QUASI-UNIFORMITY OF MINIMAL WEIGHTED ENERGY POINTS ON COMPACT METRIC SPACES

D. P. HARDIN, E. B. SAFF, AND J. T. WHITEHOUSE

Abstract. For a closed subset K of a compact metric space A possessing an α-regular measure μ with $\mu(K) > 0$, we prove that whenever $s > \alpha$, any sequence of weighted minimal Riesz s-energy configurations $\omega_N = \{x_{i,N}^{(s)}\}_{i=1}^N$ on K (for 'nice' weights) is quasi-uniform in the sense that the ratios of its mesh norm to separation distance remain bounded as N grows large. Furthermore, if K is an α-rectifiable compact subset of Euclidean space (α an integer) with positive and finite α-dimensional Hausdorff measure, it is possible to generate such a quasi-uniform sequence of configurations that also has (as $N \to \infty$) a prescribed positive continuous limit distribution with respect to α-dimensional Hausdorff measure. As a consequence of our energy related results for the unweighted case, we deduce that if A is a compact C^1 manifold without boundary, then there exists a sequence of N-point best-packing configurations on A whose mesh-separation ratios have limit superior (as $N \to \infty$) at most 2.

1. Introduction

Let A be a compact infinite metric space with metric $m : A \times A \to [0, \infty)$ and let $\omega_N = \{x_i\}_{i=1}^N \subset A$ denote a configuration of $N \geq 2$ points in A. We are chiefly concerned with two 'quality' measures of ω_N; namely, the *separation distance of* ω_N defined by

$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} m(x_i, x_j), \tag{1.1}$$

and the *mesh norm of* ω_N *with respect to* A defined by

$$\rho(\omega_N, A) := \max_{y \in A} \min_{1 \leq i \leq N} m(y, x_i). \tag{1.2}$$

This quantity is also known as the *fill radius* or *covering radius* of ω_N relative to A. The optimal values of these quantities are also of interest and we consider, for $N \geq 2$, the N*-point best-packing distance on* A given by

$$\delta_N(A) := \max\{\delta(\omega_N) \colon \omega_N \subset A,\, |\omega_N| = N\},$$

and the N*-point mesh norm* of A given by

$$\rho_N(A) := \min\{\rho(\omega_N, A) \colon \omega_N \subset A,\, |\omega_N| = N\},$$

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where $|S|$ denotes the cardinality of set S.

In the theory of approximation and interpolation (for example, by splines or radial basis functions (RBFs)), the separation distance is often associated with some measure of 'stability' of the approximation, while the mesh norm arises in the error of the approximation. In this context, the *mesh-separation ratio* (or *mesh ratio*)

$$\gamma(\omega_N, A) := \rho(\omega_N, A)/\delta(\omega_N),$$

can be regarded as a 'condition number' for ω_N relative to A. If $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of N-point configurations such that $\gamma(\omega_N, A)$ is uniformly bounded in N, then the sequence is said to be *quasi-uniform on* A. Quasi-uniform sequences of configurations are important for a number of methods involving RBF approximation and interpolation (see [9, 15, 17, 19]).

We remark that in some cases it is easy to obtain positive lower bounds for the mesh-separation ratio. For example, if A is connected, then $\gamma(\omega_N, A) \geq 1/2$. Furthermore, letting

$$B(x, r) = \{y \in A : m(y, x) \leq r\}$$

be the closed ball in A with center x and radius r, then $\gamma(\omega_N, A) \geq \beta/2$ for any N-point configuration $\omega_N \subset A$ whenever A and $\beta \in (0, 1)$ have the property that for any $r \in (0, \operatorname{diam}(A)]$ and any $x \in A$, the annulus $B(x, r) \setminus B(x, \beta r)$ is nonempty. The diameter of A is defined by

$$\operatorname{diam}(A) := \max\{m(x, y) \colon x \in A,\ y \in A\}.$$

In this paper we consider the separation distance and mesh norm of finite point configurations in A that minimize certain weighted energy functionals. We call $w : A \times A \to [0, \infty)$ an *SLP weight on* A if it is symmetric and lower semi-continuous on $A \times A$ and is positive on the diagonal, $D(A)$, of $A \times A$. For $s > 0$ and a collection of $N \geq 2$ distinct points $\omega_N = \{x_1, \ldots, x_N\} \subset A$, the *$(s, w)$-energy of ω_N* (also known as the *weighted Riesz s-energy*) is

$$E_s^w(\omega_N) := \sum_{i \neq j} \frac{w(x_i, x_j)}{m(x_i, x_j)^s} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x_i, x_j)}{m(x_i, x_j)^s}, \tag{1.3}$$

and we denote the *minimal N-point (s, w)-energy of A* by

$$\mathcal{E}_s^w(N, A) := \inf\{E_s^w(\omega_N) : \omega_N \subset A,\ |\omega_N| = N\}. \tag{1.4}$$

Since A is compact and the energy $E_s^w(\omega_N)$ is lower semi-continuous, there exists at least one N-point configuration $\omega_N^* \subset A$ such that $E_s^w(\omega_N^*) = \mathcal{E}_s^w(N, A)$. We refer to such an ω_N^* as an *N-point (s, w)-energy minimizing configuration on A*. The asymptotics as $N \to \infty$ of N-point (s, w)-energy minimizing configurations and their energies are investigated in [1, 10] for d-rectifiable sets $A \subset \mathbb{R}^p$ and $s > d$ (see further discussion in the next section).

In our results we shall require that A is either α-regular or upper α-regular as we next describe. For a positive Borel measure μ supported on A and $\alpha > 0$, we say that μ is *upper α-regular* if there is some finite constant C_0 such that

$$\mu(B(x, r)) \leq C_0\, r^{\alpha} \qquad (x \in A,\ 0 < r \leq \operatorname{diam}(A)), \tag{1.5}$$

and we say that μ is *lower α-regular* if there is some positive constant c_0 such that

$$c_0^{-1}\, r^{\alpha} \leq \mu(B(x, r)) \qquad (x \in A,\ 0 < r \leq \operatorname{diam}(A)). \tag{1.6}$$

We shall refer to A as an *upper α-regular metric space* if there exists an upper α-regular measure $\bar{\mu}$ on A such that $\bar{\mu}(A) > 0$ and shall refer to A as a *lower α-regular metric space* if there exists a lower α-regular measure $\underline{\mu}$ on A such that $\underline{\mu}(A) < \infty$. (Obviously, if A is upper α-regular then A has infinitely many points.) If A supports a measure that is both upper and lower α-regular, then we say that A is an *α-regular metric space*. If A is α-regular, then it is not difficult to show that the Hausdorff dimension of A, $\dim_{\mathcal{H}} A$, equals α (cf. [12]). Furthermore, the α-dimensional Hausdorff measure of A, $\mathcal{H}_\alpha(A)$, is positive and finite.

Many of the constants appearing in this paper, either explicitly or implicitly involve the upper and lower regularity constants C_0 and c_0 appearing in (1.5) and (1.6). However, in certain cases we are interested in 'local' regularity estimates (i.e., for r small) which can substantially improve our explicit estimates for particular metric spaces of interest (e.g., A is the sphere S^d with the Euclidean metric). Specifically, if $\bar{\mu}$ is an upper α-regular measure, $\underline{\mu}$ is a lower α-regular measure and $r^* > 0$, we define

$$\begin{aligned} C_0(r^*) &:= \sup\{\bar{\mu}(B(x,r))/r^\alpha \colon x \in A,\ 0 < r \leq r^*\}, \\ c_0(r^*)^{-1} &:= \inf\{\underline{\mu}(B(x,r))/r^\alpha \colon x \in A,\ 0 < r \leq r^*\}. \end{aligned} \tag{1.7}$$

We note that both $C_0(r^*)$ and $c_0(r^*)$ are increasing in r^*, and we make the definitions

$$\begin{aligned} C_0(0) &:= \lim_{r^* \to 0^+} C_0(r^*), \\ c_0(0) &:= \lim_{r^* \to 0^+} c_0(r^*). \end{aligned} \tag{1.8}$$

Furthermore, if A is a compact (i.e., without boundary), C^1, d-dimensional manifold and $\mu = \mathcal{H}_d$, then $C_0(0) \cdot c_0(0) = 1$. For the largest length scale of interest, with a slight abuse of notation, the global constants for $\bar{\mu}$ and $\underline{\mu}$, respectively, are related by $C_0 = C_0(\operatorname{diam}(A))$ and $c_0 = c_0(\operatorname{diam}(A))$.

One may obtain simple upper bounds for $\delta_N(A)$ (respectively, lower bounds for $\rho_N(A)$) in the case that A is lower (respectively, upper) α-regular. Specifically, if A is lower α-regular then there is a constant $c_A < \infty$ such that

$$\delta_N(A) \leq c_A N^{-1/\alpha}, \qquad (N \geq 2), \tag{1.9}$$

while if A is upper α-regular then there is a constant $\tilde{c}_A > 0$ such that

$$\rho_N(A) \geq \tilde{c}_A N^{-1/\alpha}, \qquad (N \geq 2). \tag{1.10}$$

The bound (1.9) is a consequence of the facts that the balls $\{B(x, \delta(\omega_N)/2) \colon x \in \omega_N\}$ are pairwise disjoint and that there exists a lower α-regular measure $\underline{\mu}$ with $\underline{\mu}(A) < \infty$. Similarly, if A is upper α-regular, then the bound (1.10) follows from the covering property of the balls $\{B(x, \rho(\omega_N, A)) \colon x \in \omega_N\}$ and the existence of an upper α-regular measure $\bar{\mu}$ with $\bar{\mu}(A) > 0$.

The main result of this paper, given in Theorem 5, is that a sequence of N-point (s, w)-energy minimizing configurations on an α-regular compact metric space A is quasi-uniform on A whenever $s > \alpha$. As an application, we deduce that, if $A \subset \mathbb{R}^p$ is d-rectifiable for some integer $0 < d \leq p$ with $\mathcal{H}_d(A) > 0$, then a quasi-uniform sequence of N-point configurations on A can be found that has a prescribed bounded positive density on A (see Corollary 6 and the preceding discussion). Furthermore we show that if A is a smooth closed manifold, then there exists a sequence of N-point best-packing configurations $\{\nu_N\}_{N=2}^{\infty}$ on A for which $\limsup_{N\to\infty}\gamma(\nu_N, A) \leq 2$ (see Corollary 8).

2. Main Results

We first consider the separation distance of (s, w)-energy minimizing configurations on an upper α-regular compact metric space A. For these separation results, we consider symmetric weight functions w such that $\|w(\cdot, x)\|_{L_p(\mu)}$ is uniformly bounded on A for some $1 < p \leq \infty$. Here we use the standard notation,

$$\|f\|_{L_p(\mu)} := \begin{cases} \left(\int_A |f|^p \, d\mu\right)^{1/p}, & 1 \leq p < \infty, \\ \mu\text{-ess sup } |f|, & p = \infty, \end{cases}$$

where μ is a positive Borel measure and f is a Borel measurable function on A.

The following theorem extends a result [1, Theorem 4] to a more general class of weight functions and to more general compact metric spaces.

Theorem 1. *Let A be a compact, upper α-regular metric space with respect to $\bar{\mu}$ and let w be an SLP weight on A such that $\|w(\cdot, x)\|_{L_{p_0}(\bar{\mu})}$ is uniformly bounded on A for some $1 < p_0 \leq \infty$. Suppose $1 < p \leq p_0$, $s > \alpha(1 - 1/p)$, and $N \geq 2$. If ω_N^* is an N-point (s, w)-energy minimizing configuration on A, then*

$$\delta(\omega_N^*) \geq C_1 \, N^{-\left(\frac{1}{\alpha} + \frac{1}{sp}\right)} \qquad (N \geq 2), \tag{2.1}$$

where C_1 is a constant independent of N indicated below in (3.13).

Taking w bounded and setting $p = \infty$ in Theorem 1 produces the following result.

Corollary 2. *Suppose A is a compact, upper α-regular metric space and w is a bounded SLP weight on A, and let $s > \alpha$. If ω_N^* is an N-point (s, w)-energy minimizing configuration on A, then*

$$\delta(\omega_N^*) \geq C_2 N^{-1/\alpha} \qquad (N \geq 2), \tag{2.2}$$

where C_2 is a constant independent of N. Consequently,

$$\delta_N(A) \geq C_2 N^{-1/\alpha} \qquad (N \geq 2). \tag{2.3}$$

For the unweighted case $w \equiv 1$, the constant C_2 satisfies

$$C_2 \geq \left[\frac{\bar{\mu}(A)}{C_0}\left(1 - \frac{\alpha}{s}\right)\right]^{1/\alpha} \left(\frac{\alpha}{s}\right)^{1/s}, \tag{2.4}$$

where $C_0 = C_0(\mathrm{diam}(A))$.

We note that if A in Corollary 2 is α-regular, then by inequality (1.9) we see that N-point (s, w)-energy minimizing configurations on A have the best possible order of separation as $N \to \infty$.

With respect to the separation constant of (2.4), if $d \geq 2$ and $A = \mathbb{S}^d$ with σ_d denoting the uniform probability distribution on $\mathbb{S}^d$, then we can get an explicit lower bound for C_2 by calculating the regularity constant C_0. As stated in [13], for $x \in \mathbb{S}^d$, $0 \leq r \leq 2$, and

$$\gamma_d := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d/2)\Gamma(1/2)}, \tag{2.5}$$

there holds

$$\sigma_d(r) := \sigma_d(B(x,r)) = \gamma_d \int_{1-r^2/2}^{1} (1-t^2)^{d/2-1} dt$$

from which it follows that

$$\sigma_d(r) \le \frac{\gamma_d}{d} r^d,$$

and, as $r \to 0^+$,

$$\sigma_d(r) = \frac{\gamma_d}{d} r^d + \mathcal{O}(r^{d+2}).$$

Therefore, for the uniform probability distribution on $\mathbb{S}^d$, the global upper regularity constant is

$$C_0 = \sup_{0<r\le 2} \frac{\sigma_d(r)}{r^d} = \frac{\gamma_d}{d}, \tag{2.6}$$

and when applied to (2.4) we obtain

$$C_2 \ge \left(\frac{d}{\gamma_d}\right)^{1/d} \left(1 - \frac{d}{s}\right)^{1/d} \left(\frac{d}{s}\right)^{1/s}. \tag{2.7}$$

With this lower bound for C_2, (2.2) becomes

$$\delta(\omega_N^*) \ge \left(\frac{d}{\gamma_d}\right)^{1/d} \left(1 - \frac{d}{s}\right)^{1/d} \left(\frac{d}{s}\right)^{1/s} N^{-1/d} \qquad (N \ge 2,\ s > d), \tag{2.8}$$

and, on letting $s \to \infty$, we deduce for the N-point best-packing distance

$$\delta_N(\mathbb{S}^d) \ge \left(\frac{d}{\gamma_d}\right)^{1/d} N^{-1/d} \qquad (N \ge 2,\ s > d).$$

A less explicit lower bound for the separation constant of minimal energy points for $s > d$ on $\mathbb{S}^d$ was obtained in [13, Corollary 4].

We next consider the mesh norm of (s,w)-energy minimizing configurations on an α-regular compact metric space A. In this case we require that the weight function w be bounded.

Theorem 3. *Let A be a compact, α-regular metric space with respect to the measure μ and $K \subset A$ be a compact set of positive μ-measure. Let w be a bounded SLP weight on K. If $s > \alpha$ and ω_N^* is an N-point (s,w)-energy minimizing configuration on K, then*

$$\rho(\omega_N^*, K) \le C_3\, N^{-1/\alpha} \qquad (N \ge 2), \tag{2.9}$$

where C_3 is a constant independent of N given below in (3.41).

Theorem 3 substantially extends a result of [6] that holds for unweighted energy minimizing point configurations when $K \subset \mathbb{R}^p$ is restricted to be the finite union of bi-Lipschitz images of compact sets in $\mathbb{R}^d$.

We remark that for K and A as in Theorem 3, the set K need not inherit the lower α-regularity of A. However, since $\mu(K) > 0$, we do have that K is an upper α-regular metric space and, consequently, there is a constant $\tilde{c}_K > 0$ such that (1.10) holds with A replaced by K. Hence, the inequality (2.9) has the best possible order with respect to N.

Taking $w \equiv 1$ in Theorem 3 immediately yields the following.

Corollary 4. *Let A be a compact, α-regular metric space with respect to the measure μ and let $K \subset A$ be a compact set of positive μ-measure. Then there exists a constant C_4 such that*

$$\rho_N(K) \leq C_4 \, N^{-1/\alpha} \qquad (N \geq 2).$$

Combining Corollary 2 and Theorem 3 we obtain our main result.

Theorem 5. *Let A be a compact, α-regular metric space with respect to the measure μ and let $K \subset A$ be a compact set of positive μ-measure. Furthermore, let w be a bounded SLP weight on K, and for $s > \alpha$ and $N \geq 2$, let ω_N^* be an N-point (s, w)-energy minimizing configuration on K. Then $\{\omega_N^*\}_{N=2}^{\infty}$ is quasi-uniform on K.*

We remark that there are α-regular sets A and values of $s < \alpha$ for which (unweighted) $(s, 1)$-energy minimizing configurations on A have a mesh-separation ratio that goes to ∞ with N. One such example given in [3] is a 'washer' A obtained by revolving a certain rectangle about an axis parallel to one of its sides, where it turns out that for $s < 1/3$, the support of the limit distribution of the $(s, 1)$-energy minimizing configurations on A omits an open subset of A. Also, for the logarithmic energy which corresponds to $s = 0$, it is shown in [11] that, for $w \equiv 1$, the support of the limit distribution of the log-energy minimizing configurations on a torus in $\mathbb{R}^3$ is only supported on the positive curvature portion of the torus, so that the mesh-separation ratio for such configurations is again unbounded as $N \to \infty$. Examples also abound in one dimension. For the logarithmic energy, it is well-known [21, Sections 6.7 and 6.21] that for $A = [-1, 1]$ and $w \equiv 1$ the minimum energy points are zeros of Jacobi orthogonal polynomials (together with ± 1) that have separation distance of precise order $1/N^2$ and mesh norm of precise order $1/N$, so that the mesh-separation ratio grows like N.

One of our main motivations for considering weighted minimum energy configurations is that for a large class of sets A one can design a weight function w so that a sequence of N-point (s, w)-energy minimizing configurations have a specified limiting density on A as $N \to \infty$. The following result is a consequence of Theorem 5 and [1, Corollary 2]. Recall that a set in $\mathbb{R}^p$ is *d-rectifiable* if it is the Lipschitz image of a bounded set in $\mathbb{R}^d$.

Corollary 6. *Let $d \leq p$ and $A \subset \mathbb{R}^p$ be a compact, infinite set that is d-rectifiable and lower d-regular with respect to $\mathcal{H}_d$ for some integer d. Suppose σ is a probability density on A that is continuous almost everywhere with respect to $\mathcal{H}_d$ and is bounded above and below by positive constants. Let $s > d$ and $w : A \times A \to [0, \infty)$ be given by*

$$w(x, y) := (\sigma(x)\sigma(y))^{-s/2d}. \tag{2.10}$$

For $N \geq 2$, let ω_N^* be an N-point (s, w)-energy minimizing configuration on A. Then $\{\omega_N^*\}_{N=2}^{\infty}$ is quasi-uniform on A and the sequence of normalized counting measures associated with the ω_N^*'s converges weak-star (as $N \to \infty$) to $\sigma \, \mathrm{d}\mathcal{H}_d$.

For A an infinite, compact, metric space and $s > 0$, let ω_N^s be an N-point $(s, 1)$-energy minimizing configuration on A. Furthermore, let ν_N be a cluster point (in the product topology on A^N) of ω_N^s as $s \to \infty$. As we now show, ν_N must be an

N-point best-packing configuration on A, that is, $\delta(\nu_N) = \delta_N(A)$. For this purpose, let $\tilde{\omega}_N$ be an N-point best-packing configuration on A. Then we have

$$\delta(\omega_N^s)^{-s} \le \mathcal{E}_s^1(N, A) \le E_s^1(\tilde{\omega}_N) \le N(N-1)\delta_N(A)^{-s},$$

and so

$$(N(N-1))^{-1/s}\delta_N(A) \le \delta(\omega_N^s) \le \delta_N(A),$$

which gives

$$\lim_{s\to\infty} \delta(\omega_N^s) = \delta_N(A). \tag{2.11}$$

Since $\omega_N^{s_j} \to \nu_N$ for some subsequence $s_j \to \infty$, it follows from (2.11) and continuity that $\delta(\nu_N) = \delta_N(A)$ and so ν_N is an N-point best-packing configuration on A.

In general, it is not true that a sequence of N-point best-packing configurations in A is quasi-uniform on A (e.g., if A is the classical (1/3)-Cantor set in [0,1] together with any point outside this interval). However, for A as in Theorem 5, it turns out that by using $(s,1)$-energy minimizing configurations on A and taking $s \to \infty$ we can construct a sequence of N-point best-packing configurations in A that is also quasi-uniform on A.

Theorem 7. *Let A be a compact, α-regular metric space with respect to the measure μ and let $K \subset A$ be a compact set of positive μ-measure. For $N \ge 2$, let ν_N be a cluster point of a family of N-point $(s,1)$-energy minimizing configurations on K as $s \to \infty$. Then $\{\nu_N\}_{N=2}^\infty$ is a sequence of N-point best-packing configurations on K that is also quasi-uniform on K.*

Furthermore, the mesh-separation ratios satisfy

$$\limsup_{N\to\infty} \gamma(\nu_N, K) \le \ 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha} [c_0(0)\, C_0(0)]^{1/\alpha}, \tag{2.12}$$

where $c_0(0)$ and $C_0(0)$ are given in (1.8) *for the set A.*

We note that the constant on the right-hand side of (2.12) is at least 2 per (1.7) and (1.8). One can also establish an analogous result concerning the existence of quasi-uniform sequences of *weighted* best-packing configurations (cf. [2]). We leave this extension to the reader.

When applied to a compact C^1 manifold without boundary, Theorem 7 has the following corollary.

Corollary 8. *If A is a compact d-dimensional C^1 manifold without boundary and $K \subset A$ is a compact subset of positive $\mathcal{H}_d$ measure, then there exists a sequence of N-point best-packing configurations $\{\nu_N\}_{N=2}^\infty$ on K such that*

$$\limsup_{N\to\infty} \gamma(\nu_N, K) \le 2\left(\frac{\mathcal{H}_d(A)}{\mathcal{H}_d(K)}\right)^{1/\alpha}. \tag{2.13}$$

In particular, if $K = A$ we have

$$\limsup_{N\to\infty} \gamma(\nu_N, A) \le 2. \tag{2.14}$$

The proof of Corollary 8 simply amounts to noting that for a C^1 manifold without boundary A, and $\mathcal{H}_d|_A =: \mu$, the constants $c_0(0)$ and $C_0(0)$ of (1.8) are reciprocals.

In comparison with (2.14), we remark that it is easy to see that there is always an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers such that for each k there is an N_k-point best-packing configuration $\hat{\nu}_{N_k}$ on A such that $\gamma(\hat{\nu}_{N_k}, A) < 2$ for all k; indeed, this follows from the observation that if ω_N is any N-point configuration on A such that $\gamma(\omega_N, A) \geq 2$ then there is some $x \in A \setminus \omega_N$ such that $\delta(\omega_N \cup \{x\}) = \delta(\omega_N)$.

We conclude this section with further references to related results. Separation theorems for the case $s \leq d = \dim_{\mathcal{H}}(A)$ have been established only for rather special sets and values of s. Dahlberg [5] proved that (unweighted) optimal $((p-2), 1)$-energy configurations ω_N^* on A are *well-separated* (i.e., they satisfy $\delta(\omega_N^*) \geq CN^{-1/d}$ for some positive constant C) if $A \subset \mathbb{R}^p$ $(p \geq 3)$ is a smooth $d = p-1$ dimensional closed surface in $\mathbb{R}^p$ that separates $\mathbb{R}^p$ into two components. For the critical value $s = d$ and A a d-rectifiable subset of a smooth d-dimensional manifold in $\mathbb{R}^p$, it is shown in [1] that the following weaker separation result holds

$$\delta(\omega_N^*) \geq C(N \log N)^{-1/d}, \tag{2.15}$$

for some positive constant C.

For the case that $A = \mathbb{S}^d$, the d-dimensional unit sphere in $\mathbb{R}^{d+1}$, well-separation was proved in [14] for the range of values $d-1 < s < d$ and further extended by Dragnev and Saff [8] to the range $d-2 < s < d$ with explicit estimates for the separation constant C. Well-separation for $s = d-2$ and $d \geq 3$ was established in [6].

Thus, for the important case of $A = \mathbb{S}^2$ it is known that optimal s-energy configurations on $\mathbb{S}^2$ are well-separated for all nonnegative values of $s \neq 2$ (well-separatedness for $s = 0$ was established in [18]; see also [7]); for the critical value $s = 2$, the only known separation results are of the weak form given in (2.15).

Much less is known with regard to covering (mesh norm) theorems in the case that $s \leq d$ (see [20, Sec. 1.3]).

3. Proofs

In the proofs we shall need that an SLP weight w is bounded below in a neighborhood of the diagonal $D(A)$. Indeed, the positivity and lower semi-continuity of w on $D(A)$ and the compactness of A imply that there are positive numbers η and κ such that

$$w(x,y) \geq \eta \qquad (x, y \in A,\ m(x,y) \leq \kappa). \tag{3.1}$$

Proof of Theorem 1. The initial part of this argument proceeds as in [13]. Let $N \geq 2$ be fixed and let $\omega_N^* = \{x_1, \dots, x_N\} \subset A$ be a fixed (s,w)-energy minimizing configuration in A. For $x \in A$ and $1 \leq i \leq N$, let

$$U_i(x) := \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x, x_j)}{m(x, x_j)^s}.$$

Since ω_N^* is a minimizing configuration we have the lower bound

$$U_i(x_i) \leq U_i(x) \text{ for all } x \in A. \tag{3.2}$$

Fix $r_1 \leq \operatorname{diam}(A)$ such that

$$\bar{\mu}\left(\bigcup_{j=1}^{N} B(x_j, r_1)\right) \geq \bar{\mu}(A). \tag{3.3}$$

The radius r_1 can clearly be chosen independent of N, for example $r_1 = \operatorname{diam}(A)$, and we note for future reference that it suffices to take $r_1 > \rho(\omega_N^*, A)$. For the rest of this proof we fix $r_1 = \operatorname{diam}(A)$.

Now let $0 < \theta < 1$ and define

$$r_0 := \left(\frac{\theta\bar{\mu}(A)}{N\, C_0(r_1)}\right)^{1/\alpha}, \tag{3.4}$$

where $C_0(r_1) = C_0$ is the upper regularity constant of $\bar{\mu}$ as in (1.7). We note that $r_0 < r_1$ as can be seen from the fact that $\bar{\mu}(A) \leq C_0(r_1) r_1^{\alpha}$.

For $B(x, r_0, r_1) := B(x, r_1) \setminus B(x, r_0)$, let

$$D := \bigcup_{j=1}^{N} B(x_j, r_0, r_1).$$

Using the upper regularity of $\bar{\mu}$ and (3.3) we see that

$$\bar{\mu}(D) \geq \bar{\mu}(A) - \sum_{j=1}^{N} \bar{\mu}(B(x_j, r_0)) \geq (1-\theta)\bar{\mu}(A) > 0,$$

and thus by inequality (3.2) we have

$$U_i(x_i) \leq \frac{1}{\bar{\mu}(D)} \int_D U_i(x)\, d\bar{\mu}(x) \leq \frac{1}{(1-\theta)\bar{\mu}(A)} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{B(x_j, r_0, r_1)} \frac{w(x, x_j)}{m(x, x_j)^s}\, d\bar{\mu}(x). \tag{3.5}$$

Applying Hölder's inequality with $1/q = 1 - 1/p$ we obtain

$$U_i(x_i) \leq \frac{1}{(1-\theta)\bar{\mu}(A)} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|w(\cdot, x_j)\|_{L_p(\bar{\mu})} \left(\int_{B(x_j, r_0, r_1)} \frac{1}{m(x, x_j)^{sq}}\, d\bar{\mu}(x)\right)^{1/q}. \tag{3.6}$$

Converting the integral on the right-hand side of (3.6) to the appropriate integral of the distribution function, and noting that $sq > \alpha$ by assumption, we have

$$\begin{aligned}
\int_{B(x_j, r_0, r_1)} \frac{1}{m(x, x_j)^{sq}}\, d\bar{\mu}(x) &= \int_0^{\infty} \bar{\mu}\left(\{x \in B(x_j, r_0, r_1) : m(x_j, x)^{-sq} > t\}\right) dt \\
&\leq \int_{r_1^{-sq}}^{r_0^{-sq}} \bar{\mu}\left(B(x_j, t^{-1/sq})\right)\, dt \\
&\leq \frac{C_0(r_1)\, sq}{sq - \alpha}\, r_0^{\alpha - sq} \\
&= \frac{C_0(r_1)\, sq}{sq - \alpha} \left(\frac{\theta\bar{\mu}(A)}{N\, C_0(r_1)}\right)^{1-(sq)/\alpha},
\end{aligned} \tag{3.7}$$

which, combined with (3.6), gives

$$\begin{aligned} U_i(x_i) &\le \frac{\|w\|_{p,\infty}}{(1-\theta)\bar\mu(A)} \left(\frac{C_0(r_1)\,sq}{sq-\alpha}\right)^{1/q} (N-1) \left(\frac{\theta\bar\mu(A)}{N\,C_0(r_1)}\right)^{1/q-s/\alpha} \\ &< \frac{1}{\bar\mu(A)} \left(\frac{C_0(r_1)}{\bar\mu(A)}\right)^{s/\alpha} \left(\frac{\|w\|_{p,\infty}}{(1-\theta)\theta^{s/\alpha-1/q}}\right) \left(\frac{sq\bar\mu(A)}{sq-\alpha}\right)^{1/q} N^{1/p+s/\alpha}, \end{aligned} \tag{3.8}$$

where $\|w\|_{p,\infty} := \sup_{x\in A} \|w(\cdot,x)\|_{L_p(\bar\mu)} < \infty$.

Choosing

$$\theta_0 := \frac{sq-\alpha}{sq-\alpha+\alpha q} = \left(\frac{s}{\alpha}-\frac{1}{q}\right)\left(\frac{s}{\alpha}+\frac{1}{p}\right)^{-1} < 1, \tag{3.9}$$

which minimizes the right-hand side of (3.8) with respect to θ, we obtain

$$U_i(x_i) \le c_1 N^{s/\alpha+1/p}, \tag{3.10}$$

where after a bit of arithmetic we have

$$c_1 := \|w\|_{p,\infty} \left(\frac{C_0(r_1)}{\bar\mu(A)}\,\frac{s/\alpha+1/p}{s/\alpha-1/q}\right)^{s/\alpha} \left(\frac{s/\alpha+1/p}{\bar\mu(A)}\right)^{1/p} (s/\alpha)^{1/q}. \tag{3.11}$$

Next, select the indices $1 \le i_s \ne j_s \le N$ so that $\delta(\omega_N^*) = m(x_{i_s}, x_{j_s})$ and let κ and η be as in (3.1). If $\delta(\omega_N^*) \le \kappa$, then

$$\frac{\eta}{\delta(\omega_N^*)^s} \le \frac{w(x_{i_s},x_{j_s})}{m(x_{i_s},x_{j_s})^s} \le U_{i_s}(x_{i_s}) \le c_1 N^{s/\alpha+1/p}, \tag{3.12}$$

and therefore

$$\delta(\omega_N^*) \ge \left(\frac{\eta}{c_1}\right)^{1/s} N^{-\frac{1}{\alpha}-\frac{1}{sp}}.$$

Hence, (2.1) holds with

$$C_1 := \min\{\kappa,\ (\eta/c_1)^{1/s}\}. \tag{3.13}$$

□

We remark that for the case when $w \equiv 1$ and $p = \infty$, we can take $\kappa = \infty$, $\eta = 1$, and so from (3.13) we deduce the separation estimate

$$\delta(\omega_N^*) \ge C_2 N^{-1/\alpha} \qquad (N \ge 2),$$

where

$$C_2 := \left[\frac{\bar\mu(A)}{C_0(r_1)}(1-\alpha/s)\right]^{1/\alpha} (\alpha/s)^{1/s},\ r_1 = \mathrm{diam}(A). \tag{3.14}$$

For the proof of Theorem 3, we utilize the following.

Lemma 9. *Let A be a compact, infinite, lower α-regular metric space with lower α-regular measure $\underline{\mu}$, $w : A \times A \to [0,\infty)$ be an SLP weight on A, and $s > \alpha$. Then there exists a positive integer N_0 independent of s, such that*

$$\mathcal{E}_s^w(N,A) \ge C_5 N^{1+s/\alpha} \qquad (N \ge N_0), \tag{3.15}$$

where C_5 is a constant independent of N given below in (3.19).

Proof. Let κ and η be as in (3.1) and let $0 < r_2 \le \kappa$. Since A is compact, there is some M such that the M-point best-packing distance satisfies

$$\delta_M(A) \le r_2. \tag{3.16}$$

Let $N > M$ and let $\omega_N = \{x_1, \dots, x_N\} \subset A$ be an arbitrary N-point configuration of distinct points. For $1 \le i \le N$, let $y_i \in \omega_N$ be a fixed nearest neighbor to x_i in the configuration ω_N, and set

$$\delta_i := m(x_i, y_i) = \min_{\substack{1\le j\le N\\ j\neq i}} m(x_i, x_j) > 0.$$

We assume an ordering on ω_N so that $\delta_i \le \delta_{i+1}$ for $i = 1, \dots, N-1$. We note that $\omega_N \setminus \{x_1, \dots, x_{N-M}\}$ is of cardinality M and thus for all $i \le N' := N - M$ we have that $\delta_i \le r_2 \le \kappa$.

The energy of ω_N then has the lower bound

$$\begin{aligned} E_s^w(\omega_N) &\ge \sum_{i=1}^{N'} \frac{w(x_i, y_i)}{\delta_i^s} \ge \sum_{i=1}^{N'} \eta \left(\frac{1}{\delta_i^\alpha}\right)^{s/\alpha} \ge \eta \left(\sum_{i=1}^{N'} \frac{1}{\delta_i^\alpha}\right)^{s/\alpha} (N')^{1-s/\alpha} \\ &\ge \eta \left(\sum_{i=1}^{N'} \delta_i^\alpha\right)^{-s/\alpha} (N')^{1+s/\alpha} = \eta 2^{-s} \left(\sum_{i=1}^{N'} \left(\frac{\delta_i}{2}\right)^\alpha\right)^{-s/\alpha} (N')^{1+s/\alpha}. \end{aligned} \tag{3.17}$$

where the last inequality in the first line follows from Jensen's inequality and the subsequent inequality follows from the harmonic-arithmetic mean inequality.

Let $\Lambda > 1$ and $N_0 := M\Lambda/(\Lambda - 1)$. Then $N' = N - M \ge \Lambda^{-1}N$ for $N \ge N_0$. Noting that the balls $B(x_i, \delta_i/2)$ are pairwise disjoint, we may apply the lower regularity of $\underline{\mu}$ (with regularity constant $c_0(r_2)$) to obtain

$$\begin{aligned} E_s^w(\omega_N) &\ge \eta 2^{-s} \left(c_0(r_2) \sum_{i=1}^{N'} \underline{\mu}\left(B(x_i, \frac{\delta_i}{2})\right)\right)^{-s/\alpha} (N')^{1+s/\alpha} \\ &\ge \frac{\eta}{(2^\alpha\, c_0(r_2)\, \underline{\mu}(A))^{s/\alpha}} (N')^{1+s/\alpha} \\ &\ge \Lambda^{-1-s/\alpha} \frac{\eta}{(2^\alpha\, c_0(r_2)\, \underline{\mu}(A))^{s/\alpha}} N^{1+s/\alpha} \end{aligned} . \tag{3.18}$$

Since (3.18) holds for arbitrary N-point configurations $\omega_N \subset A$ with $N \ge N_0$, we obtain that (3.15) holds with

$$C_5 := \Lambda^{-1-s/\alpha}\, \eta\, 2^{-s}\, (c_0(r_2)\, \underline{\mu}(A))^{-s/\alpha}. \tag{3.19}$$

We remark that N_0 depends on Λ and r_2, but is independent of s. □

Proof of Theorem 3. Appealing to the generality provided by Theorem 1 and Lemma 9, we can substantially extend and improve upon the arguments used in the proof of Theorem 3.6 in [6].

Let $\omega_N^* = \{x_1 \dots, x_N\}$ be an N-point (s, w)-energy minimizing configuration for the compact set K, and, for $y \in K$, consider the function

$$U(y) := \frac{1}{N} \sum_{i=1}^{N} \frac{w(y, x_i)}{m(y, x_i)^s}. \tag{3.20}$$

For fixed $1 \le j \le N$, the function $U(y)$ can be decomposed as

$$U(y) = \frac{1}{N}\frac{w(y,x_j)}{m(y,x_j)^s} + \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{w(y,x_i)}{m(y,x_i)^s}, \tag{3.21}$$

and, since ω_N^* is a minimizing configuration on K, the point x_j minimizes the sum over $i \neq j$ on the right-hand side of equation (3.21). Thus for each fixed j and $y \in K$

$$U(y) \ge \frac{1}{N}\frac{w(y,x_j)}{m(y,x_j)^s} + \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{w(x_j,x_i)}{m(x_j,x_i)^s}. \tag{3.22}$$

Summing over j gives

$$NU(y) \ge \frac{1}{N}\sum_{j=1}^{N}\frac{w(y,x_j)}{m(y,x_j)^s} + \frac{1}{N}\sum_{j=1}^{N}\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{w(x_j,x_i)}{m(x_j,x_i)^s} \tag{3.23}$$

$$= U(y) + \frac{1}{N}\mathcal{E}_s^w(N,K), \tag{3.24}$$

and thus

$$U(y) \ge \frac{1}{N(N-1)}\mathcal{E}_s^w(N,K) \ge \frac{\mathcal{E}_s^w(N,K)}{N^2} \qquad (y \in K). \tag{3.25}$$

Since K is compact, there exists a point $y^* \in K$ such that

$$\min_{1\le i\le N} m(y^*,x_i) = \rho(\omega_N^*,K) =: \rho(\omega_N^*). \tag{3.26}$$

Using the fact that a function is lower semi-continuous if and only if it is the limit of an increasing sequence of continuous functions, it is not difficult to show that since w is a bounded SLP weight on K, it may be extended to a bounded SLP weight on A. Then, by Lemma 9, there are constants N_0 and $C_5 > 0$ such that

$$\mathcal{E}_s^w(N,K) \ge \mathcal{E}_s^w(N,A) \ge C_5 N^{1+s/\alpha} \qquad (N \ge N_0). \tag{3.27}$$

We note that the constant C_5 of (3.27) does not depend on K, but rather on A (specifically on the lower regularity constant of A and on $\mu(A)$) as well as on the extended weight w.

Since (3.25) holds for the point y^* of (3.26), we combine (3.25) with (3.27) to obtain

$$U(y^*) \ge \frac{\mathcal{E}_s^w(N,K)}{N^2} \ge C_5 N^{s/\alpha-1} \qquad (N \ge N_0). \tag{3.28}$$

Next we determine an upper bound for $U(y^*)$ using the α-regularity of the superset A. Since A is upper α-regular, we see that K is also because $\mu(K) > 0$. Hence, Corollary 2 applied to K implies that there is some $C_2 > 0$ such that $\delta(\omega_N^*) \ge C_2 N^{-1/\alpha}$ for $N \ge 2$. We note that the constant C_2 here depends on K, specifically $\mu(K)$.

Let $\mathcal{N}$ consist of those $N \ge N_0$ such that

$$\rho(\omega_N^*) \ge \frac{C_2}{2}N^{-1/\alpha}. \tag{3.29}$$

If $\mathcal{N}$ is empty (or finite) then we are done. Assuming that $\mathcal{N}$ is nonempty, let $N \in \mathcal{N}$ be fixed.

For $0 < \epsilon < 1/2$, let

$$r_0 = r_0(N, \epsilon) := \epsilon\, C_2 N^{-1/\alpha}. \tag{3.30}$$

Note that any two of the balls $B(x_i, r_0) \subset A$, for $1 \le i \le N$, do not intersect since $r_0 < \delta(\omega_N^*)/2$.

For any $x \in B(x_i, r_0)$, inequalities (3.26) and (3.29) imply

$$\begin{aligned} m(x, y^*) &\le m(x, x_i) + m(x_i, y^*) \le r_0 + m(x_i, y^*) \\ &\le 2\epsilon\, \rho(\omega_N^*) + m(x_i, y^*) \le (1 + 2\epsilon) m(x_i, y^*). \end{aligned} \tag{3.31}$$

For fixed $1 \le i \le N$, using (3.31) and taking an average value on $B(x_i, r_0)$ we obtain

$$\begin{aligned} \frac{w(x_i, y^*)}{m(x_i, y^*)^s} &\le \frac{\|w\|_\infty (1 + 2\epsilon)^s}{\mu(B(x_i, r_0))} \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s} \\ &\le \frac{\|w\|_\infty\, (1 + 2\epsilon)^s\, c_0(r_0)}{r_0^\alpha} \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s}, \end{aligned} \tag{3.32}$$

where $\|w\|_\infty$ denotes the sup-norm of w on $A \times A$ and $c_0(r_0)$ is the localized constant of (1.7) for the set A.

Inequality (3.29) and definition (3.30) imply $2\epsilon\rho(\omega_N^*) \ge r_0$ and thus, for $x \in B(x_i, r_0)$, we obtain

$$\begin{aligned} m(x, y^*) &\ge m(x_i, y^*) - m(x, x_i) \ge m(x_i, y^*) - r_0 \\ &\ge m(x_i, y^*) - 2\epsilon\, \rho(\omega_N^*) \ge (1 - 2\epsilon)\rho(\omega_N^*). \end{aligned} \tag{3.33}$$

Inequality (3.33) implies

$$\bigcup_{i=1}^{N} B(x_i, r_0) \subset A \setminus B(y^*, (1 - 2\epsilon)\rho(\omega_N^*)),$$

and since the left-hand side is a disjoint union, averaging the inequalities of (3.32) we have

$$\begin{aligned} U(y^*) &\le \frac{\|w\|_\infty\, (1 + 2\epsilon)^s\, c_0(r_0)}{N\, r_0^\alpha} \sum_{i=1}^{N} \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s} \\ &\le \frac{\|w\|_\infty\, (1 + 2\epsilon)^s\, c_0(r_0)}{N\, r_0^\alpha} \int_{A \setminus B(y^*, (1 - 2\epsilon)\rho(\omega_N^*))} \frac{d\mu(x)}{m(x, y^*)^s}. \end{aligned} \tag{3.34}$$

For fixed $\tau \ge 1$ we define the radius $R(N) := \tau(1 - 2\epsilon)\rho(\omega_N^*)$, and the constant

$$\tilde{C}_0(\tau) := C_0(R(N))(1 - \tau^{\alpha - s}) + C_0 \tau^{\alpha - s}. \tag{3.35}$$

Note that if $\tau = 1$, then $\tilde{C}_0(1) = C_0$. (We retain τ as a parameter in our estimates as an option for the reader to optimize C_3 for a fixed s.) Now we break the integral on the right-hand side of (3.34) into two terms and proceed as in (3.7) to obtain

(3.36)

$$\begin{aligned}\int_{A\setminus B(y^*,(1-2\epsilon)\rho(\omega_N^*))} & \frac{d\mu(x)}{m(x,y^*)^s} \\ &= \int_{B(y^*,(1-2\epsilon)\rho(\omega_N^*),R(N))} \frac{d\mu(x)}{m(x,y^*)^s} + \int_{A\setminus B(y^*,R(N))} \frac{d\mu(x)}{m(x,y^*)^s} \\ &\leq C_0(R(N)) \int_{R(N)^{-s}}^{[(1-2\epsilon)\rho(\omega_N^*)]^{-s}} t^{-\alpha/s}dt + C_0 \int_0^{R(N)^{-s}} t^{-\alpha/s}dt \\ &= \frac{\tilde{C}_0(\tau)}{(1-\alpha/s)(1-2\epsilon)^{s-\alpha}} \rho(\omega_N^*)^{\alpha-s}.\end{aligned}$$

It is convenient to define the quantity

$$\beta(\epsilon) := \frac{\|w\|_\infty (1+2\epsilon)^s}{(1-\alpha/s)(1-2\epsilon)^{s-\alpha}(\epsilon C_2)^\alpha}, \tag{3.37}$$

and we note that for fixed $s > \alpha$ it is minimized as a function of ϵ for

$$\epsilon_0 := \frac{1}{2(2(s/\alpha)-1)} < \frac{1}{2}, \tag{3.38}$$

with minimal value

$$\beta_0 := \beta(\epsilon_0) = \frac{\|w\|_\infty}{(1-\alpha/s)^{s-\alpha+1}} \left(\frac{4s}{\alpha C_2}\right)^\alpha. \tag{3.39}$$

Using ϵ_0 and combining inequality (3.34) with inequality (3.36) we obtain

$$U(y^*) \leq c_0(r_0)\beta_0 \tilde{C}_0(\tau)\rho(\omega_N^*)^{\alpha-s}. \tag{3.40}$$

If $N \in \mathcal{N}$, then (3.40) and (3.28) imply

$$\rho(\omega_N^*) \leq \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)} N^{-1/\alpha}.$$

If $N \notin \mathcal{N}$, then either $N \leq N_0$ or $\rho(\omega_N^*) < \frac{C_2}{2}N^{-1/\alpha}$. Hence (2.9) holds with

$$C_3 := \max\left\{\operatorname{diam}(A)N_0^{1/\alpha},\ \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)}, \frac{C_2}{2}\right\}. \tag{3.41}$$

We note that if $N > N_0$, then it suffices to take

$$C_3 = \max\left\{\left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)}, \frac{C_2}{2}\right\} \tag{3.42}$$

□

Proof of Theorem 7. Starting with Theorem 3 we shall employ a bootstrapping argument whereby the constants C_2, C_5, and subsequently C_3 are redefined so as to depend on N.

We begin by noting that if $s \geq 2\alpha$, then the constant C_3 of (3.41) has a uniform upper bound in s; indeed, with $\kappa = \infty, C_2$ as defined in (3.14) and C_5 as defined in (3.19) (with $\eta = 1$), each of the three terms appearing in braces in (3.41) is uniformly bounded above. Thus there exists a constant C^* independent of $N \geq 2$ and of $s \geq 2\alpha$ such that $\rho(\omega_N^{(s)}, K) < C^* N^{-1/\alpha}$, where $\omega_N^{(s)}$ is any N-point $(s,1)$-energy minimizing configuration on K.

We next note that $C_0(0)$ of (1.8) is finite and positive, and utilizing the constant c_A of (1.9) we fix

$$C^{**} := \max\left\{C^*,\ c_A,\ \left(\frac{\mu(K)}{C_0(0)}\right)^{1/\alpha}\right\}, \tag{3.43}$$

and we now redefine the radius r_1 to be a function of N,

$$r_1(N) := C^{**}\, N^{-1/\alpha} \quad (N \geq 2). \tag{3.44}$$

Returning to the proof of Theorem 1, we note that $r_1(N) > \rho(\omega_N^{(s)}, K)$, and so inequality (3.3) holds. Furthermore, by the choice of C^{**} we have that for $0 < \theta_0 < 1$ as in (3.9)

$$r_0(N) := \left(\frac{\theta_0 \mu(K)}{N C_0(0)}\right)^{1/\alpha} < r_1(N).$$

Taking $r_0 = r_0(N)$ in the proof and remembering that $q = 1$ in the current context, we see that with A replaced by K the penultimate term on right-hand side of (3.7) becomes

$$\frac{sC_0(r_1(N))}{s-\alpha}\left(\frac{\theta_0\, \mu(K)}{N\, C_0(0)}\right)^{1-s/\alpha},$$

and thus

$$\begin{aligned}\int_{B(x_j, r_0(N), r_1(N))} \frac{d\mu(x)}{m(x,x_j)^s} &\leq \frac{sC_0(r_1(N))}{s-\alpha}\left(\frac{\theta_0\, \mu(K)}{N\, C_0(0)}\right)^{1-s/\alpha} \\ &\leq \frac{s}{s-\alpha}\left(\frac{\theta_0\, \mu(K)}{N}\right)^{1-s/\alpha} C_0(r_1(N))^{s/\alpha},\end{aligned} \tag{3.45}$$

where the last inequality follows from the fact that $C_0(0) \leq C_0(r_1(N))$ and $s > \alpha$.

For $w \equiv 1$, the constant C_2 of (3.14) with $r_1 = r_1(N)$ becomes

$$C_2(N) := \left(\frac{\alpha}{s}\right)^{1/s}\left(\frac{1-\alpha/s}{C_0(r_1(N))}\right)^{1/\alpha} \mu(K)^{1/\alpha}, \tag{3.46}$$

where $C_0(r_1(N))$ is the local upper regularity constant of (1.7), and we have

$$\delta(\omega_N^{(s)}) \geq C_2(N) N^{-1/\alpha} \qquad (N \geq 2,\ s \geq 2\alpha).$$

Furthermore, allowing the radius r_2 appearing in (3.16) to depend on $N \geq 2$ by taking $r_2 := r_1(N)$, we see via (1.9) and (3.43) that

$$r_1(N) \geq \delta_N(A) \quad (N \geq 2),$$

and there is no need to designate the integer M in the proof of Lemma 9. Thus we can take $\Lambda = 1$ in (3.19), and it follows (with $\eta = 1$) that

$$E_s^1(\omega_N^{(s)}) \geq C_5(N) N^{1+s/\alpha} \qquad (N \geq 2,\ s \geq 2\alpha),$$

where

$$C_5(N) := \frac{1}{2^s[c_0(r_1(N))\mu(A)]^{s/\alpha}}. \tag{3.47}$$

We remark that $C_2(N)$ clearly depends on the subset K, whereas $C_5(N)$ depends on the superset A.

We now return to the proof of Theorem 3 utilizing the constants $C_2(N)$ and $C_5(N)$. For β_0 as in (3.39), we see that

$$\rho(\omega_N^{(s)}, K) \leq C_3(N)N^{-1/\alpha} \ \ (N \geq N_0, \ s \geq 2\alpha),$$

where N_0 is as in Lemma 9, and by (3.42) (choosing $\tau = 1$, so that $\tilde{C}_0(\tau) = C_0$)

$$C_3(N) := \max\left\{\left[\frac{c_0(r_0)\beta_0 C_0}{C_5(N)}\right]^{1/(s-\alpha)}, \frac{C_2(N)}{2}\right\}. \tag{3.48}$$

With equations (3.46)-(3.48) in mind, we are ready to complete the proof of Theorem 7. The argument leading to equation (2.11) shows that ν_N is an N-point best-packing configuration on K for each $N \geq 2$. We now need to determine the limits of the constants $C_2(N)$ of (3.46) and $C_3(N)$ of (3.48) as $s \to \infty$. Fixing N in (3.46) yields

$$\lim_{s\to\infty} C_2(N) = \left(\frac{\mu(K)}{C_0(r_1(N))}\right)^{1/\alpha} =: \hat{C}_2(N). \tag{3.49}$$

Since $c_0(r_0)$ and C_0 are independent of s and $\lim_{s\to\infty} \beta_0^{1/(s-\alpha)} = 1$, it follows, that for fixed N

$$\begin{aligned} \lim_{s\to\infty} C_3(N) &= \max\left\{\frac{\hat{C}_2(N)}{2}, \lim_{s\to\infty} C_5(N)^{1/(\alpha-s)}\right\} \\ &= \max\left\{\frac{1}{2}\left(\frac{\mu(K)}{C_0(r_1(N))}\right)^{1/\alpha}, 2[c_0(r_1(N))\mu(A)]^{1/\alpha}\right\}. \\ &:= \hat{C}_3(N) \end{aligned} \tag{3.50}$$

From the continuity of $\delta(\cdot)$ and $\rho(\cdot, K)$ on K^N we deduce that

$$\delta(\nu_N) \geq \hat{C}_2(N)N^{-1/\alpha} \quad \text{and} \quad \rho(\nu_N, K) \leq \hat{C}_3(N)N^{-1/\alpha} \qquad (N \geq N_0).$$

Taking the ratio of these two quantities we have that

$$\frac{\rho(\nu_N, K)}{\delta(\nu_N)} \leq \frac{\hat{C}_3(N)}{\hat{C}_2(N)} = \max\left\{\frac{1}{2}, 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(r_1(N))\, C_0(r_1(N))]^{1/\alpha}\right\}, \tag{3.51}$$

and hence for $N \geq N_0$

$$\limsup_{N\to\infty} \frac{\rho(\nu_N, K)}{\delta(\nu_N)} \leq \max\left\{\frac{1}{2}, 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(0)\, C_0(0)]^{1/\alpha}\right\} \tag{3.52}$$

$$= 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(0)\, C_0(0)]^{1/\alpha} < \infty. \tag{3.53}$$

Therefore, the sequence of configurations $\{\nu_N\}_{N=2}^{\infty}$ is quasi-uniform on K. $\square$

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D. P. Hardin, E. B. Saff, and J. T. Whitehouse: Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: `Doug.Hardin@Vanderbilt.Edu`
E-mail address: `Edward.B.Saff@Vanderbilt.Edu`
E-mail address: `Tyler.Whitehouse@gmail.com`