RESEARCH STATEMENT

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1. Introduction

My research has been primarily focusing on geometric analysis and free boundary problems. I am particularly interested in the following two topics.

(i) The well-posedness problem of geometric flows and diffusion equations on non-compact manifolds, or even manifolds with singularities. Along this line of research, I have proved that the Yamabe flow may start with a weak initial regularity condition, and thus can admit an initial metric with unbounded curvature. Similar results for two well-known nonlinear diffusion models, the porous medium equation and p-Laplacian equation, are also obtained on singular manifolds. The theory of differential equations on singular manifolds can also be applied to degenerate equations on domains with higher order degeneracy at the boundary. In Section 2, a survey on these results will be presented.

(ii) The regularity of geometric flows and free boundary problems. I developed a technique to study regularity of solutions to differential equations on manifolds by means of maximal regularity theory. As an application, I have proved regularity, especially analyticity, of solutions to several geometric flows, including the mean curvature flow, the Ricci flow and the Willmore flow. The aforementioned technique was generalized to study the regularity of free boundaries occurring in phase transitions. A detailed discussion is contained in Section 3.

At the end of this article, for the reader's convenience, I list my main results and some future projects.

2. Evolution equations on manifolds with singularities

An interesting problem in geometry is whether one can find a "standard model" in each class of metrics on some manifold, thus reducing topological questions to differential geometry ones. One of the well-known representatives of this kind is the Yamabe problem. On a compact manifold (M, g), this problem aims at finding a metric conformal to g with constant scalar curvature. I will give a brief historical account of this problem in Section 2.2.

If we already have a metric, as a natural question, one might want to ask how we can "drive" the prescribed metric into a "standard" one, or at least improve it. For example, the Ricci flow tends to "flatten out" or "round out" a manifold depending on its initial "shape". Another example of evolution of metrics is the Yamabe flow, which will be discussed in details in Section 2.2. Evolution problems of this kind comprise an important class of so-called geometric flows, or geometric evolution equations.

Geometric flows have been studied in depth on compact manifolds. However, many of the strategies are limited in the case of on non-compact manifolds, in particular, manifolds with singularities. Indeed, in general, we do not know how to start some geometric flows, even just for a short time, without imposing further conditions. Hence one approach to the investigation of geometric flows on non-compact manifolds is trying to find curvature conditions, at least for local existence.

Another way to understand geometric flows, mainly in the case of manifolds with singularities, is to generalize the ideas for (degenerate or singular) elliptic differential operators. One way to view manifolds with singularities is so-called ringed spaces, that is to view a manifold M as a pair consisting of a punctured manifold $\mathsf{M} \setminus \{\mathsf{p}\}$ (by removing the singularity p) and a subalgebra of differential operators degenerating at a specified rate near the singularity, rather than just M itself, see [44]. These considerations lead to the study of degenerate or singular differential operators on manifolds with singularities. In Section 2.1, I first present a brief history of this line of research, and then introduce the geometric framework and theoretic basis of my research in the field of evolution equations on manifolds with singularities.

2.1. **A historical account and general framework.** The study of differential operators on manifolds with singularities is motivated by a variety of applications from applied mathematics, geometry and topology. All of the work is related, in one way or another, to the seminal paper by V.A. Kondrat'ev [36]. There is a tremendous amount of literature on pseudo-differential calculus of differential operators of Fuchs type, which has been introduced independently by R.B. Melrose[41, 42] and B.-W. Schulze[44, 49, 50, 51]. One branch of these activities is connected with the b-calculus on manifolds with cylindrical ends. See [41, 42]. Research along another line, known as cone differential operators, studies differential operators on conic manifolds. The investigation of conic singularities was initiated by J. Cheeger in [16, 17, 18]. In both directions, many authors have been very active and made remarkable contributions. The amount of research in this field is enormous, and thus it is literally impossible to list all of the achievements.

To distinguish these two lines of work, for instance, we look at a compact Riemannian manifold (M, g) with boundary $(\partial\mathsf{M}, g_{\partial\mathsf{M}})$. R. B. Melrose makes a collar neighborhood of $\partial\mathsf{M}$ into a cylindrical end by associating $\partial\mathsf{M}$ with the singular metric $(dt/t)^2 + g_{\partial\mathsf{M}}$. Instead, B.-W. Schulze uses the degenerate metric $dt^2 + t^2 g_{\partial\mathsf{M}}$. The associated Laplacian-Beltrami operators with respect to these two metrics are

$$((t\partial_t)^2 + \Delta_{g_{\partial\mathsf{M}}}), \quad \text{and} \quad t^{-2}((t\partial_t)^2 + \Delta_{g_{\partial\mathsf{M}}}),$$

respectively. Many efforts have been made to generalize research in these directions to more complicated types of singularities, e.g., edge and corner pseudodifferential calculus. However, for higher order singularities, the corresponding algebra becomes significantly more complicated. Therefore, a natural question to ask is whether we can find a general approach, which is less sensitive to the geometric structure near the singularities, to analysing differential equations on manifolds with singularities.

In 2012, H. Amann [2, 3] introduced a geometric framework, called *singular manifolds*, to study differential equations on non-compact manifolds, or even manifolds with singularities. As a starting point, he first looked at a class of possibly non-compact manifolds, called *uniformly regular Riemannian manifolds*. Roughly speaking, a manifold (M, g) is *uniformly regular* if its differential structure is induced by an atlas whose local patches are of comparable sizes, and all transition maps and curvatures have uniformly bounded derivatives. In the boundaryless case, (M, g) is *uniformly regular* iff it is complete and of bounded geometry, see [53]. (M, g) is a *singular manifold* if $(\mathsf{M}, g/\rho^2)$ is *uniformly regular* for some $\rho \in C^\infty(\mathsf{M}, (0, \infty))$, i.e., it is conformally *uniformly regular*. Conventionally, we write M, g, ρ as a 3-tuple $(\mathsf{M}, g; \rho)$. This concept covers the aforementioned examples of manifolds with singularities, including cylinder, cone, edge and corner ends.

The framework of *singular manifolds*, as we can observe from its definition, does not depend on the order of the singularities or specific geometric structure near the singularities. Moreover, function space theory, for instance, interpolation, embedding and trace theorems, is well established for *singular manifolds* in [2, 3]. It is known that although most function spaces, like Sobolev-Slobodeckii spaces, can be defined invariantly on non-compact manifolds, the corresponding function space theory does not hold in this generality. These preparations and the uniform structure of *singular manifolds* enable us to build up the theory of maximal regularity for differential operators on *singular manifolds*, with singularities of arbitrarily high dimensions and of various structures. Maximal regularity theory has proven itself a powerful tool in the study of nonlinear evolution equations in recent decades, especially for higher order differential equations, see [1], [6]-[8], [22], [23], [26]-[30], [38], [46], [47], [61] for example. On *uniformly regular Riemannian manifolds*, L_p and continuous maximal regularity theories are established by H. Amann, G. Simonett and the author, see [4, 60]. The differential operators studied in these papers include the ones associated with R. B. Melrose's b-calculus. The corresponding theory for degenerate or singular differential operators on *singular manifolds* have been set up in H. Amann's and the author's papers [4, 57].

In virtue of a theorem by G. Da Prato and P. Grisvard [21] and S.B. Angenent [7], continuous maximal regularity theory gives rise to local existence and uniqueness for quasilinear or even fully nonlinear equations. In the theory of continuous maximal regularity on *singular manifolds*, in particular, I work in a weighted *Hölder* framework. We denote the weighted *little Hölder* space of order $s \geq 0$ and weight $\vartheta \in \mathbb{R}$ by $bc^{s,\vartheta}(\mathsf{M})$.

To keep this article at a reasonable length, I would like to refer the reader to [57, Section 2.2] for the precise definition of these spaces. To briefly illustrate my work, for example, we take a look at a model quasilinear bilaplacian problem on a *singular manifold* (M, g)

$$\begin{cases} \partial_t u + u^n \Delta_g^2 u = 0, \\ u(0) = u_0. \end{cases}$$

Here Δ_g is the Laplace-Beltrami operator with respect to g. This model is closely related to the thin film equation. Suppose that U is a properly chosen open subset of some *little Hölder* space $bc^{s,\vartheta}(\mathsf{M})$. If, for every $u \in U$, the principal symbol of the operator $u^n \Delta_g^2$ satisfies

$$u^n (g^*(-i\xi, -i\xi))^2 \sim \rho^4 |\xi|_{g^*}^4, \quad \xi \in T^*\mathsf{M},$$

where g^* is the cotangent metric induced by g, then the above equation has a unique classical solution in weighted *little Hölder* spaces for any $u_0 \in U$. More general results also hold true for differential operators acting on tensor fields.

2.2. **The Yamabe flow.** In 1960, H. Yamabe [68] attempted to solve the Yamabe problem by using calculus of variations and elliptic partial differential equations. But his proof contained a error, which was discovered by N.S. Trudinger. The proof for Yamabe problem was later completed by N.S. Trudinger [63], T. Aubin [9], R. Schoen [48].

As an alternative approach to the Yamabe problem, R. Hamilton introduced the normalized Yamabe flow, which asks whether a metric, driven by this flow converges conformally to one with constant scalar curvature. The normalized Yamabe flow reads as follows.

$$\begin{cases} \partial_t g = (s_g - R_g) g & \text{on} \quad \mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}, \end{cases} \tag{2.1}$$

where R_g is the scalar curvature with respect to the evolving metric g, and s_g is the average of R_g. Here $\mathsf{M}_T := (0,T) \times \mathsf{M}$ for $T \in (0, \infty]$. On an m-dimensional closed compact manifold (M, g_0), an easy computation shows that the normalized Yamabe flow is equivalent to the (unnormalized) Yamabe flow.

$$\begin{cases} \partial_t g = -R_g g & \text{on} \quad \mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{2.2}$$

This parabolic equation has a history of its own. The concept of closed manifolds in this article refers to manifolds with boundary, not necessarily compact. On closed compact manifolds, global existence and regularity of the solutions to equation (2.2) has been well established. R.G. Ye [69] proved that the unique solution to (2.2) exists globally and smoothly for any smooth initial metric. Fix a background metric $\tilde{g}$ such that $(\mathsf{M}, \tilde{g})$ is compact. In particular, we can take $\tilde{g} = g_0$. Let $g = u^{\frac{4}{m-2}} \tilde{g}$ for some $u > 0$. By rescaling the time variable, equation (2.2) is equivalent to

$$\begin{cases} \partial_t u = u^{-\frac{4}{m-2}} L_{\tilde{g}} u & \text{on} \quad \mathsf{M}_T; \\ u(0) = u_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{2.3}$$

with a positive function u_0. In addition, $L_{\tilde{g}} u := \Delta_{\tilde{g}} u - \frac{m-2}{4(m-1)} R_{\tilde{g}} u$ is the conformal Laplacian operator with respect to the metric $\tilde{g}$. By the compactness of $(\mathsf{M}, \tilde{g})$, uniform ellipticity of the operator $u_0^{-\frac{4}{m-2}} L_{\tilde{g}}$ is guaranteed. That is why well-posedness is relatively easy in the compact case.

Convergence of solutions to (2.1) is another important topic in the study of the Yamabe flow. R. Hamilton conjectured that on a compact Riemannian manifold solutions to (2.1) converge to a metric of constant scalar curvature as $t \to \infty$. B. Chow [19] commenced the study of Hamilton's conjecture and proved convergence in the case when (M, g_0) is locally conformally flat and has positive Ricci curvature. Later, this result was improved by R.G. Ye [69], wherein the author removed the restriction on the positivity of Ricci curvature by lifting the flow to a sphere, and deriving a Harnack inequality. In the case that $3 \leq m \leq 5$, H. Schwetlick and M. Struwe [52] showed that the normalized Yamabe flow evolves any initial metric to one with constant scalar curvature as long as the initial Yamabe energy is small. In [12], S. Brendle was able to remove the smallness assumption on the initial Yamabe energy. A convergence result is stated in [13] by the same author for higher dimension cases.

Nevertheless, the theory for the Yamabe flow on non-compact manifolds is far from being settled. Even local well-posedness is only established for restricted situations. Its difficulty can be observed from the fact that, losing the compactness of (M, g_0), equation (2.3) can exhibit degenerate and singular behaviors simultaneously. The investigation of the Yamabe flow on non-compact manifolds was initiated by L. Ma and Y. An in [39]. Later, conditions on extending local in time solutions were explored in [40]. In [39], the authors showed that for a complete closed non-compact Riemannian manifold (M, g_0) with Ricci curvature bounded from below and with a uniform bound on the scalar curvature in the sense that:

$$\mathrm{Ric}_{g_0} \geq -K g_0, \quad |R_{g_0}| \leq C,$$

then equation (2.2) has short time solution on $\mathsf{M} \times [0, T(g_0)]$ for some $T(g_0) > 0$. If in addition $R_{g_0} \leq 0$, then this solution is global. Here Ric_{g_0} is the Ricci curvature tensor with respect to g_0. The proof is based on the widely used technique consisting of exhausting M with a sequence of compact manifolds with boundary and studying the solutions to a a sequence of initial boundary value problems. Then uniform estimates of these solutions and their gradients are obtained by means of the maximum principle on manifolds with Ricci curvature bounded from below in the sense given above. The uniform boundedness of the scalar curvature plays an indispensable role in the proof for local well-posedness, although this has not been pointed out explicitly. This is a remarkable feature of the Yamabe flow and related geometric flows. Indeed, it can be illustrated by an example in [32].

Theorem 2.1 (G. Giesen, P.M. Topping [32])**.** *Suppose that T^2 is a torus equipped with an arbitrary conformal structure and $\mathsf{p} \in \mathsf{T}^2$. Let h be the unique complete, conformal, hyperbolic metric on $\mathsf{T}^2 \setminus \{\mathsf{p}\}$. Then there exists a smooth Ricci flow $g(t)$ on T^2 for $t > 0$ such that $g(t) \to h$ smoothly locally on $\mathsf{T}^2 \setminus \{\mathsf{p}\}$ as $t \to 0$.*

Hence a Ricci flow with unbounded curvature may pull "points at infinity" to within finite distance in finite time. This phenomena counters the classic situation. While starting a Ricci flow with an initial metric with bounded curvature and evolving it with bounded curvature, any curve heading to infinity retains its infinite length. It is well known that in dimension two, the Yamabe flow agrees with the Ricci flow. The above observation points out part of the difficulty in evolving a metric with a unbounded curvature. However, in a recent paper, I was able to prove the following result. Assume that $(\mathsf{M}, \tilde{g}; \rho)$ is a *singular manifold* without boundary and $g_0 \in [\tilde{g}]$, the conformal class of $\tilde{g}$. Let $g = u^{\frac{4}{m-2}} \tilde{g}$.

Theorem 2.2 (Y. Shao [57])**.** *Suppose that $u_0 \in U^s_\vartheta := \{u \in bc^{s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ with $s \in (0,1)$, and $\vartheta = (m-2)/2$. Then equation (2.3) has a unique smooth local positive solution with*

$$\hat{u} \in C(J(u_0), U^s_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$ for some $T(u_0) > 0$.

Here $bc^{s,\vartheta}(\mathsf{M})$ is the weighted *little Hölder* space mentioned in Section 2.1. Firstly, under the conformal change $g_0 = u_0^{\frac{4}{m-2}} \tilde{g}$, we have

$$R_{g_0} = -\frac{4(m-1)}{m-2} u_0^{-\frac{m+2}{m-2}} L_{\tilde{g}} u. \tag{2.4}$$

To show that in Theorem 2.2 we may start with a metric with unbounded scalar curvature, we take $\rho = \mathbf{1}_{\mathsf{M}}$ for computational brevity, i.e., $(\mathsf{M}, \tilde{g})$ to be *uniformly regular*. Then there is some $C > 1$ such that

$$1/C \leq \|u_0^{-\frac{m+2}{m-2}}\|_\infty \leq C, \quad \|R_{\tilde{g}}\|_\infty \leq C.$$

But at the same time, there are ample examples of $u_0 \in U^s_\vartheta$ with unbounded derivatives. In view of formula (2.4), it is not hard to create g_0 with unbounded scalar curvature. Therefore, the Yamabe flow can admit a unique smooth solution while starting at a metric with unbounded curvature, and these solutions evolve into one with bounded curvature instantaneously. This is a quite unexpected observation, as explained before.

Secondly, based on [33], in every conformal class we can find a metric $\tilde{g}$ making $(\mathsf{M}, \tilde{g})$ *uniformly regular.*

Hence, Theorem 2.2 applies to every conformal class.

Concerning the convergence of solutions in the non-compact case, very little is known. Convergence of a family of explicit solutions, so-called cigar manifolds, is studied in [14].

Another line of work on the Yamabe flow is to extend the problem onto manifolds with boundary. S. Brendle in [11] commenced the exploration of existence and convergence results on compact manifolds with conditions making the boundary minimal. Let $\mathcal{H}_g$ be the mean curvature of $\partial\mathsf{M}$.

Theorem 2.3 (S. Brendle [11])**.** *Suppose that (M, g_0) is a compact manifold with boundary. Assume that*

(i) $[g_0]$ *contains a metric with negative scalar curvature and vanishing mean curvature, or*

(ii) $[g_0]$ *contains a metric with vanishing scalar curvature and vanishing mean curvature, or*

(iii) $[g_0]$ *contains a metric with positive scalar curvature and vanishing mean curvature. Furthermore, g_0 is conformally flat and $\partial\mathsf{M}$ is umbilic.*

Then the initial boundary value problem

$$\begin{cases} \partial_t g = (s_g - R_g)g & \text{on} \quad \mathsf{M}_\infty; \\ \mathcal{H}_g = 0 & \text{on} \quad \partial\mathsf{M} \times [0,\infty); \\ g(0) = g_0 & \text{on} \quad \mathsf{M} \end{cases} \tag{2.5}$$

has a unique smooth solution converging to a metric with constant scalar curvature.

Under the conformal change $g = u^{\frac{4}{m-2}} g_0$, $\mathcal{H}_g$ is related to $\mathcal{H}_{g_0}$ by

$$\mathcal{H}_g = \frac{2}{m-2} u^{-\frac{m}{m-2}} \left(\frac{\partial u}{\partial \nu_0} + \frac{m-2}{2} \mathcal{H}_{g_0} u \right),$$

where ν_0 is the unit outward normal vector with respect to the metric g_0. Then the boundary condition in (2.5) becomes the homogeneous Neumann condition $\frac{\partial u}{\partial \nu_0} = 0$.

Along this research line, I am particularly interested in establishing well-posedness of the Yamabe flow on *singular manifolds* with boundary under inhomogeneous boundary conditions.

$$\begin{cases} \partial_t g = -R_g g & \text{on} \quad \mathsf{M}_T; \\ \mathcal{H}_g = f & \text{on} \quad \partial\mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}. \end{cases}$$

Suppose $(\mathsf{M}, \tilde{g}; \rho)$ is a singular manifold with boundary. Put $g = u^{\frac{4}{m-2}} \tilde{g}$. Then the above equation is equivalent to

$$\begin{cases} \partial_t u - u^{-\frac{4}{m-2}} L_{\tilde{g}} u = 0 & \text{on} \quad \mathsf{M}_T; \\ \frac{2}{m-2} u^{-\frac{m}{m-2}} \left(\frac{\partial u}{\partial \tilde{\nu}} + \frac{m-2}{2} \mathcal{H}_{\tilde{g}} u \right) = f & \text{on} \quad \partial\mathsf{M}_T; \\ u(0) = u_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{2.6}$$

Here $\tilde{\nu}$ is the unit outward normal vector with respect to the metric $\tilde{g}$.

Conjecture 2.4. *Suppose that $u_0 \in U^{1+s}_\vartheta := \{u \in bc^{1+s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ with $0 < s < 1$, and $\vartheta = (m-2)/2$. Then for every $f \in bc^{s,-m^2/4}(\mathsf{M})$ satisfying the compatibility condition*

$$\frac{2}{m-2} u_0^{-\frac{m}{m-2}} \left(\frac{\partial u_0}{\partial \tilde{\nu}} + \frac{m-2}{2} \mathcal{H}_{\tilde{g}} u_0 \right) = f,$$

equation (2.6) has a unique smooth positive local solution with

$$\hat{u} \in C(J(u_0), U^{1+s}_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$ for some $T(u_0) > 0$.

Certainly, we can also allow f to evolve with respect to time. Similar to the case of closed manifolds, it can still be shown that in every conformal class there exists some metric $\tilde{g}$ making $(\mathsf{M}, \tilde{g})$ *uniformly regular*. Hence if Conjecture 2.4 is true, it applies to every conformal class.

Our strategy to prove Conjecture 2.4 is divided into two parts. First, as a theoretical basis, we prove the continuous maximal regularity theory for an abstract system of a linear initial boundary value problem. Secondly, using the aforesaid continuous maximal regularity theory, we intend to prove an existence and uniqueness theorem for an abstract system of quasilinear initial boundary value problem with nonlinear boundary condition, which covers equation (2.6), by means of the contraction mapping principle. This is one of my long term projects.

Remark 2.5. In the L_p-framework, I have proved in [58] generation of analytic semigroup for a class of degenerate/singular elliptic operators on *singular manifolds* with flexible rate of degeneracy or singularity while approaching singular ends. A precise characterization for the domains of these operators is obtained. In the future, I aim at obtaining a more solid understanding of the L_p-theory for nonlinear differential equations, and then showing regularization phenomena of the metric driven by other geometric flows, e.g., the Ricci-DeTurck flow, similar to that of the Yamabe flow on *singular manifolds*, i.e., evolving a metric with unbounded curvature into one with bounded curvature.

2.3. **Nonlinear degenerate and singular diffusion equations.** Diffusion equations have proven themselves a strong tool for the study of numerous phenomenon, like population growth, heat transportation, fluid dynamics. In the recent decade, lots of efforts have been taken to study various diffusion models on Riemannian manifolds, mainly because of their close relationship with many geometric flows, for example the Ricci flow. On non-compact manifolds, in particular those with singularities, diffusion models are usually expected to have degenerate or singular behaviors.

To obtain a good understanding of nonlinear degenerate and singular diffusion equations on singular manifolds, probably the best starting point is the two well-known relatives of the heat equation, namely, the porous medium equation (PME) and the parabolic p-Laplacian equation. In this subsection, I will focus on PME,

$$\begin{cases} \partial_t u = \Delta_g u^n & \text{on} \quad \mathsf{M}_T; \\ u(0) = u_0 & \text{on} \quad \mathsf{M}, \end{cases} \tag{2.7}$$

mainly because of its close relationship with the Yamabe flow, although similar results concerning the parabolic p-Laplacian equation can also be found in my recent paper [57].

On compact manifolds, the theory of PME and the techniques used to prove this theory make no huge difference from those in Euclidean spaces. Existence, uniqueness and L_1-stability of non-negative weak solutions to Cauchy problems for PME on compact manifolds can be found in J.L. Vázquez's book [65]. Various aspects of PME on Riemannian manifolds, e.g., Harnack inequalities and comparison principles for local radial symmetric solutions, have been explored by many mathematicians. See [15, 24, 35, 45, 67, 70] for example. To the best of my knowledge, research in this direction is all restricted to the case of complete, or even compact, manifolds.

An obvious barrier to the study of PME on manifolds with singularities is reflected by the fact that we need to deal with two different types of degeneracies at the same time: the degeneracy while approaching the singular ends, and the degeneracy caused by the vanishing set of the initial data. In [57], an existence and uniqueness result similar to that formulated for the Yamabe flow has been proved for (2.7). This seems to be the first one concerning PME on manifolds with singularities. This result serves as the stepstone to a more general theory for PME. Recently, I was able to prove the following comparison principle for the solutions to (2.7).

Theorem 2.6 (Y. Shao [59])**.** *Suppose that* $U^{1+s}_\vartheta := \{u \in bc^{1+s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ *with* $0 < s < 1$, $\vartheta = -2/(n-1)$, *and* $f \in bc^{s,\vartheta}(\mathsf{M})$. *Assume that* u *and* $\hat{u}$ *are solutions to equation* (2.7) *with initial data* $u_0, \hat{u}_0 \in U^{1+s}_\vartheta$, *respectively, on* $[0,T]$. *If* $u_0 \le \hat{u}_0$, *then* $u \le \hat{u}$ *on* $[0,T]$.

If the initial datum vanishes somewhere on M, in general, we cannot expect to have a classical solution to (2.7). Based on Theorem 2.6, I am planning to prove existence and regularity of solutions to (2.7) with more general initial data, say without the restriction $\inf \rho^{\vartheta} u_0 > 0$.

2.4. **Differential operators with higher order degeneracy at the boundary.** Many problems arising in mathematical finance, fluid mechanics, population dynamics can be modelled by degenerate differential equations on domains with specific rate of degeneration at the boundary. In [5], it is shown that C^k-domains with compact boundaries can be viewed as *singular manifolds*. (The open half space $\mathbb{H}^m = \mathbb{R}^+ \times \mathbb{R}^{m-1}$ is also a *singular manifold*.) In virtue of this observation, the theory of differential equations on *singular manifolds* can be applied to degenerate equations on domains. I proved a local existence and uniqueness theorem for a class of differential equations, analogous to the model shown in Section 2.1, with rate of degeneration larger than or comparable to the square of the distance function to the boundary of the domain. See [57, Sections 3.2, 4.4]. This result extends the work in [31, 66] to unbounded domains and to higher order elliptic operators. We can apply the aforementioned theorem to the thin film equation, the following equation occurring in studying the ribbon flares caused by magnetic turbulence on the solar surface

$$u_t = \frac{1}{\eta} u^2 (\frac{\partial^2 u}{\partial x^2} + u^2), \quad x \in [0,1], \quad t \in [0,T],$$

and of course many other examples. I expect to find more interesting applications of this theorem.

3. **Regularity of geometric flows and free boundary problems**

In [56], I developed a technique relying on a family of parameter-dependent diffeomorphism, maximal regularity theory, and the implicit function theorem to prove regularity of solutions to geometric flows. This technique has been generalized in [54] to show regularity of free boundary problems.

The idea of establishing regularity of solutions to differential equations by means of the implicit function theorem in conjunction with a translation argument was originated by S.B. Angenent in [6] to prove the analyticity of the free boundary in one dimensional porous medium equations, and has proved itself a useful tool later in many publications. See for example [8, 29, 30, 38, 47]. More precisely, to study the regularity of the solution to a differential equation, one introduces parameters representing translation in space and time into the solution to the given differential equation. Then we study the parameter-dependent equation satisfied by this transformed solution. The implicit function theorem yields the smooth dependence upon the parameters of the solution to the parameter-dependent problem. This regularity property is then inherited by the original solution. An advantage of this technique is reflected by its power to prove analyticity of solutions to differential equations, which cannot be attained through the classical method of bootstrapping.

However, applying this technique to differential equations on manifolds causes an essential challenge, considering for instance the usual translation $(t, x) \mapsto (t + \lambda, x + \mu)$, because of the global nature of translations. Hence we desire an alternative that only shifts the variables “locally”. The idea of localizing translations is first put to use in [28] to study regularity of solutions to elliptic and parabolic equations in Euclidean space. The basic building block of [28] consists of rescaling translations by some cutoff function. This idea paves the way to introducing truncated translations on Riemannian manifolds by means of a smooth atlas, which induces a family of parameter-dependent diffeomorphisms acting on functions or tensor fields.

I would also like to mention that the method to combine the implicit function theorem and translations on compact closed manifolds is also addressed in [27] by J. Escher and G. Prokert. The authors first construct a family of global real analytic vector fields, which induces a one-parameter group of diffeomorphisms to transform the solutions to differential equations. It is known that the existence of such a one-parameter group is valid on compact closed manifolds, but is not guaranteed in general.

3.1. **Regularity of solutions to geometric flows.** I have applied the previously mentioned parameter-dependent diffeomorphisms to prove regularity, especially analyticity, of solutions to several geometric flows in the three papers [55, 56, 60], including

- the averaged/unaveraged mean curvature flow [56];

- the Ricci flow [56];
- the surface diffusion flow [56];
- the Willmore flow [55];
- the normalized/unnormalized Yamabe flow [60].

Among geometric flows on the evolution of surfaces driven by their curvatures, one famous example is the Willmore flow, where the moving surface $\Gamma(t)$ obeys the law:

$$V(t) = -\Delta_{\Gamma(t)}\mathcal{H}_{\Gamma(t)} - 2\mathcal{H}_{\Gamma(t)}(\mathcal{H}^2_{\Gamma(t)} - K_{\Gamma(t)}).$$

Here $V(t)$ denotes the velocity in the normal direction of Γ at time t, $\mathcal{H}_{\Gamma(t)}$ and $K_{\Gamma(t)}$ stand for the mean curvature and the Gaussian curvature of $\Gamma(t)$, respectively. $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator on $\Gamma(t)$.

Theorem 3.1 (Y. Shao [55])**.** *If the initial hypersurface Γ_0 belongs to the class $C^{2+\alpha}$ for some $\alpha \in (0,1)$, then the Willmore flow immediately becomes analytic, that is, inside its maximal interval of existence $[0,T)$,*

$$\mathcal{M} := \bigcup_{t\in(0,T)} (\{t\} \times \Gamma(t))$$

is a real analytic hypersurface in $\mathbb{R}^4$. In particular, each manifold $\Gamma(t)$ is real analytic for $t \in (0,T)$.

Similar results on analyticity of solutions have been proved for the mean curvature flow and the surface diffusion flow in [56].

Another kind of geometric flow concerns the evolution of metrics, as was mentioned in Section 2. In [60], we proved the following regularity result for the Yamabe flow.

Theorem 3.2 (Y. Shao, G. Simonett [60])**.** *Suppose that $(\mathsf{M},\tilde{g})$ is a* uniformly regular Riemannian manifold *without boundary. Let $g_0 = u_0^{\frac{4}{m-2}}\tilde{g}$. For each*

$$u_0 \in U^s := \{u \in bc^s(\mathsf{M}) : \inf u > 0\}, \quad s \in (0,1),$$

the Yamabe flow (2.2) *has a unique local solution g on some interval $J(u_0) := [0, T(u_0))$. Moreover,*

$$g \in C^\infty((0,T(u_0)) \times \mathsf{M}, T^0_2\mathsf{M}).$$

In particular , if $\tilde{g}$ is real analytic, then

$$g \in C^\omega((0,T(u_0)) \times \mathsf{M}, T^0_2\mathsf{M}).$$

Here ω is the symbol for real analyticity. Similar results for singular manifolds were proved in [57]. For any closed manifold M, by [43, Theorem 1.4, Corollary 1.5], in every conformal class containing a C^k-metric with $k \in \{\infty, \omega\}$ there exists a C^k-metric $\tilde{g}$ making $(\mathsf{M},\tilde{g})$ *uniformly regular*. Therefore the solution obtained in the above theorem in every conformal class containing at least one real analytic metric is analytic.

Not surprisingly, for the normalized Yamabe flow (2.1), analogous results hold. In particular, if $\tilde{g}$ is real analytic, then the equilibria in $[\tilde{g}]$, that is to say, those of constant scalar curvature, must be analytic. This implies that solutions to the Yamabe problem in any conformal class possessing at least one real analytic metric are analytic, see [60].

In virtue of the DeTurck's trick [25], in [56], temporal analyticity of the Ricci flow is also obtained formulated by means of a fixed atlas. See [37] for a related result. I shall point out that, with an arbitrary C^ω-atlas, spatial analyticity of solutions to the Ricci flow with smooth initial metric in general cannot be true. See [56, Section 1] for a justification. This contrasts with S. Bando's result in [10].

By using the same technique for the Ricci flow with minor modifications, to prove analyticity of the renormalization group flow seems to be another promising project.

3.2. **Regularity of the free boundary in phase transitions and two-phase fluids problems.** A well-known approach to free boundary problems is to transform the original problems with a moving boundary, or separating interface, which we denote by $\Gamma(t)$, into one with a fixed reference manifold Σ by means of the Hanzawa transformation, see [34]. Then the problem of establishing the regularity of the free boundary $\Gamma(t)$ is transferred into that of the height function parameterizing $\Gamma(t)$ over Σ. But in view of the physical quantities in the bulk phases, e.g., the temperature function in the case of phase transition, or the velocity of the flow in the case of two-phase fluids, adjacent to $\Gamma(t)$, we need introduce a localized translation not only on the fixed reference surface Σ, but also in a neighborhood of Σ. This adds one more degree of complexity to the aforementioned technique for geometric flows. In [54], we have built up a complete theory of parameter-dependent diffeomorphisms for the free boundary problem arising in two-phase transition problems. Of course, this theory applies without difficulty to one-phase problems. In [54], we study the regularity of the free boundary to a thermodynamically consistent two-phase Stefan problem with surface tension.

Suppose that $\Omega \subset \mathbb{R}^{m+1}$ is occupied by a material that can undergo phase changes: at time t, phase i occupies the subdomain $\Omega_i(t)$ of Ω, respectively, with $i = 1, 2$. We assume that $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$. The *Stefan problem with surface tension* consists in finding a family of closed compact hypersurfaces $\{\Gamma(t)\}_{t\geq 0} := \partial\Omega_1(t)$ and an appropriately smooth function $\theta : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \kappa(\theta)\partial_t\theta - \operatorname{div}(d(\theta)\nabla\theta) = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ \partial_{\nu_\Omega}\theta = 0 & \text{on} \quad \partial\Omega \\ [\![\theta]\!] = 0 & \text{on} \quad \Gamma(t) \\ [\![\psi(\theta)]\!] + \sigma\mathcal{H} = \gamma(\theta)V & \text{on} \quad \Gamma(t) \\ [\![d(\theta)\partial_\nu\theta]\!] = (l(\theta) - \gamma(\theta)V)V & \text{on} \quad \Gamma(t) \\ \theta(0) = \theta_0 & \\ \Gamma(0) = \Gamma_0. & \end{cases} \tag{3.1}$$

Here $\theta(t)$ denotes the (absolute) temperature, $\nu(t)$ the outer normal field of $\Omega_1(t)$, $V(t)$ the normal velocity of $\Gamma(t)$, $\mathcal{H}(t) = \mathcal{H}_{\Gamma(t)}$ the mean curvature of $\Gamma(t)$, and $[\![v]\!] = v_2|_{\Gamma(t)} - v_1|_{\Gamma(t)}$ the jump of a quantity v across $\Gamma(t)$. Several quantities are derived from the free energies $\psi_i(\theta)$: $\kappa_i(\theta) > 0$ the heat capacity; and $l(\theta)$ the latent heat. Furthermore, $d_i(\theta) > 0$ denotes the coefficient of heat conduction in Fourier's law, $\gamma(\theta) \geq 0$ is the coefficient of kinetic undercooling, and $\sigma > 0$ is the coefficient of surface tension. It is understood that if a quantity ϵ is defined in both phase then $\epsilon := \epsilon_1\chi_{\Omega_1(t)} + \epsilon_2\chi_{\Omega_2(t)}$.

We have established in [54] the regularity of the free boundary $\Gamma(t)$: for $k \in \mathbb{N} \cup \{\infty, \omega\}$, if $\psi \in C^{k+3}(0, \infty)$, $d \in C^{k+1}(0, \infty)$, and $\gamma \in C^{k+2}(0, \infty)$, then under mild initial regularity conditions, in the interior of the maximal interval of existence $J = [0, T)$,

$$\mathcal{M} := \bigcup_{t\in(0,T)} \{\{t\} \times \Gamma(t)\}$$

is a C^k-manifold in $\mathbb{R}^{m+2}$. In particular, each manifold $\Gamma(t)$ is C^k for $t \in (0, T)$.

One of the best known models in describing the motion of viscous fluids is the Navier-Stokes system. I am interested in proving regularity of the free boundary arising in the two-phase Navier-Stokes systems with surface tension. More precisely, $\Omega_i(t)$, $i = 1, 2$, are occupied by two incompressible, immiscible, viscous fluids. All notations have the same meaning as in (3.1). The motion of the fluids is governed by the following law:

$$\begin{cases} \rho(\partial_t u + (u|\nabla)u) - \mu\Delta u + \nabla q = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ \operatorname{div} u = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ [\![u]\!] = 0 & \text{on} \quad \Gamma(t) \\ -[\![S(u, q)\nu]\!] = \sigma\mathcal{H}\nu & \text{on} \quad \Gamma(t) \\ V = (u|\nu) & \text{on} \quad \Gamma(t) \\ u(0) = u_0 & \\ \Gamma(0) = \Gamma_0. & \end{cases} \tag{3.2}$$

Here $(\cdot|\cdot)$ is the inner product in $\mathbb{R}^{m+1}$. The unknowns of (3.2) are the velocity field $u(t,\cdot):\Omega(t)\to\mathbb{R}^{m+1}$, the pressure field $q(t,\cdot):\Omega(t)\to\mathbb{R}$, and the free boundary $\Gamma(t)$. The constants ρ_i and μ_i denote the densities and viscosities of the corresponding fluids. $S(u,q)$ is the stress tensor.

Conjecture 3.3. *Let $p>m+3$. Suppose that $u_0\in W_p^{2-2/p}(\Omega_0,\mathbb{R}^{m+1})$ and the initial hypersurface Γ_0 belongs to the class $W_p^{3-2/p}$. In addition, the initial data (u_0,Γ_0) satisfies an appropriate compatibility condition. Then (3.2) has a unique solution on $J=[0,T)$. Moreover,*

$$\mathcal{M}:=\bigcup_{t\in(0,T)}\{\{t\}\times\Gamma(t)\}$$

is a real analytic manifold in $\mathbb{R}^{m+2}$.

4. Summary of main results and future projects

For the reader's convenience, a list of my my main contributions obtained in published/submitted papers, and some future projects, mentioned in earlier sections, is given below.

4.1. Main results.

- Continuous maximal regularity theory on *uniformly regular Riemannian manifolds* and *singular manifolds* [57, 60];
- generation of analytic semigroup for second order degenerate/singular elliptic operators on singular manifolds in L_p-framework with a precise characterization of domains for these operators [58];
- local well-posedness and uniqueness theorem for the Yamabe flow on *singular manifolds* (the solution may start with an initial metric with unbounded curvature) [57];
- local well-posedness and uniqueness theorem for the porous medium equation and the evolutionary p-Laplacian equation on *singular manifolds* [57];
- local existence and uniqueness of solutions to a generalized thin film equations, including a discussion of the waiting-time phenomena [57];
- a regularization technique to prove regularity of solutions to geometric flows and free boundary problems [54, 56];
- regularity of the averaged/unaveraged mean curvature flow [56];
- regularity of the Ricci flow [56];
- regularity of the surface diffusion flow [56];
- regularity of the Willmore flow [55];
- regularity of the normalized/unnormalized Yamabe flow [60];
- analyticity of the Yamabe problem [60];
- regularity of the free boundary arising in a thermodynamically consistent two-phase Stefan problem with surface tension [54];
- the equivalence of the class of *uniformly regular Riemannian manifolds* without boundary to the class of complete manifolds with bounded geometry [53].

4.2. **Future projects.**

- local existence and uniqueness of the Yamabe flow on *singular manifolds* with boundary (with inhomogeneous boundary condition) (long term project);
- global existence of the Yamabe flow on *singular manifolds* (long term project);
- well-posedness of geometric flows, e.g., Ricci-DeTurck flow, with initial metric of unbounded curvature on *singular manifolds*, similar to the pheonomenon of the Yamabe flow (long term project);
- existence, uniqueness and regularity of solutions to degenerate or singular diffusion equation on *singular manifolds*, for instance, the porous medium equations, with more general initial data, especially with non-empty vanishing set;
- analyticity of the solutions to the renormalization group flow;
- regularity of the free boundary arising in the two-phase Navier-Stokes equations with surface tension.

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