslash\{\top\}$ and

$$(x,\top)\cdot(y,\top) = (xy, \top/x\wedge y\backslash\top) = (xy,\top\wedge\top) = (xy,\top).$$

Hence, $\{(x,\top) : x\in C_S\backslash\{\top\}\}$ is closed under multiplication.

We know that $E, D\in L_S$. Therefore,

$$\begin{aligned}
D\cdot D &= (\top,e)\cdot(\top,e) = (\top, e/\top\wedge\top\backslash e) = (\top,d_0)\in L_S \quad \text{and}\\
D/(\top,d_0) &= (d_0,\top)\in L_S.
\end{aligned}$$

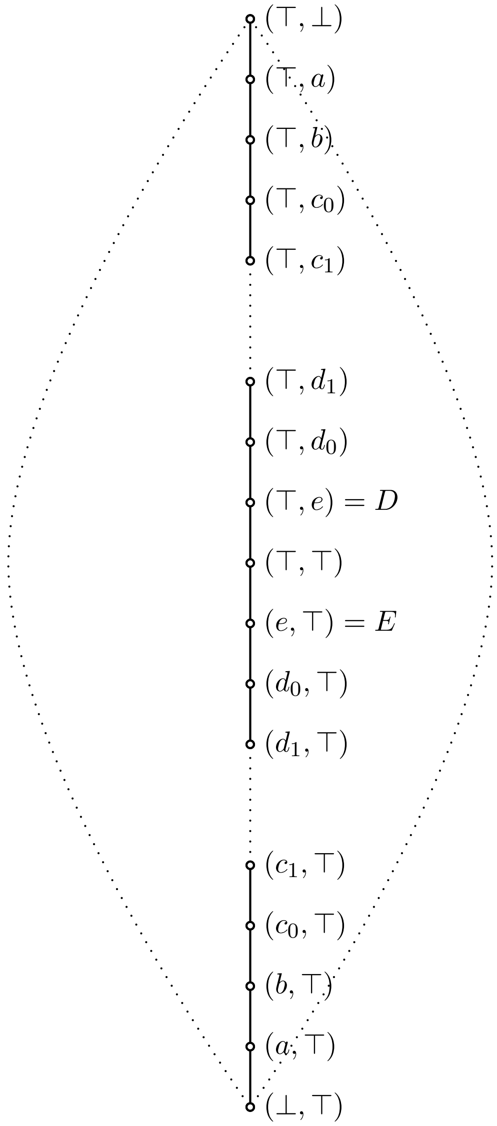

Figure 1. $\tilde{\mathbf{C}}_S$

Furthermore,

$$\begin{aligned}
(d_0,\top)^3 &= (\bot,\top)\in L_S,\\
D/(\bot,\top) &= (\top,\bot)\in L_S,\\
(\bot,\top)/(d_0,\top) &= (\bot/d_0\wedge\top\backslash\top, d_0\cdot\top) = (c_0,d_0)\in L_S,\\
(c_0,d_0)\wedge E &= (c_0,\top)\in L_S \quad \text{and}\\
D/(c_0,\top) &= (\top,c_0)\in L_S.
\end{aligned}$$

Since $\bot/d_i = c_i$ and $\bot/c_i = d_{i+1}$, for all $i\in\omega$, we have

$$\begin{aligned}
(\bot,\top)/(d_i,\top)\wedge E &= (c_i,d_i)\wedge E = (c_i,\top), \quad \text{and}\\
(\bot,\top)/(c_i,\top)\wedge E &= (d_{i+1},c_i)\wedge E = (d_{i+1},\top).
\end{aligned}$$

Hence, $(d_0,\top)\in L_S$ and $(c_0,\top)\in L_S$ imply $(d_i,\top)\in L_S$ and $(c_i,\top)\in L_S$, for all $i\in\omega$, and because of $D\in L_S$, also $(\top,d_i)\in L_S$ and $(\top,c_i)\in L_S$, for all $i\in\omega$.

Finally,

$$\begin{aligned}
(d_0,\top)\cdot(d_0,\top) &= (b,\top)\in L_S,\\
D/(b,\top) &= (\top,b)\in L_S,\\
(c_0,\top)\cdot(d_0,\top) &= (a,\top)\in L_S, \quad \text{and}\\
D/(a,\top) &= (\top,a)\in L_S.
\end{aligned}$$

Thus, $\{(x,\top) : x\in C_S\backslash\{\top\}\} \subseteq L_S$ and $\{(\top,x) : x\in C_S\backslash\{\top\}\} \subseteq L_S$. ■

Note, that $(\bot,\top)$ is the bottom element and $(\top,\bot)$ is the top element of $\mathbf{L}_S$.

The next result is not needed in the proof of the main theorem, but it is of some independent interest.

Proposition 3.8. $(\top,\top)\notin L_S$.

Proof. Let $A,B\in\tilde{\mathbf{C}}_S$ with $A=(a,x)$ and $B=(b,y)$. We show that, if $A\cdot B=(\top,\top)$, $B/A=(\top,\top)$, $A\backslash B=(\top,\top)$, $A\vee B=(\top,\top)$ or $A\wedge B=(\top,\top)$, then either $A=(\top,\top)$ or $B=(\top,\top)$. This, of course, implies that $(\top,\top)\notin L_S$.

Let $(\top,\top)=A\cdot B=(a\cdot b,\ y/a\wedge b\backslash x)$. Then, without loss of generality $a=\top$ and $b\in\{e,\top\}$. Since $e\backslash y=\top$ implies $y=\top$ and $\top\backslash x=\top$ implies $x=\top$, it follows that $A=(\top,\top)$.

Let $(\top,\top) = B/A = (b/a \wedge y\backslash x,\ a\cdot y)$. Then either $a = \top$ and $y \in \{e,\top\}$ or else $a \in \{e,\top\}$ and $y = \top$. In the first case, $y\backslash x = \top$ implies $x = \top$. In the second case $b/a = \top$ implies $b = \top$. Hence, $A = (\top,\top)$ or $B = (\top,\top)$. The case $(\top,\top) = A\backslash B$ is handled similarly.

Let $(\top,\top) = A \vee B = (a \vee b, x \wedge y)$. Then $x = y = \top$. Furthermore, $a = \top$ or $b = \top$, hence $A = (\top,\top)$ or $B = (\top,\top)$. The case $(\top,\top) = A \wedge B$ is handled similarly. ■

3.4. The atomic structure of $\mathbf{L}(\mathcal{I}n\mathcal{RL})$

We conclude the proof by showing that the algebras $\mathbf{L}_S$ generate distinct atoms in the subvariety lattice. We start with two auxiliary results. The first result was proved in [5] in the setting of residuated lattices. However, the proof presented here is a routine adaptation of N. Galatos's original proof.

PROPOSITION 3.9. (Compare with Theorem 3.1 in [5]). Let $\mathbf{L}$ be a strictly simple algebra in a variety $\mathcal{V}$. We assume that $\mathbf{L}$ has a residuated lattice reduct and possesses a least element $\bot$ (that is not necessarily among its nullary operations). We assume further that there exists a unary term $q_\bot$ such that $q_\bot(x) = \bot$ for all $x \in L$ with $x \neq e$. Then the variety generated by $\mathbf{L}$ is an atom in the subvariety lattice of $\mathcal{V}$.

PROOF. Note that $\mathbf{L}$ has a greatest element, namely $\top = \bot\backslash e$. Let $\mathcal{U}$ be the variety generated by $\mathbf{L}$ and let $\mathbf{M}$ be a non-trivial subdirectly irreducible algebra in $\mathcal{U}$. By Jónsson's Lemma, [10], for congruence distributive varieties, there exists an ultrapower $\mathbf{B}$ of $\mathbf{L}$ and a subalgebra $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{M}$ is the image of $\mathbf{C}$ under an epimorphism ϕ.

The term description of $\bot$, along with the fact that $\mathbf{L}$ is strictly simple, implies that for each $b \in L$ there exists a unary term q_b satisfying $q_b(x) = b$, for all $x \in L$ with $x \neq e$. Thus, the operation tables for $\mathbf{L}$ can be described by universal formulas and hence they are preserved by any ultrapower of $\mathbf{L}$. More specifically, if f is a fundamental ν-ary operation of $\mathbf{L}$ and $c, c_0, \dots, c_{\nu-1} \in L$, then the the equality "$c = f(c_0, \dots, c_{\nu-1})$" is captured by the universal formula "$(\forall x)(x \neq e \implies q_c(x) = f(q_{c_0}(x), \dots q_{c_{\nu-1}}(x))$". It follows that if $x \neq e \neq y$ in $\mathbf{B}$ and $c \in L$, then $q_c(x) = q_c(y)$. Denote this common value by c', for every $c \in L$. Now the universal formula $(\forall x, y, z)\,(x, y \neq e\ \rightarrow\ q_\bot(x) = q_\bot(y) \leq z)$ holds in $\mathbf{L}$, since $\bot$ is the least element. Thus, $\bot'$ is the least element and $\top' = \bot'\backslash e'$ the greatest element of any non-trivial subalgebra of $\mathbf{B}$. If $\mathbf{D}$ is the subalgebra of $\mathbf{C}$ generated by $\bot'$, then the map $c \mapsto q_c(\bot')$ $(c \in L)$ is an isomorphism between **L** and **D**. We claim that the subalgebra $\phi(\mathbf{D})$ of **M** is isomorphic to **L**. Indeed, since **D** is strictly simple, the restriction of ϕ on **D** is either injective or else collapses all elements of D to a single element. The latter case cannot occur, since it would also send $\top'$ to the same element and force the image of **C** under ϕ to be trivial, contrary to the assumption. Thus, every non-trivial subdirectly irreducible algebra in $\mathcal{U}$ has a subalgebra isomorphic to **L**, and hence $\mathcal{U}$ is an atom in the subvariety lattice. ∎

The preceding result immediately yields he following:

Corollary 3.10. Let **L** be a strictly simple residuated lattice (respectively, involutive residuated lattice) with least element $\bot$. If there is a unary term q such that $q(x) = \bot$ for all $x \in L$ with $x \neq e$, then the variety generated by **L** is an atom in the subvariety lattice $\mathbf{L}(\mathcal{RL})$ (respectively, $\mathbf{L}(\mathcal{I}n\mathcal{RL})$).

Lemma 3.11. For a residuated lattice or an involutive residuated lattice **L** with least element $\bot$, the following statements are equivalent:

1. There exists a unary term p such that $p(x) = \bot$, for all $x \in L$ with $x < e$.
2. There exists a unary term p such that $p(x \wedge e) \wedge p(e/x \wedge e) = \bot$, for all $x \in L$ with $x \neq e$.
3. There exists a unary term q such that $q(x) = \bot$, for all $x \in L$ with $x \neq e$.

Further, if q is a term in the language of residuated lattices, then so is p.

Proof. The equivalence of (2) and (3) is clear. It is also clear that (2) implies (1). To prove that (1) implies (2), consider $x \in L$. If $x \ngeq e$, then $x \wedge e < e$, and hence, by (1), $p(x \wedge e) = \bot$. It follows that $p(x \wedge e) \wedge p(e/x \wedge e) = \bot$. If, on the other hand, $x > e$, then $e/x < e$. Thus, invoking (1) once again, we get successively $p(e/x \wedge e) = \bot$ and $p(x \wedge e) \wedge p(e/x \wedge e) = \bot$.

Lastly, the preceding discussion makes clear that if q is a term in the language of residuated lattices, then so is p. ∎

We now have all the necessary information to complete the proof of the main result.

Theorem 3.12. The subvariety lattice $\mathbf{L}(\mathcal{I}n\mathcal{RL})$ of $\mathcal{I}n\mathcal{RL}$ contains uncountably many atoms.

PROOF. In light of Corollary 3.10, the theorem will be established if we prove the following:

1. Each $\mathbf{L}_S$ $(S \subseteq \omega)$ is a strictly simple algebra.
2. There exists a unary term q such that $q(X) = (\bot, \top)$, for all $X \in L_S$ with $X \neq E$.
3. If S_1 and S_2 are distinct subsets of ω, then there exists an identity that holds in $\mathbf{L}_{S_1}$ but not $\mathbf{L}_{S_2}$, and vice versa.

We first prove (2). Let $X \in L_S$ with $X < E$. Then $X = (x, \top)$ with $x < e$. In $\mathbf{C}_S$, we have $x^3 = \bot$, for all $x \notin \{e, \top\}$. Hence, $X^3 = (x, \top)^3 = (x^3, \top) = (\bot, \top)$. Thus, (2) is a consequence of Lemma 3.11.

We next prove (1). Note that $\mathbf{L}_S$ has no proper subalgebras since it is generated by $\{E, D\}$. To show that it does not possess any non-trivial proper congruences, let us recall that the congruences of $\mathbf{L}_S$ are those of the residuated lattice reduct, $\mathbf{R}_S$, of the corresponding dualizing residuated lattice. By Proposition 2.1, the congruences of $\mathbf{R}_S$ correspond to its convex normal subalgebras. Let $\mathbf{A}$ be any nontrivial convex normal subalgebra of $\mathbf{R}_S$. Then, there exists an element $Y \in \mathbf{A}$ with $Y \neq E$. In light of (2), there exists a unary term q, in the language of residuated lattices, such that $q(X) = (\bot, \top)$, for all $X \in L_S$ with $X \neq e$. In particular, $q(Y) = (\bot, \top)$ and hence $(\bot, \top) \in A$. Furthermore, since $(\bot, \top)/(\bot, \top) = (\top, \bot)$, both the bottom and top elements of R_S are in A. By convexity, we can infer that $A = R_S$. We have established that $\mathbf{R}_S$ has no nontrivial convex normal subalgebras, and therefore $\mathbf{L}_S$ is strictly simple.

We lastly prove (3). Let S_1 and S_2 be two distinct subsets of ω and let $\mathbf{L}_{S_1}$ and $\mathbf{L}_{S_2}$ be the corresponding involutive dualizing residuated lattices. By Lemma 3.7, $\{(x, \top) : x \in \mathbf{C}_{S_1} \backslash \{\top\}\} \subseteq \mathbf{L}_{S_1}$ and $\{(x, \top) : x \in \mathbf{C}_{S_2} \backslash \{\top\}\} \subseteq \mathbf{L}_{S_2}$. Furthermore these sets are closed under multiplication. Without loss of generality, we may assume that there exists $i \in \omega$ such that $i \in S_1$ and $i \notin S_2$. Then,

$$\begin{array}{rcll} (c_{i+1}, \top) \cdot (d_i, \top) & = & (a, \top) & \text{in } \mathbf{L}_{S_1}, \text{ but} \\ (c_{i+1}, \top) \cdot (d_i, \top) & = & (b, \top) & \text{in } \mathbf{L}_{S_2}. \end{array}$$

Since $\mathbf{L}_{S_1}$ and $\mathbf{L}_{S_2}$ are generated by their term defined bottom elements, one can find an equation that holds in one of the algebras but not in the other. In more detail, using the notation of Proposition 3.9 and starting with the element $a = (\bot, \top)$, the equations in question are $q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(a,\top)}$ and $q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(b,\top)}$. ■

It is worth mentioning that [6] presents two constructions for embedding a residuated lattice into an involutive residuated lattice. However, they cannot be used for the purposes of this paper. Indeed, even if one starts with a strictly simple residuated lattice $\mathbf{L}$, neither the constructed involutive residuated lattice $\mathbf{L}^*$ is simple, nor there exists a clear association of $\mathbf{L}^*$ with a strictly simple involutive residuated lattice.

For each $S \subseteq \omega$, let $\mathcal{U}_S$ denote the subvariety of $\mathcal{RL}$ generated by $\mathbf{C}_S$ and let $\mathcal{V}_S$ denote the subvariety of $\mathcal{I}n\mathcal{RL}$ generated by $\mathbf{L}_S$. It was shown in [9] that each variety $\mathcal{U}_S$ satisfies the identity $x^3 \approx x^4$. The next result shows that the varieties $\mathcal{V}_S$ satisfy an analogous identity.

Proposition 3.13. Each variety $\mathcal{V}_S$ satisfies the identity $x^4 \approx x^5$.

Proof. Recall that $x^3 = \bot$, for all $x \in C_S$ with $x < e$. Furthermore, we have $(x, y)^2 = (x^2, y/x \wedge x\backslash y)$ and

$$(x, y)^4 = (x^4, (y/x \wedge x\backslash y)/x^2 \wedge x^2\backslash(y/x \wedge x\backslash y)).$$

Because of $\top \cdot x = x \cdot \top = x$, for all $x \in C_S$ with $x \neq e$, it follows that

$$x \neq e \text{ and } x \leq y \quad \text{imply} \quad y/x \wedge x\backslash y = \top \tag{i}$$

and therefore,

$$x < e \text{ and } x^2 \leq y/x \wedge x\backslash y \quad \text{imply} \quad (x, y)^4 = (x, y)^5, \tag{ii}$$

where $(x, y)^4 = (\bot, \top)$.

$x = \top$: If $x = \top$ and $y \neq e$ we have $(\top, y)^2 = (\top, y)$. Furthermore, if $x = \top$ and $y = e$, it follows that $(\top, e)^4 = (\top, d_0)^2 = (\top, d_0) = (\top, d_0) \cdot (\top, e) = (\top, e)^5$.

$x = e$: If $x = e$, then $(e, y)^2 = (e^2, y/e \wedge e\backslash y) = (e, y)$, for all $y \in C_S$.

$x < e$: If $x < e$, for each $y \in C_S$, it will be shown in the following, that $x^2 \leq y/x \wedge x\backslash y$ which implies, by (ii), $(x, y)^4 = (x, y)^5$.

If $x < e$ and $x \leq y$, then, by (i), $y/x \wedge x\backslash y = \top$, hence, $x^2 \leq y/x \wedge x\backslash y$.

If $x = d_i$, with $i \in \omega$, and $x > y$, it follows that $x^2 = b$. Furthermore, since $b \cdot d_i = d_i \cdot b = \bot$, for all $i \in \omega$, we have $b \leq y/d_i \wedge d_i\backslash y$, for all $y \in C_S$ and $i \in \omega$. Hence, $x^2 \leq y/x \wedge x\backslash y$.

If $x < d_i$, for all $i \in \omega$, then $x^2 = \bot$, thus, $x^2 \leq y/x \wedge x\backslash y$, for all $y \in C_S$. ■

We remark that none of the varieties $\mathcal{V}_S$ satisfies the identity $x^3 \approx x^4$. Indeed, by Lemma 3.7, for all $S \subseteq \omega$, $(\bot, \top), (a, \top) \in L_S$. Note further that $(\bot, \top)/(a, \top) = (\bot\backslash a \wedge \top/\top, a \cdot \top) = (d_0, a)$, thus $(d_0, a) \in L_S$. Hence, $(d_0, a)^3 = (\bot, d_0)$, but $(d_0, a)^4 = (\bot, \top)$.

A particularly significant subvariety of $\mathcal{RL}$ is the variety $\mathcal{RepRL}$ of representable residuated lattices. This is simply the subvariety of $\mathcal{RL}$ generated by all totally ordered residuated lattices. Refer to [4] or [9] for an equational description of this variety. Correspondingly, we have the variety, $\mathcal{RepInRL}$, of representable involutive residuated lattices. It is generated by all totally ordered involutive residuated lattices. Note that each variety $\mathcal{U}_S$ is a subvariety of $\mathcal{RepRL}$. On the other hand, none of the varieties $\mathcal{V}_S$ is contained in $\mathcal{RepInRL}$, since their generators, $\mathbf{L}_S$, are simple – and hence, subdirectly irreducible – but not totally ordered. Thus, the question remains whether there exist uncountably many atoms in the subvariety lattice of $\mathcal{RepInRL}$. We remark, in this connection, that the variety of Boolean algebras is the only minimal variety below the variety of integral involutive residuated lattices. Recall that an involutive residuated lattice is *integral*, if it satisfies the law $x \wedge e \approx e$.

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Constantine Tsinakis
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240, U.S.A.
constantine.tsinakis@vanderbilt.edu

Annika M. Wille
Department of Mathematics
Technische Universität Darmstadt
Schlossgartenstrasse 7
64289 Darmstadt, Germany
awille@mathematik.tu-darmstadt.de