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PARTIAL CONTACT OF A SMOOTH ELASTIC WAVY STRIP PRESSED BETWEEN TWO FLAT SURFACES

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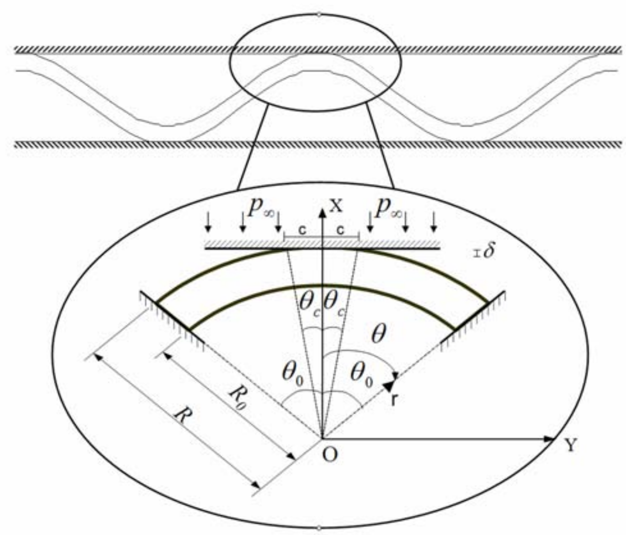
ABSTRACT

The contact of a smooth elastic wavy strip pressed between two surfaces is solved by considering a curved beam in contact with a rigid half-space. In assuming so, the Michell Fourier series expansion for elastic bodies is shown to satisfy the resulting mixed boundary value problem. When the contact region is small compared to the radius of curvature of the beam, semi-analytical solutions are obtained by exploiting dual series equation techniques. The relation between the level of loading and the extent of contact, as well as stress on the surface and in the beam, are found. Various boundary conditions on the ends, which arise as lower order terms in the Michell solution, are considered. This semi-analytical solution may prove useful in analyzing the contact of a corrugated seal.

INTRODUCTION

This research aims to extend the wavy surface half-space solution [1] to consider a finite beam geometry, i.e. the contact between a rigid half-space and finite wavy elastic smooth strip. The solution technique is similar to that found in [2], in which a stress function approach—here the Michell solution—changes the mixed boundary value problem into dual series equations. The equations are solved with an integral of the Abel type. The beam geometry results in a Fourier series whose terms approach a constant, which requires a numerical scheme often encountered in finite body problems. As the strip width goes to infinity, the solution for an elastic cylinder in contact with a half-space is recovered. Thin beam approximations are also calculated.

CONTACT OF A SMOOTH, ELASTIC WAVY STRIP PRESSED BETWEEN TWO SURFACES



Consider a smooth wavy elastic wavy strip, mathematically treated as two regular, infinite sine waves evenly spaced apart. The contact of a smooth elastic wavy strip pressed between two surfaces is approximated here with a curved beam in contact with a rigid half-space. Assuming so allows the use of the Michell solution [3]. For linear elasticity to hold, it is assumed here that the contact region, 2c, is small compared to the radius of curvature, R. Also, $\sin\theta_0 \ll 1$ ensures that the beam is shallow so that its deformations are small. The thickness of the beam, however, is not constrained.

To model a wavy strip, the moments are chosen to be zero at the ends of the beam, corresponding to the location of zero curvature. The natural choice of $\tau_{r\theta} = 0$ and $u_\theta = 0$ at the ends is consistent with the physics of the problem. The normal forces at the ends of the beam are chosen to be zero.

FORMULATION OF THE ELASTICITY PROBLEM

The symmetric mixed boundary conditions on the top surface suggest using the symmetric Michell solution. The lower order terms in the Michell solution correspond to moments ($n = 0$) and normal forces with moments ($n = 1$) at the ends of a curved beam. For the physics of the contact problem, only the $n = 2^+$ order terms that represent "solutions for shearing and normal forces… acting on the inner and outer boundaries of a circular ring" need to be considered [3]. Equilibrium and compatibility are automatically satisfied. The Michell solution assumes that the geometry is a complete ring; for a partial ring, i.e. a curved beam, the eigenfunctions in θ are adjusted by the factor π / θ_0 giving

$$\phi = \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + c_n r^{-n} + d_n r^{-n+2}) \cos \frac{n\pi\theta}{\theta_0} \quad (1)$$

The boundary conditions for $c \ll R$ are

$$u_\theta = 0,\ \tau_{r\theta} = 0, \qquad 0 \le |\theta| = \pi,\ R_0 \le r \le R \quad (2)$$

$$\sigma_{rr} = \tau_{r\theta} = 0, \qquad 0 \le |\theta| < \pi,\ r = R_0 \quad (3)$$

$$\sigma_{rr} = \tau_{r\theta} = 0, \qquad \theta_c \le |\theta| < \pi,\ r = R_0 \quad (4)$$

$$\frac{\partial u_r}{\partial \theta} = R \sin\theta,\ \tau_{r\theta} = 0, \quad 0 \le |\theta| < \theta_c,\quad r = R \quad (5)$$

The mixed boundary condition (5) results in the dual series equations of the *form*

$$\sum_{n=2}^{\infty} a_n R^{n-2} \sin n\theta\, f_n(\alpha, \nu, \mu) = \sin\theta, \qquad 0 \le |\theta| < \theta_c \quad (6)$$

$$\sum_{n=2}^{\infty} a_n R^{n-2} \cos n\theta\, g_n(\alpha, \nu, \mu) = 0, \qquad \theta_c \le |\theta| < \pi \quad (7)$$

where $\alpha = R_0 / R$, μ is shear modulus and ν is Poisson's ratio. A procedure for solving the dual series is delineated in [4]. However, the Fourier coefficients do not converge to zero and is indicative of the finite geometry. Consequently, numerical calculations using a Gauss quadrature scheme are employed.

EVALUATION OF THE RESULTS

The normal stress on the surface for $0 \le |\theta| < \theta_c$ is calculated for various beam thickness parameter α. Figure 1 shows the stress distribution for an intermediate value of $\alpha = 0.5$

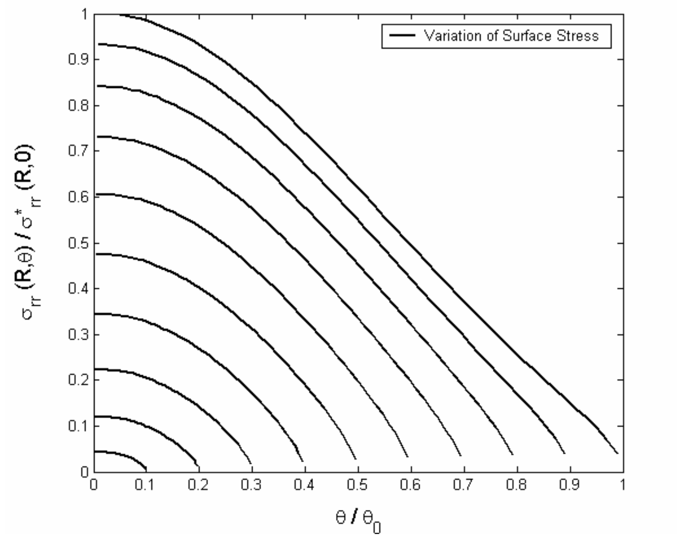


Figure 1. Normal surface stress for $\alpha = 0.5$ for various extents of contact

Asymptotic results for $\alpha \to 0$ (an elastic cylinder in contact with a rigid half-space) and $\alpha \to 1$ (a thin, shallow arch) are also calculated. Both results agree well with the literature [3, 5]. Furthermore, the solution can be altered to include moments or forces at the end of the beams by including the lower order terms in the Michell solution [3].

ACKNOWLEDGMENTS

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