Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups

Cornelia Druțu and Mark Sapir

$\mathbb{R}$-trees

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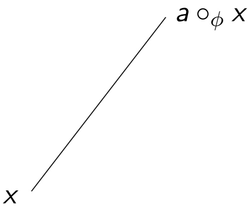

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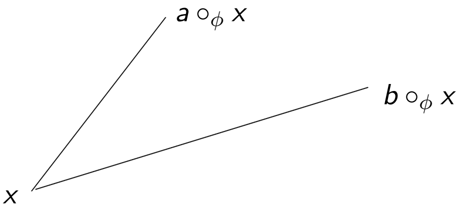

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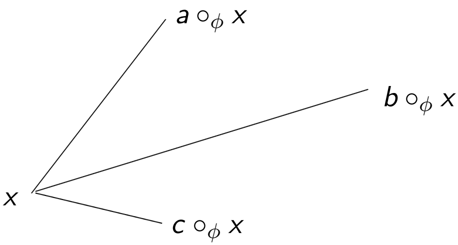

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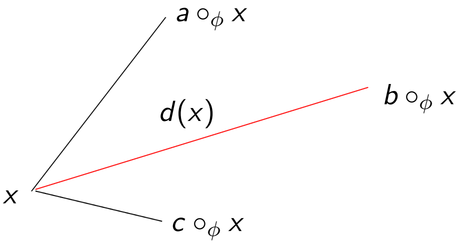

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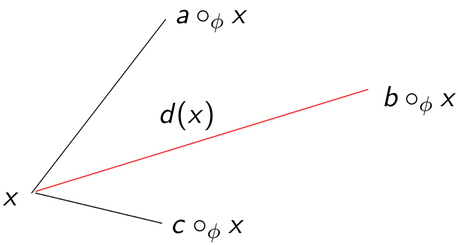

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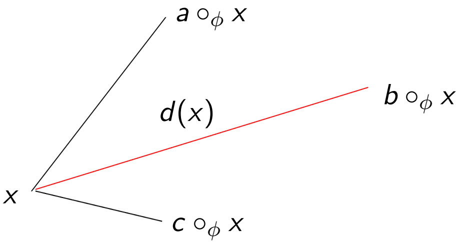

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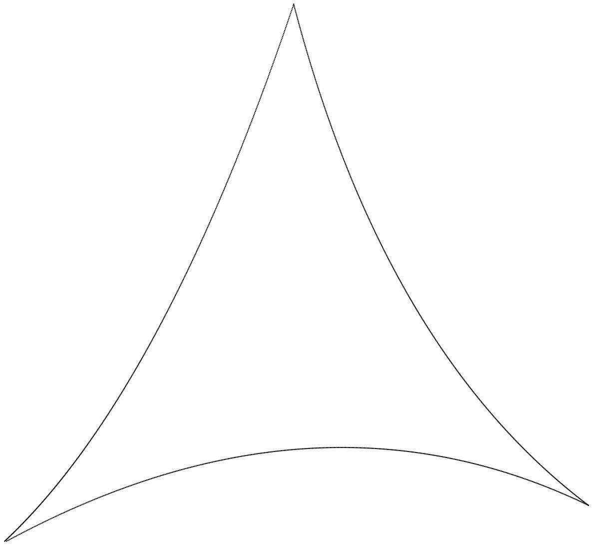

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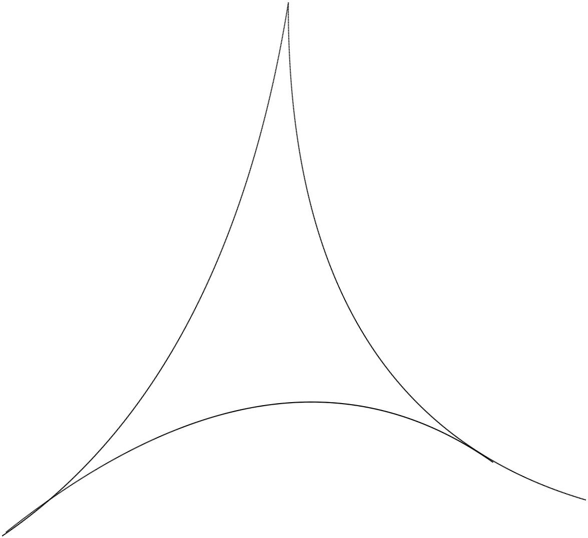

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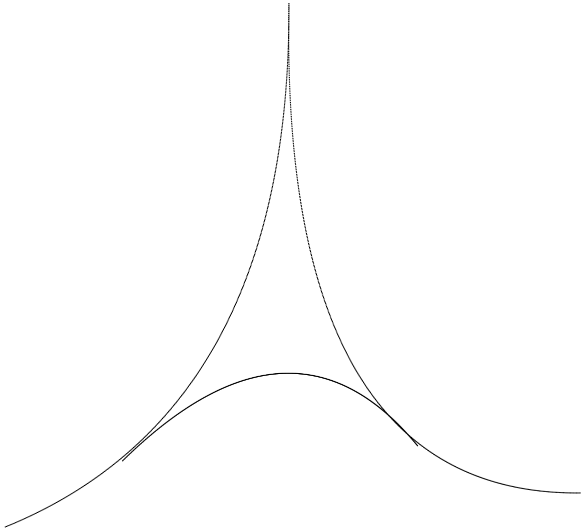

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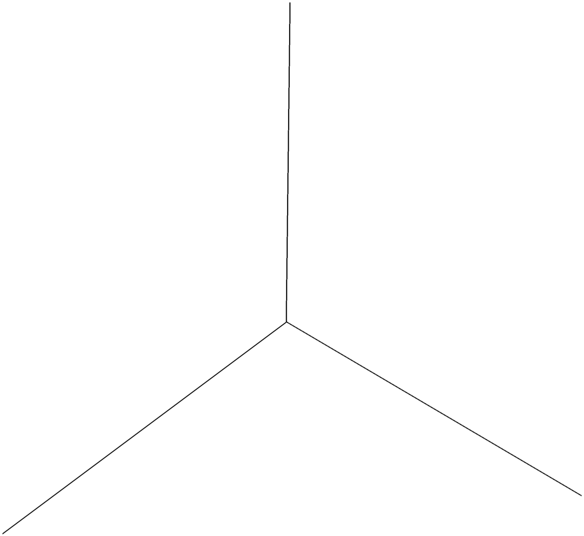

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But in many cases they are tree-graded spaces. Recall the definition.

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Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

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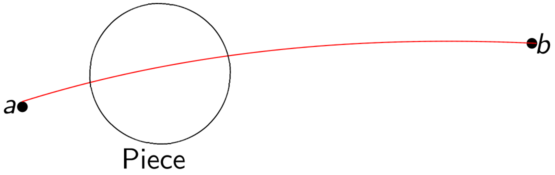

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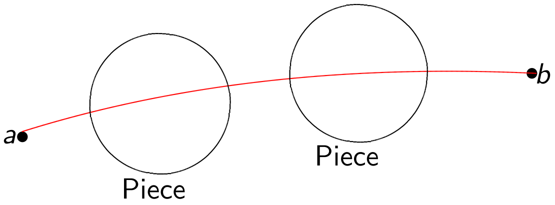

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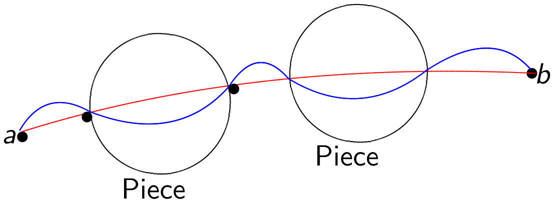

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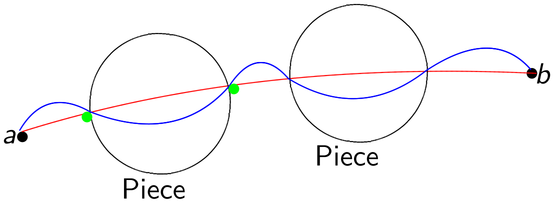

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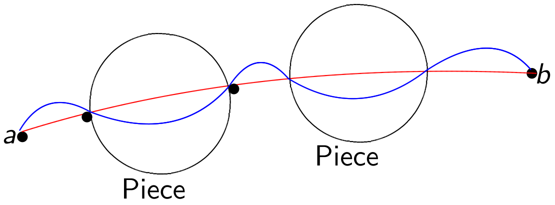

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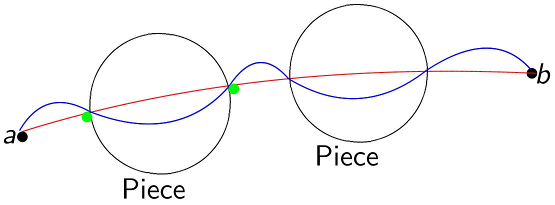

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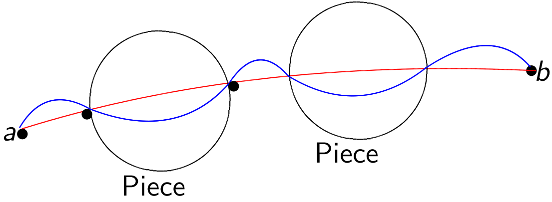

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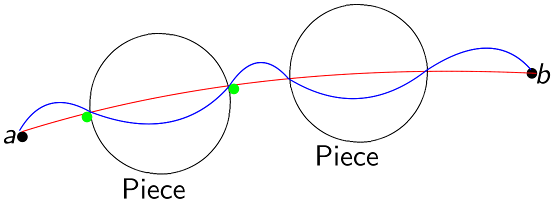

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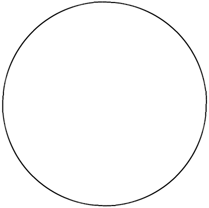

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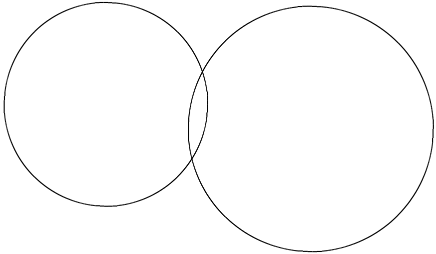

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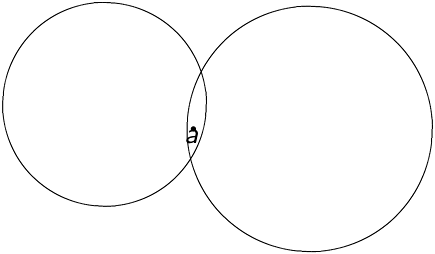

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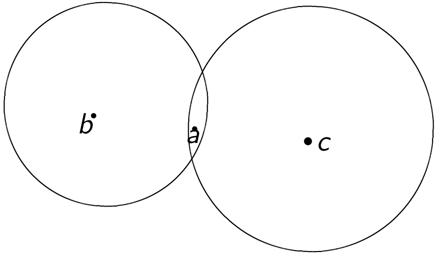

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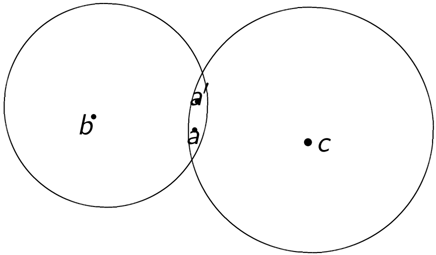

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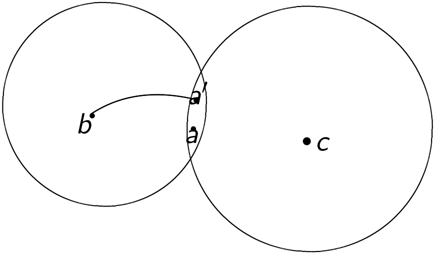

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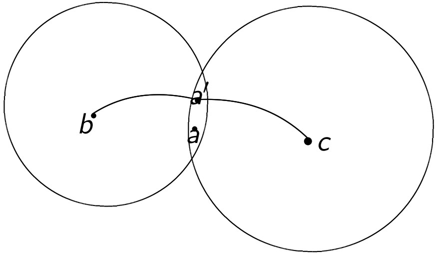

Cut points and tree-graded structures

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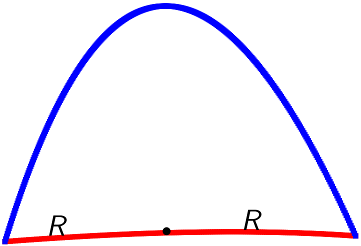

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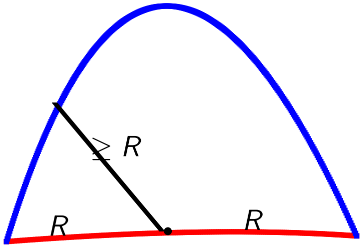

The length of the blue arc should be $> O(R)$.

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Cut points and tree-graded structures

Recall that hyperbolicity $\equiv$superlinear divergence of any pair of geodesic rays with common origin.

Transversal trees

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The geodesics $[x, y]$ from transversal trees are called *transversal geodesics*.

Transversal trees, an example

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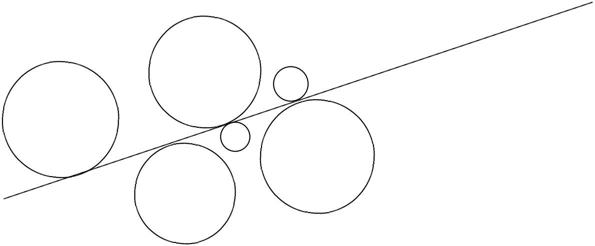

A tree-graded space. Pieces are the circles and the points on the line.

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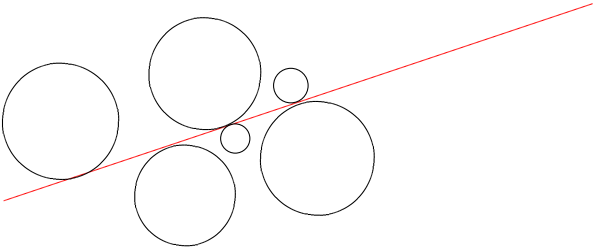

A tree-graded space. Pieces are the circles and the points on the line.

The line is a transversal tree, the other transversal trees are points on the circles.

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Actions on tree-graded spaces

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Our main result shows that a group acting “nicely” on a tree-graded space also acts “nicely” on an $\mathbb{R}$-tree.

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(3) T is a line and G has a subgroup of index at most 2 that is an extension of the kernel of that action by a finitely generated free Abelian group.

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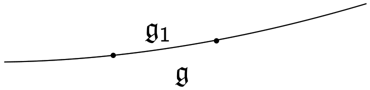
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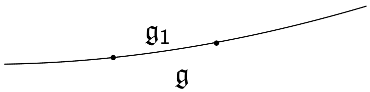

$H = \mathrm{Stab}(\mathfrak{g})$

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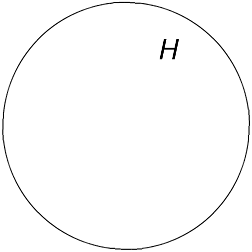

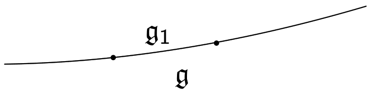

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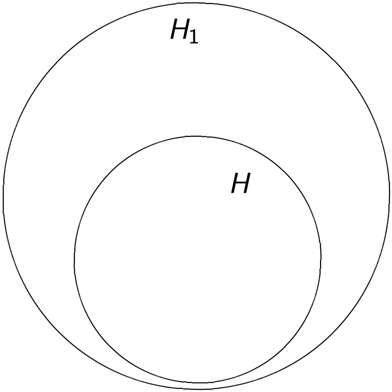

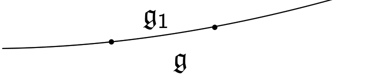

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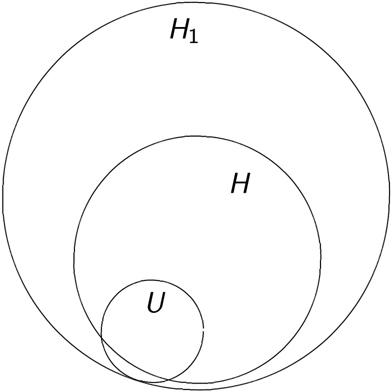
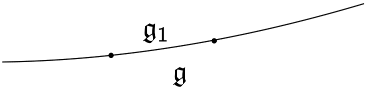


H_1/U is Abelian, $|U| \leq D$.



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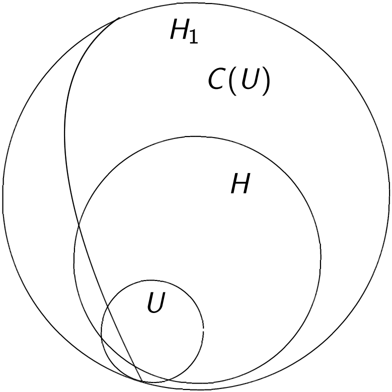

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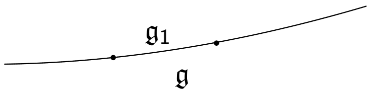

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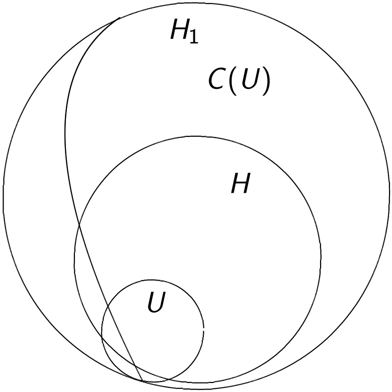

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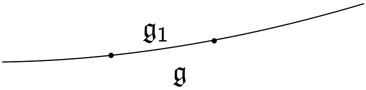

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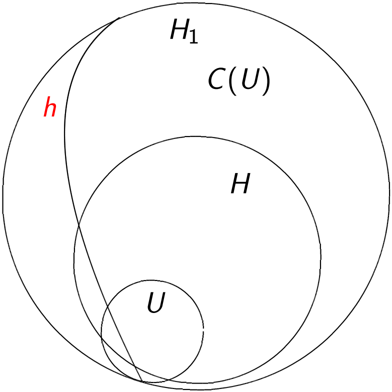

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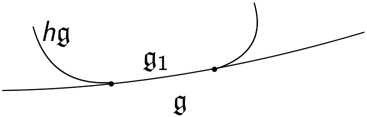

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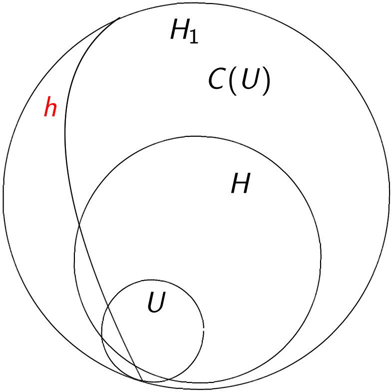

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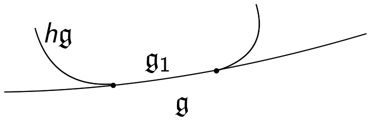

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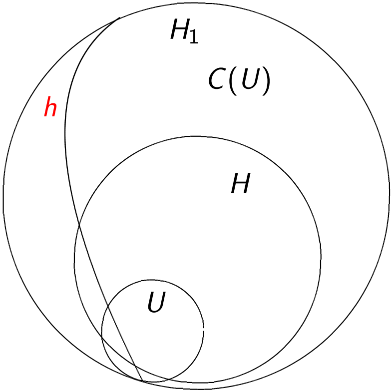

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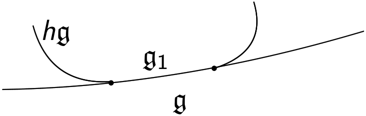

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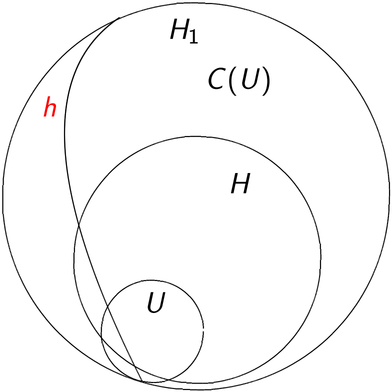

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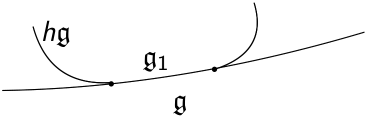

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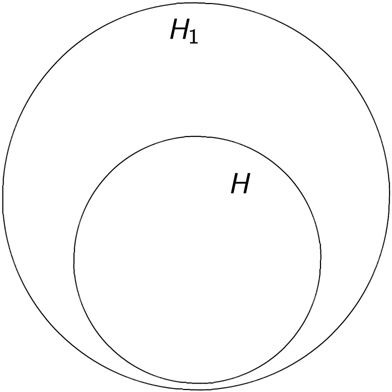

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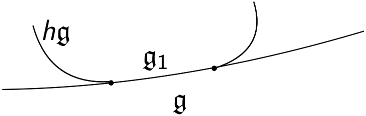

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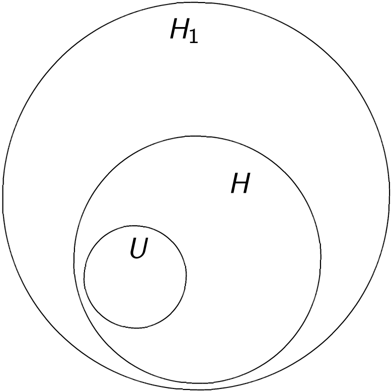

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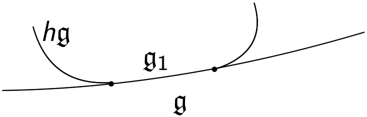

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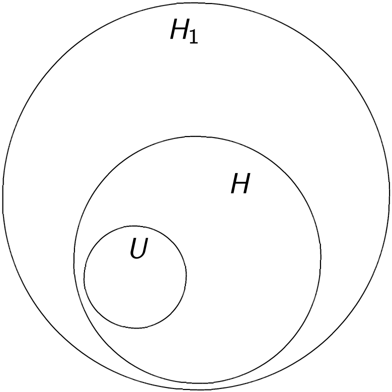

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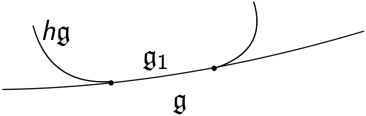

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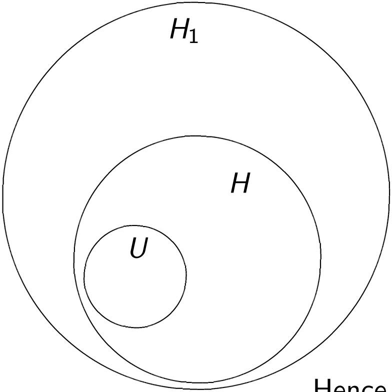

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Theorem (Dahmani) If Λ is finitely presented, and G is relatively hyperbolic then there are finitely many subgroups of G, up to conjugacy, that are images of Λ in G by homomorphisms without accidental parabolics.

Homomorphisms into groups

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Note that if a group G splits over an Abelian subgroup C, say, $G = A *_C B$, then it typically has many outer automorphisms that are identity on A and conjugate B by elements of C. Hence we need to modify the definition of accidental parabolics as follows.

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Theorem Let Λ be a finitely generated group, G be a relatively hyperbolic group and parabolic subgroups are small (no free non-Abelian subgroups).

Then the number of pairwise non-conjugate in G injective homomorphisms $\Lambda \to G$ without weakly accidental parabolics is finite.

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Relatively hyperbolic groups with infinite $\mathrm{Out}(G)$ and non-co-Hopf relatively hyperbolic groups have been studied extensively (Paulin, Rips-Sela, T.Delzant-L.Potyagailo, D. Groves and I. Belegradek - A. Szczepański.)

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- G splits over a virtually cyclic subgroup;
- G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;
- G can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of G.

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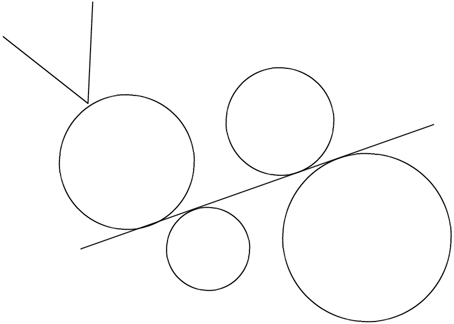

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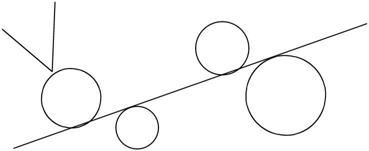

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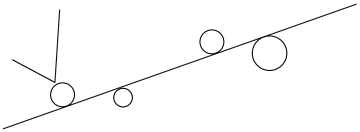

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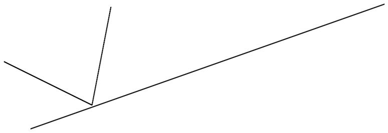

An example

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Note that pieces do not intersect.

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Indeed, every arc in T contains an arc from a transverse tree of $\mathbb{F}$.

Thus in this case G acts non-trivially on an $\mathbb{R}$-tree with arc stabilizers from $\mathcal{C}_2$.

Case B.

Suppose that G fixes a point in T.

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The corresponding $\approx$-class is a union of pieces and is a tree-graded space $(R, \mathcal{R})$ with trivial transversal trees. G acts on R.

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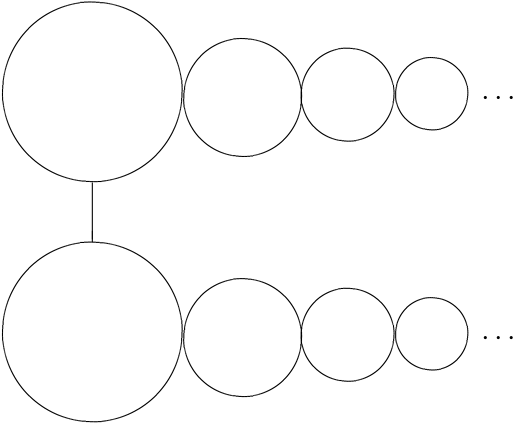

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It must stabilize at $\mathcal{P}_\alpha$.

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Then we define a simplicial tree having pieces of $\mathcal{P}_{\delta-1}$ and intersections of these pieces as vertices, and edges connecting a piece and a vertex inside it.

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We define the structure of a pre-tree (Bowditch) on X.

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- (PT3) xzy and $z \neq w$ then $(xzw \vee yzw)$.

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We apply a version of Levitt's theorem and complete the proof.