The sediment particle flux: An illustration of the pronounced consequences of the small numbers of particles involved in rarefied (non-continuum) transport conditions

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1 Rarefied sediment transport

For context and clarity this section provides a qualitative overview of rarefied transport conditions. Consequences of the small numbers of particles involved in rarefied transport are then described in the next section, specifically in relation to the sediment particle flux.

Sediment transport in many natural and experimental settings involves rarefied conditions in which moving particles are at low concentrations. Rarefied conditions occur with transport of soil particles by rain splash (Furbish et al., 2007, 2009, 2017b; Dunne et al., 2010), the skittering of rockfall material over scree slopes (Gerber and Scheidegger, 1974; Kirkby and Statham, 1975; Statham, 1976; Dorren, 2003; Tesson et al., 2020; Furbish et al., 2021a; Williams and Furbish, 2021) and the raveling of particles on hillslopes following disturbances or release from temporary storage behind obstacles such as vegetation (Roering and Gerber, 2005; Lamb et al., 2011, 2013; DiBiase and Lamb, 2013; DiBiase et al., 2017; Doane, 2018; Doane et al., 2018, 2019; Roth et al., 2020). Rarefied conditions occur with transport of bed load particles, notably at moderate to small transport stages (Ancey et al., 2008; Ancey, 2010; Furbish et al., 2012b; Roseberry et al., 2012; Fathel et al., 2015; Ashley et al., 2021; Chartrand et al., 2022; Pierce et al. 2022; Ancey and Recking, 2023), and with ordinary dilute transport of sand by wind (Ungar and Haff, 1987).

In the situations summarized above the interactions between particles and the surface over which they move are far more important in determining the dynamical behavior of the particles than are their interactions with other moving particles (Furbish et al., 2021a; Furbish and Doane, 2021; Allemand et al., 2023), akin to granular shear flows at high Knudsen number (Risso and Cordero, 2002; Kumaran, 2005, 2006). And, when particle motions are coupled with surrounding fluid motions, particle–surface interactions combined with alternating states of motion and rest hold a central role in particle transport (Einstein, 1937, 1950; Nikora et al., 2002; Furbish et al., 2012a; Roseberry et al., 2012; Fathel et al., 2015; Ballio et al., 2018; Pierce and Hassan, 2020a; Pierce, 2021; Pierce et al., 2022; Wyssmann et al., 2023; Li et al., 2023). Rarefied particle motions are therefore distinct from granular flows, suspensions and ordinary granular gases; and when fluid-shear driven, infrequent dissipative collisions between moving particles cannot support the particles against gravity as envisioned with intense sheet flows involving a granular kinetic behavior (Jenkins and Hanes, 1998; Pasini and Jenkins, 2005; Berzi and Fraccarollo, 2016).

In addition, when viewed with respect to their small number concentrations, the motions and transport of a well-defined group of sediment tracer particles amidst a sea of unmarked particles involve rarefied conditions. This includes natural tracer particles in soils, for example, particles identified by their cosmogenic nuclide concentrations or luminescence ages or intensities (Furbish et al., 2018a, 2018b; Gray et al., 2020), and the movement of artificial or tagged tracer particles in natural rivers (e.g. Ferguson and Wathen, 1998) and experimental rivers (e.g. Hill et al, 2010).

Here is a point of reference. For sediment particles with diameter D we can define a dimensionless particle activity as $\gamma_n^* = \langle \gamma_n \rangle D^2$, where $\langle \gamma_n \rangle$ is the average number of *moving* particles per unit area of a streambed or the land surface. A value $\gamma_n^* \sim 1$ coincides with a close-packed monolayer or a layer of thickness D within a granular flow. With rain splash transport during a heavy rainstorm the activity $\gamma_n^* \sim 10^{-3}$ and is much smaller for ordinary storm intensities. With transport of sand and gravel as bed load within flume experiments and in natural rivers, typically the activity $\gamma_n^* \sim 10^{-4}$–10^{-1} for varying flow conditions and is much smaller for tracer particles. In soils there might ideally be only 10–100 natural tracer particles within a cubic cm. These numbers are far smaller than those required for continuum conditions, and particle dynamics bear little resemblance to continuum behavior (Appendix).

Nonetheless, descriptions of sediment transport, including rarefied conditions, conventionally adopt a continuum-like framework (e.g. Bagnold, 1966; Paola and Voller, 2005; also see Coleman and Nikora, 2009). Yet rarefied conditions are entirely at odds with continuum formulations of transport (Furbish et al., 2012a, 2012c; 2017a, 2017b, 2021a; Fathel et al., 2015; Ancey, 2020a, 2020b; Furbish and Doane, 2021; Williams and Furbish, 2021; Pierce, 2021; Hassan et al., 2022). As Furbish and Doane (2021, p. 633) note:

> The continuum hypothesis... stands as a triumph of the physical sciences... [allowing] us to envision many solid and fluid materials at our ordinary macroscopic scale of observation as being continuous things whose properties and behavior can be described using that part of the calculus given to continuously differentiable functions — even though when we focus our attention on the scale of the elements of a "continuous" material, that is, at the particle scale, we discover that it is decidedly discontinuous. That said, this lovely continuum siren is to be avoided as a de facto starting point in descriptions of sediment motions and transport.

Furbish and Doane (2021) use the example of rarefied particle motions on hillslopes to outline philosophical and technical aspects of pursuing a statistical mechanics description of rarefied transport without assuming a continuum behavior at the outset. This is within the context of a broader effort. For example, Furbish and Doane (2021) offer a sample of 81 papers representing recent work on probabilistic elements of sediment motions and transport in five topical areas: bed load particle motions and transport; bed load tracer particle motions, including effects of particle–bed exchanges; nonlocal sediment transport on hillslopes; particle motions in soils, including tracer particles; and rain splash transport. Among these, Schumer et al. (2009) provide a valuable primer on using advection–dispersion equations to describe tracer particle transport and Ancey (2020a, 2020b) provides a timely review of the state of research efforts focused on bed load transport. Importantly, the probabilistic nature of this body of work, harking back to that of Einstein (1937), is not conditioned by a continuum framework, and it in part reflects insights gained from increasing access to advanced computations and high-speed measurement techniques, notably imaging. Much of this work is focused on the kinematics of particle motions and transport, but with increasing efforts to explicitly incorporate mechanics. For example, this includes efforts to couple bed load particle motions with fluid motions using a Langevin-like equation (Ancey and Heyman, 2014; Fan et al., 2014; Pierce, 2021; Pierce et al., 2022; Furbish and Williams, 2025) and consideration of the energetics of particle motions on hillslopes (Furbish et al., 2021a, 2021b, 2021c; Williams and Furbish, 2021).

Here we step back to the kinematics of particle motions. We focus on attributes of the sediment particle flux, noting that this quantity is ideal for illustrating pronounced consequences of the small numbers of particles involved in rarefied transport. We use the examples of rain splash and bed load transport because the geometry of particle trajectoriess are in each case reasonably well-constrained, yet distinctive, as revealed by high-speed imaging, and because the mechanisms of particle excitation offer a key contrast for points we want to make.

2 Rain splash and bed load transport

Consider the transport of sediment by rain splash on the land surface or as bed load on a streambed, and let $q_x(x)$ [$\mathrm{L}^2\ \mathrm{T}^{-1}$] denote the one-dimensional particle volumetric flux at the coordinate position x. Then, in the case of rain splash we might propose that

$$q_x(x) = f(R, S_x, \mathbf{S}, ...)\,, \tag{1}$$

where R denotes rainfall properties (e.g. rainfall intensity, drop sizes), S_x denotes the local land-surface slope parallel to x and $\mathbf{S}$ represents a set of soil properties (e.g. particle sizes, moisture content). In the case of bed load we might typically assume a semiempirical relation,

$$q_x(x) = f(\tau_\mathrm{b}, \mathbf{S}, ...)\,, \tag{2}$$

where τ_b denotes the macroscopically defined stress imposed by the fluid on the streambed and now $\mathbf{S}$ represents factors describing the sediment (e.g. particle sizes) and streambed conditions. The ellipses in (1) and (2) acknowledge possible effects of unspecified factors.

The flux $q_x(x)$ in (1) or (2) conventionally is treated as a deterministic quantity that is continuously differentiable with respect to space and time. Indeed, upon letting $\eta(x,t)$ denote the local elevation of the land surface or streambed with volumetric particle concentration c_b, then when we write the divergence of the flux to give the Exner equation,

$$c_\mathrm{b}\frac{\partial \eta(x,t)}{\partial t} = -\frac{\partial q_x}{\partial x}\,, \tag{3}$$

we are undebatably adopting a deterministic continuum-like framework in which the flux $q_x(x)$ is treated as a local instantaneous quantity specified by the form and ingredients of (1) or (2). This of course means we are assuming the derivatives $\partial f(R, S_x, ...)/\partial x$ and $\partial f(\tau_\mathrm{b}, ...)/\partial x$ are well defined, specifically the implied derivatives of one or more controlling factors in the functions. For rarefied conditions, however, this framework is entirely misleading, as follows.

Consider Figure 1, which shows high-fidelity numerical simulations of individual realizations of the particle number flux $\hat{q}_{nx}(\Delta t)$ [$\mathrm{L}^{-1}\ \mathrm{T}^{-1}$] associated with rain splash on a horizontal surface ($S_x = 0$) during steady rainfall. Each realization is calculated as the net number $N(\Delta t)$ of particles crossing a position x per length Δy normal to x, per averaging interval Δt, namely, $\hat{q}_{nx}(\Delta t) = N(\Delta t)/\Delta y \Delta t$. Moreover, each realization arises from *precisely the same* controlling factors: the rainfall intensity, surface slope, particle size and so on. Indeed, these are examples of an infinite set

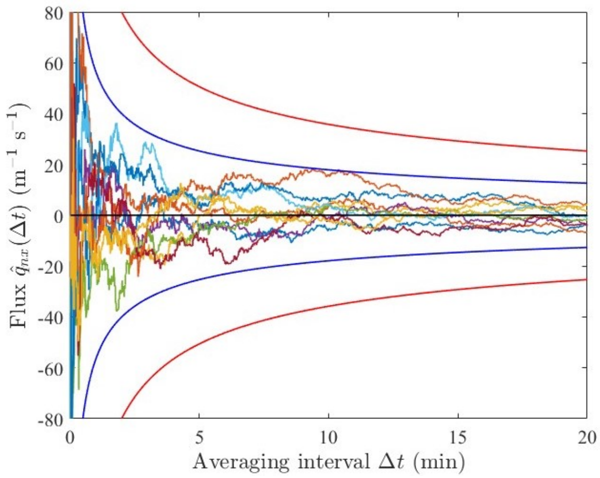

Figure 1: Plot of 10 realizations of the number flux $\hat{q}_{nx}(\Delta t)$ showing (black line) expected flux $\langle q_{nx} \rangle = 0$ together with (blue lines) ± 1 and (red lines) ± 2 standard deviations in the values of realizations $\hat{q}_{nx}(\Delta t)$ about the expected value. Values of the particle volumetric flux $\hat{q}_x(\Delta t)$ are obtained by multiplying by the particle volume V_p, namely, $\hat{q}_x(\Delta t) = V_\mathrm{p} \hat{q}_{nx}(\Delta t)$. Similar results are obtained when the expected flux $\langle q_{nx} \rangle$ is finite with nonzero surface slope S_x. Numerical simulations are based on the theory and experiments of Furbish et al. (2007).

of possible realizations for the same controlling factors. Hence, the figure also shows the ± 1 and ± 2 standard deviations in the values of possible realizations $\hat{q}_{nx}(\Delta t)$ as their magnitudes generally decrease with increasing averaging interval Δt.

Consider Figure 2, which shows measured realizations of the particle number flux $q_{nx}(\Delta t)$ for

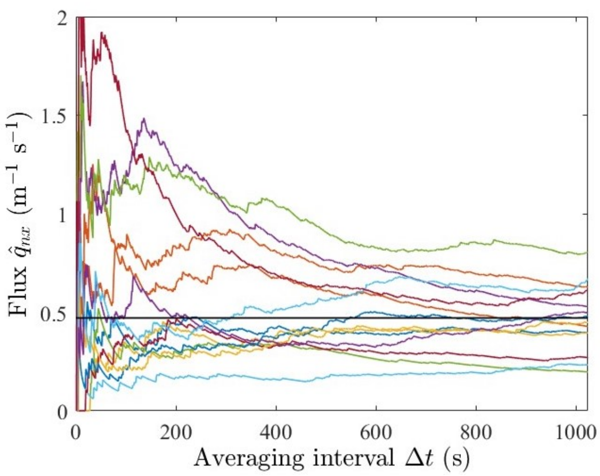

Figure 2: Plot of 14 realizations of the number flux $\hat{q}_{nx}(\Delta t)$ with (black line) expected flux $\langle q_{nx} \rangle$ for D = 11.3–16 mm particles. Values of the particle volumetric flux $\hat{q}_x(\Delta t)$ are obtained by multiplying by the particle volume V_p, namely, $\hat{q}_x(\Delta t) = V_\mathrm{p} \hat{q}_{nx}(\Delta t)$. The expected value $\langle q_{nx} \rangle$ is estimated as the time average of the full time series. Similar results occur for other sizes in the gravel mixture. Data come from experiments reported by Chartrand (2017).

the volumetric modal size of a gravel mixture transported as bed load during steady conditions. As with the rain splash example, each realization is calculated as $\hat{q}_{nx}(\Delta t) = N(\Delta t)/\Delta y \Delta t$. Each realization arises from the same controlling factors: the flume discharge, water-surface slope, the mixture of particle sizes on the streambed and so on. Thus, as with rain splash, these are examples of an infinite set of possible realizations for the same controlling factors.

Notice the particle numbers, per averaging width, in the ordinates of Figure 1 and Figure 2. One might imagine, for example, that 10^7–10^8 particles entrained by raindrops within a square meter during one minute of a heavy rainstorm (e.g. Furbish et al., 2017b) represents a large number, and that with averaging the law of large numbers yields rapid convergence of the particle flux to its ensemble expected value. Similarly, one might imagine that 10^4–10^5 gravel particles of various sizes crossing the end of a half-meter wide flume during one hour at a moderate transport stage (Chartrand, 2017) represents a large number. But in fact these are tiny numbers of particles compared to the numbers involved in continuum transport conditions. Individual realizations of the flux and its divergence therefore can involve large fluctuations about ensemble expected values. The law of large numbers is at work, but not on the space and time scales that we associate with continuum behavior.

It thus should be apparent from these examples that the flux is not an instantaneous quantity; it is a time-averaged quantity that can involve large fluctuations depending on the averaging interval Δt. Equally important, the flux is not a deterministic quantity that is continuously differentiable with respect to space and time; it is a random variable described by a probability distribution $f_{\hat{q}_x}(\hat{q}_x; \Delta t)$ with expected value $\langle q_x \rangle$, where angle brackets denote an ensemble average. To emphasize this point we write

$$\hat{q}_x(x; \Delta y, \Delta t) = \langle q_x \rangle + \varepsilon(\Delta y, \Delta t) \,, \tag{4}$$

where the circumflex highlights that this is a deviation about the expected value $\langle q_x \rangle$ with uncertainty ε. The appearance of Δy and Δt in this expression reminds us that the value $\hat{q}_x$ and the associated uncertainty ε depend on the spatial resolution Δy and the averaging interval Δt.

Here is a key point. Uncertainty is not synonymous with error. That is, one must not interpret ε in (4) as representing an error in the estimate of the expected value $\langle q_x \rangle$. To paraphrase Ancey (2020a) with a slight respin, whereas a deterministic perspective typically views the "noise" represented by ε as the outcome of stochastic (unexplained) influences and thus error, the probabilistic perspective outlined here views this noise as being an inherent feature of the transport process. In a real setting, only one of an infinite set of possible realizations occurs. Yet with a specific set of values of the controlling factors, any one of the infinite set *could* occur, each entirely consistent with the physics involved. Uncertainty then simply refers to a description of the likelihood of possible values $\hat{q}_x$ relative to the expected value $\langle q_x \rangle$, as illustrated in Figures 1 and 2. And, based on a suitable form of the Exner equation, it is straightforward to show how realizations of the land-surface elevation and the streambed elevation similarly fluctuate about expected changes in elevation.

Here is a second key point. Inasmuch as the objective is to associate an observed (measured) value of the flux with specific (nominally known) controlling factors that predict the expected value $\langle q_x \rangle$, then we may view $\hat{q}_x$ as an estimate of the expected value $\langle q_x \rangle$ conditioned on the length Δy and averaging interval Δt. This is in effect an experimental perspective, and is at the heart of interpreting experimental measurements to go with formulations of the functional relations embodied in (1) and (2). The length Δy and the averaging interval Δt therefore loom large in describing the uncertainty involved in such efforts.

In formulating descriptions of the particle flux under rarefied conditions there is value in starting with an unambiguous probabilistic definition of the expected flux $\langle q_x \rangle$. For example, for both rain splash transport and bed load transport the expected flux under nominally uniform, steady conditions can be expressed kinematically as

$$\langle q_x \rangle = E \langle L_x \rangle \,, \tag{5}$$

where E [L T^{-1}] is the particle volumetric entrainment rate and $\langle L_x \rangle$ [L] is the ensemble average particle displacement parallel to x measured start to stop. Notwithstanding uncertainty in the specific values of E and $\langle L_x \rangle$, this is the correct description of the ensemble expected value $\langle q_x \rangle$ so long as the average $\langle L_x \rangle$ is finite. The ingredients E and $\langle L_x \rangle$ are well defined with a clear physical interpretation and probabilistic basis (Einstein, 1950) deriving from a master equation (Furbish et al., 2012a, 2017a, 2017b; see also Pierce et al., 2022).

In the case of rain splash transport, much effort has been given to quantifying particle entrainment by drop impacts and particle displacements (see Furbish et al., 2007). Importantly, the stochastic structure of the entrainment rate E is known; this quantity is not merely an empirical rate constant. It specifically represents a compound Poisson process so long as we accept that the drop impact rate is Poissonian (e.g. Uijlenhoet et al., 1999; Jameson and Kostinski, 2002; Larsen et al., 2005) and that the number of entrained particles per drop impact is a random variable with defined mean and variance. These statistical mechanical elements in turn provide a formal basis for describing the inherent uncertainty represented by realizations $\hat{q}_x$ about the expected flux $\langle q_x \rangle$ (Figure 1).

The problem is harder for bed load transport. Nonetheless, (5) offers physical clarity. Much effort has been given to quantifying particle displacements leading to the ensemble average displacement $\langle L_x \rangle$, although this mostly has been centered on kinematic descriptions that are not yet fully connected to macroscopic flow conditions. Moreover, the probabilistic conceptualization of entrainment by Einstein (1950), elaborated by others (e.g. Fernadez Luque and Van Beek, 1976; Tsujimoto, 1978; Nakagawa and Tsujimoto, 1980; van Rijn, 1984, 1986; Lu and Cheng, 2025), lends itself for treating the entrainment rate E as more than a semi-empirical rate constant. What is needed is a description of the specific stochastic structure of E, as with rain splash, whose mechanical basis incorporates effects of moving particles (e.g. Ancey et al., 2008; Ancey and Heyman, 2014; Ma et al., 2014) and any feedbacks with changing streambed conditions (e.g. Singh et al., 2009; Masteller and Finnegan, 2017; Yager et al., 2018; Pierce and Hassan, 2020b) in relation to macroscopic flow conditions. Because (5) sets up the problem as a counting process, it provides a useful starting point. Nonetheless, inasmuch as entrainment depends on effects of moving particles, then the problem likely requires incorporating elements of the more conventional activity form of the flux involving the product of the particle activity and the ensemble averaged particle velocity (e.g. Lajeunesse et al., 2010; Furbish et al., 2012a). Achieving a clear understanding and description of particle entrainment is at the forefront of problems in sediment transport research.

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Adopting a continuum framework to describe rarefied transport imposes a deterministic style of thinking consisting of the concepts and language of continuum mechanics. Such descriptions, however, are entirely misleading. Expected values (e.g. the sediment transport rate) are assumed to be matched to specified controlling factors, and fluctuations in behavior for fixed controlling factors are then viewed as “noise” due to unexplained perturbations about the nominally preferred state. The effort thus translates to the creation of deterministic expectations of behavior that are inconsonant with the inherent, stochastic variability of complex systems — a hallmark of such systems.

In contrast, the probabilistic view outlined here fully embraces the mechanics of sediment particle motions and transport, but is entirely agnostic to the presence or absence of continuum conditions. This view acknowledges, relative to continuum conditions, the small numbers of particles involved during transport, and the pronounced consequences of these small numbers in concert with the idiosyncrasies of the geometry of sediment particle motions. Time series of quantities are treated as realizations of stochastic processes, with a focus on counting processes. Then, for example, the variability in realizations of the particle transport rate (Figure 1, Figure 2) is viewed as an inherent feature of the transport process where, for a specific set of controlling factors, any one of an ensemble of possible realizations *could* occur, each entirely consistent with the physics involved. Likewise, the variability in granular surfaces produced by transport represents the inherent richness of possible surface configurations. Expectations of behavior are thus probabilistic, where variability about an average state is just as important as the average in terms of characterizing how the process works.

Appendix

To reinforce the point that rarefied sediment transport has little to do with ordinary continuum behavior, here we briefly compare four features of particle behavior in a continuum fluid with features of sediment particle motions, as in the standard practice of comparing ordinary gases as a canonical reference point with granular gases and denser flows (Jaeger et al., 1996; Brilliantov and Pöchel, 2004; Delannay et al., 2017; Furbish et al., 2021a; Berzi, 2024). To be clear, whereas these differences in particle behavior are well known, we nonetheless suggest that we have not yet paid sufficient attention to the implications and consequences of these differences. Specifically:

1) *Particle numbers:* With respect to a granular–fluid interface, a square cm monolayer of sediment particles with diameter $D = 1$ mm at close packing contains about 10^2 particles, and a square cm monolayer of $D = 0.1$ mm sediment particles contains about 10^4 particles. A square cm "monolayer" of air particles (molecules) at ordinary pressure–temperature conditions contains about 10^{11} particles, even though the mean free path is about 10^3 times the effective particle diameter. With respect to particles composing a dense granular material, a cubic cm of sediment particles with diameter $D = 1$ mm at close packing contains about 10^3 particles; and for a soil formed on granitic bedrock a cubic cm might contain only 10–100 tracer particles marked by ^{10}Be atoms (Furbish et al., 2018b). A cubic cm of air particles contains about 10^{19} particles. As a consequence of the great difference between particle numbers in fluid and sediment systems embodied in these simple examples, the law of large numbers yields rapid convergence to ensemble expected values of attributes of fluid systems (e.g. pressure, velocity) at the continuum scale, but a similar convergence does not occur for sediment systems (Figures 1 and 2, Section 2). Quantities that are considered deterministic in continuum fluid systems — notably particle fluxes — are stochastic quantities with decided uncertainty in sediment systems.

2) *Collision frequencies:* Frequent elastic particle–particle collisions — about 10^{28} per second in a cubic cm of air at ordinary pressure–temperature conditions — maintain uniform particle number densities in continuum conditions at a scale larger than the mean free path. This mechanism of homogenization does not occur in rarefied sediment systems where effects of dissipative

particle–surface interactions (Schmeeckle et al., 2001) in concert with fluctuating fluid forces dominate particle behavior, giving a decided patchiness in particle movement reminiscent of clustering in granular gases (e.g. Roseberry et al., 2012; Furbish et al., 2017b), or patterns in movement associated with systematic spatial variations in particle excitation. Moreover, frequent particle–particle collisions in gases are the source of thermodynamic quantities such as pressure, temperature and viscosity (Jeans, 1940; Meyer, 1971) — quantities that we do not ascribe to rarefied sediment systems such as bed load and rain splash transport.

3) *Particle trajectories:* Random particle motions in a continuum fluid are statistically isotropic with speeds (10^2–10^3 m s^{-1}) that typically are much larger than local continuum fluid velocities. Bed load particle trajectories are strongly unidirectional and therefore anisotropic in the horizontal plane, and the magnitudes of velocity fluctuations ($\leq 10^{-1}$ m s^{-1}) are similar to or smaller than the mean motion (e.g. Lajeunesse et al., 2010; Roseberry et al., 2012; Seizilles et al. 2014; Liu et al., 2019; Williams, 2024). Intermittent, rarefied particle motions on hillslopes similarly exhibit pronounced downslope versus transverse asymmetry (Williams and Furbish, 2021). In relation to the first item above, this anisotropy combined with small particle speeds and intermittency of motions, unlike a continuum fluid, strongly influences the counting of particles that cross a specified coordinate position during two-dimensional transport and therefore the rate at which the component fluxes converge to their ensemble averages. In contrast, the radial symmetry of particle trajectories during rain splash transport lends an insensitivity to this counting relative to the mean motion and the orientation of the coordinate system.

4) *Particle excitation:* Continuous particle motions in a continuum fluid guarantee that the particle flux can be resolved at all positions and times as macroscopic flow conditions change due to, for example, a change in the pressure field. In stark contrast, aside from chronic athermal granular creep (Deshpande et al., 2021), sediment particles alternate between states of motion and rest. Like granular gases, sediment particles are excited externally and their activity varies over space and time. In relation to the first item above, the spatial and temporal resolution and the associated uncertainty of the particle flux vary as the numbers of moving particles change.

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