SOME REMARKS ON UNIFORMLY REGULAR RIEMANNIAN MANIFOLDS

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Abstract. We establish the equivalence between the family of closed uniformly regular Riemannian manifolds and the class of complete manifolds with bounded geometry.

1. Introduction

In 2012, H. Amann introduced a class of (possibly noncompact) manifolds, called *uniformly regular Riemannian manifolds*. Roughly speaking, an m-dimensional Riemannian manifold (M, g) is *uniformly regular* if its differentiable structure is induced by an atlas such that all its local patches are of approximately the same size, all derivatives of the transition maps are bounded, and the pull-back metric of g in every local coordinate is comparable to the Euclidean metric g_m. The precise definition of *uniformly regular Riemannian manifolds* will be presented in Section 2 below.

In this article, the concept of closed manifolds refers to manifolds without boundary, not necessarily compact. The main objective of this short note is to prove that the family of closed *uniformly regular Riemannian manifolds* coincides with the class of complete manifolds with bounded geometry in the following sense. A closed *uniformly regular Riemannian manifold* is geodesically complete, has positive injectivity radius, and all covariant derivatives of the curvature tensor are bounded. The second and third conditions are usually referred to as of bounded geometry. The precise definitions of positive injectivity radius and bounded geometry will be given later in this introductory section.

Nowadays, there is rising interest in studying differential equations on non-compact manifolds, see [18, 20, 21, 29, 30, 32, 33] for instance. It is a well-known fact that many well established analytic tools in Euclidean space fail, in general, on Riemannian manifolds. For instance, for $u \in C^2(\mathbb{R}^m)$ with $u^* := \sup u < \infty$, we can always find a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that

$$u(x_k) > u^* - 1/k, \quad |\nabla u(x_k)| < 1/k, \quad \Delta u(x_k) < 1/k.$$

Nevertheless, it is known that this maximum principle does not always hold true on non-compact Riemannian manifolds. Indeed, there are counterexamples provided by H. Omori [23].

A corner stone in the study of differential equations is the theory of function spaces. In order to study this theory on Riemannian manifolds, it is natural to impose extra geometric conditions, most likely certain restrictions on the curvatures. Among all efforts made to find proper assumptions, one extensively studied category is the class of complete manifolds with bounded geometry. A closed manifold (M, g) is (geodesically) complete if all geodesics are infinitely extendible with respect to arc length. The Hopf-Rinow theorem states that this is equivalent to asserting that M is complete as a metric space with respect to the intrinsic metric induced by g. A closed manifold (M, g) is said to

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have a positive injectivity radius if there exists a positive number $\iota(\mathsf{M})$ such that the exponential map at $\mathsf{p} \in \mathsf{M}$, exp_{p}, is a diffeomorphism from

$$\mathbb{B}^m(0, \iota(\mathsf{M})) := \{\xi \in \mathbb{R}^m : |\xi|_{g_m} < \iota(\mathsf{M})\} \quad \text{onto} \quad O_{\mathsf{p}}(\iota(\mathsf{M})) := exp_{\mathsf{p}}(\mathbb{B}^m(0, \iota(\mathsf{M})))$$

for all $\mathsf{p} \in \mathsf{M}$. A manifold (M, g) is of bounded geometry if it has a positive injectivity radius and all covariant derivatives of the curvature tensor are bounded, i.e.,

$$\| |\nabla_g^k R|_g \|_\infty \le C(k), \quad k \in \mathbb{N}_0, \tag{1.1}$$

where R is the Riemannian curvature tensor, and ∇_g denotes the extension of the Levi-Civita connection over $\mathcal{T}^\sigma_\tau \mathsf{M}$ with $\sigma, \tau \in \mathbb{N}_0$. Let $T\mathsf{M}$ and $T^*\mathsf{M}$ be the tangent and the cotangent bundle, respectively. Then $\mathcal{T}^\sigma_\tau \mathsf{M}$ is the $C^\infty(\mathsf{M})$-module of all smooth sections of $T^\sigma_\tau \mathsf{M} := T\mathsf{M}^{\otimes \sigma} \otimes T^*\mathsf{M}^{\otimes \tau}$, the (σ, τ)-tensor bundles. $|\cdot|_g$ is the (vector bundle) norm induced by the extension of the Riemannian metric g from the tangent bundle to $T^\sigma_\tau \mathsf{M}$. Condition (1.1) can be replaced by the boundedness of all covariant derivatives of the sectional curvature. See [9, Section 5.14] for a justification. This condition can also be formulated equivalently by asking that in all normal geodesic coordinates of every local chart $(O_{\mathsf{p}}(\iota(\mathsf{M})), exp_{\mathsf{p}}^{-1})$, we have

$$\det[(g_{ij})_{ij}] \ge c, \quad \max_{|\alpha| \le k} \|\partial^\alpha g_{ij}\|_\infty \le C(k), \quad k \in \mathbb{N}_0,$$

where $(g_{ij})_{ij}$ is the local matrix expression of the metric tensor g, for some constants $C(k), c > 0$. See [31, Section 7.2.1]. The amount of literature on differential equations on complete manifolds with bounded geometry is vast. Most of the work concerns heat kernel estimates and spectral theory. See, for example, [14, 17] and the references therein. With additional restrictions like nonnegative Ricci curvature, L_q-L_p maximal regularity theory is established for second order elliptic operators. See [18, 21].

To illustrate some of our recent results for differential equations on *uniformly regular Riemannian manifolds*, we look at linear differential operators $\mathcal{A} : \mathcal{T}^\sigma_\tau \mathsf{M} \to \Gamma(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ of order l acting on (σ, τ)-tensor fields, defined by

$$\mathcal{A} = \mathcal{A}(\mathfrak{a}) := \sum_{r=0}^{l} a_r \bullet (\nabla_g^r \cdot),$$

where $a_r \in \Gamma(\mathsf{M}, T^{\sigma+\tau+r}_{\tau+\sigma} \mathsf{M})$, the set of all sections of $T^{\sigma+\tau+r}_{\tau+\sigma} \mathsf{M}$, and $\bullet$ denotes complete contraction. See [28, Section 2] for a detailed discussion. We consider the following initial value problem on a closed *uniformly regular Riemannian manifold* (M, g) in Hölder spaces:

$$\begin{cases} \partial_t u + \mathcal{A} u = f & \text{on} \quad \mathsf{M}_T; \\ \quad u(0) = u_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{1.2}$$

Here $\mathsf{M}_T := \mathsf{M} \times (0, T)$ for $T \in (0, \infty]$.

For $k \in \mathbb{N}_0$ and $\sigma, \tau \in \mathbb{N}_0$, we define $\|u\|_{k,\infty} := \max_{0 \le i \le k} \| |\nabla_g^i u|_g \|_\infty$ and

$$BC^k(\mathsf{M}, T^\sigma_\tau \mathsf{M}) := (\{u \in C^k(\mathsf{M}, T^\sigma_\tau \mathsf{M}) : \|u\|_{k,\infty} < \infty\}, \|\cdot\|_{k,\infty}).$$

We also set $BC^\infty(\mathsf{M}, T^\sigma_\tau \mathsf{M}) := \bigcap_k BC^k(\mathsf{M}, T^\sigma_\tau \mathsf{M})$, endowed with the conventional projective topology. Then

$$bc^k(\mathsf{M}, T^\sigma_\tau \mathsf{M}) := \text{the closure of } BC^\infty(\mathsf{M}, T^\sigma_\tau \mathsf{M}) \text{ in } BC^k(\mathsf{M}, T^\sigma_\tau \mathsf{M}).$$

Let $k < s < k + 1$. Now the *little Hölder* space $bc^s(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ is defined by

$$bc^s(\mathsf{M}, T^\sigma_\tau \mathsf{M}) := (bc^k(\mathsf{M}, T^\sigma_\tau \mathsf{M}), bc^{k+1}(\mathsf{M}, T^\sigma_\tau \mathsf{M}))^0_{s-k,\infty}.$$

Here $(\cdot, \cdot)^0_{\theta,\infty}$ is the continuous interpolation method, see [2, Example I.2.4.4]. In [3, 4], the theory of function spaces, including the *little Hölder* space, is studied.

A linear differential operator $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ is said to be *normally elliptic* if there exists some constant $\mathcal{C}_{\mathfrak{e}} > 0$ such that for every pair $(\mathsf{p}, \xi) \in \mathsf{M} \times \Gamma(\mathsf{M}, T^*M)$ with $|\xi(\mathsf{p})|_{g^*(\mathsf{p})} = 1$ for all $\mathsf{p} \in \mathsf{M}$, the principal symbol of $\mathcal{A}$ defined by

$$\hat{\sigma}\mathcal{A}^{\pi}(\mathsf{p}, \xi(\mathsf{p})) := (a_l \bullet (-i\xi)^{\otimes l})(\mathsf{p}) \in \mathcal{L}(T_{\mathsf{p}}\mathsf{M}^{\otimes\sigma} \otimes T^*_{\mathsf{p}}\mathsf{M}^{\otimes\tau})$$

satisfies $S := \Sigma_{\pi/2} := \{z \in \mathbb{C} : |\arg z| \le \pi/2\} \cup \{0\} \subset \rho(-\hat{\sigma}\mathcal{A}^{\pi}(\mathsf{p}, \xi(\mathsf{p})))$, and

$$(1 + |\mu|)\|(\mu + \hat{\sigma}\mathcal{A}^{\pi}(\mathsf{p}, \xi(\mathsf{p})))^{-1}\|_{\mathcal{L}(T_p\mathsf{M}^{\otimes\sigma} \otimes T_p^*\mathsf{M}^{\otimes\tau})} \le \mathcal{C}_{\mathfrak{e}}, \quad \mu \in S.$$

In the above, g^* is the contravariant metric induced by g. We readily check that a normally elliptic operator must be of even order. $\mathcal{A}$ is called *s-regular* if

$$a_r \in bc^s(\mathsf{M}, T^{\sigma+\tau+r}_{\tau+\sigma}\mathsf{M}), \quad r = 0, 1, \cdots, l.$$

The following continuous maximal regularity theorem has been established by two of the authors.

Theorem 1.1 (Y. Shao, G. Simonett [28])**.** *Let (M, g) be a closed uniformly regular Riemannian manifold, $\gamma \in (0, 1]$, and $s \notin \mathbb{N}_0$. Suppose that $\mathcal{A}$ is a $2l$-th order normally elliptic and s-regular differential operator acting on (σ, τ)-tensor fields. Then for any*

$$(f, u_0) \in C_{1-\gamma}([0, \infty); bc^s(\mathsf{M}, T^\sigma_\tau\mathsf{M})) \times bc^{s+2l\gamma}(\mathsf{M}, T^\sigma_\tau\mathsf{M}),$$

equation (1.2) *has a unique solution*

$$u \in C_{1-\gamma}([0, \infty); bc^{s+2l}(\mathsf{M}, T^\sigma_\tau\mathsf{M})) \cap C^1_{1-\gamma}([0, \infty); bc^s(\mathsf{M}, T^\sigma_\tau\mathsf{M})).$$

Equivalently, $\mathcal{A}$ generates an analytic semigroup on $bc^s(\mathsf{M}, T^\sigma_\tau\mathsf{M})$ and has the property of continuous maximal regularity.

Here $C^k_{1-\gamma}([0, \infty); X)$ is some weighted $C^k((0, \infty); X)$-space for a given Banach space X, see [28, Section 3] for a precise definition.

One may observe from the statement of Theorem 1.1 that no additional geometric assumption is needed. So it generalizes the existing results on complete manifolds with bounded geometry, see [14, 17, 31]. By means of results of G. Da Prato, P. Grisvard [13], S. Angenent [7] and P. Clément, G. Simonett [12], Theorem 1.1 gives rise to existence and uniqueness of solutions to many quasilinear or even fully nonlinear differential equations on *uniformly regular Riemannian manifold*. See [27, 28] for example.

A similar result to Theorem 1.1 in an L_p-framework can be found in [5] for second order initial boundary value problems.

In [6, Theorem 4.1], it is shown that a complete manifold with bounded geometry is *uniformly regular*. We aim at establishing the other inclusion, i.e., a *uniformly regular Riemannian manifold* is complete and of bounded geometry. In Section 3, we prove the following theorem.

Theorem 1.2. *Suppose that (M, g) is a closed uniformly regular Riemannian manifold. Then*

(a) *(M, g) is complete;*

(b) *all covariant derivatives of the curvature tensor are bounded, i.e.,*

$$\||\nabla^k_g R|_g\|_\infty \le C(k), \quad k \in \mathbb{N}_0,$$

where R is the Riemannian curvature tensor;

(c) *(M, g) has positive injectivity radius.*

Remark 1.3. The concept of *uniformly regular Riemannian manifolds* with boundary can be defined in a similar way to Section 2 below, see [3, 4]. The proof of Theorem 1.2(a)-(b) in Section 3 still holds true for manifolds with boundary.

However, the concept of positive injectivity radius for manifolds with boundary needs to be defined separately for $\mathsf{p} \in \mathring{\mathsf{M}}$ and $\mathsf{p} \in \partial\mathsf{M}$. See [24] for precise definitions of interior and boundary injectivity radius. The idea of the proof for Theorem 1.2(c) can still be applied to *uniformly regular Riemannian manifolds* in the case of non-empty boundary with necessary modifications. But, in this case, these manifolds are no longer geodesically complete, but only complete as a metric space. Therefore, roughly speaking, *uniformly regular Riemannian manifolds* with boundary are complete as a metric space, with positive interior and boundary injectivity radius, and have bounded derivatives of the curvature tensor.

To avoid too much technicality, we will focus on the class of closed *uniformly regular Riemannian manifolds* in this article. Interested readers may make use of the technique in the proof of Theorem 1.2(c) for more general situations.

This article is organized as follows. In Section 2, we give the precise definition of closed *uniformly regular Riemannian manifolds* and present several examples of this class. This concept can be extended to manifolds with boundary, see [3, 4]. In Section 3, we provide a proof for Theorem 1.2.

2. Uniformly regular Riemannian manifolds

Let (M, g) be a closed C^∞-Riemannian manifold of dimension m endowed with g as its Riemannian metric such that its underlying topological space is separable. An atlas $\mathfrak{A} := (\mathsf{O}_\kappa, \varphi_\kappa)_{\kappa\in\mathfrak{K}}$ for M is said to be normalized if $\varphi_\kappa(\mathsf{O}_\kappa) = \mathbb{B}^m$. Here $\mathbb{B}^m$ is the unit ball centered at the origin in $\mathbb{R}^m$. We put $\psi_\kappa := \varphi_\kappa^{-1}$.

The atlas $\mathfrak{A}$ is said to have *finite multiplicity* if there exists $N \in \mathbb{N}$ such that any intersection of more than N coordinate patches is empty. Put

$$\mathfrak{N}(\kappa) := \{\tilde{\kappa} \in \mathfrak{K} : \mathsf{O}_{\tilde{\kappa}} \cap \mathsf{O}_\kappa \neq \emptyset\}.$$

The *finite multiplicity* of $\mathfrak{A}$ and the separability of M imply that $\mathfrak{A}$ is countable.

An atlas $\mathfrak{A}$ is said to fulfil the *uniformly shrinkable* condition, if it is normalized and there exists $r \in (0,1)$ such that $\{\psi_\kappa(r\mathbb{B}^m) : \kappa \in \mathfrak{K}\}$ is a cover for M.

Following H. Amann [3, 4], we say that (M, g) is a *uniformly regular Riemannian manifold* if it admits an atlas $\mathfrak{A}$ such that

- (R1) $\mathfrak{A}$ is uniformly shrinkable and has finite multiplicity. If M is oriented, then $\mathfrak{A}$ is orientation preserving.
- (R2) $\|\varphi_\eta \circ \psi_\kappa\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$, $\eta \in \mathfrak{N}(\kappa)$, and $k \in \mathbb{N}_0$.
- (R3) $\psi_\kappa^* g \sim g_m$, $\kappa \in \mathfrak{K}$. Here g_m denotes the Euclidean metric on $\mathbb{R}^m$ and $\psi_\kappa^* g$ denotes the pull-back metric of g by ψ_κ.
- (R4) $\|\psi_\kappa^* g\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$.

Here $\|u\|_{k,\infty} := \max_{|\alpha|\le k} \|\partial^\alpha u\|_\infty$, and it is understood that a constant $c(k)$, like in (R2), depends only on k. An atlas $\mathfrak{A}$ satisfying (R1) and (R2) is called a *uniformly regular atlas.* (R3) reads as

$$|\xi|^2/c \le \psi_\kappa^* g(x)(\xi,\xi) \le c|\xi|^2, \quad \text{for any } x \in \mathbb{B}^m,\ \xi \in \mathbb{R}^m,\ \kappa \in \mathfrak{K} \text{ and some } c \ge 1. \tag{2.1}$$

In the following, we will present several examples of *uniformly regular Riemannian manifolds.*

Example 2.1.

(a) *Suppose that* (M, g) *and* $(\tilde{\mathsf{M}}, \tilde{g})$ *are both uniformly regular Riemannian manifolds. Then so is* $(\mathsf{M} \times \tilde{\mathsf{M}}, g + \tilde{g})$.

(b) *Let $f : \tilde{\mathsf{M}} \to \mathsf{M}$ be a diffeomorphism of manifolds. If (M, g) is a uniformly regular Riemannian manifold, then so is $(\tilde{\mathsf{M}}, f^*g)$.*

(c) $(\mathbb{R}^m, g_m)$ *is uniformly regular.*

(d) *Every compact closed manifold is uniformly regular.*

(e) *Every complete manifold with bounded geometry is uniformly regular.*

(f) *Let $J := (1, \infty)$, and $R_\alpha(t) : J \to J$: $t \mapsto t^\alpha$ for $\alpha \in [0, 1]$. Assume that (B, g_B) is a d-dimensional compact closed submanifold of $\mathbb{R}^n$. The (model) (R_α, B)-funnel $F(R_\alpha, B)$ on J is defined by*

$$F(R_\alpha, B) = F(R_\alpha, B; J) := \{(t, R_\alpha(t)y) : t \in J,\, y \in B\} \subset \mathbb{R}^{1+n}.$$

It is a $(1 + d)$-dimensional submanifold of $\mathbb{R}^{1+n}$. The map

$$\phi : F \to J \times B : \quad (t, R_\alpha(t)y) \to (t, y)$$

is a diffeomorphism. Suppose that $\{V_0, V_1\}$ is an open covering of (M, g) such that (V_1, g) is isometric to some (R_α, B)-funnel, $(F(R_\alpha, B), \phi^*(dt^2 + g_B))$, and V_0, $V_0 \cap V_1$ are relatively compact in M. Then (M, g) is uniformly regular.

(g) *(M, g) is a stretched corner manifold, that is, $\{V_0, V_1\}$ is an open covering of (M, g) with V_0, $V_0 \cap V_1$ relatively compact in M, and (V_1, g) is isometric to $(C(B), (dt/t)^2 + g_B + (ds/(ts)^2)$. Here (B, g_B) is an d-dimensional compact closed submanifold embedded in the unit sphere of $\mathbb{R}^{d+1}$. Both dt^2 and ds^2 denote the standard metric of $(0, 1]$. The stretched (model) corner end $C(B)$ is defined by*

$$C(B) := \{(ts, tsy, s) : (t, y, s) \in (0, 1] \times B \times (0, 1]\} \subset \mathbb{R}^{d+2}.$$

Then (M, g) is uniformly regular.

(h) *The open unit ball $\mathbb{B}^m$ in the Euclidean space $\mathbb{R}^m$ equipped with the Poincaré disk metric, that is,*

$$(\mathbb{B}^m, 4dx^2/(1 - |x|^2)^2),$$

is uniformly regular.

(i) *(M, g) is a "$\mathfrak{b}$-Riemannian manifold". To be more precise, let B be a compact manifold with boundary ∂B. We extend ∂B by a stretched conic end, that is, $\mathsf{M} := B \cup X$, where X is diffeomorphic to $(0, 1] \times \partial B$ and equipped with the metric $g = (dt/t)^2 + g_{\partial B}$. This metric g is called an exact b-metric. Then (M, g) is uniformly regular.*

With the help of the first two examples, we can construct more complicated *uniformly regular Riemannian manifolds* based on Example 2.1(c)-(i).

In order to prove some of the statements in Examples 2.1, it is convenient to introduce the concept of *singular manifolds*. Roughly speaking, a manifold $(M, \hat{g})$ is called a *singular manifold* in the sense defined in [3, Section 2], if there exists some $\rho \in C^\infty(\mathsf{M}, (0, \infty))$ such that $(\mathsf{M}, \hat{g}/\rho^2)$ is *uniformly regular*. The function ρ is called a *singularity datum* of $(\mathsf{M}, \hat{g})$. A *singular manifold* is *uniformly regular* if $\rho \sim \mathbf{1}_{\mathsf{M}}$. If two real-valued functions f and g are equivalent in the sense that $f/c \leq g \leq cf$ for some $c \geq 1$, then we write $f \sim g$.

Remarks 2.2.

(i) The concept of stretched corner manifolds is used in [11]. Note that the conventional corner manifolds, see [26], are not *uniformly regular*, but indeed are *singular manifolds.*

(ii) The concept of "b-Riemannian manifolds" was introduced by R.B. Melrose, see [22, Chapter 2] for a detailed discussion. Example 2.1(f) implies that Theorem 1.1 on *uniformly regular Riemannian manifolds* can be considered as some extension of the theory of b-calculus.

Proof of Example 2.1.

(a) Example 2.1(a) is a special case of [6, Theorem 3.1].

(b) [6, Lemma 3.4] implies Example 2.1(b).

(c) Example 2.1(c) follows from [6, formula (3.3)].

(d) See [6, Corollary 4.3].

(e) See [6, Theorem 4.1], and also [8, Lemma 2.2.6] and [15].

(f) See [6, Theorem 1.2].

(g) [6, Example 5.1, Lemmas 5.2 and 6.1] imply that

$$(C(B), s^2dt^2 + (ts)^2 g_B + ds^2)$$

is a *singular manifold* with *singulartiy datum* $R(ts, tsy, s) = ts$. Then the assertion follows from [6, Lemma 3.3].

(h) We put $J_1 := [0, 1)$. By [6, Lemma 5.2], we have $((0, 1]; dt^2)$ is a *singular manifold* with singular datum $R_2(t) := t^2$. [6, Lemma 3.4] implies that $(J_1; dt^2)$ is a *singular manifold* with singular datum $1 - R_2(t)$. The unit sphere $\mathbb{S}^{m-1}$ is *uniformly regular* by Example 2.1(d). Then by [6, Theorems 3.1, 8.1 and Lemma 3.4],

$$(\mathbb{B}^m, dt^2 + (1 - R_2(t))^2 g_S)$$

is a *singular manifold* with *singulartiy datum* $1 - R_2(t)$, where g_S is the metric on $\mathbb{S}^{m-1}$ induced by g_m. It is easy to see that $(\mathbb{B}^m, dt^2 + (1 - R_2(t))^2 g_S)$ is diffeomorphic to $(\mathbb{B}^m, g_m)$. Now the statment of Example 2.1(h) follows from the definition of *singular manifolds* and [6, Lemma 3.4].

(i) To see that Example 2.1(i) holds true, by [6, Lemma 3.3] we only need to show that $(X, (dt/t)^2 + g_{\partial B})$ is *uniformly regular*. [6, Example 5.1, Lemmas 5.2 and 6.1] imply that $(X, dt^2 + t^2 g_{\partial B})$ is a *singular manifold* with *singularity datum* $R(t) = t$. By definition, $(X, (dt/t)^2 + g_{\partial B})$ is *uniformly regular*.

$\square$

3. Proof of the main theorem

Proof of Theorem 1.2.

(a) By the Hopf-Rinow Theorem, it suffices to show that (M, g) is complete as a metric space in the intrinsic metric induced by g. The *uniformly shrinkable* condition implies that there exists a number $r \in (0, 1)$ such that $(\widetilde{\mathsf{O}}_\kappa)_\kappa := (\psi_\kappa(r\mathbb{B}^m))_\kappa$ still forms an open cover of M.

We claim that there exists some constant $c \geq 1$ such that

$$\mathbb{B}_g(\psi_\kappa(x_\kappa), \delta/c) \subset \psi_\kappa(\mathbb{B}^m(x_\kappa, \delta)) \subset \mathbb{B}_g(\psi_\kappa(x_\kappa), c\delta) \tag{3.1}$$

for any $\kappa \in \mathfrak{K}$, $x_\kappa \in r\mathbb{B}^m$, and $\delta < 1 - r$. Here $\mathbb{B}_g(\mathsf{p}, \mathrm{R})$ denotes all the points on M with distance less than R to p with respect to the intrinsic metric. Indeed, put $\mathsf{p}_\kappa := \psi_\kappa(x_\kappa)$, and let $d_g(\cdot, \cdot)$ denote the distance function, that is,

$$d_g(\mathsf{p}, \mathsf{q}) := \inf\{L(\gamma) : \gamma : [0, 1] \to \mathsf{M} \text{ piecewise } C^\infty\text{-curve with } \gamma(0) = \mathsf{p},\ \gamma(1) = \mathsf{q}\},$$

where $L(\gamma)$ is the length of γ with respect to the intrinsic metric induced by g, see [19, p. 15]. For any $\mathsf{p} \notin \psi_\kappa(\mathbb{B}^m(x_\kappa,\delta))$ and on any piece-wise smooth curve $\gamma : [0,1] \to \mathsf{M}$ connecting p_κ and p, i.e., $\gamma(0) = \mathsf{p}_\kappa$ and $\gamma(1) = \mathsf{p}$, we take t_γ to be the first escape time of γ out of $\psi_\kappa(\mathbb{B}^m(x_\kappa,\delta))$. Then letting $\mathsf{p}_\gamma := \gamma(t_\gamma)$, one can compute by means of (R3)

$$\begin{aligned} d_g(\mathsf{p},\mathsf{p}_\kappa) &\ge \inf_\gamma \int_0^{t_\gamma} \sqrt{g(\gamma'(t),\gamma'(t))}\,dt \\ &= \inf_\gamma \int_0^{t_\gamma} \sqrt{\psi_\kappa^* g(\psi_\kappa^*\gamma'(t),\psi_\kappa^*\gamma'(t))}\,dt \\ &\ge \inf_\gamma \frac{1}{c}\int_0^{t_\gamma} \sqrt{g_m(\psi_\kappa^*\gamma'(t),\psi_\kappa^*\gamma'(t))}\,dt = \frac{1}{c}|\varphi_\kappa(\mathsf{p}_\gamma) - x_\kappa| = \delta/c. \end{aligned}$$

Here the constant c is the same as in (2.1), and $\gamma : [0,1] \to \mathsf{M}$ runs over all piecewise smooth curves connecting p_κ and p. This proves the first part of inequality (3.1). The second part of (3.1) follows in a similar manner.

Assume $(\mathsf{p}_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in the intrinsic metric. Then for any $\varepsilon \le \delta/c$, there exists some $N_\varepsilon \in \mathbb{N}$ such that $\mathsf{p}_l \in \mathbb{B}_g(\mathsf{p}_n,\varepsilon)$ for all $n, l \ge N_\varepsilon$. Since $(\widetilde{\mathsf{O}}_\kappa)_\kappa$ covers M, for any $n \ge N_\varepsilon$, we can find some $\kappa = \kappa(n)$ with $\mathsf{p}_n \in \widetilde{\mathsf{O}}_\kappa$. Then for any $l \ge N_\varepsilon$, we have

$$\mathsf{p}_l \in \mathbb{B}_g(\mathsf{p}_n,\varepsilon) \subset \mathbb{B}_g(\mathsf{p}_n,\delta/c) \subset \psi_\kappa(\mathbb{B}^m(\varphi_\kappa(\mathsf{p}_n),\delta)) \subset \mathsf{O}_\kappa.$$

Since the maps φ_κ and ψ_κ are continuous between $(\mathbb{B}^m, g_m)$ and (O_κ, d_g), it is easy to see that we actually have

$$\psi_\kappa(\mathbb{B}^m(\varphi_\kappa(\mathsf{p}_n),\delta)) \subset\subset \mathsf{O}_\kappa.$$

Hence there exists a convergent subsequence $\mathsf{p}_{l_k} \to \mathsf{p}$ for some $\mathsf{p} \in \mathsf{O}_\kappa$. Since $(\mathsf{p}_l)_{l\in\mathbb{N}}$ is Cauchy, this implies that $\mathsf{p}_l \to \mathsf{p}$ in the intrinsic metric.

(b) To prove part (b), we need to first introduce some notations. Let $\sigma, \tau \in \mathbb{N}_0$. For abbreviation, we set $\mathbb{J}^\sigma := \{1, 2, \ldots, m\}^\sigma$, and $\mathbb{J}^\tau$ is defined alike. Given local coordinates $\varphi = \{x^1, \ldots, x^m\}$, $(i) := (i_1, \ldots, i_\sigma) \in \mathbb{J}^\sigma$ and $(j) := (j_1, \ldots, j_\tau) \in \mathbb{J}^\tau$, we set

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad \partial^{(i)} := \partial^{i_1} \circ \cdots \circ \partial^{i_\sigma}, \quad dx^{(j)} := dx^{j_1} \otimes \cdots \otimes dx^{j_\tau}$$

with $\partial^i = \frac{\partial}{\partial x^i}$. The local representation of $a \in \Gamma(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)}$$

with coefficients $a^{(i)}_{(j)}$ defined on O_κ. We can also extend the Riemannian metric $(\cdot|\cdot)_g$ from the tangent bundle to any (σ,τ)-tensor bundle $T^\sigma_\tau \mathsf{M}$ such that $(\cdot|\cdot)_g := (\cdot|\cdot)_{g^\tau_\sigma} : T^\sigma_\tau\mathsf{M} \times T^\sigma_\tau\mathsf{M} \to \mathbb{K}$ by

$$(a|b)_g = g_{(i)(\tilde{i})} g^{(j)(\tilde{j})} a^{(i)}_{(j)} \bar{b}^{(\tilde{i})}_{(\tilde{j})}$$

in every coordinate with $(i), (\tilde{i}) \in \mathbb{J}^\sigma$, $(j), (\tilde{j}) \in \mathbb{J}^\tau$ and

$$g_{(i)(\tilde{i})} := g_{i_1 \tilde{i}_1} \cdots g_{i_\sigma \tilde{i}_\sigma}, \quad g^{(j)(\tilde{j})} := g^{j_1 \tilde{j}_1} \cdots g^{j_\tau \tilde{j}_\tau}.$$

In addition,

$$|\cdot|_g := |\cdot|_{g^\tau_\sigma} : \mathcal{T}^\sigma_\tau \mathsf{M} \to C^\infty(\mathsf{M}),\ a \mapsto \sqrt{(a|a)_g}$$

is called the (vector bundle) norm induced by g.

The Riemannian curvature tensor, in every coordinate, reads as

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \sum_r (\Gamma^r_{jk}\Gamma^l_{ir} - \Gamma^r_{ik}\Gamma^l_{jr}),$$

where $R = R^l_{ijk}\frac{\partial}{\partial x^l}\otimes dx^i \otimes dx^j \otimes dx^k$, and Γ^k_{ij} are the Christoffel symbols with respect to the Levi-Civita connection ∇_g, see [16, p. 134]. For $l \in \mathbb{J}^1$, $(i) \in \mathbb{J}^3$ and $(r) \in \mathbb{J}^k$,

$$\nabla^k_g R = (\nabla_{(r)} R^l_{(i)}) \frac{\partial}{\partial x^l} \otimes dx^{(i)} \otimes dx^{(r)}, \tag{3.2}$$

where $\nabla_{(r)} R^l_{(i)} = \partial^{(r)} R^l_{(i)} + b^l_{(i;r)}$ with $b^l_{(i;r)}$ a linear combination of the elements of

$$\{\partial^\alpha R^{\tilde{l}}_{(\tilde{i})} : |\alpha| < k,\, \tilde{l} \in \mathbb{J}^1,\, (\tilde{i}) \in \mathbb{J}^3\},$$

the coefficients being polynomials in the derivatives of the Christoffel symbols of order at most $k - 1 - |\alpha|$. See [3, formula (3.18)]. It follows from [3, formula (3.19)] that

$$\|\psi^*_\kappa \Gamma^l_{ij}\|_{k,\infty} \le C(k), \quad k \in \mathbb{N}_0, \quad i, j, l \in \{1, 2, \cdots, m\}. \tag{3.3}$$

One can easily infer from (R4) that

$$\|\psi^*_\kappa g^*\|_{k,\infty} \le C(k), \quad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}_0, \tag{3.4}$$

where g^* is the induced contravariant metric. Now combing with (R4), (3.2)-(3.4) imply the asserted statement.

(c) It is instructive to comment on the intuition behind our proof of positive injective radius, as well as some of its geometric underpinnings. The breakdown of injectivity of the exponential map at a point p will happen, as it is known, at the first time when any two geodesics leaving p intersect each other. Whether or not this happens, and when it happens, is dictated by the curvature of the manifold, which governs the focusing of such geodesics, and with large curvature values contributing for potentially smaller injectivity radius. In our case, the uniform bounds on derivatives of the metric assure uniform bounds on the curvature. Thus, intuitively, it is not unexpected that uniformly regular manifolds posses a lower bound on the injectivity radius. We can turn this idea into a more geometric appealing proof, which we now state.

Given any point $\mathsf{p} \in \mathsf{M}$, there exists some $\kappa \in \mathfrak{K}$ such that $\mathsf{p} \in \widetilde{\mathsf{O}}_\kappa$, where $(\widetilde{\mathsf{O}}_\kappa)_\kappa = (\psi_\kappa(r\mathbb{B}^m))_\kappa$ is an open cover of M, which is defined in part (a). Set $\delta := (1-r)/2$ and $W_\kappa := \psi_\kappa((r+\delta)\bar{\mathbb{B}}^m)$. $W_\kappa \setminus \psi_\kappa(0)$ can be identified with a collar neighborhood of $\partial\widetilde{\mathsf{O}}_\kappa$ by a diffeomorphism

$$\Phi_\kappa : W_\kappa \setminus \psi_\kappa(0) \to \partial\widetilde{\mathsf{O}}_\kappa \times (0, 1] : \mathsf{p} \mapsto (\psi_\kappa(r\varphi_\kappa(\mathsf{p})/|\varphi_\kappa(\mathsf{p})|), |\varphi_\kappa(\mathsf{p})|/(r+\delta)).$$

We can attach a neck, $\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]$, to W_κ along its boundary, obtaining the new manifold

$$W_\kappa \cup_{\Phi_\kappa} (\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]),$$

see Figure 1. Here the notation $\cup_{\Phi_\kappa}$ means that we take the union of W_κ and $\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]$, and Φ_κ defines an equivalence relation between ∂W_κ and one component of the boundary of $\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]$, i.e., $\partial\widetilde{\mathsf{O}}_\kappa \times \{1\}$.

Let $W_{\kappa,c}$ be a copy of W_κ. As before, we can identify $W_{\kappa,c} \setminus \psi_\kappa(0)$ with $\partial\widetilde{\mathsf{O}}_\kappa \times [2, 3)$ by a diffeomorphism Ψ_κ. Attach $W_{\kappa,c}$ to the other end of the neck, $\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]$, to construct a new closed compact manifold M_κ (see Figure 1 again), i.e.,

$$\mathsf{M}_\kappa := W_\kappa \cup_{\Phi_\kappa} (\partial\widetilde{\mathsf{O}}_\kappa \times [1, 2]) \cup_{\Psi_\kappa} W_{\kappa,c}.$$

By means of the diffeomorphism Ψ_κ, we can identify $\{W_\kappa \setminus \psi_\kappa(0)\} \cup \widetilde{\mathsf{O}}_\kappa \times [1, 2)$ with $\partial\widetilde{\mathsf{O}}_\kappa \times (0, 2)$. Based on this observation, we can introduce a differentiable structure on M_κ. Such a smooth differentiable

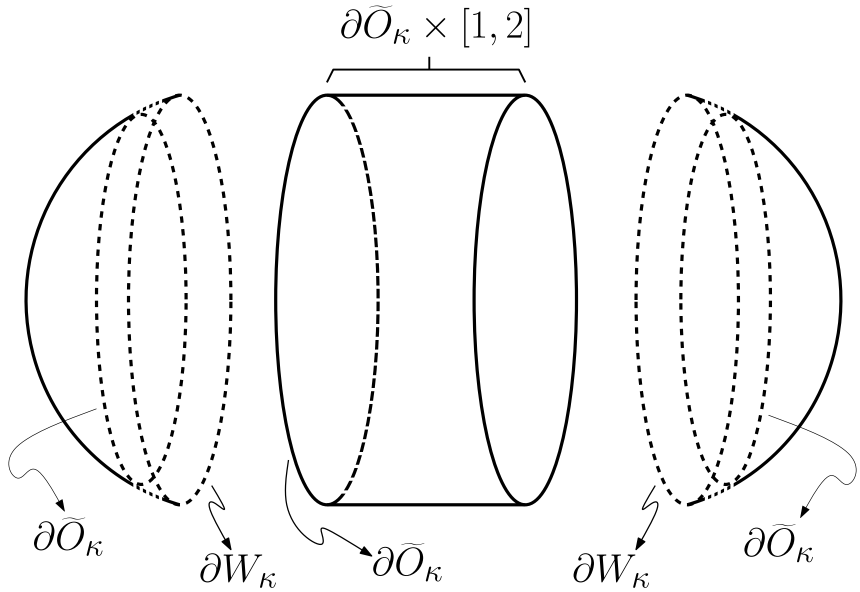

FIGURE 1. Illustration of how M_κ is constructed out of gluing the boundary of two balls to a common cylindrical neck.

structure on M_κ exists as long as there is a diffeomorphism $f : \partial W_\kappa \to \partial\widetilde{\mathsf{O}}_\kappa \times \{1\}$. The reason for us to introduce the maps Φ_κ and Ψ_κ explicitly is to show that we can construct a metric on the new manifold M_κ satisfying some uniform boundedness conditions on several geometric quantities of M_κ, as shown in the condition (3.5) below.

Choose smooth cutoff functions

$$\chi : \mathring{W}_\kappa \to [0,1], \quad \chi|_{\psi_\kappa(\mathbb{B}(0,r+\delta/2))} \equiv 1,$$

and

$$\zeta : \partial\widetilde{\mathsf{O}}_\kappa \times (2r/(1+r), 3-2r/(1+r)) \to [0,1], \quad \zeta|_{\partial\widetilde{\mathsf{O}}_\kappa \times [(1+3r)/2(1+r), 3-(1+3r)/2(1+r)]} \equiv 1.$$

We denote the copy of χ on $W_{\kappa,c}$ by χ_c. Endow M_κ with the metric

$$\widehat{g} = \chi g + \chi_c g + \zeta h.$$

Here, h is the standard product metric on $\partial\widetilde{\mathsf{O}}_\kappa \times (0,3)$.

We can choose $\psi_\kappa^*\chi$ and ζ to be independent of p and $\kappa \in \mathfrak{K}$. Then there exists a constant C satisfying

$$1/C \leq D(\mathsf{M}_\kappa) \leq C, \quad 1/C \leq V(\mathsf{M}_\kappa) \leq C, \quad |K_{\widehat{g}}|_{\widehat{g}} \leq C. \tag{3.5}$$

Here $D(\mathsf{M}_\kappa)$, $V(\mathsf{M}_\kappa)$ and $K_{\widehat{g}}$ are the diameter, volume and sectional curvature of $(\mathsf{M}_\kappa, \widehat{g})$, respectively. Indeed, the first two inequalities are straightforward. To verify the third inequality, we first check that, in the neighborhood $U_\kappa = \psi_\kappa((r+\delta/2)\mathbb{B}^m \setminus (r/2)\bar{\mathbb{B}}^m)$ of $\partial\widetilde{\mathsf{O}}_\kappa$, the metric g satisfies

$$\det[\widetilde{\Phi}_\kappa^* g] \geq M, \quad \|\widetilde{\Phi}_\kappa^* g|_{W_\kappa}\|_{k,\infty} \leq M(k), \quad k \in \mathbb{N}_0, \tag{3.6}$$

with M and $M(k)$ independent of p and $\kappa \in \mathfrak{K}$. Here

$$\widetilde{\Phi}_\kappa : W_\kappa \setminus \psi_\kappa(0) \to r\mathbb{S}^{m-1} \times (0,1] : \mathsf{p} \mapsto (r\varphi_\kappa(\mathsf{p})/|\varphi_\kappa(\mathsf{p})|, |\varphi_\kappa(\mathsf{p})|/(r+\delta)).$$

This assertion follows easily from (R3), (R4) and direct computation. Since the pull back metric of h under the map

$$f : \partial\widetilde{\mathsf{O}}_\kappa \times (0,3) \to r\mathbb{S}^{m-1} \times (0,3) : \quad (\mathsf{q}, a) \mapsto (\psi_\kappa(\mathsf{q}), a)$$

satisfies an analog of (3.6), the boundedness of $K_{\widehat{g}}$ follows from a similar argument to the proof of part (b).

Now it follows from [10, Theorem 5.8] that M_κ has a lower bound on its injectivity radius, i.e., $\iota(\mathsf{M}_\kappa) > R$ for some $R > 0$. Restricting back to $W_\kappa \subset \mathsf{M}_\kappa$, it follows from (3.1) that there exists some t^* independent of p such that any normal geodesic, i.e., geodesics with unit speed, starting from p cannot leave W_κ in $[0, t^*)$, and thus will not close in M within $[0, T^*)$, where $T^* = \min\{2\iota(\mathsf{M}_\kappa), t^*\}$. This proves the injectivity radius of p is larger than or equal to $\min\{\iota(\mathsf{M}_\kappa), t^*/2\}$. See [25, Section 6.6].

Since the constant C is independent of p and $\kappa \in \mathfrak{K}$, so is $\iota(\mathsf{M}_\kappa)$. Therefore we have proved the asserted statement. □

Remark 3.1. The concept of *uniformly regular Riemannian manifolds* with boundary is defined by modifying the *normalization* condition of the atlas $\mathfrak{A}$ as follows. For those local patches $\mathsf{O}_\kappa \cap \partial\mathsf{M} \neq \emptyset$, $\varphi_\kappa(\mathsf{O}_\kappa) = \mathbb{B}^m \cap \mathbb{H}^m$, where $\mathbb{H}^m$ is the closed half space $\bar{\mathbb{R}}^+ \times \mathbb{R}^{m-1}$. See [3, 4]. As we comment in Remark 1.3, the above proof for Theorem 1.2(a) and (b) works for *uniformly regular Riemannian manifolds* with non-empty boundary as well.

Geodesic completeness of a closed *uniformly regular Riemannian manifold* (M, g) can also be established directly by considering the equations for geodesics. There exists a number $r \in (0, 1)$ such that $(\widetilde{\mathsf{O}}_\kappa)_\kappa := (\psi_\kappa(r\mathbb{B}^m))_\kappa$ still forms an open cover of M. Given any $\mathsf{p} \in \mathsf{M}$ and $X_\mathsf{p} \in T_\mathsf{p}\mathsf{M}$, we assume that $\mathsf{p} \in \widetilde{\mathsf{O}}_\kappa$. Then the equation of the geodesic starting from $\mathsf{p} \in \mathsf{M}$ with initial velocity $X_\mathsf{p} \in T_\mathsf{p}\mathsf{M}$ in the local coordinate $(\mathsf{O}_\kappa, \varphi_\kappa)$ reads as

$$\begin{cases} \ddot{C}^k(t) = -\Gamma^k_{ij}(C(t))\dot{C}^i(t)\dot{C}^j(t) \\ C(0) = \varphi_\kappa(\mathsf{p}) =: x_\mathsf{p} \\ \dot{C}(0) = d\varphi_\kappa(\mathsf{p})X_\mathsf{p} =: V_\mathsf{p}, \end{cases} \tag{3.7}$$

where $C(t) = C^i(t)e_i \in \mathbb{R}^m$. Without loss of generality, we may assume that $|V_\mathsf{p}|_{g_m} = 1$.

Setting $Z(t) = \dot{C}(t)$, equation (3.7) can be rewritten as follows.

$$\begin{cases} \dot{C}^k(t) = Z^k(t) \\ \dot{Z}^k(t) = -\Gamma^k_{ij}(C(t))Z^i(t)Z^j(t) \\ C(0) = x_\mathsf{p} \\ Z(0) = V_\mathsf{p}. \end{cases} \tag{3.8}$$

Let $W(t) := (C(t), Z(t))$, and F_κ be so defined that

$$F_\kappa(W(t)) = \left(Z^1(t), \cdots, Z^m(t); -\Gamma^1_{ij}(C(t))Z^i(t)Z^j(t), \cdots, -\Gamma^m_{ij}(C(t))Z^i(t)Z^j(t)\right).$$

Equation (3.8) is equivalent to

$$\begin{cases} \dot{W}(t) = F_\kappa(W(t)) \\ W(0) = (x_\mathsf{p}, V_\mathsf{p}). \end{cases} \tag{3.9}$$

By (3.3), there exists a constant M uniform in all indices and κ, such that

$$\|\psi_\kappa^*\Gamma^k_{ij}\|_\infty < M.$$

We fix some $\delta \in (0, 1 - r)$. Then $\bar{\mathbb{B}}(x_\mathsf{p}, \delta) \subset \mathbb{B}^m$.

Lemma 3.2. *Given any,* $(x_\mathsf{p}, V_\mathsf{p}) \in r\mathbb{B}^m \times \mathbb{S}^{m-1}$*,* (3.9) *has a unique nonextendible solution* $W \in C^1(J; \mathbb{B}^m \times \mathbb{R}^m)$ *on* $J = J(x_\mathsf{p}, V_\mathsf{p}) := [0, T)$*, where* $T = T(x_\mathsf{p}, V_\mathsf{p}) \geq \tau^* := \min\{\delta/(4\sqrt{m}), 1/(8M)\}$*.*

Proof. By the Picard-Lindelöf Theorem, (3.9) has a unique local in time solution on a maximal interval of existence $J = J(x_{\mathsf{p}}, V_{\mathsf{p}}) := [0, T)$ with $T = T(x_{\mathsf{p}}, V_{\mathsf{p}})$. Suppose that for some $(x_{\mathsf{p}}, V_{\mathsf{p}})$, $T(x_{\mathsf{p}}, V_{\mathsf{p}}) < \tau^*$. For $k = 1, 2, \cdots, m$, we have

$$Z^k(t) = Z^k(0) + \int_0^t \dot{Z}^k(s)\, ds = Z^k(0) - \int_0^t \Gamma_{ij}^k(C(s)) Z^i(s) Z^j(s)\, ds.$$

Let $\alpha(t) := \max\limits_{i \in \{1, \cdots, m\}} \max\limits_{s \in [0,t]} |Z^i(s)|$ for $t \in J = [0, T)$. Then we infer from the above equation that

$$\alpha(t) \le 1 + tM\alpha^2(t), \quad t \in J. \tag{3.10}$$

Let $\alpha_{1,2}(t)$ be the solution to the equation $\alpha = 1 + tM\alpha^2$. It is easy to see that

$$\alpha_1(t) = \frac{2}{1 + \sqrt{1 - 4tM}} \in [1, 2), \quad \alpha_2(t) = \frac{2}{1 - \sqrt{1 - 4tM}} \in (2, \infty).$$

(3.10) implies that $\alpha(t) \in [0, \alpha_1(t)] \cup [\alpha_2(t), \infty)$. We will show that in fact $\alpha(t) \in [0, \alpha_1(t)]$. Let

$$E := \{t \in J : \alpha(t) \le \alpha_1(t)\}.$$

At $t = 0$, we have $\alpha(0) \le 1 = \alpha_1(0)$. Hence, E is nonempty. By the continuity of $\alpha(t)$ and $\alpha_1(t)$ in t, E is closed in J. For the same reason, $J \setminus E = \{t \in J : \alpha(t) \ge \alpha_2(t)\}$ is also closed in J. Then E is open, and thus $E = J$. Therefore $\alpha(t) \le \alpha_1(t) < 2$ for $t \in J$. It implies that

$$|Z(t)|_{g_m} < 2\sqrt{m}, \quad t \in J. \tag{3.11}$$

For any $t \in J$, this yields $|C(t) - x_{\mathsf{p}}| \le 2\sqrt{m}\tau^* < \delta/2$. We thus infer that

$$C(t) \in \mathbb{B}(x_{\mathsf{p}}, r + \delta/2) \subset\subset \mathbb{B}^m, \quad t \in J. \tag{3.12}$$

(3.11) and (3.12) contradict [1, Theorem 7.6]. This proves the uniform lower bound for $T(x_{\mathsf{p}}, V_{\mathsf{p}})$, i.e., $T \ge \tau^*$. □

Lemma 3.2 implies that, given any positive constant C, for any initial velocity $|V_{\mathsf{p}}|_{g_m} \le C$, the maximal interval of existence $J = J(x_{\mathsf{p}}, V_{\mathsf{p}}) = [0, T(x_{\mathsf{p}}, V_{\mathsf{p}}))$ for the solution to (3.9) is uniform, i.e., $T(x_{\mathsf{p}}, V_{\mathsf{p}}) \ge \tau$ for some τ independent of $(x_{\mathsf{p}}, V_{\mathsf{p}})$.

Given $\mathsf{p} \in \mathsf{M}$, any geodesic $G(t)$ starting from p with initial velocity $X_{\mathsf{p}} \in T_{\mathsf{p}}\mathsf{M}$ fulfilling $|X_{\mathsf{p}}|_{g(p)} = 1$ satisfies equation (3.7) in the local coordinate $(\mathsf{O}_\kappa, \varphi_\kappa)$ with $\mathsf{p} \in \psi_\kappa(r\mathbb{B}^m)$. In view of (R3), $V_{\mathsf{p}} := \varphi_\kappa^* X_{\mathsf{p}}$ fulfils $|V_{\mathsf{p}}| \le C$ for some C independent of $(\mathsf{p}, X_{\mathsf{p}})$. Therefore, $G(t)$ exists on some $[0, T^*]$, where T^* is independent of $(\mathsf{p}, X_{\mathsf{p}})$. Since geodesics are parameterized with respect to arc length, any geodesic is infinitely extendible. This gives an alternative proof for the geodesic completeness of (M, g).

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