Statistical equilibrium transport of bed load sediment: The role of particle velocity, acceleration and jerk

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1 Context

From a statistical mechanics point of view, thermodynamic equilibrium of an ordinary gas coincides with a condition in which the Maxwell-Boltzmann distributions of particle energies and speeds are stationary. This is manifest macroscopically as fixed thermodynamic state variables — pressure and temperature. In pursuing a statistical mechanics description of sediment particles transported as bed load, an intriguing possibility is that an analogue of thermodynamic equilibrium exists. Currently our simplest description of equilibrium bed load transport is that the particles collectively experience zero acceleration. In addressing this we of course must keep in mind fundamental differences between gas particles and sediment particles, where particle collisions in a gas are elastic and with bed load the particle-surface interactions and occasional particle-particle collisions are highly dissipative. We highlight other differences below.

Here we present a brief qualitative description of this problem, with limited explanation. A quantitative version is to be presented elsewhere. We begin with the idea of a phase space involving sediment particle motions. Our description of these motions starts with a Lagrangian perspective, then transitions to an Eulerian perspective.

2 Velocity–acceleration phase space

Sediment particles transported as bed load in rivers have a distinctive behavior: they start and stop. Due to fluid forces exerted on them, particles accelerate from a state of rest and then during motion experience fluctuating forces due to turbulence and due to interactions with the streambed, including collisions. The particles eventually decelerate and by chance come to rest. Travel times T_{p} are mostly short, less than a tenth of a second. But sometimes particles take longer hops with travel times on the order of seconds, depending on overall flow conditions and streambed roughness.

Focusing just on motion in the streamwise (downstream) direction x, high-speed imaging of bed load particles reveals that after a particle is accelerated from rest its velocity u_{p} typically fluctuates in response to turbulence and to particle-streambed interactions before returning to zero with deposition (Figure 1, left). Variations in particle velocity represent streamwise accelerations, $a_x = \mathrm{d}u_{\mathrm{p}}/\mathrm{d}t$, positive and negative. These accelerations are represented by the local slope of the velocity signal. In turn, variations in the local slope of the signal, and thus variations in the acceleration, represent streamwise particle jerk, $j_x = \mathrm{d}a_x/\mathrm{d}t$.

The particle velocity signals in Figure 1 (left) are noisy. So before considering them further, let us first examine an idealized, smooth particle motion, formed as the sum of sinusoids, that mimics essential features of natural particles (Figure 1, right top). The motion starts and ends in states of rest, with variations in velocity during motion. Now let us ignore time and plot the particle motion in a velocity-acceleration phase space (Figure 1, right bottom), where we also can plot the particle jerk. The velocity-acceleration phase trajectory starts and ends at $u_{\mathrm{p}} = 0$ and involves clockwise loops within the phase space. Each loop represents an interval of increasing-

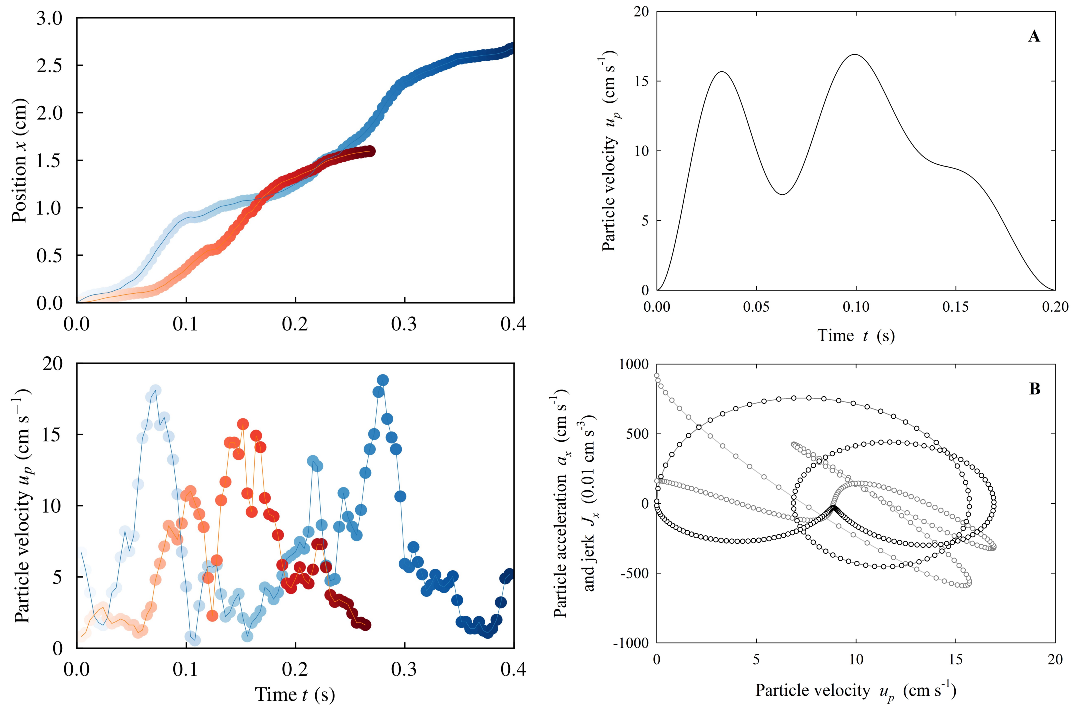

Figure 1: **Left:** plots of streamwise particle position x versus time t (top) showing two examples of relatively long motions in which most travel times are much less than 0.4 s, and mostly less than 0.1 s; and streamwise particle velocity u_p versus time t (bottom) showing fluctuations in velocities over different durations. Data are from Roseberry et al. (2012). **Right:** plots of particle velocity u_p versus time t (top) for an idealized smooth motion ($T = 0.2$ s) consisting of the sum of three sinusoids; and particle acceleration a_x (black circles) and jerk j_x (gray circles) versus velocity u_p (bottom) for smooth velocity signal; jerk j_x is multiplied by factor 0.01 so that it appears on the plot. In this example the acceleration trajectory, starting at $u_\mathrm{p} = 0$, everywhere progresses in a clockwise manner except where it approaches $a_x = 0$ from below and is then jerked into another clockwise trajectory before reaching $u_\mathrm{p} = 0$; and the jerk trajectory, starting at $u_\mathrm{p} = 0$, progresses to its maximum negative value at $u_\mathrm{p} \approx 16$ cm s^{-1} before looping through four local extrema. Circles on the trajectories are plotted at 0.001 s intervals.

then-decreasing velocity in the velocity-time plot (Figure 1, right top). Variations in acceleration represent jerk, positive and negative. Positions where $\mathrm{d}a_x/\mathrm{d}u_\mathrm{p} = 0$ locally represent constant acceleration and thus a fixed net force with zero jerk. Trajectories must cross the horizontal axis $a_x = 0$ vertically, representing zero acceleration and thus a zero net force with momentarily constant velocity and a local extremum of jerk. Returning to natural particles, although the signals are noisy due to limitations of temporal resolution of the high-speed imaging, the clockwise loops in the phase trajectories are apparent (Figure 2, left).

The time-velocity, velocity-acceleration and velocity-jerk plots above offer a decidedly Lagrangian perspective of motion in which the velocity, acceleration and jerk "belong" to the particles and vary with time. Now let us briefly turn to an Eulerian perspective. Imagine a great number of particle motions, that is, a great number of looping phase trajectories, each starting and ending at $u_\mathrm{p} = 0$. These involve varying loop sizes associated with different particle travel times T with distinct velocity signals. Now imagine taking a snapshot of the velocity-acceleration phase space at some instant. This reveals a cloud of points in a velocity-acceleration diagram (Fig-

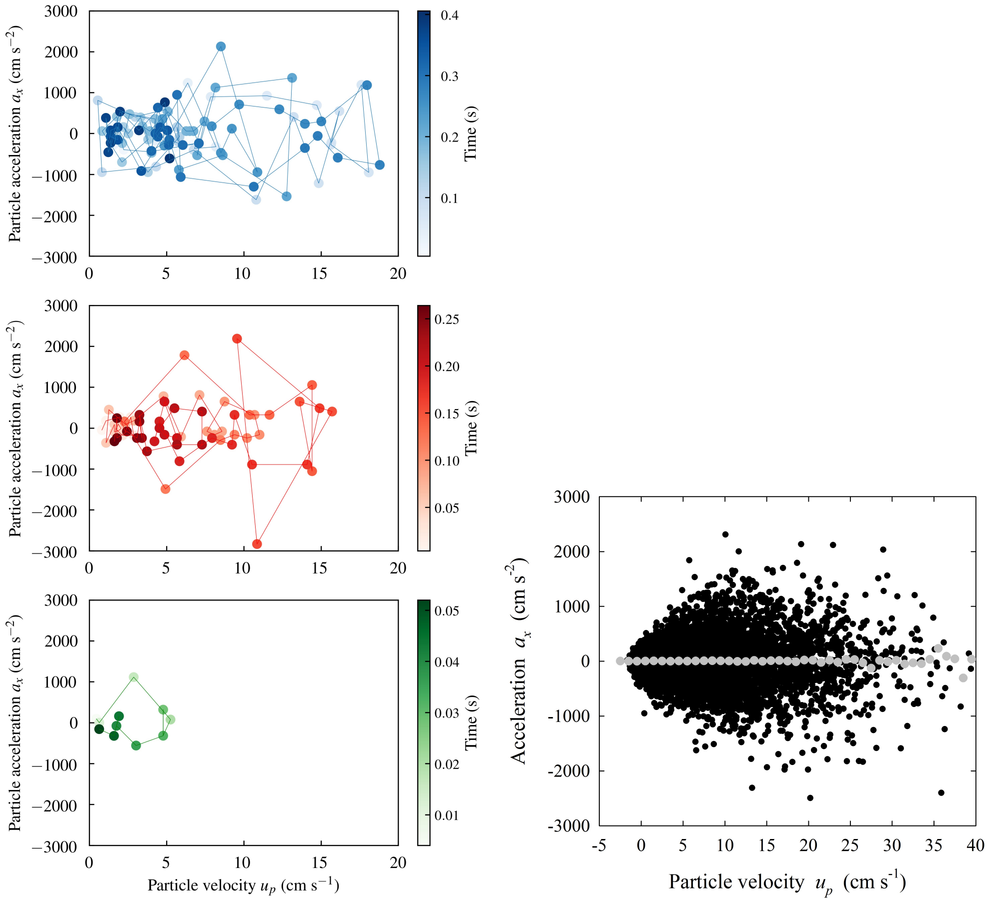

Figure 2: **Left:** plots of particle acceleration a_x versus velocity u_p showing irregular phase trajectories involving clockwise motions over the domain; accelerations in top two panels are based on particle velocities depicted in Figure 1 (left), where color time stamps coincide. Lines that double back on themselves reflect limits of resolution in the finite difference approximations of u_p and a_x. **Right:** plot of streamwise acceleration a_x versus velocity u_p estimated from high-speed imaging; gray dots are average values of the accelerations calculated for 1 cm s^{-1} intervals of velocity.

ure 2, right) with varying point density over the phase space. In this example the point density is high at small velocities and accelerations, and decreases with increasing velocities and accelerations, positive and negative. This reflects that most particle motions are of short duration with relatively small looping phase trajectories. If we take a snapshot at any other instant, the positions of individual points in the plot would be different due to trajectory motions, but the point density of the cloud and variations in this density over the phase space would be similar. Now envision an animation of the trajectory motions, blurring our eyes just right so that we do not focus on individual points. Although individual points are locally moving in many directions, akin to molecules in a fluid, we can envision a steady, collective clockwise circulation of the points. This circulation of points starts at $u_\mathrm{p} = 0$ with $a_x > 0$ and ends at $u_\mathrm{p} = 0$ with $a_x < 0$. Moreover, this collective motion is qualitatively similar to a fluid circulation, where we can imagine that the speed of the circulation varies with the (u_p, a_x) coordinate position. If with sufficient points we define the point number density as $n(u_\mathrm{p}, a_x)$ — a steady scalar field — then this offers an Eulerian perspective of particle motions in the phase space. This steady field in fact characterizes equilibrium bed load transport, analogous to thermodynamic equilibrium as decribed in statistical mechanics.

Recall that in classical mechanics the state of a particle is defined by its position and velocity. This is particularly well suited to gas particles which move through a vacuum between successive collisions and experience only the steady force of gravity between these collisions but no fluctuating forces. In contrast, bed load particles experience fluctuating fluid forces at all times together with impulses associated with particle-streambed interactions between states of rest. In this view of particle behavior, particle positions are unimportant. Choosing the velocity-acceleration phase space allows us to describe effects of fluid forces that continuously act on particles. Also note that the experimental imaging time resolution in Figure 2 does not fully reveal the magnitudes of particle accelerations. Numerical simulations at high temporal resolution suggest that accelerations are not necessarily symmetrical about $a_x = 0$, and their magnitudes can be much larger than those shown, particularly accelerations $a_x < 0$ involving particle-streambed collisions.[1]

References

[1] Roseberry et al. (2012) *Journal of Geophysical Research – Earth Surface*, 117, F03032, doi: 10.1029/2012JF002353

[2] Schmeekle, M. W. (2014) *Journal of Geophysical Research – Earth Surface*, 119, 1240–1262, doi: 10.1002/2013JF002911.

[1] These simulations were conducted by Mark Schmeeckle at Arizona State University using a coupled LES-DEM model as described in: Schmeeckle, M. W. (2014) *Journal of Geophysical Research – Earth Surface*, 119, 1240–1262, doi: 10.1002/2013JF002911. When accelerations are numerically resolved at the same temporal resolution of the experimental high-speed imaging, the resulting velocity-acceleration plots mimic the data in Figure 2 (right)