Bed load particle velocities: The essential role of particle–bed collisions in the Langevin-like equation

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1 Context

The intermittent tumbling and hopping of sediment particles along a streambed in response to fluid forces and particle–bed interactions — bed load transport — naturally occurs under rarefied (non-continuum) conditions (Furbish et al., 2012a; Roseberry et al., 2012; Fathel et al., 2015). Here, the ensemble distribution $f_{v_x}(v_x)$ of streamwise particle velocities v_x (Fathel et al., 2015) is a key element of descriptions of bed load transport. This includes, for example, the definitions of the activity forms of the particle flux and the Exner equation (Furbish et al., 2012a), and explanations of particle phase trajectories in the velocity–acceleration phase space (Furbish et al., 2012b) during equilibrium transport.[1] Based on high-speed imaging of particle motions, various forms of the velocity distribution $f_{v_x}(v_x)$ have been reported for different sediment and flow conditions; among these are exponential, Gaussian and gamma-like distributions (see Pierce et al., 2022). Clarifying the statistical mechanical basis of the form of the velocity distribution $f_{v_x}(v_x)$ and its parametric values is a key challenge in sediment transport research.

One particularly interesting approach to this problem involves appealing to a Langevin-like equation to describe the distribution $f_{v_x}(v_x)$ of velocities v_x. The Langevin equation stands as a key landmark of early 20th century statistical mechanics. It is a stochastic differential equation used to describe how a system — its state — responds to combined deterministic and fluctuating forces, where the characteristic response time of the deterministic part is much larger than the time scale of the fluctuations. Although originally used to describe the velocity state of a Brownian particle subjected to viscous forces and the fluctuating forces of collisions with surrounding particles (Langevin, 1908), the resulting formalism and style of analysis is much broader and thus applicable to a wide range of problems.

In this essay we start with a brief primer on Brownian particle motion and the Langevin equation, highlighting the ergodic behavior of the particle in relation to thermal equilibrium. We then illustrate an application of a Langevin-like equation to describe the velocity distribution of bed load particles involving continuous particle motions. This highlights the essential role of particle–bed collisions in modulating particle velocities (Furbish et al., 2012b; Pierce, 2021; Williams, 2024). We then briefly turn to the problem of discontinuous, non-ergodic particle motions. In this case, owing to effects of turbulence and nonlinear particle–fluid coupling in concert with discontinuous

[1]See explanation in the essay: Statistical equilibrium transport of bed load sediment: The role of particle velocity, acceleration and jerk

motions, it might be possible to obtain a description of the distribution of particle velocities based on a Fokker–Planck equation, but not a Langevin-like equation.

2 Brownian motion and the Langevin equation

Focusing on the one-dimensional motion of a Brownian particle with mass m, let $x(t)$ denote its position and let $v_x(t)$ denote its velocity. The Langevin equation is a statement of Newton's second law,

$$m\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = -\gamma v_x(t) + \eta(t)\,. \tag{1}$$

Here, the term $-\gamma v_x(t)$ is a viscous drag force given by Stokes's law where $\gamma = 6\pi\mu R$ with dynamic viscosity μ and particle radius R. The quantity $\eta(t)$ denotes a randomly fluctuating force due to collisions with surrounding fluid particles. It is Gaussian with the expected value $\mathrm{E}(\eta) = \langle\eta(t)\rangle = 0$ and covariance $\langle\eta(t)\eta(t')\rangle = B\delta(t-t')$ with strength B. We can immediately identify an e-folding time, or correlation time, as $t_e = m/\gamma$, which represents viscous relaxation to the mean state.

As written, (1) is problematic, as it involves a mixture of deterministic and stochastic parts. The fluctuating force $\eta(t)$, as a white noise, is uncorrelated from one instant to the next, so it is not clear that the velocity $v_x(t)$ is differentiable. In fact, (1) is purely symbolic, what van Kampen (1981) refers to as a "pre-equation" that cannot be manipulated until a convention for doing this is specified. To illustrate this point we multiply (1) by the integrating factor e^{t/t_e}, define $z = v_x e^{t/t_e}$ so that $\mathrm{d}z = e^{t/t_e}\mathrm{d}v_x + e^{t/t_e}\gamma v_x \mathrm{d}t = e^{t/t_e}\eta(t)\mathrm{d}t$, then integrate and use the definition of z to obtain

$$v_x(t) = v_x(0)e^{-t/t_e} + \int_0^t e^{-(t-s)/t_e}\eta(s)\mathrm{d}s\,, \tag{2}$$

with initial velocity $v_x(0)$. This seems reasonable, but the meaning of the integral is unclear given that the noise $\eta(t)$ is essentially discontinuous. We therefore back up and rewrite (1) with $\eta(t)\mathrm{d}t = \mathrm{d}W(t)$, where $W(t)$ denotes a Wiener process.[2] Namely,

$$\mathrm{d}v_x(t) = -\frac{\gamma}{m}v_x(t)\mathrm{d}t + \frac{1}{m}\mathrm{d}W(t)\,. \tag{3}$$

This formally is a stochastic differential equation, often referred to as an Itô process.

With $\mathrm{d}x(t) = v_x(t)\mathrm{d}t$, integrating each term in (3) gives

$$v_x(t) - v_x(0) = -\frac{\gamma}{m}[x(t)-x(0)] + \frac{1}{m}[W(t)-W(0)]\,. \tag{4}$$

Focusing on the last term, here we are only assuming that $W(t)$ is continuous at time t. Imagine dividing W into n small increments, $W(t_1)-W(0), W(t_2)-W(t_1), ..., W(t_n)-W(t_{n-1})$. These increments are longer than the particle collision time, but shorter than the viscous relaxation time. If these increments are independent and Gaussian, then the sum of n increments is Gaussian. With $\eta\mathrm{d}t = \mathrm{d}W$ we may therefore write (2) as

$$v_x(t) = v_x(0)e^{-t/t_e} + \int_0^t e^{-(t-s)/t_e}\mathrm{d}W\,, \tag{5}$$

[2]Prof. Bernhard Mehlig of the Statistical Physics of Complex Systems group at the University of Gothenburg, Sweden (http://gu-statphys.org/) provides a straightforward demonstration of the result that $\eta(t)\mathrm{d}t = \mathrm{d}W$.

where the integral is now interpreted as a weighted sum of Gaussian increments to time t.

Let us now take an average over an ensemble of nominally identical systems, that is, the set of all possible realizations. This gives

$$\begin{aligned}\langle v_x(t)\rangle &= \left\langle v_x(0)e^{-t/t_e}\right\rangle + \left\langle \int_0^t e^{-(t-s)/t_e}\mathrm{d}W\right\rangle \\ &= \langle v_x(0)\rangle e^{-t/t_e} + \int_0^t e^{-\gamma(t-s)/t_e}\langle \mathrm{d}W\rangle \\ &= \langle v_x(0)\rangle e^{-t/t_e}\,, \end{aligned} \tag{6}$$

where $\langle v_x(0)\rangle$ is an average over the distribution of initial states, and we are using the fact that $\langle \mathrm{d}W\rangle = 0$. The last line in (6) shows that the average particle velocity approaches zero at long times t. Now consider the second moment of just the deterministic part of (5). This gives

$$\langle v_x^2(t)\rangle = \langle v_x^2(0)\rangle e^{-2t/t_e}\,, \tag{7}$$

which similarly shows that the average squared velocity approaches zero at long times t. However, thermal equilibrium precludes this outcome. The *equipartition theorem* for translational kinetic energy says that if $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ denotes the particle speed, then the total energy is partitioned equally among the three components. That is, $\langle (1/2)mv^2\rangle = (3/2)k_\mathrm{B}T$ giving $\langle v_x^2\rangle = k_\mathrm{B}T/m$, where k_B is the Boltzmann constant and T is temperature. Thus, the randomly fluctuating forces embodied in the stochastic part of (1) are *essential* for thermal equilibrium. This points to the *fluctuation–dissipation theorem* (Callen and Welton, 1951). Viscous drag dissipates thermal kinetic energy. But this in turn involves a conversion of thermal energy to kinetic energy of the Brownian particle.

We can write the Itô process (3) in terms of a Fokker–Planck equation describing the time evolution of the distribution $f_{v_x}(v_x)$ of velocities v_x. It is then possible to show that the competition between viscous damping and thermal excitation leads to a stationary Gaussian distribution $f_{v_x}(v_x)$ with expected value $\mathrm{E}(v_x) = \mu_{v_x} = 0$ and variance $\mathrm{Var}(v_x) = \sigma_{v_x}^2 = k_\mathrm{B}T/m$. In turn, because the particle position $\mathrm{d}x(t) = v_x(t)\mathrm{d}t$, it is possible to show that the distribution $f_x(x,t)$ of particle positions x is Gaussian with expected value $\mathrm{E}(x) = \mu_x = 0$ and variance $\mathrm{Var}(x) = \sigma_x^2 = 2\kappa t$ with particle diffusivity $\kappa = k_\mathrm{B}T/\gamma = k_\mathrm{B}T/6\pi\mu R$.

Turning to applications at the sediment particle scale, the approach is conceptually identical in describing the forces acting on a particle. This includes specifying deterministic forces that are fixed or change slowly relative to the time scale of fluctuating forces. In the absence of fluid forces, the fluctuating forces involve particle–surface collisions (e.g. Furbish et al., 2021; Williams and Furbish, 2021), and in the case of bed load particles the fluctuating forces involve effects of turbulence combined with particle–surface interactions. For example, Fan et al. (2014) formulate a simplistic Langevin-like equation for bed load particle velocities having the form,

$$m\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x + \eta(t)\,. \tag{8}$$

Here, F_x is conceived as a fixed average force due to fluid drag and particle–bed friction, assumed to be Coulomb-like, and fluctuations in these forces are combined into $\eta(t)$, a Gaussian white noise. In turn, Pierce et al. (2022) consider the situation where particles alternate between states of motion and rest. The particle velocity is described in terms of a stochastic process according to

$\mathrm{d}x(t)/\mathrm{d}t = v_x(t)\sigma(t)$ with dichotomous noise that randomly switches between the rest state ($\sigma = 0$) and the active state ($\sigma = 1$). During the active state the velocity is described by a Langevin-like equation,

$$m\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = [F_x(v_x) + \eta(t)]\sigma(t)\,, \tag{9}$$

where $F_x(v_x)$ is a deterministic force whose structure varies with the assumptions regarding the factors producing it, and $\eta(t)$ is a Gaussian white noise. The authors examine the outcome of assuming that $F_x = \gamma[V - v_x(t)]$ for fixed velocity V, neglecting characteristic transient accelerations of particles from and to states of rest (Roseberry et al., 2012). By neglecting rest times ($\sigma = 1$), the formulation reduces to that of Ancey and Heyman (2014) for moving particles.

In the next section we consider continuous particle motions in a laminar flow to highlight the essential role of particle–bed collisions in modulating particle velocities. Because the formulation of Fan et al. (2014) involves particle–bed interactions, we return to this work in Section 4 to make a point about discontinuous (non-ergodic) particle motions in turbulent flow.

3 Particle velocities with continuous motions

Consider a bed load particle whose motion continues indefinitely, and which is subject to statistically time-homogeneous forces — like a Brownian particle. This situation, involving ergodic conditions, can be approximated experimentally (Williams, 2024). Namely, consider a bed load particle with mass m and diameter D moving over a surface within an approximately laminar shear flow with dynamic viscosity μ. The tumbling motion of the particle continues indefinitely and involves frequent collisions with the bed. Focusing just on motion parallel to the downstream x coordinate, an appropriate Langevin-like equation starts with

$$m\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_\mathrm{D} + F_\mathrm{c} + \eta(t)\,, \tag{10}$$

where $v_x(t)$ denotes the particle velocity, F_D denotes a drag force on the particle, F_c denotes a collisional friction force and $\eta(t)$ denotes a Gaussian white noise with zero mean.[3]

Assuming a Stokesian regime we set

$$F_\mathrm{D} = \frac{m}{t_e}[V - v_x(t)]\,, \tag{11}$$

where $t_e = m/\langle k_1\rangle\mu D$ denotes the e-folding response time of the particle and V denotes a characteristic flow velocity at a position that is on the order of one particle diameter from the bed. For a sphere within an unbounded domain the coefficient $k_1 = 3\pi$ according to Stokes's law. For a sphere moving over a relatively smooth surface, k_1 may be considered a fixed value, although not equal to 3π. For a tumbling angular particle, k_1 fluctuates as the detailed velocity boundary layer surrounding the particle varies, so this coefficient must be treated as a random variable. Fluctuations in drag also occur with small vertical movements within the boundary layer. Note that we are assuming k_1 and v_x are uncorrelated. Here it is important to note that by using the average $\langle k_1\rangle$

[3] Written as an Itô process, (10) becomes

$$\mathrm{d}v_x = \frac{1}{m}(F_\mathrm{D} + F_\mathrm{c})\mathrm{d}t + \frac{1}{m}\mathrm{d}W\,.$$

within the response time t_e in (11) we are assuming the fluctuations in this quantity are shorter than the response time of the particle to variations in the velocity v_x. This places effects of the fluctuations within the noise $\eta(t)$.

Collisional friction is appropriately treated energetically. Following the work of Furbish et al. (2021) and Williams and Furbish (2021), let $E = (m/2)v_x^2$ denote the particle kinetic energy measured parallel to x. During collisions part of this energy is transferred to other modes: transverse and vertical components of translational kinetic energy, three components of rotational kinetic energy, vibrational kinetic energy, and the irreversible production of heat with particle and surface deformation.[4] Each of these represents a loss of E. We then assume that the change in streamwise kinetic energy during a collision is $-(1-\epsilon^2)E$. Here, ϵ is akin to a coefficient of restitution, and it must be viewed as a random variable (Gunkelmann et al., 2014; Serero et al., 2015; Williams and Furbish, 2021). In turn, the *expected* apparent change in streamwise particle momentum during a collision is $-\sqrt{2m}(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle$, where we are assuming that ϵ and E are independent. The expected apparent change in streamwise momentum per unit distance is $-k_2\sqrt{2m}(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle/D^*$, where D^* is the average distance between collisions (see below) and k_2 is a dimensionless coefficient on the order of unity. In turn, the expected rate of change in momentum is

$$F_{\mathrm{c}} = -\frac{k_2\sqrt{2m}(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*}v_x\,, \tag{12}$$

Note that the quantity $k_2\sqrt{2m}(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle/D^*$ may be considered a zeroth-order friction factor. It has the same dimensions as m/t_e in (11), giving the appearance that friction is Newtonian (i.e. Stokes-like).

Here it is important to note that we are treating the friction factor in (12) as a fixed expected value rather than an instantaneous value, assuming the collision time is much shorter than the response time of the particle to the fluid drag. This anticipates that steady conditions in the mean exist, and as with the drag force F_{D} it places fluctuations in the collisional force about the expected value within the noise $\eta(t)$ in (10). In actuality the separation of time scales is not as clear as in the classical use of the Langevin equation to describe Brownian motion. Indeed, this is an example of where we hope the fluctuating forces are sufficiently uncorrelated that a white noise approximation is reasonable (see Section 5).

The bed is either smooth, or it consists of quasi-randomly spaced roughness elements with diameter d. We treat these as end-member cases. If the bed is rough and the movement of the particle requires navigating around and over the roughness elements, we assume that $D^* \sim d$. This represents a bottom-up control on collisional friction. Note that this can be modified to acknowledge effects of the relative roughness d/D, although we do not elaborate this point in this essay. If the bed is smooth and the tumbling motion of the particle involves collisions due to its angularity, then we assume that $D^* \sim D$. This represents a top-down control on collisional friction (Williams, 2024). If such motions consist of short saltations then this relation can be altered to $D^* \sim (\langle v_x\rangle/v_{\mathrm{s}})D$, where v_{s} denotes the Stokesian settling speed. In the experiments described below the ratio $\langle v_x\rangle/v_{\mathrm{s}}$ is likely unity or less.

[4] Let us note the interesting work of Fernández et al. (2024) showing that during collisions angular particles can transfer rotational kinetic energy into translational kinetic energy giving the appearance of a coefficient of restitution larger than one.

We now write the Langevin-like equation as

$$\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = \frac{1}{t_e}(V - v_x) - \sqrt{\frac{2}{m}} \frac{k_2(1 - \langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*} v_x + \frac{1}{m}\eta(t)\,. \tag{13}$$

Taking the ensemble average then yields

$$\frac{1}{t_e}(V - \langle v_x\rangle) - \sqrt{\frac{2}{m}} \frac{k_2(1 - \langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*} \langle v_x\rangle = 0\,. \tag{14}$$

Solving for the average $\langle v_x\rangle$,

$$\langle v_x\rangle = \frac{V}{1 + t_\mathrm{e}\sqrt{\frac{2}{m}}\frac{k_2(1-\langle\epsilon\rangle)\langle\sqrt{E}\rangle}{D^*}}\,. \tag{15}$$

Notice that if friction is "turned off" by setting $\langle\epsilon\rangle = 1$ the expected value $\langle v_x\rangle = V$. Moreover, because motion is unidirectional we may use the definition of the kinetic energy $E = (m/2)v_x^2$ and solve the quadratic to give

$$\langle v_x\rangle = \frac{-D^* + \sqrt{D^*[4t_e V k_2(1 - \langle\epsilon\rangle) + D^*]}}{2t_e k_2(1 - \langle\epsilon\rangle)}\,. \tag{16}$$

Owing to collisional friction the expected velocity $\langle v_x\rangle$ is less than the characteristic fluid velocity V. Whereas (16) gives $\langle v_x\rangle \to V$ in the limit of $\langle\epsilon\rangle \to 1$ as expected, in the limit of $\langle\epsilon\rangle \to 0$,

$$\langle v_x\rangle \to \frac{-D^* + \sqrt{D^*[4t_e V k_2 + D^*]}}{2t_e k_2}\,. \tag{17}$$

This represents the expected velocity for the situation in which each collision brings the particle to rest.

Let us now use (13) to write the probability current of the Fokker–Planck equation. If $f_{v_x}(v_x)$ denotes the stationary probability distribution of the velocities v_x, then

$$-\kappa_x \frac{\mathrm{d}f_{v_x}(v_x)}{\mathrm{d}v_x} + \left[\frac{1}{t_e}(V - v_x) - \sqrt{\frac{2}{m}} \frac{k_2(1 - \langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*} v_x\right] f_{v_x}(v_x) = 0\,. \tag{18}$$

where κ_x is an unconstrained diffusion coefficient. This yields a Gaussian distribution,

$$f_{v_x}(v_x) = \frac{1}{\sqrt{2\pi\kappa_x/(a+b)}} \exp\left[-\frac{(v_x - \mu_{v_x})^2}{2\kappa_x/(a+b)}\right]\,, \tag{19}$$

with mean $\mu_{v_x} = \langle v_x\rangle$ given by (16) and variance $\sigma_{v_x}^2 = \kappa_x/(a+b)$, where

$$a = \frac{1}{t_e} \quad \text{and} \quad b = \sqrt{\frac{2}{m}} \frac{k_2(1 - \langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*}\,. \tag{20}$$

Thus, the mean velocity is strongly conditioned by particle–bed interactions as the particles are responding to the flow. The variance depends on fluctuations associated with both particle–fluid interactions and particle–bed interactions. In the absence of collisional friction such that $\langle\epsilon\rangle = 1$, then (19) becomes

$$f_{v_x}(v_x) = \frac{1}{\sqrt{2\pi\kappa_x t_e}} \exp\left[-\frac{(v_x-\mu_{v_x})^2}{2\kappa_x t_e}\right], \tag{21}$$

with mean $\mu_{v_x} = V$ and variance $\kappa_x t_e$. That is, the process is entirely mean-reverting to V. The variance depends only on fluctuations associated with particle–fluid interactions.

Turning to transverse velocities v_y, the Langevin-like equation is

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -\frac{1}{t_e}v_y - \sqrt{\frac{2}{m}}\frac{k_2(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*}v_y + \frac{1}{m}\eta(t)\,. \tag{22}$$

which is a classical Ornstein–Uhlenbeck process that is mean-reverting to $\langle v_y\rangle = 0$, where drag and friction contribute equally to the mean reversion. Because motion is bidirectional the expected energy $\left\langle\sqrt{E}\right\rangle$ is to be considered a finite parametric value. If $f_{v_y}(v_y)$ denotes the stationary probability distribution of the velocities v_y, then the probability current is

$$-\kappa_y\frac{\mathrm{d}f_{v_y}(v_y)}{\mathrm{d}v_y} - \left[\frac{1}{t_e}v_y + \sqrt{\frac{2}{m}}\frac{k_2(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*}v_y\right]f_{v_y}(v_y) = 0\,. \tag{23}$$

where κ_y is an unconstrained diffusion coefficient. This yields a Gaussian distribution,

$$f_{v_y}(v_y) = \frac{1}{\sqrt{2\pi\kappa_y/(a+b)}}\exp\left[-\frac{v_y^2}{2\kappa_y/(a+b)}\right], \tag{24}$$

with mean $\mu_{v_y} = \langle v_y\rangle = 0$ and variance $\sigma^2_{v_y} = \kappa_y/(a+b)$. In the absence of friction (24) becomes

$$f_{v_y}(v_y x) = \frac{1}{\sqrt{2\pi\kappa_y t_e}}\exp\left[-\frac{v_y^2}{2\kappa_y t_e}\right], \tag{25}$$

with variance $\kappa_y t_e$.

Let us return to the Langevin-like equation (13) and write it as

$$\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = aV - (a+b)v_x + \frac{1}{m}\eta(t)\,. \tag{26}$$

The solution of (00) is

$$v_x(t) = v_x(0)e^{-(a+b)t} + \frac{aV}{a+b}\left[1-e^{-(a+b)t}\right] + e^{-(a+b)t}\int_0^t e^{(a+b)s}\eta(s)\,\mathrm{d}s\,. \tag{27}$$

Taking the ensemble average over all possible realizations,

$$\langle v_x(t)\rangle = \langle v_x(0)\rangle e^{-(a+b)t} + \frac{aV}{a+b}\left[1-e^{-(a+b)t}\right], \tag{28}$$

where $\langle v_x(0)\rangle$ is the average over all possible initial states. In the limit of $t\to\infty$ the expected value $\langle v_x\rangle \to V/(1+b/a)$, as described above. But returning to (27), let us calculate the variance of just the deterministic part. Namely,

$$\mathrm{Var}[v_x(t)] = \left\langle[v_x(t)-\langle v_x\rangle]^2\right\rangle = \left\langle\left(v_x(0)e^{-(a+b)t} + \frac{aV}{a+b}\left[1-e^{-(a+b)t}\right]\right)^2\right\rangle - \left(\frac{V}{1+b/a}\right)^2. \tag{29}$$

Upon expanding the parenthetical term it is straightforward to show that in the limit of $t \to \infty$ the variance $\mathrm{Var}(v_x) \to 0$. However, observations tell us this is incorrect.

Recall from Section 2 our similar analysis of the classical Langevin equation applied to Brownian motion, leading to the conclusion that the equipartition theorem for translational kinetic energy precludes the outcome of zero variance. The noise $\eta(t)$ is essential to satisfy thermodynamic equilibrium conditions. With bed load particles we do not have the analogue of an equipartition theorem. Nonetheless, we do have a principle that precludes the outcome of zero variance in the presence of particle–bed interactions. Because particle motions are driven externally (unlike gas particles), then entirely analogous to motions on a Galton board (Williams and Furbish, 2021; Williams, 2024), particle spreading in both the streamwise and transverse directions (thus involving fluctuations in velocity) cannot exist without streamwise advective motion as particles navigate roughness elements on the surface. Similarly, for top-down behavior involving the wobbly trajectories of angular particles as they tumble over a surface, spreading in both the streamwise and transverse directions cannot occur without advective motion. This directly leads to the appearance of the quantities a and b in the variances $\sigma^2_{v_x}$ and $\sigma^2_{v_y}$, where effects of fluctuations are contained in the noise $\eta(t)$.

Consider the experiments of Williams (2024). These involve the movement of both spheres and natural angular particles over smooth and rough surfaces within an approximately laminar flow. The experiments are designed to create ergodic conditions in which particles remain in motion in response to an approximately fixed near-bed flow velocity V. Consistent with the analysis above, distributions of streamwise velocities v_x are approximately Gaussian (Figure 1). Likewise,

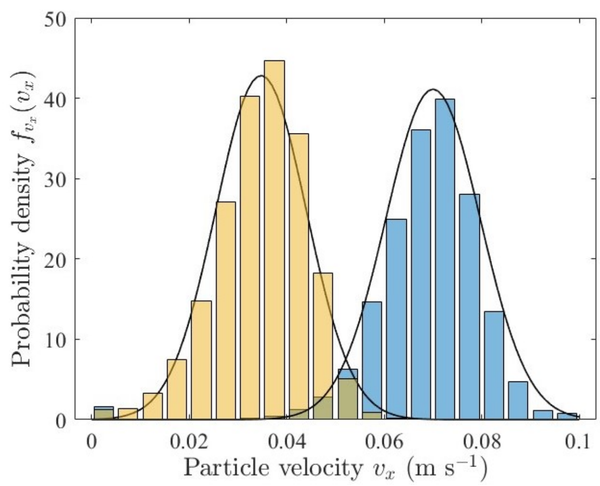

Figure 1: Histograms of streamwise velocities v_x of coarse sand-sized angular particles moving continuously with a laminar flow over rough (yellow) and smooth (blue) surfaces, as reported by Williams (2024). Rough surface is 80-grit sandpaper. Flow Reynolds number $Re = 420$ and Froude number $Fr = 0.22$. Imaging with Chronos 2.1-HD High Speed Camera with a Sigma 24-70 mm f/2.8 Nikon F zoom lens at 60 fps.

cross-stream velocities v_y are Gaussian with zero mean. These histograms qualitatively illustrate a strong bottom-up influence of bed roughness on the frequency and intensity of collisions, and a weaker top-down influence (Williams and Furbish, 2021) as angular particles tumble over a smooth surface. Figure 1 suffices for the purpose of this essay, noting that the experiments involve numerous ancillary measurements, data sets and analyses (Williams, 2024).

Whereas these results are satisfying, the formulation is unlikely correct in detail. In particular the noise $\eta(t)$ in (13) is not likely white nor Gaussian. Unlike particle–particle collisions during simple Brownian motion, this noise reflects a mixture of things: fluctuations in the fluid forces on a particle due to its irregular geometry and tumbling motion, and fluctuations due to particle–bed collisions. Moreover, the separation of scales between the deterministic (expected) and fluctuating forces is not as clear as with Brownian motion. In addition, we cannot (yet) constrain the values of various coefficients in the Langevin-like equation (13).

4 Particle velocities with discontinuous motions

When particles alternate between states of motion and rest — a hallmark of natural bed load particle motions — statistically time-homogeneous conditions cannot exist. Particles experience widely varying travel times and associated hop distances, and motions are fundamentally non-ergodic. As above, we can start with a Langevin-like equation to describe the motion of a particle, but then we must proceed differently. Whereas we average over an ensemble of particles, we cannot view this as an average over statistically similar realizations that evolve indefinitely in time.

To be clear, the ensemble average is over all possible velocity states v_x. This can be interpreted two ways. First, because a sediment particle alternates between states of motion and rest, the average is taken over all velocity states experienced by the particle only during periods of motion. In this case we must envision that over a long period of time the particle explores all possible velocity states in the proportions given by the steady distribution $f_{v_x}(v_x)$. In effect this is a time average obtained in the limit of $t \to \infty$. The more straightforward way of interpreting the ensemble average is to view it as an average of all possible velocity states v_x experienced by an ensemble of independent but nominally identical particles at any instant, where each particle is in a state of motion representing all instants over all possible travel times.

Consider the formulation of Fan et al. (2014), which is mechanically incorrect, but contains important lessons. This formulation aims specifically at an exponential distribution of velocities (Roseberry et al., 2012; Fathel et al., 2015) and starts with a Langevin-like equation,

$$m \frac{\mathrm{d}v_x}{\mathrm{d}t} = F_\mathrm{D} - \mathrm{sgn}(v_x) F_\mathrm{C} + \eta(t) \,, \tag{30}$$

where F_D denotes a fixed (average) fluid drag force, F_C denotes a fixed friction force that is taken to be Coulomb-like wherein the normal force is given by the buoyant weight of the particle, and $\eta(t)$ is a Gaussian white noise. Note that the friction force is fashioned after a conceptualization of transport attributable to the work of Bagnold (1966),[5] and that neither the drag force nor the friction force depends on the velocity state v_x. A mean-reverting behavior is therefore precluded.

As in Section 2 we write $\eta(t)\mathrm{d}t = \mathrm{d}W(t)$, rearrange (30) and integrate to give

$$v_x(t) = v_x(0) + \frac{1}{m}[F_\mathrm{D} - \mathrm{sgn}(v_x) F_\mathrm{C}]t + \frac{1}{m}\int_0^t \mathrm{d}W \,. \tag{31}$$

[5]As described in Section 3, particle friction is collisional, not Coulomb-like, and the normal force during collisions is a random variable whose average is not set by the buoyant weight of the particle. Among other problems in Bagnold's formulation, using the buoyant weight in a Coulomb-like manner implies that neutrally buoyant particles are frictionless despite particle–bed and particle–particle collisions, giving the nonphysical result that the drag force $F_\mathrm{D} = 0$ for steady motion. More generally, a Coulomb-like model with dynamic friction coefficient has no relevance to rarefied bed load particle motions.

Let us now momentarily imagine that particle motion continues indefinitely, as in the preceding section, and take the ensemble average over all possible realizations to give

$$\langle v_x(t)\rangle = \langle v_x(0)\rangle + \frac{1}{m}[F_\mathrm{D} - \mathrm{sgn}(v_x)F_\mathrm{C}]t \,. \tag{32}$$

This says that the expected particle velocity varies linearly with time if $F_\mathrm{D} - \mathrm{sgn}(v_x)F_\mathrm{C} \neq 0$. That is, a system of independent particles (representing the ensemble) is at any instant accelerating. In this situation a steady-state distribution of particle velocities does not exist.

With equilibrium conditions the particle system experiences zero acceleration. Upon taking an ensemble average over all possible velocity states v_x we therefore must conclude that $F_\mathrm{D} - \mathrm{sgn}(v_x)F_\mathrm{C} = 0$. This is consistent with the expectation that the ensemble averaged force on the particles must be zero for steady transport conditions (Furbish et al., 2012b). But as a consequence the expected velocity $\langle v_x\rangle$ cannot be mechanically specified and (30) reduces to

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = \eta(t)\,. \tag{33}$$

This says that for steady conditions the particle velocity fluctuates as a white noise about an unspecified average. The ensemble average of (33) is zero.

Continuing with the formulation of Fan et al. (2014), the Langevin-like equation (30) satisfies a Fokker–Planck equation. The associated probability current must be zero everywhere over the state space v_x with steady-state conditions, and Fan et al. (2014) write this as

$$\kappa_x\frac{\mathrm{d}f_{v_x}(v_x)}{\mathrm{d}v_x} + F_x f_{v_x}(v_x) = 0 \tag{34}$$

where κ_x denotes a diffusion coefficient and F_x denotes the (incorrectly) assumed finite force per unit mass arising from fluid drag and Coulomb-like friction. This gives the desired outcome, an exponential distribution $f_{v_x}(v_x)$ of velocities v_x. Because F_x is assumed to be finite (rather than zero), this solution requires a nonphysical step change in the average force F_x at $v_x = 0$ to accommodate the occurrence of negative particle velocities (e.g. Roseberry et al., 2012), and it provides no constraint on the global net force. By correctly letting $F_x \to 0$ the solution of (34) is an unrealistic uniform distribution over all v_x.

van Kampen (1981) points out that for some problems it might be preferable to directly formulate a Fokker–Planck equation based on physical arguments instead of starting with a Langevin equation. For example, following Furbish et al. (2012b) we may start with a Fokker–Planck equation with zero drift and state-dependent diffusion coefficient $\kappa_x(v_x)$. In this case the probability current at steady state is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v_x}[\kappa_x(v_x)f_{v_x}(v_x)] &= 0\\ \frac{\mathrm{d}\kappa_x(v_x)}{\mathrm{d}v_x}f_{v_x}(v_x) + \kappa_x(v_x)\frac{\mathrm{d}f_{v_x}(v_x)}{\mathrm{d}v_x} &= 0\,.\end{aligned} \tag{35}$$

Because this does not involve a drift term, it acknowledges that the ensemble averaged force on the particles is zero. However, it does not explicitly reveal elements of the fluid and collisional forces involved, as with a Langevin-like equation. Instead, the state-dependent diffusion coefficient $\kappa_x(v_x)$ gives a term involving the derivative $\mathrm{d}\kappa_x(v_x)/\mathrm{d}v_x$, which represents a well-known *apparent* drift (van Kampen, 1981). The coefficient $\kappa_x(v_x)$ represents the rate of change in the particle kinetic energy, so the apparent drift term in (35) in effect represents a probability flux from velocity states with large kinetic energy toward states with small kinetic energy (Furbish et al., 2012b). This coincides with a flux toward small velocity states.

If the stationary distribution $f_{v_x}(v_x)$ is indeed exponential, then according to (35),

$$\begin{aligned} \frac{\mathrm{d}\kappa_x(v_x)}{\mathrm{d}v_x} e^{-v_x/\mu_{v_x}} - \kappa_x(v_x)\frac{1}{\mu_{v_x}} e^{-v_x/\mu_{v_x}} &= 0 \\ \frac{\mathrm{d}\kappa_x(v_x)}{\mathrm{d}v_x} - \frac{1}{\mu_{v_x}}\kappa_x(v_x) &= 0 \end{aligned} \tag{36}$$

which requires that $\kappa_x(v_x) = \kappa_{x0}e^{v_x/\mu_{v_x}}$, where κ_{x0} denotes the value at $v_x = 0$. This state-dependent diffusion coefficient is entirely consistent with the strongly heteroscedastic velocity–acceleration phase behavior reported by Furbish et al. (2012b) and Wu et al. (2020; Figure 1 therein) associated with an exponential distribution of velocities, where negative accelerations are dominated by particle–bed collisions.

The Fokker–Planck equation with probability current given by (35) is likely an oversimplification of the conditions in a turbulent flow that yield an exponential distribution of particle velocities v_x, as observed from high-speed imaging (Roseberry et al., 2012; Fathel et al., 2015). Nonetheless, the analysis strongly points to the occurrence of state-dependent diffusion in this problem based on the velocity–acceleration phase behavior (Furbish et al., 2012b). In work to be presented elsewhere, numerical simulations indicate that the frequency and intensity of particle–bed collisions strongly modulate particle velocities. In contrast to the Langevin equation (13) as applied to ergodic conditions in Section 3, the simulations allow for transitions between rest and active states. The fixed velocity V in (13) instead varies, $V \to V(t)$, as a red noise in the simulations to mimic turbulence fluctuations, and the drag force is nonlinear rather than Stokesian. The simulations give a systematic variation in the form of the velocity distribution — exponential to gamma-like to Gaussian — with decreasing collision frequency coinciding with increasing flow strength. The simulations also correctly mimic the particle velocity–acceleration phase behavior (Furbish et al., 2012b) and the nonlinear variation in particle hop distances with increasing travel times, as reported by Wu et al. (2020).

With the idea of state-dependent diffusion in place, let us briefly return to the case of laminar flow (Section 3), albeit involving discontinuous particle motions. For conditions with relatively small mean velocity, we assume that to first order the diffusion coefficient varies as $\kappa_x(v_x) = \alpha v_x$. The probability current then involves a diffusive term, $-\mathrm{d}[\alpha v_x f_{v_x}(v_x)]/\mathrm{d}v_x$, and (18) becomes

$$-\alpha v_x \frac{\mathrm{d}f_{v_x}(v_x)}{\mathrm{d}v_x} - \alpha f_{v_x}(v_x) + \left[\frac{1}{t_e}(V - v_x) - \sqrt{\frac{2}{m}}\frac{k_2(1-\langle\epsilon\rangle)\left\langle\sqrt{E}\right\rangle}{D^*}v_x\right] f_{v_x}(v_x) = 0\,, \tag{37}$$

which involves an apparent drift equal to $-\alpha f_{v_x}(v_x)$. The solution of (37) for $v_x > 0$ is

$$f_{v_x}(v_x) = \frac{1}{\Gamma(aV/\alpha)}\left(\frac{a+b}{\alpha}\right) v_x^{aV/\alpha - 1} e^{-(a+b)v_x/\alpha}\,. \tag{38}$$

This is a gamma distribution with shape parameter aV/α, scale parameter $\alpha/(a+b)$, and mean $\mu_{v_x} = aV/(a+b)$, where a and b are defined by (20). Notice that when $aV/\alpha = 1$, (38) reduces to an exponential distribution with mean $\mu_{v_x} = \alpha/(a+b)$. We presume that a clearer understanding and formulation of how diffusion varies with the velocity state v_x would reveal a smooth transition between the gamma and Gaussian behaviors, (38) and (19).

5 Concluding remarks

The Langevin-like equation applied to continuous particle motions (Section 3) provides a clear view of the essential role of particle–bed collisions in modulating particle velocities, a statistical physics that is entirely incompatible with Coulomb-like (continuum) conceptualizations of friction during rarefied bed load transport. In considering discontinuous non-ergodic particle motions, a clearer understanding is needed regarding how the diffusion coefficient in the Fokker–Planck equation varies with the velocity state, possibly leading to a systematic transition in the form of the velocity distribution with increasing flow strength, from exponential to gamma-like to Gaussian.

In contrast to the classic problem of Brownian motion, the separation of scales between the persistent (deterministic) and the fluctuating forces influencing bed load particle motion is not as clear. Moreover, we must assume the noise representing the fluctuations is sufficiently uncorrelated that white noise is a forgiving approximation. In the case of sediment particles subjected to a range of turbulence frequencies in concert with intermittent particle–surface collisions, fluctuating forces might be more appropriately treated as a correlated (colored) noise, and are likely state-dependent. In addition, we do not have the analogue of an equipartition theorem. Nonetheless, we do have a principle concerning the consequences of particles moving over rough surfaces, which tells us that differential motions guarantee a finite variance in particle velocities.

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