Statistical Physics of Rarefied Sediment Particle Motions and Transport

Applications to Hillslopes and Rivers

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Preface

The objective of this book is to provide a framework for describing sediment particle motions and transport in a manner that integrates principles and methods of probability with mechanical considerations of particle motions. Indeed, there is a growing interest in probabilistic descriptions of sediment motions and transport. This interest undoubtedly stems from the recognition that particle motions are inherently stochastic, and therefore that the concepts and language of probability are well suited to the problem of describing these motions. It also is clear that experimental measurement techniques, notably including high-speed imaging, are increasingly providing the means to directly observe details of particle behavior that admit probabilistic descriptions, as well as to test and constrain theories of this behavior. Similarly, advanced computational methods that treat the physics of granular materials, notably including the physics of coupled fluid–particle systems, are revealing things about particle behavior that cannot be observed in experiments.

One cannot miss noticing that this interest in probabilistic descriptions of sediment motions harks back to important early work, for example, that of G. I. Taylor concerning particle diffusion in turbulent flows, the pioneering work of H. A. Einstein, whose 1937 doctoral thesis is entitled, "Bedload transport as a probability problem," the contributions of H. Nakagawa and T. Tsujimoto, who extended Einstein's work to an entrainment form of the Exner equation, and to the equally pioneering work of W. E. H. Culling, who, in the 1960s, envisioned soil creep as a particle diffusion-like process. It is interesting, however, that studies of sediment particle motions and transport, at least in the Earth-surface processes community, did not linger on this earlier work, but instead mostly moved toward a deterministic and largely continuum-based point of view.

In retrospect it seems clear that this move was a misstep of sediment transport research in the 20th century. Sediment transport in many natural and experimental settings involves rarefied conditions that are entirely at odds with a continuum framework. Except for a handful of phenomena, for example, debris flows and possibly creeping soil, transport conditions do not satisfy the continuum hypothesis. And, by adopting a continuum framework, this point of view in turn imposed a deterministic style of thinking consisting of the concepts and language of continuum mechanics: a language in which quantities of interest — variables — follow the ordinary rules of algebra, where quantities and their relationships are expressed as continuously differentiable functions with respect to space and time, and where system evolution proceeds in a deterministic manner according to the imposed equations. Concurrently, such descriptions of transport systems are often based on a premise invoking the existence of preferred states ("equilibria") or configurations that are mechanically matched to specified external controlling factors, centered on the notion that these systems tend to be mean-reverting. Fluctuations in behavior for fixed controlling factors, for example varying transport rates, are then viewed as stochastic perturbations about the nominally preferred state. The effort thus translates to the creation of deterministic expectations of behavior that are inconsonant with the inherent, stochastic variability of complex systems — a hallmark of such systems.

A probabilistic view, on the other hand, fully embraces the mechanics of sediment particle motions and transport, but is entirely agnostic to the presence or absence of continuum conditions. This view acknowledges, relative to continuum conditions, the small numbers of particles involved during transport, and the pronounced consequences of these small numbers in concert with the idiosyncrasies of the geometry of sediment particle motions. Quantities of interest — variables — are treated as *random* variables characterized by probability distributions, where relationships among the quantities follow the algebraic rules of random variables. Time series of quantities are treated as realizations of stochastic processes, with a focus on counting processes. Then, for example, the variability in realizations of the particle transport rate is viewed as an inherent feature of the transport process where, for a specific set of controlling factors, any one of an ensemble of possible realizations *could* occur, each entirely consistent with the physics involved. Likewise, the variability in granular surfaces produced by transport represents the inherent richness of possible surface configurations. Expectations of behavior are thus probabilistic, where variability about an average state is just as important as the average in terms of characterizing how the process works.

This represents a *style* of thinking that decidedly leans on lessons from the explosion of 20th century work in probabilistic physics, growing from the triumphs of statistical mechanics in the late 19th and early 20th centuries led by J. C. Maxwell, L. Boltzmann and J. W. Gibbs. However, in drawing from this body of work to develop a probabilistic framework for describing sediment particle motions and transport, we must avoid a second misstep: uncritically adopting formalism from statistical physics. No doubt, the style of analysis and formalism have much to offer. But here it becomes essential to recognize, at the outset, the fundamental differences between dissipative sediment particle systems and, say, conservative gas particle systems as treated in classical statistical mechanics, and importantly, the implications and consequences of these differences. The statistical physics of sediment particle motions and transport therefore must be approached and developed in its own right.

The book contains fifteen chapters plus appendixes. The first chapter provides a summary of the material covered, and offers a philosophical statement of why this material matters. It outlines the contrast between probabilistic and deterministic perspectives, and the implications of rarefied versus continuum conditions. The remaining chapters are presented in two parts entitled *Foundational Material* and *Applications*.

The second and third chapters, *Probabilistic Concepts I* and *Probabilistic Concepts II*, present a selection of topics from probability that are well suited to the analysis of sediment particle motions and transport, streamlined for our purposes, and are mostly aimed at readers starting to learn the principles and language of probability. These are lengthy chapters that merit special comment. Namely, it is unusual for students in Earth and environmental sciences programs in the U.S. to be formally exposed to the theory and practice of probability and statistics.[1] This is unfortunate. Many of the systems we study are messy, and our descriptions of these systems and explanations of how they work necessarily involve approximations and uncertainty. Probability and statistics are the natural language of uncertainty, and in many situations an explanation of how something works requires a probabilistic description; no other possibility exists. Yet our tepid relationship with probability and statistics means that we mostly learn about these topics in an ad hoc formulaic manner centered on data analysis rather than as a formal part of describing the stochastic qualities of systems and their behavior. For these reasons I settled on devoting significant space to coverage of basic material in these two chapters. In addition to providing a foundation for the material that follows, it is my hope that the style of presentation in these chapters, when combined with the rest of the book, serves to illustrate the essence of probabilistic thinking and analysis as a cornerstone of doing science.

The backbone of the foundational material is then covered in the chapters entitled *Basic Stochastic*

[1] As a masters student my advisor encouraged me to read the classic paper by H. E. Daniels, "The statistical theory of the strength of bundles of threads. I" (*Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 183, 405–435, 1945; https://www.jstor.org/stable/97823). At the time (1980) I was working on the problem of how tree roots contribute to soil strength. This is the paper that first revealed to me the stunning beauty and implications of the mathematics of probability. I had previously gained only a vague understanding of a probability distribution in a biostatistics course. When reading the Daniels paper, the idea and implications of a probability density function immediately clicked upon seeing it embedded within its language of calculus.

Processes, *Particle Motions*, *Kinematics of the Particle Flux*, *Master Equations*, *Particle Advection and Diffusion*, and *Entropy*. The second part of the book given to applications includes six chapters entitled *Rarefied Transport on Hillslopes*, *Rain Splash Transport*, *Bed Load Transport*, *Surface Evolution*, *Particles in Soils* and *Considerations of Entropy*.

This book likely is more suitable as a "reference" to be placed judiciously on one's bookshelf than as a text for class instruction. Nonetheless, I have tried to write it in the style of a text in the sense of offering pedagogical clarification of certain (sometimes basic) topics rather than just assuming the underpinnings or implications of these topics are readily apparent. At numerous points I present full derivations of mathematical material, sometimes with purposeful redundancy. This is with the view that, as autodidacts, too frequently we read texts or papers in which equations are presented and we muse: "So, I think I grasp what this equation purports to represent, but where did it come from? What is its basis?" Such derivations also can be helpful in developing mathematical skills, and sometimes they can point to ways for thinking about separate problems. The ordering of the material is intended to build on itself — although I accept that few are likely to read the material in any particular order that I might have imagined.

Here are three final points regarding my intentions in writing this book. First, the material covered in this book does not lend itself to a synthesis of an established body of knowledge aimed at a particular readership. Unlike writing a text on, say, fluid physics, there are no models for assessing essential topics and material to be covered. Indeed, it took me three tries to organize the material in the form of a reasonably coherent body of knowledge rather than as a compendium of published material on the topic. I therefore wrote this book in a way that I thought would appeal to smart people who will fix my misconceptions, build on the ideas presented, and carry us forward. Second, I hope that the material convincingly illustrates that much of the probabilistic language used to describe sediment transport on hillslopes and in rivers is basically the same, and that these often separated areas of study therefore have much to learn from each other. Third, this text is as much about a philosophical style of thinking about sediment transport as it is about the technical material involved. I cannot promise that this style offers immediate answers to many interesting questions about sediment transport, but I can promise a quirky intellectual adventure.

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Chapter 1

Introduction

1.1 Preamble

This book is about sediment particle motions and transport. Equally, it is about a style of thinking — a framework for describing particle motions and transport in a manner that integrates principles and methods of probability with mechanical considerations of particle motions. As context for developing this framework, here we summarize the material covered in the book and offer a philosophical statement regarding why this material matters. This starts with a summary of the aims and purpose of statistical physics as applied to sediment systems, outlining the contrast between probabilistic and deterministic perspectives, and the challenges that are involved in adapting concepts from statistical physics to sediment systems. We then summarize the meaning and implications of rarefied versus continuum conditions. The final section offers a brief preview of how material covered in the book translates to descriptions of particle transport using the examples of rain splash and bed load transport.

1.2 Statistical Physics and Sediment Systems

When we say statistical physics, we are referring to a relatively broad area of physics marked by its probabilistic style of thinking and analysis inspired by the triumphs of statistical mechanics in the late 19th and early 20th centuries, led by J. C. Maxwell, L. Boltzmann and J. W. Gibbs. Gibbs (1902) coined the term statistical mechanics, and more than one author has pointed out that Gibbs probably should have called it probabilistic mechanics, to distinguish the approach of this field from the purposes and methodologies of statistics.[1] Nonetheless, the name statistical physics persists in the broader scope of things beyond classical statistical mechanics, perhaps in honor of Gibbs's word choice, and so as to retain the ring of statistical mechanics. To be sure, there is no consensus on how statistical mechanics and statistical physics differ. Some view these as being synonymous.[2] Others

[1]Numerous texts on statistical mechanics are available. In addition to perusing material in one or more of these, I recommend examining four classic books. The 1902 book by Gibbs, *Elementary Principles of Statistical Mechanics*, is considered his masterpiece. Gibbs introduces his conceptualization of an ensemble of systems and the implications that emerge from this. The 1938 text by Tolman, *The Principles of Statistical Mechanics*, is still considered to be the most authoritative text on statistical mechanics available, covering both classical and quantum statistical mechanics. The 1958 text by Kittel, *Elementary Statistical Physics*, is particularly accessible. Kittel was a teacher at heart, and he offers numerous simple examples explaining the essence of demanding material. The 1952 text by Schrödinger, *Statistical Thermodynamics* is essentially a synopsis of the subject, and includes Schrödinger's views on the profound impact of the idea of a Gibbs ensemble. The 2023 text by James Sethna, *Statistical Mechanics: Entropy, Order Parameters, and Complexity*, provides a clear view of the astonishing scope and applications of modern statistical physics. For those who like online presentations, I recommend that of Peter Eastman at Stanford University.

[2]This is reflected by the titles of texts. Despite covering essentially the same material, some authors choose statistical mechanics and others statistical physics. And, some authors prefer statistical thermodynamics.

view statistical mechanics as being represented by topics resting directly on the foundational principles provided by Boltzmann and Gibbs, perhaps emphasizing connections with thermodynamics. Then, statistical physics is broader, to include statistical mechanics, kinetic theory (classical and as applied to granular gases), the physics of stochastic processes, and numerous other topics developed after the time of Boltzmann and Gibbs, including the probabilistic physics of granular materials, which are not necessarily keyed into connections with thermodynamics.[3] Here we adopt this latter view to emphasize the importance of adapting the probabilistic style of analysis rather than the details of statistical mechanics, notably in relation to sediment systems where we cannot appeal to most of the foundational ingredients of statistical mechanics and thermodynamics centered on the behavior of large numbers of gas particles, classical or quantum.[4]

Statistical mechanics, thence statistical physics, holds clear scientific distinction. As Frigg and Werndl (2023) point out:

> Statistical Mechanics is the third pillar of modern physics, next to quantum theory and relativity theory. Its aim is to account for the macroscopic behaviour of physical systems in terms of dynamical laws governing the microscopic constituents of these systems and probabilistic assumptions. Like other theories in physics, statistical mechanics raises a number of foundational and philosophical issues. But... [u]nlike theories such as quantum mechanics, [Statistical Mechanics] has not yet found a generally accepted theoretical framework or a canonical formalism. What we encounter in [Statistical Mechanics] is a plurality of different approaches and schools of thought, each with its own mathematical apparatus and foundational assumptions.

The final sentence in this paragraph speaks to a point offered in the *Preface*. Namely, whereas the style of analysis and formalism of statistical mechanics have much to offer, the statistical physics of sediment particle motions and transport must be approached and developed in its own right. I suspect this type of attention to the contexts and idiosyncrasies of the great breadth of systems currently examined using methods of statistical mechanics will likely defer if not preclude a single canonical formalism.

As context for pursuing a statistical physics framework for describing sediment systems, consider some details of early work with reference to the quoted paragraph above. In classical statistical mechanics focused on gases, the microscopic constituents are atoms and molecules, and the "dynamical laws governing the microscopic constituents" are the laws of classical Hamiltonian mechanics (neglecting quantum statistical mechanics). The microscopic behavior consists of conservative particle–particle collisions (again neglecting quantum effects) between which the particles move through a vacuum, perhaps within a gravitational or electromagnetic field. Then, the "macroscopic behavior" essentially consists of the laws of thermodynamics — behavior emerging at a scale much larger than the mean free path of the particles in the thermodynamic limit of large numbers of particles.

To be clear, the conventional explanation for appealing to the methods of statistical mechanics is essentially operational (Tolman, 1938). Whereas the dynamical behavior of classical gas particles is entirely deterministic, we can never know the initial conditions nor solve the vast set of equations given the great numbers of particles involved. Hence, the problem must be approached statistically, where the large numbers of particles become an advantage owing to consequences of the law of large

[3] A perusal of the titles and abstracts of papers appearing on the research-sharing platform arXiv in the area of Statistical Mechanics (cond-mat.stat-mech) reveals the astonishing breadth of topics that are nominally associated with the field of statistic mechanics — a breadth I view as reflecting that of statistical physics.

[4] Indeed, we do not have analogues of a Boltzmann constant, a canonical ensemble, a thermodynamic limit, and so on. As we will see, our analogue of an equipartition theorem is a principle related to the consequences of particle motions on a rough surface; and our version of entropy is aligned more with the Shannon entropy of information theory than with the Gibbs entropy.

numbers. The list of successes is long:[5] the Boltzmann distribution of energy states and the Boltzmann transport equation; Gibbs's explanation of entropy; the Maxwell–Boltzmann distributions of particle momenta, speeds, kinetic energies and related distributions; probabilistic explanations of Brownian motion and particle diffusion more generally, including state-dependent diffusion; transport theory, with explanations of fluid viscosity and other transport coefficients; explanations of fluid behavior under both continuum and rarefied conditions; explanations of material compressibility, thermal expansion and adiabatic processes; explanations of phase transitions; and so on. To these successes we should add results of related work in statistical physics: the development of various master equations, including the Fokker–Planck equation, describing the evolution of the probability distribution of a quantity of interest; the Langevin equation and related descriptions of stochastic dynamics; and the development of theory of stochastic processes more generally. All fields of science, including the life sciences, have been strongly impacted by the implications and results of statistical mechanics and closely related work.

Compare this now with sediment systems. At the scale of the "microscopic constituents" — the sediment particles — we are already at a scale that is much larger than the Knudsen scale of the fluids in which the particles are immersed. (The Knudsen number is the ratio of the molecular mean free path and a characteristic length, which defines the scale at which a fluid satisfies the continuum hypothesis.) Sediment particles within sufficiently viscous fluids are subject to the macroscopic behavior of these fluids; and although the particles might interact via dissipative collisions, they are otherwise always mechanically coupled to varying degrees with the surrounding fluid and thus are generally subjected to fluctuating forces, unlike gas particles moving through a vacuum between collisions. Moreover, as fully elaborated below (Section 1.4), in most settings the interactions between sediment particles and the granular surfaces over which they move are far more important in determining the dynamical behavior of the particles than are their interactions with other moving particles — a characteristic of rarefied (non-continuum) conditions. These particle–surface interactions, dominated by dissipative collisions, are decidedly probabilistic given the complexity of natural particle shapes, trajectories, surface textures and collisional micro-physics. Thus, although the "dynamical laws governing the microscopic constituents" certainly consist of Newton's laws, we rarely can use them in a reductionist manner akin to using Hamilton's equation to describe particle behavior in classical statistical mechanics. Rather, we must invoke coarser descriptions that are necessarily abstractions of physics occurring at the particle scale and smaller, and sometimes larger.

For example, if our objective is to describe rain splash transport in relation to land-surface configuration (e.g. surface slope) at rainstorm time scales, we do not focus on the fluid–granular micro-physics occurring during raindrop impacts leading to particle entrainment thence parabolic particle trajectories lasting less than one-tenth of a second. (This phenomenon is an active area of research in granular physics.) Instead, we might use experimental measurements to describe the probabilistic behavior of particle trajectories in relation to raindrop and sediment properties, clarify how these trajectories translate to transport rates, then add effects of known stochastic characteristics of rainfall to our description. Similarly, in aiming at a description of bed load particle motions, we do not attempt to describe the fluctuating fluid forces on a particle within a turbulent shear flow by integrating the fluid stress over the particle according to the Navier–Stokes equation. Indeed, even if we could specify the fluctuating fluid velocity field surrounding a moving particle over time, the equations describing the forces acting on it (Maxey and Riley, 1983; Schmeeckle et al., 2007) are too complex to be practically

[5]Despite the successes of classical statistical mechanics, physicists had not yet reached consensus on the existence of atoms when Maxwell, Boltzmann and Gibbs pursued their work. Nonetheless, evidence for the existence of atoms was compelling, and the kinetic theory of gases was well established. The field of atomic physics then exploded near the turn of the century with the discovery of radioactive decay by Henri Becquerel in 1896, followed by the discovery of radium and polonium by Marie and Pierre Curie, thence punctuated by the descriptions of Brownian motion provided by Einstin (1905) and von Smoluchowski (1906), which helped to confirm the existence of atoms.

useful outside of specialized numerical simulations. Instead we must simplify, and we might choose to focus on the dominant force, the drag, then describe particle behavior in terms of a Langevin-like equation, adding a random "noise" term to represent fluctuations in the drag. (This is in fact a macroscopic version of classical treatments of Brownian particle motion; Chapter 12.) At a coarser scale involving many particle motions, we might in turn use what we learn about motions of individual particles as the basis for appealing to a specific counting process from stochastic process theory to describe the particle flux at a specified coordinate position.

Here, "noise" and "stochastic" merit clarification, as we use these words many times in this and later chapters. Within the mechanical context of sediment systems, noise is a convenient word to describe the detailed things that are occurring according to Newton's second law at scales we cannot resolve or describe due to the complexity involved, and which produce an unpredictability in a measurable phenomenon at our scale of interest. To be sure, when a raindrop impacts a soil surface and ejects many particles into parabolic trajectories, every detail from the fluid continuum scale to the soil particle scale to the deforming drop scale to the particle trajectory scale is entirely deterministic — just as we envision gas particle motions in classical statistical mechanics. When observed at the trajectory scale, however, these underlying deterministic details escape us. We instead see an outcome that is only statistically predictable, manifest as consistency in particle trajectories and the associated probability distribution of particle displacements. And because we prefer to avoid implications of the word random — given that there *is* a certain (statistical) predictability to what we observe — we instead call the outcome "stochastic." Thus, a noise driven system — a stochastic process — is in fact a deterministic physics driven system. The underlying deterministic physics is giving behavior that is observable (measurable) at larger scales, but which involves a degree of unpredictability or stochasticity given the infinite set of possible outcomes.

Thus, at the particle scale we must simplify and approximate our descriptions of behavior in a manner that allows us to focus on the outcome of the detailed physics — for example particle trajectories and displacements — then extend our descriptions of transport upward from this scale. In fact, we have no choice but to appeal to coarsened physics, often with semi-empirical closures. As we will see, this in turn means that descriptions of particle behavior often start at a kinematic level grounded in what we know to be correct based on probabilistic expressions of conservation (Section 1.3.5, Chapter 7), where we then attempt to unfold the ingredients of the kinematics based on defensible probabilistic assumptions regarding the dynamics. This, then, points to a reason we say statistical physics rather than statistical mechanics — to indicate the coarser resolution required in many problems. Nonetheless, the aim is the same: an explanation of emergent macroscopic behavior based on clarifying the stochastic mechanical behavior of the constituents of a system subject to imposed external constraints.

So what, then, constitutes the framework — the style of thinking and analysis — that we intend to pursue? In essence this consists of appealing to ideas, principles and methods of statistical physics that are *transferable* to other particle systems, particularly across scales, despite differences in the nature and behavior of the particles. To be sure, we will examine classical statistical mechanics treatments of particle motions and behavior even though the details might not be directly applicable to sediment systems. We will see that this provides essential context for understanding what *is* transferable across systems and scales.

We start by acknowledging that quantities we previously might have envisioned and treated as being deterministic are in fact random variables characterized by probability distributions that reflect the *inherent variability* in these quantities due to the stochasticity of the underlying dynamics, rather than just uncertainty of (unknown) factors influencing them. Among these quantities are: the numbers of particles entrained, disentrained and deposited within specified intervals of time and space; particle velocity, acceleration and energy states during transport; the flux of particles at a specified coordinate position due to, for example, raindrop impacts on a sloping surface or fluid forces in a turbulent flow; the local fluctuating elevation of the land surface or a streambed; and so on. This positions us to appeal to conservation of probability, and, in certain circumstances, leads to some form of master equation describing the time evolution of the probability distribution of the quantity of interest. In this manner we pay attention to the idea that the variability of a quantity about an expected (average) state is just as important as the average in terms of characterizing how the system works.

At the outset we adopt probabilistic descriptions of conservation that are indifferent to the existence of rarefied versus continuum conditions. These have the form of master equations, formulated at the particle scale rather than at any imagined continuum-like scale, starting with conservation of particle numbers (i.e. mass). Here we will discover that the ideas of advection and diffusion, familiar topics in deterministic continuum mechanics, are actually probabilistic constructs deriving from conservation. And, because a master equation is a statement of conservation of probability, then as suggested above we can apply the formalism to other quantities — the fluctuating elevation states of the land surface or a steambed, the fluctuating numbers of bed load particles within a control volume, and so on. Similarly, we can use this formalism to examine conservation of particle energy and momentum states as described by the probability distributions of these quantities.

Our approach involves appealing to counting processes to directly address the important implications and consequences of the small numbers of particles involved in rarefied (non-continuum) transport, relative to the large numbers of particles treated in classical statistical mechanics that give the continuum conditions and behavior of ordinary fluids and solids. This in effect is an extension of choosing probabilistic descriptions of particle behavior that are agnostic to the existence of rarefied versus continuum conditions. And, this includes paying attention to the equally important effects of the idiosyncrasies of the geometry of particle motions — the radial symmetry of rain splashed particle trajectories versus the strongly unidirectional motions of bed load particles — in contrast to the statistically isotropic motions of particles in a gas.

Our approach involves aiming at ensemble descriptions of quantities described as random variables. This implies, to the extent possible, leaning on the classical interpretation of probability to deduce the probability distribution if not the sample space of a quantity of interest: the set of all possible (accessible) values of the quantity. Here we will examine the ordinary idea of ensemble expectations as used in distribution theory, to mean the theoretically "true" moments of a distribution, and the less familiar idea of a Gibbs ensemble, to mean a great number of independent but nominally identical systems, and then show how these conceptualizations of ensembles are related. In turn, we will examine the implications of ensemble averaging versus time averaging, the central topic of ergodic theory, with relevance to classical versus frequentist interpretations of probability and with decided practical implications for data analysis.

Our use of probability distributions to characterize physical quantities is grounded in the idea that we should insist on physical interpretability of the distributions we choose. Indeed, an essential strategy of statistical physics is to deduce the specific distribution of a quantity of interest, such that its parametric values have a clear physical basis in the behavior of the constituents of the system. This is at the heart of Boltzmann's formulation of the distribution of energy states, and the formulation of the Maxwell–Boltzmann distributions. To be sure, because of the necessary coarsening of the physics in describing sediment systems, we must in many situations accept empirical closures. As a consequence we cannot achieve "first principles" results as in classical statistical mechanics. Nonetheless, we can insist on results that are consistent with our understanding of mechanical principles at our scale of interest, where dimensions and dimensional homogeneity always matter.

With this preview of things to come later in the book, let us now step back and briefly look at our epistemological experiences, at least for many of us in the Earth sciences.

In learning how to describe the behavior of mechanical systems, mostly we are initially exposed to deterministic examples. We study Newton's laws as these pertain to simple particle systems, and then move on to the behavior of solids and fluids treated as continuous materials, wrapping our heads around Lagrangian versus Eulerian perspectives. The formalism is unambiguous, and describing the behavior of a well constrained system is in principle straightforward. Indeed, much of the legacy of geophysics resides in the determinism of continuum mechanics. Perhaps it is therefore natural that we might envision that a mechanistic description of the behavior of a system implies that such a description ought to be, or perhaps only can be, a deterministic one. Such a perception represents a lost opportunity. The most elegant counterpoint example, as outlined above, is the field of classical statistical mechanics — devoted specifically to the probabilistic (i.e. non-deterministic) treatment of the behavior of gas particle systems in order to justify the principles of thermodynamics — yet which is no less mechanical in its conceptualization of this behavior than, say, the application of Newton's laws to the behavior of a deterministic system consisting of harmonic oscillators constructed from Hookean springs and dashpots, or involving the motion of a Newtonian fluid subject to specific initial and boundary conditions.

Once steeped in the language of mechanics, we understandably take comfort in mechanistic descriptions of system behavior. Specifically, we invest trust in the underlying foundation, and implied rigor, provided by classical mechanics. This is a good thing. But given the complexity and the uncertainty in describing the behavior of sediment systems, here it is essential to consider the idea that the concepts and language of probability are well suited to the problem of describing this behavior — precisely because of the complexity and uncertainty involved. This involves relaxing our expectations that a deterministic-like relationship exists between, say, the flux of bed load sediment and the fluid stress imposed on the streambed, or the flux of sediment on a hillslope and the local land-surface slope — particularly when these involve noise driven processes. This idea of leaning on probability to describe the behavior of sediment systems, as elaborated herein, is not as straightforward as describing the behavior of idealized gas particle systems. Nonetheless, the objective is the same: to be mechanistic, yet probabilistic. These worldviews are entirely compatible.

To be sure, the extent to which the tools of probability can be fruitfully brought to bear to characterize particle motions and transport vary with the specific process considered, and the information we have available to constrain any particular probabilistic description of motions. For example, we know far more about the probabilistic qualities of bed load sediment transport in shear flows based on flume experiments than, say, soil particle transport and mixing associated with bioturbation and athermal granular creep. The objective therefore is to aim at probabilistic descriptions of sediment particle motions and transport that lean on the *style of thinking* of statistical mechanics, recognizing that this endeavor — to reinforce points above — involves tailoring descriptions of transport to the process, the scales of interest and the techniques of observation and measurement used. The examples provided throughout the book illustrate these points.

1.3 Overview of Material

1.3.1 Probabilistic Concepts

As mentioned in the *Preface*, the first two chapters of the first part of the book (Foundational Material) offer a selection of topics from probability that are well suited to the analysis of sediment particle motions and transport. In addition to providing a foundation for topics presented in later chapters, the material presented in these two chapters, when combined with the rest of the book, is aimed at illustrating the essence of probabilistic thinking and analysis as a cornerstone of doing the science of sediment transport.

The starting point is a sketch of 20th century affairs in the field of probability, noting that there are numerous philosophical as well as scientific questions that remain open. This includes different interpretations of the meaning of probability and how it is defined (Spiegelhalter, 2024), evolving perspectives regarding the use and methods of statistics in ongoing "statistics wars" (Mayo, 2018), and the intriguing (noncontroversial) idea that quantum randomness associated with Heisenberg's uncertainty principle is fully manifest at our ordinary macroscopic scales such that quantum uncertainty can in fact dominate observable probabilistic behavior (Raymond, 1967; Albrecht and Phillips, 2014). Within this context we summarize the classical and frequentist views of probability — in their barest essence — as context for adapting principles of probability to the analysis of sediment particle motions and transport.

The idea of treating a physical quantity as a random variable that is characterized by a probability distribution is a central concept throughout this book. In Chapter 2 we focus on the definitions of a probability mass function and a probability density function of a single random variable, and the distribution of a function of a random variable. In addition we consider several special distributions, for example, heavy-tailed distributions and mixture distributions. The final part of this chapter is given to definitions and explanations of expectations, focusing on the moments (e.g. mean and variance) of probability distributions, and ensemble averaging versus time averaging. This serves to initially highlight ensemble averaging, a particularly important idea examined throughout the book.

In Chapter 3 we turn to probability distributions involving two or more random variables, and the algebra of random variables. Our coverage starts with joint probability distributions and conditional distributions. We then examine the covariance of two random variables and the autocovariance of random variables manifest as time series processes. This is followed by an introduction to the algebra and calculus of expectations. Here we highlight the idea that the basic algebra of random variables is the same as the familiar symbolic algebra of ordinary (deterministic) variables, but the *rules* of manipulating random variables differ in relation to quantities defined in terms of the probability distributions of random variables, for example, expected values and higher moments, covariances, and so on. Indeed, descriptions of sediment particle motions and transport that involve random variables and which either do not recognize or acknowledge the differences in algebra can badly misrepresent the physical situation. Our coverage includes rules of expectations associated with algebraic operations involving random variables, for example, sums and products of random variables. We finish the chapter by examining one of the most important results of probability theory — the central limit theorem — including immediate implications for sediment transport.

1.3.2 Basic Stochastic Processes

Starting in Chapter 4 we lean into the idea of describing ingredients of sediment transport — particle entrainment, disentrainment and the particle flux — as counting processes. A counting process is a stochastic process that is concerned with the numbers of events that occur over time or space. Thus, we will envision counting the number of raindrop impacts that occur within a specified area on the land surface and the number particles that are ejected by these impacts; the number of particles that are entrained at the crest of a hillslope then move downward over its surface; and the number of particles that are entrained from a sediment surface and deposited onto it within a turbulent shear flow. Then, for each of these we will envision counting the number of particles that cross a specified coordinate position — the essential basis of defining the particle flux at the coordinate position. To be clear, here an event refers to the occurrence of something: a raindrop impact, the entrainment or deposition of a particle, the crossing of a coordinate position, and so on.

In this counting of events we are in effect envisioning a time series consisting of the set of locations of the events on a timeline. Or, we are envisioning a spatial series consisting of the locations of the events on a coordinate axis. In turn, we will want to characterize the stochastic (probabilistic) structure of the series. That is, we wish to describe, for example, the average rate of occurrence of events and the distribution of the numbers of events that are likely to occur within a specified interval of time or space. This leads to the more general idea of stochastic processes, certain ones of which may be considered counting processes.

A stochastic process may be denoted as

$$\{z(s); s \in S\}. \tag{1.1}$$

Here, z denotes the quantity of interest, a random variable, that varies with time or space s, where S denotes what mathematicians refer to as the index set, which may be discrete or continuous. We will focus on continuous-time and continuous-space processes in which case a stochastic process over time might be denoted as $\{z(t); t \geq 0\}$ and a stochastic process over the spatial coordinate x might be denoted as $\{z(x); x \geq 0\}$. The curly braces indicate that $z(t)$ is to be viewed as a time series and $z(x)$ is to be viewed as a space series. We mostly will be concerned with time series, although at several points we address spatial processes.

Note that $z(t)$ and $z(x)$ may represent integer-valued or real-valued quantities. If $z(t)$ and $z(x)$ represent integer numbers of events, as in the examples above, then these are counting processes. Usually a counting process also is defined as one in which the integer value is non-decreasing with t or x. But the idea of a stochastic process is more general than this; not all stochastic processes are counting processes. Stochastic processes that are not counting processes abound. Examples include the coordinate position of a particle or its velocity as these vary with time, and the local elevation of a streambed or the land surface.

Focusing on time, a stochastic process $z(t)$ proceeds over time according to specific probabilistic rules. In Chapter 4, and then again in Chapter 7, we will focus on a small set of continuous-time processes. The starting point is a Poisson process. Because of its special properties (for example, a Poisson process is "memoryless") and its appearance in so many natural systems, a Poisson process is the foundation for examining other stochastic processes. Perhaps the quintessential example of a Poisson process is a time series consisting of radioactive decay events, where each event is entirely independent of previous decay events. We also will see, for example, that raindrop impacts are Poissonian, as are photon arrivals on a surface.

Closely related to a Poisson process are renewal processes, which generally are more varied in their stochastic structure. Certain renewal processes produce "bursty" time series of events. We will examine, for example, the appearance of bursty series of crossing events defined by the instants that bed load particles cross the end of an experimental flume — a form of intermittency in transport. In addition, Poisson and renewal processes may be either homogeneous or inhomogeneous, qualitatively to mean that the expected intensity of events may be fixed or vary in some manner with time.

We will examine simple Markov processes, focusing on what are referred to as birth–death Markov processes. A Markov process is defined as one in which the transition to a new state with the occurrence of an event depends only on the current state $z(t)$ attained with the occurrence of the previous event. As the name implies, stochastic birth–death processes are often adopted in descriptions of population dynamics. A natural connection with sediment systems is when we view a particle entrainment event as a birth and a deposition event as a death. These processes thus can be viewed together to examine the outcome of erosion and deposition on a granular surface.

In relation to this we will briefly consider simple Lévy processes consisting of the difference of two stochastic processes, for example, the difference of two Poisson processes. This is important in certain problems involving a competition between processes. For example, this occurs with rain splash transport when counting particle that cross a coordinate position in both positive and negative directions, the outcome of which defines the net particle flux at the coordinate position. Or, this competition occurs with entrainment and deposition of particles within a specified area on a streambed or the land surface, with implications for interpreting the Exner equation.

Let us note that a stochastic process as outlined above characterizes the probabilistic structure of a time series or space series of events — for example entrainment, deposition and crossing events — but in itself reveals nothing directly about what produces these events. In some situations we might have a mechanistic theory that points to a specific stochastic process. But in some situations, a mechanistic explanation is unlikely. For example, in relation to rain splash transport we might qualitatively appeal to the randomness of raindrop nucleation and growth in describing raindrop impacts as a Poisson process (Chapter 11), but going much beyond this qualitative level of explanation would be difficult (and a task for atmospheric physics).

In contrast, we also will examine select stochastic processes that are conceptualized mechanistically at the outset. This includes processes described by a Langevin equation (Chapter 4), a stochastic differential equation expressed in the form of Newton's second law, originally used to describe Brownian particle motion, but with applications at macroscopic scales. In particular we will use a Langevin-like equation to highlight the role of particle–bed collisions in modulating bed load particle velocities (Chapter 12). Similarly, we will examine the process of particle disentrainment with respect to both time and space based on *survival analysis* (also referred to as *reliability theory*). When viewed as a mechanically informed stochastic process, this type of analysis consists of a probabilistic description of the rate of particle disentrainment, which determines the probability distribution of particle travel times or displacements measured start to stop. When viewed kinematically, the analysis provides a central ingredient of our formulation of the entrainment forms of the sediment particle flux and the Exner equation (Chapters 6 and 7).

1.3.3 Particle Motions

Sediment particles move in response to forces exerted on them by moving fluids, by particle–particle interactions during transport, by the actions of terrestrial and aquatic biota, or simply in response to gravity. In focusing on the mechanisms that produce these motions, we typically organize our thinking in terms of the specific transport process: the transport of bed load in a turbulent shear flow, the aeolian transport of sand, the transport and mixing of particles associated with bioturbation, the process of dry particle ravel down a rough inclined surface, and so forth. In Chapter 5, however, we momentarily step back and consider basic attributes of sediment particle motions and transport more or less independently of the processes involved. This is aimed at providing clarity in our descriptions of motions and the precise meaning of quantities defined from measurements of particle motions, for example, those obtained from high-speed imaging.

Starting with well-known definitions of the particle velocity, acceleration and jerk, we highlight how these quantities characteristically fluctuate in response to imposed forces, with implications for the behavior of particle phase trajectories in the velocity–acceleration phase space associate with bed load transport. In subsequent sections we consider the nature of particle–surface interactions, highlighting the consequences of collisions, with implications for the statistical mechanics of particle disentrainment, modulation of particle velocities during transport, and the randomization of particle trajectories in relation to particle diffusion. We consider definitions of particle rest times and wait times, highlighting the implications of distinguishable versus indistinguishable particles. And, we consider definitions of the ages and residence times of particles in relation to their movement through a control volume, showing later in Chapter 14 that these are strongly influenced by the geometry of the volume, the location and proximity of the faces through which the particles move, and the intensity of particle mixing within the volume.

We then turn to measurements of particle motions, starting with a description of the inherent uncertainty that goes with image resolution in time series (videos) of motions. This includes how finite differencing of measured particle positions influences the fidelity of approximations of particle velocity and acceleration. We end Chapter 5 with a description of the censorship of motions associated with measurements over finite experimental intervals ("windows") of space and time, and the implications for estimating the distributions of particle travel distances and associated travel times.

1.3.4 Kinematics of the Particle Flux

Among the many items included within the general topic of sediment transport, perhaps historically most attention has been given to descriptions of the particle flux — the number (or mass or volume) of particles moving through a control surface at a specified coordinate position, per unit area (or width), per unit time. In Chapter 6 we examine essential kinematic elements of the sediment particle flux. Because descriptions of the particle flux under rarefied conditions are inextricably conditioned on the lengths and time intervals used in averaging, we present the flux as a counting process to highlight this conditioning, and to illustrate the consequences of the small numbers of particles involved in rarefied transport — in stark contrast with continuum conditions.

We then turn to definitions of the particle flux, focusing on the *activity form* and the *entrainment form* of these definitions. The activity form is continuum-like in its conceptualization, and involves the product of a measure of the number of moving particles and a suitable average particle velocity. The entrainment form is focused on the motions of particles that are entrained from a surface then return to it according to a distribution of travel distances. These definitions are centered on the movement of particles through a control surface, so they are immediately adaptable to descriptions of the Exner equation involving the divergence of the flux (Chapter 7). Our description of the particle flux provides a preview of the idea that the flux involves both advective and diffusive parts, and the importance of distinguishing between probabilistically expected conditions versus what occurs in any realization due to the relatively small numbers of particles involved in rarefied transport.

We examine implications and consequences of time averaging. This is focused on measurements of time series of sediment transport, and how the measurement interval Δt influences our understanding and interpretation of the particle flux, including the rate of convergence of estimates of the flux to the expected value, with decided practical implications. Our descriptions of counting processes covered in Chapter 4 have a central role in this topic.

We end the chapter by examining the concepts of *local transport* versus *nonlocal transport*. This features the entrainment form of the particle flux, which embodies the idea of nonlocal transport, to mean that the flux at a position x at time t may be viewed as the outcome of conditions influencing the entrainment of particles from upstream or upslope positions at earlier instants in time as well as their subsequent travel to and arrival at x at time t. We clarify the physical versus the mathematical perspectives of local and nonlocal transport, and why all sediment transport is nonlocal when viewed from a physical perspective.

1.3.5 Master Equations

In our examination of basic probabilistic concepts (Chapter 2) we highlight the idea that a probability distribution describing a physical quantity must sum (or integrate) to unity. This is a simple but profound property. It means that probability is a conserved quantity, just as we insist that mass, momentum and energy are conserved quantities. If the physical quantity represents, say, the configuration or mechanical state of a particle or system of particles where the probability distribution of the possible states changes with time, then a *master equation* is an expression that describes this time evolution of the distribution of states. The gravitas connoted in the name master equation therefore is merited. Namely, as a formal expression of conservation of probability, a master equation is as fundamental as a local conservation law (e.g. continuity) in physics. In Chapter 7 we consider several forms of master equations.

When applied to particle configurations or motions, we can use a master equation to describe the time evolution of the probability distribution of particle states, for example, coordinate positions or momentum states or energy states. We also may consider extensive particle properties whose probability distributions change with time. The centerpiece of this form of master equation is a probability distribution describing transitions to any one state from all other possible states, and vice versa, embedded within a convolution integral over all possible states.

In formulating a master equation focused on particle coordinate positions we will examine two forms, one involving continuous particle motions and one involving discontinuous motions. The latter provides a way of taking into account that, in many settings, sediment particles are only intermittently activated and moving. The formulation then leads to an Exner equation of particle conservation that explicitly accounts for the particle activity.

When applied to counting processes in which we are interested in numbers of events — entrainment, deposition or crossing events — or the fluctuating numbers of particles within a control volume, a master equation again provides a concise way to describe transitions from one number state to another. The outcome is a description of the time evolution of the ensemble distribution of the number of interest. This type of formulation in particular provides a clear illustration of our version of a Gibbs-like ensemble of systems, and what is meant by ensemble averaging versus time averaging; and it looms large in treatments of rain splash transport (Chapter 00), bed load transport (Chapter 00) and surface evolution (Chapter 00).

As a formulation of conservation of probability, a master equation is kinematic rather than dynamical in nature. It reveals no physics. This is left to be sorted out in the particular application. Nonetheless, inasmuch as the embedded probability distribution describing transitions in state satisfies certain conditions, then the master equation naturally leads to descriptions of conservation that involve advection and diffusion. Perhaps the most celebrated of such descriptions — certainly the most widely used — is the Fokker–Planck equation, which effectively is an advection–diffusion equation. Indeed, Einstein (1905) and von Smoluchowski (1906) used the Fokker–Planck equation to describe Brownian particle motions, and Chandrasekhar (1943) used an analogous master equation — the Kramers–Chandrasekhar equation — to examine the dynamics of stellar systems.[6] Culling (1963) first introduced a master equation to the Earth-surface community in his description of soil creep as a particle diffusion-like process, building directly from Chandrasekhar's work. We will see that the Fokker–Planck equation also has a central role in characterizing the probabilistic behavior of stochastic processes described by stochastic differential equations. This includes the Langevin equation (Chapter 4) which, as mentioned above, is a stochastic differential equation expressed in the form of Newton's second law, originally used to describe Brownian particle motion, but much broader in its applications.

In many situations we are interested in the motions of sediment particles that are entrained from a surface and then return to it after traveling some distance, and we focus on motions measured parallel to the surface: the streambed or the hillslope surface. For this intrinsically two-dimensional problem, we will formulate a master equation whose centerpiece is the joint distribution of particle displacements and associated travel times, which is convolved with the particle entrainment rate over all possible starting positions. This leads to the entrainment form of the Exner equation, which in turn can be rewritten as an advection–diffusion equation analogous to the Fokker–Planck equation. The formulation is closely aligned with the idea of nonlocal transport (Chapter 6), and it is a central element of treatments of rarefied particle motions on hillslopes (Chapter 10), rain splash transport (Chapter 11), bed load transport (Chapter 12) and surface evolution (Chapter 13).

1.3.6 Particle Advection and Diffusion

When viewed at the macroscopic scale of an ordinary fluid, the ideas of advection and diffusion are familiar and qualitatively well defined. For example, consider a one-dimensional advection–diffusion equation describing the behavior of a dissolved or suspended material with concentration $c(x, t)$ in a

[6] On a personal note, the paper by Chandrasekhar (1943), in illustrating the natural extension of the formalism from Brownian motions to stellar motions, solidified my growing appreciation for the versatility of the probabilistic style of thinking involved.

moving fluid,

$$\frac{\partial c(x,t)}{\partial t} = -u\frac{\partial c}{\partial x} + \kappa\frac{\partial^2 c}{\partial x^2}\,. \tag{1.2}$$

We say that (1.2) describes the local rate of change in the concentration $c(x,t)$ [M L^{-3}] due to advection of the material at velocity u [L T^{-1}], where the advective flux $q_{m\mathrm{A}} = uc$ [M L^{-2} T^{-1}]. In turn we say that the concentration also changes due to a diffusive flux given by Fick's first law, $q_{m\mathrm{D}} = -\kappa\partial c/\partial x$ with diffusivity κ [L^2 T^{-1}]. The total mass flux q_m at any coordinate position x is $q_m = q_{m\mathrm{A}} + q_{m\mathrm{D}}$. And, this deterministic expression works well at the continuum scale. When viewed at the particle scale, however, the ideas of advection and diffusion are fundamentally probabilistic constructs, and this requires a complete restructuring of our thinking about these concepts in relation to sediment particles.

To initially illustrate this point, consider a cloud of sediment particles moving with varying velocities toward and through an elementary surface A [L^2]. When viewed at the particle scale, the instantaneous solid mass flux q_m [M L^{-2} T^{-1}] associated with the surface A is precisely defined by the surface integral of surface-normal velocities of the solid fraction, namely,

$$q_m(t) = \frac{1}{A}\int_A \rho_\mathrm{p}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A\,, \tag{1.3}$$

where ρ_p [M L^{-3}] is the particle mass density, $\mathbf{v}$ [L T^{-1}] is the *discontinuous* particle velocity field viewed at the surface A, and $\mathbf{n}$ is the unit vector normal to A. For studies of sediment transport, however, this precise definition is impractical. Except possibly using high-speed imaging of a small number of particles (Drake et al., 1988; Lajeunesse et al., 2010; Roseberry et al., 2012; Ballio et al., 2014; Houssais et al., 2015) at high resolution, the flux described by (1.3) is virtually impossible to measure, and we are far from possessing a theory of sediment motions that describes the velocity field $\mathbf{v}$ in any realistic setting. Conventional descriptions of the flux therefore instead appeal to measures of collective particle behavior, specifically averaged quantities such as the average particle velocity and number concentration, to replace the detailed information contained in the particle velocity field $\mathbf{v}$ at the surface A.

An important objective of Chapter 7 and Chapter 8 is to show that this replacement of the precise information contained in the velocity field $\mathbf{v}$ at A with averaged quantities requires a careful rendering of these averaged quantities. For example, in the case of an ordinary continuum fluid the particle number density at the Knudsen scale is statistically uniform owing to homogenization by high-intensity particle–particle collisions (Appendix A). As a consequence the particle flux viewed at the continuum scale is purely advective such that the local mass flux parallel to x is $q_m = \rho\langle v_x\rangle$ with fluid mass density ρ and locally averaged particle velocity $\langle v_x\rangle$. In contrast, the number density of dissolved or suspended particles within the surrounding sea of "unmarked" fluid particles may at any instant vary spatially. As a consequence the flux of these particles has both advective and diffusive parts, as in (1.2). Here, advection means that the average motion of the marked particles matches that of all fluid particles, and diffusion means that for probabilistic reasons marked particles on average move from locations with relatively high concentration toward locations with lower concentration.

Compare this with sediment particles transported under non-continuum conditions. Homogenization of the particle number density by particle–particle collisions does not occur, and systematic spatial variations in the particle number density can exist — a hallmark of certain sediment transport processes, notably bed load transport. As a consequence the particle flux in general has both advective and diffusive parts. Here, as in continuum theory, advection is associated with the mean particle motion. Particle diffusion is then associated with variations in particle velocities about the average velocity in the presence of spatial variations in the number density. Moreover, particle diffusion also occurs in relation to spatial variations in the rate of excitation of particles. Yet the precise description of the particle flux given by (1.3) does not distinguish between advection and diffusion. Thus, because any definition of the flux must be consistent with the definition (1.3), we must demonstrate this consistency in appealing to a description involving averaged quantities.

If we simply accept at the outset that sediment particles move with varying velocities and trajectories in a quasi-random manner as in, say, molecular systems, then the temptation is strong to approach this problem heuristically and, for example, assume a Fick-like description of particle diffusion that is added to the mean motion. Instead, however, we will approach this problem with a direct rendering of the flux based on a probabilistic description of particle motions. This will allow us to show the consistency in the formulation with the surface-integral definition of the flux, and it will reveal the geometrical basis of particle diffusion. Perhaps most importantly, this rendering will allow us to accommodate features of sediment particle motions — notably their start-and-stop behavior — that are distinct from the behavior of particles in classic thermodynamic systems.

Our phenomenological description of particle diffusion in Chapter 8 clearly reveals its probabilistic basis in relation to spatial variations in particle numbers or variations in the rate of particle excitation. We then examine measures of diffusion, starting with the canonical reference point of Gaussian diffusion (or normal diffusion or Fickian diffusion) of a Brownian particle as described by Einstein (1905) and von Smoluchowski (1906) — that particle spreading, measured by the rate of change in the variance of particle coordinate positions, increases linearly with time. This represents a key reference point for comparison with sediment particle diffusion, including the possibility of anomalous diffusion in which the mechanics give particle spreading that is either faster or slower than Gaussian diffusion.

Although we lean on kinematic descriptions and measures of diffusion originally developed for gas systems, the mechanics of sediment particle diffusion are far different from those of gas particles. Particle diffusion in a gas occurs independently of advective motion, but abiological sediment particle diffusion cannot occur separately from advection. We fully develop this idea in Chapter 8, starting with a description of particle motions on a Galton board (also known as a quincunx or bean machine).

1.3.7 Considerations of Entropy

The concept of entropy is a central element of statistical mechanics and thermodynamics, with far-reaching implications across all of the natural sciences. Indeed, a major part of the efforts of Boltzmann and Gibbs was focused on clarifying the meaning and implications of entropy. In 1948, Claude Shannon defined entropy in the context of information theory, with profound implications for informatics as well as the analysis of system complexity (Golan and Harte, 2022) in the natural sciences, engineering, social sciences, economics, and so on. The concept of entropy is, to say the least, a grand idea. And, given the complexity of sediment systems, we should not be surprised that entropy bears on questions in sediment transport.

The Gibbs entropy is defined as

$$S = -k_{\mathrm{B}} \sum_{x} p_x(x) \log p_x(x) \,, \tag{1.4}$$

with the Boltzmann constant k_{B}. The Shannon entropy is defined as

$$H = -\sum_{x} p_x(x) \log p_x(x) \,. \tag{1.5}$$

In the work of Boltzmann and Gibbs, x represents an energy state so that $p_x(x)$ represents the distribution of these states, the set of all possible ways to arrange a great number of particles into accessible energy states subject to macroscopic (thermodynamics) constraints. In the work of Shannon, x represents an element of a system of communication, for example an alphabet, so that $p_x(x)$ represents the probability distribution of the occurrence of such elements in relation to transmitted information.

The celebrated physicist and probabilist Edwin Jaynes (1957a, 1957b) elaborated the significance of the fact that the Gibbs entropy in statistical mechanics and the Shannon entropy in information theory are essentially one and the same, differing only by a constant. This similarity inspired Jaynes to champion the use of a maximum entropy criterion in choosing a probability distribution, leading to what is now known as the maximum entropy method (aka MaxEnt or MEM). The key idea of the maximum entropy method, whether viewed as a method of statistical mechanics or as one of inferential statistics, is that it provides an unbiased choice of a distribution by honoring only what is known mechanically about a system. That is, this unbiased choice is a maximally noncommittal choice that is faithful to what we do not know; it is therefore the most reasonable choice in the absence of additional information (Jaynes, 1957a; Williamson, 2010, pp. 25 and 51). Importantly, mechanical constraints imposed on the system are part of the choice of the distribution, as opposed to empirical fitting without regard to such constraints. The maximum entropy method has been applied in a remarkable variety of fields (Shore and Johnson, 1980; Ramirez and Carta, 2006; Verkley and Lynch, 2009; Singh, 2011; Peterson et al., 2013; Golan and Harte, 2022), including sediment transport (Furbish and Schmeeckle, 2013; Furbish et al., 2016, 2021c).

In Chapter 9 we examine Jaynes's view that the Shannon entropy is fundamentally a measure of uncertainty, or equally, a measure of how things are organized into possible states. We also examine the idea of *accessible* states of a system, and the idea of *typical sets* of states, with broader philosophical implications for thinking about the states and configurations of systems that we seem to repeatedly observe versus the infinite sets of states and configurations that could occur. Then, in Chapter 15 we apply the maximum entropy method in describing the energetics of rarefied particle motions on hillslopes as well as bed load particle velocities to suggest how the probability distributions of these motions reflect mechanical constraints.

1.3.8 Applications

Each of the six chapters in the second part of the book (Applications) systematically steps through the ingredients of its featured topic, connecting these ingredients with material covered in the eight chapters of the first part of the book (Foundational Material). As a preview of these connections, here we consider highlights of the featured topics.

The chapter *Rarefied Transport on Hillslopes* considers what is likely the most fundamental of all sediment transport processes imaginable — the motions of natural angular particles down rough inclined surfaces under the influence of gravity, negligibly influenced by the surrounding fluid. This topic serves to highlight the statistical mechanics of particle disentrainment in which, following entrainment, the cessation of motion within any downslope interval is probabilistic. A description of the energetics of particle–surface collisions during downslope movement yields a well-defined distribution of particle travel distances whose form and parametric values reflect a balance between the particle kinetic energy gained from gravitational potential energy and the extraction of kinetic energy by collisions. The analysis illustrates why the notion of a deterministic threshold for cessation of motion is an illusion, with implications transferable to bed load transport.[7]

The processes described in the chapter *Rain Splash Transport* are ideal for illustrating the consequences of small numbers of particles involved in rarefied transport, where unsteady conditions preclude the convergence of the particle number flux to an expected value over rainstorm time scales. This topic also highlights effects of nonlocal transport, where the particle flux is not just proportional to the land-surface slope as is usually assumed, but also depends on the surface concavity at length scales comparable to particle displacements — scales that are of particular interest in the initial development of rills by surface flows, where it is often asserted, incorrectly, that rain splash transport

[7] Similarly, particle entrainment is probabilistic, and the notion of a so-called critical shear stress marking the onset of bed load transport is an empirical expedient — an illusion of mechanics.

dampens rill development.

In the chapter *Bed Load Transport* the profound consequences of the small numbers of moving particles involved in rarefied transport, in concert with the idiosyncrasies of particle motions, are on full display. Despite an historic focus on nominally steady transport, natural conditions are mostly unsteady at multiple scales. Both steady and unsteady conditions involve an inherent variability in the numbers of moving particles and the particle flux, such that the variance of the flux varies systematically with the sampling interval, conditioned on the spatial resolution of the measurements. This variability in turn depends on the particle size in a mixture of sizes. The analysis points to transport as a stochastic process — for example a compound Poisson or renewal process — that exhibits burstiness in relation to collective entrainment and microtopographic rearrangements during transport. The statistical mechanics basis of particle motions is embodied in the velocity–acceleration phase behavior of the active particle ensemble. This is manifest in the phase space as a circulation of particle phase trajectories that reflect continuous acceleration and jerk due to fluid forces and particle–bed interactions (mostly dissipative collisions). Indeed, this phase behavior highlights the central role of particle–bed interactions during transport, which strongly modulate particle velocities and their probability distributions, as well as rates of streamwise and transverse particle diffusion.

The chapter *Surface Evolution* considers the aspects of sediment transport that change the configurations of granular surfaces — the land surface in relation to rain splash transport and streambeds in relation to bed load transport. Herein we highlight the versatility and clear physical interpretablility of the entrainment form of the Exner equation via its focus on the essential elements of the phase transitions of transport (Section 1.4), its embodiment of the idea of nonlocal transport (Chapter 6), and the clarity it provides in illustrating consequences of the small numbers of particles involved in rarefied transport. The analysis illustrates how the microtopographic smoothing conventionally attributed to rain splash transport is only a probabilistic expectation, and any smoothing that does occur is negligible over the time scales of rainstorms capable of producing surface flows and initial rill development. With rain splash transport we also examine two important forms of particle diffusion: that associated with surface concavity, whose nonlocal effect is to modulate the particle flux and its divergence; and diffusion associated with variations in particle excitation leading to surface roughening by mound growth beneath desert shrubs, which therein act as sediment capacitors (sensu Furbish et al., 2009). With bed load transport, the analysis shows why the local streambed elevation fluctuates about the expected conditions for nominally steady controlling factors. Like the rain splash problem, the initial growth of bedforms as an instability of the particle–fluid interface is a probabilistic behavior. Because the convolution form of the Exner equation explicitly incorporates the distribution of particle hop distances, the formulation fully subsumes advection and any approximation of particle diffusion, and it correctly selects a preferred bedform wavelength without appealing to the semi-empirical heuristic of a saturation length, as is conventionally done.

The chapter *Particles in Soils* systematically examines soils as complex granular materials whose bulk, creeping motions on hillslopes arise from a combination of chronic granular deformation and patchy, intermittent particle displacements associated with biological and mechanical disturbances. The analysis shows that a mechanically active soil cannot be considered continuum-like akin to a deforming dense granular material, except possibly locally. Instead, the configuration and motion of creeping soil on a hillslope, when viewed over a time scale approaching the soil residence time on the hillslope, must be conceptualized as representing ensemble averaged conditions. Then, the configuration and motion of creeping soil on a real hillslope at any instant is just one possible realization that is likely to exhibit significant variability about expected (idealized) conditions. Natural tracer particles, for example quartz grains marked with cosmogenic ^{10}Be atoms or possessing optically stimulated luminescence ages, indicate the occurrence of significant particle mixing in natural soils, and point to the key role of mechanical disturbances; such mixing cannot be achieved just with granular deformation. In view of our uncertainty regarding the complex micro-mechanics of creeping soil that occurs over time scales much longer than the human experience, parsimony points to the probabilistic simplicity of a Fokker–Planck-like equation to describe tracer particle motions and mixing. The analysis requires careful averaging to obtain an advection–diffusion–reaction equation whose form is correctly matched to the type of tracer.

The chapter *Considerations of Entropy* applies the maximum entropy method to describe the distribution of particle travel distances on hillslopes in relation to their energetics during downslope movement, and the distribution of bed load particle velocities in a turbulent shear flow in relation to the ensemble averaged particle acceleration of zero. The analysis is aimed at suggesting how these probability distributions, obtained as maximum entropy distributions, reflect mechanical constraints imposed on particle motions. For the topic of rarefied particle motions on hillslopes, the analysis also serves to clarify the meaning of nonlocal versus local transport in relation to the form of the distribution of particle travel distances. In both cases we highlight Jaynes's philosophy of appealing to mechanical constraints as part of choosing a distribution, as opposed to empirical fitting without regard to such constraints.

1.4 Rarefied Versus Continuum Conditions

The continuum hypothesis — the essential basis of continuum mechanics — stands as a triumph of the physical sciences.[8] This hypothesis allows us to envision many solid and fluid materials at our ordinary macroscopic scale of observation as being continuous things whose properties and behavior can be described using that part of calculus given to continuously differentiable functions — even though when we focus our attention on the scale of the elements of a "continuous" material, that is, at the particle scale, we discover that it is decidedly discontinuous. Indeed, many of the definitions of basic, familiar quantities describing the properties, rheology and motion of real materials — their intensive properties, thermodynamic state variables, rheological coefficients, discharges and fluxes, the divergence of these fluxes and so on — at the outset assume continuous substances and continuum behavior that involve smooth changes with respect to space and time. That said, this lovely continuum siren is to be avoided as a de facto starting point in descriptions of sediment motions and transport. Many sediment particle motions on Earth's surface are patchy, intermittent and demonstrably rarefied (Furbish et al., 2012a, 2017b, 2018c, 2021a; Ancey, 2020a, 2020b; Furbish and Doane, 2021) — conditions that are entirely at odds with continuum formulations of these motions.

From a theoretical point of view the clearest defensible starting point is to view rarefied sediment transport in terms of the phase transitions involved. Namely, in its barest essence, rarefied sediment transport consists of phase transitions of a granular material: the *entrainment* and the *disentrainment* of the material,[9] where intervening particle motions are directionally biased owing to gravity or fluid forces. This phenomenon occurs at the interface between a granular solid that contains interstitial fluid, and a fluid that intermittently contains the entrained solid, where individual particles reside only briefly in a state of motion close to the interface. In this view of things the physics of the phase transitions becomes central in describing transport, where the stochastic process of entrainment regulates the availability of particles and the stochastic process of disentrainment modulates the bias of the particle motions via particle–surface interactions.

This point of view acknowledges all attributes of particle motions, start to stop. As described in later chapters, this allows us to highlight how different features of particle motions are incorporated

[8] Let us be clear that we are referring to the version of this hypothesis as applied to descriptions of real material systems rather than to the related mathematical idea posed by Georg Cantor, that there is no set of numbers whose size falls between the two infinities associated with the natural numbers and the real numbers.

[9] In terms of phase changes in the basic states of matter, entrainment is akin to sublimation and disentrainment is akin to deposition in the case of rarefied conditions involving small concentrations of moving particles — although the physics of the phase changes is entirely different.

into distinct kinematic definitions of the particle flux and its divergence (e.g. the particle activity forms and the entrainment forms of the flux and the Exner equation; Chapters 6 and 7), it highlights the statistical mechanics of particle–surface interactions in explanations of motions at small scales, and it points to the dominant role of entrainment (i.e. particle availability) in setting transport rates when viewed at large space and time scales, where uncertainties in the factors controlling transport unavoidably increase. Importantly, these phase transitions and intervening motions do not constitute a continuum behavior.

Indeed, as mentioned in the Preface, one of the missteps of sediment transport science during the past century was to uncritically adopt a continuum framework for describing sediment particle motions and transport. This is despite our awareness of fundamental differences between sediment particle motions and those of particles composing ordinary fluid continua, as well as the vast differences in the numbers of particles involved. Because of this legacy, and because the perspective developed throughout this book is centered on a probabilistic non-continuum framework, we need to address the important differences between rarefied and continuum conditions rather than leaving this comparison unsaid. Many of the points we make concerning probabilistic versus continuum descriptions of sediment transport have been implied if not stated explicitly by others. Following the lead of van Kampen (1981, p. 176), I will therefore say them "loudly and, I hope, clearly."

1.4.1 Rarefied Conditions

As fully explained in later chapters, sediment transport in many natural and experimental settings involves rarefied conditions in which moving particles are at low concentrations. Rarefied conditions occur with transport of soil particles by rain splash (Furbish et al., 2007, 2009, 2017b; Dunne et al., 2010), the skittering of rockfall material over scree slopes (Gerber and Scheidegger, 1974; Kirkby and Statham, 1975; Statham, 1976; Dorren, 2003; Tesson et al., 2020; Furbish et al., 2021a; Williams and Furbish, 2021) and the raveling of particles on hillslopes following disturbances or release from temporary storage behind obstacles such as vegetation (Roering and Gerber, 2005; Lamb et al., 2011, 2013; DiBiase and Lamb, 2013; DiBiase et al., 2017; Doane, 2018; Doane et al., 2018, 2019; Roth et al., 2020). Rarefied conditions occur with transport of bed load particles, notably at moderate to small transport stages (Ancey et al., 2008; Ancey, 2010; Furbish et al., 2012b; Roseberry et al., 2012; Fathel et al., 2015; Ashley et al., 2021; Chartrand et al., 2022; Pierce et al. 2022; Ancey and Recking, 2023), and with ordinary dilute transport of sand by wind (Ungar and Haff, 1987).

In the situations summarized above the interactions between particles and the surfaces over which they move are far more important in determining the dynamical behavior of the particles than are their interactions with other moving particles (Furbish et al., 2021a; Furbish and Doane, 2021; Allemand et al., 2023), akin to granular shear flows at high Knudsen number (Risso and Cordero, 2002; Kumaran, 2005, 2006). And, when particle motions are coupled with surrounding fluid motions, particle–surface interactions combined with alternating states of motion and rest hold a central role in particle transport (Einstein, 1937, 1950; Nikora et al., 2002; Furbish et al., 2012a; Roseberry et al., 2012; Fathel et al., 2015; Ballio et al., 2018; Pierce and Hassan, 2020; Pierce, 2021; Pierce et al., 2022; Wyssmann et al., 2023; Li et al., 2023). Rarefied particle motions are therefore distinct from granular flows, suspensions and ordinary granular gases; and when fluid-shear driven, infrequent dissipative collisions between moving particles cannot support the particles against gravity as envisioned with intense sheet flows involving a granular kinetic behavior (Jenkins and Hanes, 1998; Pasini and Jenkins, 2005; Berzi and Fraccarollo, 2016).

In addition, when viewed with respect to their small number concentrations, the motions and transport of a well-defined group of sediment tracer particles amidst a sea of unmarked particles involve rarefied conditions. This includes natural tracer particles in soils, for example, particles identified by their cosmogenic nuclide concentrations or luminescence ages or intensities (Furbish et al., 2018b, 2018c; Gray et al., 2020), and the movement of artificial or tagged tracer particles in natural rivers (e.g. Ferguson and Wathen, 1998) and experimental rivers (e.g. Hill et al, 2010).

Here is a point of reference. For sediment particles with diameter D we can define a dimensionless particle activity as $\gamma_n^* = \langle \gamma_n \rangle D^2$, where $\langle \gamma_n \rangle$ is the average number of *moving* particles per unit area of a streambed or the land surface. A value $\gamma_n^* \sim 1$ coincides with a close-packed monolayer or a layer of thickness D within a granular flow. With rain splash transport during a heavy rainstorm the activity $\gamma_n^* \sim 10^{-3}$ and is much smaller for ordinary storm intensities (Chapter 11). With transport of sand and gravel as bed load within flume experiments and in natural rivers, typically the activity $\gamma_n^* \sim 10^{-4}$–10^{-1} for varying flow conditions and is much smaller for tracer particles (Chapter 12). In soils there might ideally be only 10–100 natural tracer particles within a cubic cm (Chapter 14). These numbers are far smaller than those required for continuum conditions. More importantly, as summarized in the next section, such particles do not possess the dynamical behavior needed to satisfy the continuum hypothesis.

Furbish and Doane (2021) highlight the points above and use the example of rarefied particle motions on hillslopes to outline philosophical and technical aspects of pursuing a statistical mechanics description of rarefied transport without assuming a continuum behavior at the outset. This is within the context of a broader effort. For example, Furbish and Doane (2021) offer a sample of 81 papers representing work on probabilistic elements of sediment motions and transport in five topical areas: bed load particle motions and transport; bed load tracer particle motions, including effects of particle-bed exchanges; nonlocal sediment transport on hillslopes; particle motions in soils, including tracer particles; and rain splash transport. Among these, Schumer et al. (2009) provide a valuable primer on using advection–dispersion equations to describe tracer particle transport and Ancey (2020a, 2020b) provides a review of the state of research efforts focused on bed load transport. Importantly, the probabilistic nature of this body of work, harking back to that of Einstein (1937), is not conditioned by a continuum framework, and it in part reflects insights gained from increasing access to advanced computations and high-speed measurement techniques, notably imaging. Much of this work is focused on the kinematics of particle motions and transport, but with increasing efforts to explicitly incorporate mechanics. For example, this includes efforts to couple bed load particle motions with fluid motions in concert with particle–surface interactions using a Langevin-like equation (Ancey and Heyman, 2014; Fan et al., 2014; Pierce, 2021; Pierce et al., 2022; Williams, 2024), and consideration of the energetics of particle motions on hillslopes (Furbish et al., 2021a, 2021b, 2021c).

Questions concerning the use of a continuum framework to describe moving granular materials are not new. In his pioneering work on granular flows as a fluid mechanical phenomenon, Haff (1983) cautions for the need to assess whether such flows actually satisfy the continuum hypothesis, and indeed Haff's cooling law is limited to the homogeneous cooling state (Brito and Ernst, 1998; Nie et al., 2002; Brilliantov and Pöschel, 2004; Dominguez and Zenit, 2007; Brilliantov et al., 2018; Yu et al., 2020). Owing to limitations of continuum-like descriptions, studies of granular gases have fully embraced numerical simulations with supporting experiments to describe gas dynamics in which inelastic collisions have a central role in clustering, collapse and aggregation behaviors (Brilliantov and Pöschel, 2004; Brilliantov et al., 2018). Efforts to explore hydrodynamic descriptions of granular gases typically focus on restrictive conditions, for example, the onset of clustering as an instability (Mitrano et al., 2014; Louge, 2014) and the appearance of hydrodynamic behavior at certain scales (Duffy and Brey, 2003). Similarly there are ongoing questions concerning the rheological behavior of dense granular flows (see reviews by Delannay et al., 2017, Guazzelli and Pouliquen, 2018 and Berzi, 2024). For example these include: the significance of nonlocal effects (Pouliquen and Forterre, 2009; Henann and Kamrin, 2013; Tang et al., 2018; Fazelpour and Daniels, 2023) and the dynamic compressibility (Heyman et al., 2017) in continuum-like models relating the effective viscosity to the inertial number; the extent to which the framework provided by kinetic theory of granular gases can be extended to dense granular flows (Berzi, 2024); and effects of boundary conditions involving varying modes of collisional friction (Jenkins, 1992; Zhang et al., 2015). Thus, the material presented in this book, centered on rarefied sediment transport, may be viewed as part of a larger effort to clarify consequences of the specifics of particle behavior during transport rather than assuming these specifics intrinsically admit a conventional continuum-like description.

1.4.2 Continuum Conditions

When we say "continuum," normally we mean this as defined in continuum mechanics: a material whose attributes — its local thermodynamic state variables, transport coefficients, velocity and so on — can be considered deterministic quantities when viewed at a scale much larger than the particle (molecular) scale, specifically much larger than the mean free path in the case of ordinary fluids. And, these quantities are justifiably treated as fields described by continuously differentiable functions. The mechanical and thermodynamic properties of such continua permit internal transfers of momentum and energy in the form of, say, shear stresses, sound waves and heat flow. When we refer to a continuum description or a continuum formulation of sediment transport, however, we do not mean a continuum in the same sense as above. Rather, we are implying that we can use the same formalism of continuum mechanics — that sediment particles in transport possess attributes and behavior that justify the use of continuously differentiable equations involving deterministic quantities analogous to a continuum fluid. Despite fundamental differences between fluid and rarefied sediment systems, this involves asserting that locally defined continuum-like velocities, fluxes and associated statements of conservation, if not state variables, can be described in the same manner. We do not concern ourselves with, say, internal energy transfers or sound propagation unless working with granular gases or dense granular flows.

Often we explicitly appeal to particle continuum conditions (e.g. Paola and Voller, 2005), and this is certainly implied in conceptualizations of bed load transport as a traction layer involving Coulomb-like behavior (e.g. Bagnold, 1966). Coleman and Nikora (2009) point out that various derivations of the Exner equation normally assume that the sediment–fluid mixture is a continuum leading to what they refer to as mixture-scale equations. But a continuum-like formulation could refer to the idea that attributes of a phenomenon are treated as continuously differentiable functions without necessarily claiming that the phenomenon involves a continuum material. For example, the Smoluchowski equation[10] (an advection–diffusion equation) describing the time evolution of the continuous probability density function of tracer particle positions is continuum-like. This equation describes the probabilistic ensemble behavior (sensu Gibbs, 1902) of an individual particle having little to do with a continuum, or it may equally apply to a great number of particles in any realization (Schumer et al., 2009; Furbish et al., 2018c; Furbish and Doane, 2021). In this latter case the set of particles is treated as a continuous ("continuum") material, as with, for example, a noncolloidal suspension (Acrivos, 1995), although we do not associate with it the mechanical and thermodynamic properties of an ordinary continuum. More generally we can write continuously differentiable master equations to describe the time evolution of the probability distributions of particle states without implying continuum conditions (Ancey et al., 2008; Ancey 2010, 2020a; Furbish et al., 2012a, 2018a, 2018c, 2021a; Heyman, 2014; Ancey and Heyman, 2014; Pierce and Hassan, 2020; Pierce, 2021; Furbish and Doane, 2021; Wu et al., 2023; Chapter 7).

In order for a fluid to be considered a continuum, the conditions that must be satisfied, and the particle dynamics associated with these conditions, are well understood (Jeans, 1940; Meyer, 1971; Acrivos, 1995; Furbish, 1997). In the case of sediment particles, however, analogous criteria have not been systematically established. In appealing to a continuum description of sediment behavior, there is the risk of assuming that particle-scale conditions during transport sufficiently mimic those in a fluid system that the continuum hypothesis is justified. Consequently key insights can be gained

[10]The Smoluchowski equation is like the more well-known Fokker–Planck equation (Risken, 1984), but it pertains to particle positions rather than particle velocities.

by comparing, at the outset, certain features of particle behavior in a continuum fluid with features of sediment particle motions, as in the standard practice of comparing ordinary gases as a canonical reference point with granular gases and denser flows (Jaeger et al., 1996; Brilliantov and Pöchel, 2004; Delannay et al., 2017; Furbish et al., 2021a; Berzi, 2024). To be clear, whereas the differences in particle behavior outlined next are well known, we nonetheless suggest that we have not yet paid sufficient attention to the implications and consequences of these differences. Specifically:

1) *Particle numbers:* With respect to a granular–fluid interface, a square cm monolayer of sediment particles with diameter $D = 1$ mm at close packing contains about 10^2 particles, and a square cm monolayer of $D = 0.1$ mm sediment particles contains about 10^4 particles. A square cm "monolayer" of air particles (molecules) at ordinary pressure–temperature conditions contains about 10^{11} particles, even though the mean free path is about 10^3 times the effective particle diameter. With respect to particles composing a dense granular material, a cubic cm of sediment particles with diameter $D = 1$ mm at close packing contains about 10^3 particles; and for a soil formed on granitic bedrock a cubic cm might contain only 10–100 tracer particles marked by ^{10}Be atoms (Furbish et al., 2018c). A cubic cm of air contains about 10^{19} particles. As a consequence of the great difference between particle numbers in fluid and sediment systems embodied in these simple examples, the law of large numbers yields rapid convergence to ensemble expected values of attributes of fluid systems (e.g. pressure, velocity) at the continuum scale (Appendix A), but a similar convergence does not occur for sediment systems (Figures 1.2 and 1.3, Section 1.5). Quantities that are considered deterministic in continuum fluid systems — notably particle fluxes — are stochastic quantities with decided uncertainty in sediment systems (Chapter 6).

2) *Collision frequencies:* Frequent elastic particle-particle collisions — about 10^{28} per second in a cubic cm of air at ordinary pressure-temperature conditions — maintain uniform particle number densities in continuum conditions at a scale larger than the mean free path (Appendix A). This mechanism of homogenization does not occur in rarefied sediment systems where effects of dissipative particle–surface interactions (Schmeeckle et al., 2001) in concert with fluctuating fluid forces dominate particle behavior, giving a decided patchiness in particle movement reminiscent of clustering in granular gases (e.g. Roseberry et al., 2012; Furbish et al., 2017b), or patterns in movement associated with systematic spatial variations in particle excitation. Moreover, frequent particle–particle collisions in gases are the source of thermodynamic quantities such as pressure, temperature and viscosity (Jeans, 1940; Meyer, 1971) — quantities that we do not ascribe to rarefied sediment systems such as bed load and rain splash transport.

3) *Particle trajectories:* Random particle motions in a continuum fluid are statistically isotropic with speeds (10^2–10^3 m s^{-1}) that typically are much larger than local continuum fluid velocities. Bed load particle trajectories are strongly unidirectional and therefore anisotropic in the horizontal plane, and the magnitudes of velocity fluctuations ($\leq 10^{-1}$ m s^{-1}) are similar to or smaller than the mean motion (e.g. Lajeunesse et al., 2010; Roseberry et al., 2012; Seizilles et al. 2014; Liu et al., 2019; Williams, 2024). Intermittent, rarefied particle motions on hillslopes similarly exhibit pronounced downslope versus transverse asymmetry (Williams and Furbish, 2021). In relation to the first item above, this anisotropy combined with small particle speeds and intermittency of motions, unlike a continuum fluid, strongly influences the counting of particles that cross a specified coordinate position during two-dimensional transport and therefore the rate at which the component fluxes converge to their ensemble averages (Chapter 12). In contrast, the radial symmetry of particle trajectories during rain splash transport lends an insensitivity to this counting relative to the mean motion and the orientation of the coordinate system (Chapter 11).

4) *Particle excitation:* Continuous particle motions in a continuum fluid guarantee that the particle flux can be resolved at all positions and times as macroscopic flow conditions change due to, for example, a change in the pressure field. In stark contrast, aside from chronic athermal granular creep (Deshpande et al., 2021), sediment particles alternate between states of motion and rest. Like granular

gases, sediment particles are excited externally and their activity varies over space and time. In relation to the first item above, the spatial and temporal resolution and the associated uncertainty of the particle flux vary as the numbers of moving particles change (Chapter 6).

Appendix A summarizes essential elements of continuum conditions as background for the four points above, including comparisons with rarefied conditions. Because of the context and contrast provided by these four points, we repeatedly refer to them throughout the book.

1.5 Preview of Several Essential Concepts

To end this first chapter, let us briefly preview how material covered in the book translates to descriptions of particle transport. We focus on attributes of the sediment particle flux, noting that this quantity is ideal for illustrating pronounced consequences of the small numbers of particles involved in rarefied transport. We use the examples of rain splash and bed load transport because the geometry of particle trajectoriess are in each case reasonably well-constrained, yet distinctive, as revealed by high-speed imaging, and because the mechanisms of particle excitation offer a key contrast for points we want to make. In later chapters we elaborate the concepts previewed here while greatly broadening our coverage of topics.

Consider the transport of sediment by rain splash on the land surface or as bed load on a streambed, and let $q_x(x)$ [$\mathrm{L}^2\ \mathrm{T}^{-1}$] denote the one-dimensional particle volumetric flux at the coordinate position x. Then, in the case of rain splash we might propose that

$$q_x(x) = f(R, S_x, \mathbf{S}, ...)\,, \tag{1.6}$$

where R denotes rainfall properties (e.g. rainfall intensity, drop sizes), S_x denotes the local land-surface slope parallel to x and $\mathbf{S}$ represents a set of soil properties (e.g. particle sizes, moisture content). In the case of bed load we might typically assume a semi-empirical relation,

$$q_x(x) = f(\tau_\mathrm{b}, \mathbf{S}, ...)\,, \tag{1.7}$$

where τ_b denotes the macroscopically defined stress imposed by the fluid on the streambed and now $\mathbf{S}$ represents factors describing the sediment (e.g. particle sizes) and streambed conditions. The ellipses in (1.6) and (1.7) acknowledge possible effects of unspecified factors.

The flux $q_x(x)$ in (1.6) or (1.7) conventionally is treated as a deterministic quantity that is continuously differentiable with respect to space and time. Indeed, upon letting $\eta(x,t)$ denote the local elevation of the land surface or streambed with volumetric particle concentration c_b, then when we write the divergence of the flux to give the Exner equation,

$$c_\mathrm{b}\frac{\partial \eta(x,t)}{\partial t} = -\frac{\partial q_x}{\partial x}\,, \tag{1.8}$$

we are undebatably adopting a deterministic continuum-like framework in which the flux $q_x(x)$ is treated as a local instantaneous quantity specified by the form and ingredients of (1.6) or (1.7). This of course means we are assuming the derivatives $\partial f(R, S_x, ...)/\partial x$ and $\partial f(\tau_\mathrm{b}, ...)/\partial x$ are well defined, specifically the implied derivatives of one or more controlling factors in the functions. For rarefied conditions, however, this framework is entirely misleading, as follows.

Consider Figure 1.1, which comes from Chapter 11. This figure shows high-fidelity numerical simulations of individual realizations of the particle number flux $\hat{q}_{nx}(\Delta t)$ [$\mathrm{L}^{-1}\ \mathrm{T}^{-1}$] associated with rain splash on a horizontal surface ($S_x = 0$) during steady rainfall. Each realization is calculated as the net number $N(\Delta t)$ of particles crossing a position x per length Δy normal to x, per averaging interval Δt, namely, $\hat{q}_{nx}(\Delta t) = N(\Delta t)/\Delta y \Delta t$. Moreover, each realization arises from *precisely the same* controlling factors: the rainfall intensity, surface slope, particle size and so on. Indeed, these are

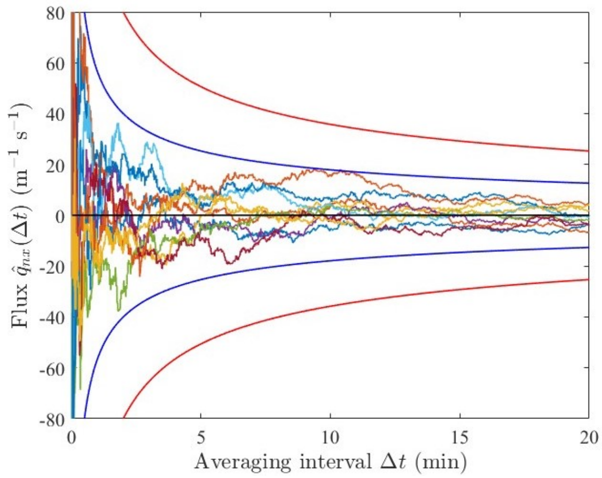

Figure 1.1: Plot of 10 realizations of the number flux $\hat{q}_{nx}(\Delta t)$ showing (black line) expected flux $\langle q_{nx} \rangle = 0$ together with (blue lines) ± 1 and (red lines) ± 2 standard deviations in the values of realizations $\hat{q}_{nx}(\Delta t)$ about the expected value. Values of the particle volumetric flux $\hat{q}_x(\Delta t)$ are obtained by multiplying by the particle volume V_p, namely, $\hat{q}_x(\Delta t) = V_\mathrm{p}\hat{q}_{nx}(\Delta t)$. Similar results are obtained when the expected flux $\langle q_{nx} \rangle$ is finite with nonzero surface slope S_x.

examples of an infinite set of possible realizations for the same controlling factors. Hence, the figure also shows the ± 1 and ± 2 standard deviations in the values of possible realizations $\hat{q}_{nx}(\Delta t)$ as their magnitudes generally decrease with increasing averaging interval Δt.

Consider Figure 1.2, which comes from Chapter 12. This figure shows measured realizations of

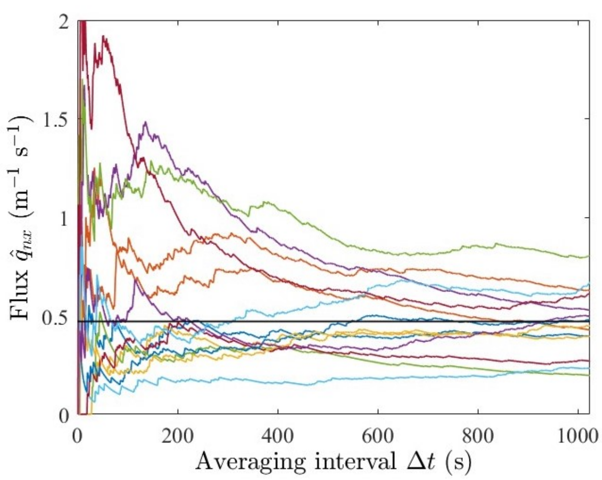

Figure 1.2: Plot of 14 realizations of the number flux $\hat{q}_{nx}(\Delta t)$ with (black line) expected flux $\langle q_{nx} \rangle$ for $D = 11.3$–16 mm particles. Values of the particle volumetric flux $\hat{q}_x(\Delta t)$ are obtained by multiplying by the particle volume V_p, namely, $\hat{q}_x(\Delta t) = V_\mathrm{p}\hat{q}_{nx}(\Delta t)$. The expected value $\langle q_{nx} \rangle$ is estimated as the time average of the full time series. Similar results occur for other sizes in the gravel mixture.

the particle number flux $q_{nx}(\Delta t)$ for the volumetric modal size of a gravel mixture transported as bed load during steady conditions. As with the rain splash example, each realization is calculated as $\hat{q}_{nx}(\Delta t) = N(\Delta t)/\Delta y \Delta t$. Each realization arises from the same controlling factors: the flume discharge, water-surface slope, the mixture of particle sizes on the streambed and so on. Thus, as with rain splash, these are examples of an infinite set of possible realizations for the same controlling factors.

Notice the particle numbers, per averaging width, in the ordinates of Figure 1.1 and Figure 1.2. One might imagine, for example, that 10^7–10^8 particles entrained by raindrops within a square meter during one minute of a heavy rainstorm (Chapter 11) represents a large number, and that with averaging the law of large numbers yields rapid convergence of the particle flux to its ensemble expected value. Similarly, one might imagine that 10^4–10^5 gravel particles of various sizes crossing the end of a half-meter wide flume during one hour at a moderate transport stage (CHapter 12) represents a large number. But in fact these are tiny numbers of particles compared to the numbers involved in continuum transport conditions. Individual realizations of the flux and its divergence therefore can involve large fluctuations about ensemble expected values. The law of large numbers is at work, but not on the space and time scales that we associate with continuum behavior (Appendix A).

It thus should be apparent from these examples that the flux is not an instantaneous quantity; it is a time-averaged quantity that can involve large fluctuations depending on the averaging interval Δt. Equally important, the flux is not a deterministic quantity that is continuously differentiable with respect to space and time; it is a random variable described by a probability distribution $f_{\hat{q}_x}(\hat{q}_x; \Delta t)$ with expected value $\langle q_x \rangle$, where angle brackets denote an ensemble average. To emphasize this point we write

$$\hat{q}_x(x; \Delta y, \Delta t) = \langle q_x \rangle + \varepsilon(\Delta y, \Delta t) , \tag{1.9}$$

where the circumflex highlights that this is a deviation about the expected value $\langle q_x \rangle$ with uncertainty ε. The appearance of Δy and Δt in this expression reminds us, as fully elaborated in Chapter 6, that the value $\hat{q}_x$ and the associated uncertainty ε depend on the spatial resolution Δy and the averaging interval Δt.

Here is a key point. Uncertainty is not synonymous with error. That is, one must not interpret ε in (1.9) as representing an error in the estimate of the expected value $\langle q_x \rangle$. To paraphrase Ancey (2020a) with a slight respin, whereas a deterministic perspective typically views the "noise" represented by ε as the outcome of stochastic (unexplained) influences and thus error, the probabilistic perspective elaborated herein views this noise as being an inherent feature of the transport process. In a real setting, only one of an infinite set of possible realizations occurs. Yet with a specific set of values of the controlling factors, any one of the infinite set *could* occur, each entirely consistent with the physics involved. Uncertainty then simply refers to a description of the likelihood of possible values $\hat{q}_x$ relative to the expected value $\langle q_x \rangle$, as illustrated in Figures 1.1 and 1.2. In later chapters we further illustrate and elaborate this interpretation of the particle flux based on time series of transport. And, based on a suitable form of the Exner equation we show how realizations of the land-surface elevation and the streambed elevation similarly fluctuate about expected changes in elevation.

Here is a second key point. Inasmuch as the objective is to associate an observed (measured) value of the flux with specific (nominally known) controlling factors that predict the expected value $\langle q_x \rangle$, then we may view $\hat{q}_x$ as an estimate of the expected value $\langle q_x \rangle$ conditioned on the length Δy and averaging interval Δt. This is in effect an experimental perspective, and is at the heart of interpreting experimental measurements to go with formulations of the functional relations embodied in (1.6) and (1.7). As elaborated in later chapters, the length Δy and the averaging interval Δt therefore loom large in describing the uncertainty involved in such efforts.

In formulating descriptions of the particle flux under rarefied conditions there is value in starting with an unambiguous probabilistic definition of the expected flux $\langle q_x \rangle$. For example, for both rain splash transport and bed load transport the expected flux under nominally uniform, steady conditions can be expressed kinematically as (Chapter 6)

$$\langle q_x \rangle = E \langle L_x \rangle , \tag{1.10}$$

where E [L T^{-1}] is the particle volumetric entrainment rate and $\langle L_x \rangle$ [L] is the ensemble average particle displacement parallel to x measured start to stop. Notwithstanding uncertainty in the specific values of E and $\langle L_x \rangle$, this is the correct description of the ensemble expected value $\langle q_x \rangle$ so long as the average $\langle L_x \rangle$ is finite. The ingredients E and $\langle L_x \rangle$ are well defined with a clear physical interpretation and probabilistic basis (Einstein, 1950) deriving from a master equation (Furbish et al., 2012a, 2017a, 2017b; Chapter 6; see also Pierce et al., 2022).

In the case of rain splash transport, much effort has been given to quantifying particle entrainment by drop impacts and particle displacements (Chapter 11). Importantly, we will add a clear description of the stochastic structure of the entrainment rate E; this quantity is not merely an empirical rate constant. It specifically represents a compound Poisson process so long as we accept that the drop impact rate is Poissonian (e.g. Uijlenhoet et al., 1999; Jameson and Kostinski, 2002; Larsen et al., 2005) and that the number of entrained particles per drop impact is a random variable with defined mean and variance. These statistical mechanical elements in turn provide a formal basis for describing the inherent uncertainty represented by realizations $\hat{q}_x$ about the expected flux $\langle q_x \rangle$ (Figure 1.1).

The problem is harder for bed load transport. Nonetheless, (1.10) offers physical clarity. Much effort has been given to quantifying particle displacements leading to the ensemble average displacement $\langle L_x \rangle$, although this mostly has been centered on kinematic descriptions that are not yet fully connected to macroscopic flow conditions (Chapter 12). Moreover, the probabilistic conceptualization of entrainment by Einstein (1950), elaborated by others (e.g. Fernadez Luque and Van Beek, 1976; Tsujimoto, 1978; Nakagawa and Tsujimoto, 1980; van Rijn, 1984, 1986), lends itself for treating the entrainment rate E as more than a semi-empirical rate constant. What is needed is a description of the specific stochastic structure of E, as with rain splash, whose mechanical basis incorporates effects of moving particles (e.g. Ancey et al., 2008; Ancey and Heyman, 2014; Ma et al., 2014) and any feedbacks with changing streambed conditions (e.g. Singh et al., 2009; Masteller and Finnegan, 2017; Yager et al., 2018; Pierce and Hassan, 2020b) in relation to macroscopic flow conditions. Because (1.10) sets up the problem as a counting process (Chapter 4), it provides a useful starting point. Nonetheless, inasmuch as entrainment depends on effects of moving particles, then the problem likely requires incorporating elements of the more conventional activity form of the flux involving the product of the particle activity and the ensemble averaged particle velocity (Chapter 6). We examine the point that achieving a clear understanding and description of particle entrainment is at the forefront of problems in sediment transport research.

Here we have highlighted basic elements of the sediment particle flux as a preview of how material covered in the book translates to descriptions of particle transport. To be sure, this topic involves much more, and we return to it numerous times in later chapters, building from foundational material in Part I of the book. In our coverage of transport processes, time scales of interest span intervals much smaller than individual particle travel times to intervals much longer than the human experience, and length scales span less than a particle diameter to the lengths of hillslopes and river reaches. We of course extend the coverage to processes beyond rain splash and bed load transport.

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