CHARACTERIZING KERNELS OF OPERATORS RELATED TO THIN PLATE MAGNETIZATIONS VIA GENERALIZATIONS OF HODGE DECOMPOSITIONS
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L. BARATCHART[†], D. P. HARDIN[†], E. A. LIMA[*], E. B. SAFF[†], AND B. P. WEISS[*]

ABSTRACT. Recently developed scanning magnetic microscopes measure the magnetic field in a plane above a thin-plate magnetization distribution. These instruments have broad applications in geoscience and materials science, but are limited by the requirement that the sample magnetization must be retrieved from measured field data, which is a generically nonunique inverse problem. This problem leads to an analysis of the kernel of the related magnetization operators, which also has relevance to the "equivalent source problem" in the case of measurements taken from just one side of the magnetization. We characterize the kernel of the operator relating planar magnetization distributions to planar magnetic field maps in various function and distribution spaces (e.g., sums of derivatives of L^p (Lebesgue spaces) or bounded mean oscillation (BMO) functions). For this purpose, we present a generalization of the Hodge decomposition in terms of Riesz transforms and utilize it to characterize sources that do not produce magnetic field either above or below the sample, or that are magnetically silent (i.e., no magnetic field anywhere outside the sample). For example, we show that a thin-plate magnetization is silent (i.e., in the kernel) when its normal component is zero and its tangential component is divergence-free. In addition, we show that compactly supported magnetizations (i.e., magnetizations that are zero outside of a bounded set in the source plane) that do not produce magnetic fields either above or below the sample are necessarily silent. In particular, neither a nontrivial planar magnetization with fixed direction (unidimensional) compact support nor a bidimensional planar magnetization (i.e., a sum of two unidimensional magnetizations) that is nontangential can be silent. We prove that any planar magnetization distribution is equivalent to a unidimensional one. We also discuss the advantages of mapping the field on both sides of a magnetization, whenever experimentally feasible. Examples of source recovery are given along with a brief discussion of the Fourier-based inversion techniques that are utilized.

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1. Introduction

The Earth's geomagnetic field is generated by convection of the liquid metallic core, a mechanism known as geodynamo. The geomagnetic field may be recorded as remanent magnetization (the large-scale, semi-permanent alignment of electron spins in matter) in geologic materials containing ferromagnetic minerals. This remanent magnetization provides records of the intensity and orientation of the ancient magnetic field. It can also be used to study processes other than those of geomagnetism, including characterizing past motions of tectonic plates and as a relative chronometer through the identification of geomagnetic reversals (periods when the Earth's north and south magnetic poles are interchanged) in rock sequences [17]. Rocks from Mars, the Moon, and asteroids are also known to contain remanent magnetization which indicates the past presence of core dynamos on these bodies [1, 7, 25]. Magnetization in meteorites may even record magnetic fields produced by the young sun and the protoplanetary disk (the primordial nebula of gas and dust around the young sun), which may have played a key role in solar system formation [25].

Until recently, nearly all paleomagnetic techniques were only capable of analyzing bulk samples (typically several centimeters in diameter). In particular, the vast majority of magnetometers currently in use in the geosciences infer the net magnetic moment of a rock sample from a set of measurements of the three components of the sample's external magnetic field taken at a fixed distance. These data can be used to uniquely measure the net moment of the sample (the integral of the magnetization distribution over the sample's volume) provided that the sample's geometry satisfies certain constraints [5]. While this approach has historically provided a wealth of geological information, much could be gained by retrieving the magnetization distribution *within* the sample. Such data could be used to directly correlate magnetization with mineralogy and textures in rock samples, which would provide powerful information on the origin and age of the magnetization. This goal has recently motivated the development of scanning magnetic microscopy methods that can extend paleomagnetic measurements to submillimeter scales.

Typical scanning magnetic microscopes map only a single component of the magnetic field measured in a planar grid at a fixed distance above a planar sample [11]. However, geoscientists are ultimately interested in determining the magnetization distribution within a sample because it is this quantity, rather than the field it produces, that provides a direct record of the ancient field intensity and direction. This inverse problem can be regarded as an *equivalent source problem* (cf. [4, Sec. 12.1.2]) with added constraints such as properties of the support or direction of the magnetization. In particular, reconstructing physically relevant magnetizations is of special interest. A key difficulty is that, in general, infinitely many magnetization patterns can produce the same magnetic field data observed outside the magnetized region. Thus, in order to retrieve a magnetization from magnetic field measurements, an ill-posed inverse problem [18] must be solved for which the characterization of magnetically silent sources is of utmost importance (cf. [12] and [3] for related investigations that focus instead on the reconstruction of current distributions in conducting materials from magnetic field data). Analyzing these silent sources is critical for determining the intrinsic limitations of this inverse problem and for devising regularization schemes needed for effective inversion algorithms.

In this paper, we use tools from modern harmonic analysis (e.g., Riesz transforms, Fourier multipliers and their distributional spaces) to provide a full characterization of silent sources for the planar equivalent source problem. We will especially emphasize the fundamental role that Riesz transforms play in this problem. A significant contribution of this work is to introduce a generalization of the classical Helmholtz-Hodge decomposition, which we call the *Hardy-Hodge decomposition*, as a key tool for characterizing silent sources.

Here we set up a framework for the recovery of infinitely thin planar magnetization distributions whose field is measured by a scanning magnetic microscope. Such distributions are relevant for typical geologic magnetic microscopy studies, which involve the analysis of rock thin sections that are usually much thinner (three orders of magnitude smaller) than their horizontal dimensions and whose fields are measured at distances exceeding 4-5 times their thicknesses [26]. This geometry means that the planar magnetization distribution is an accurate model for the sample. Therefore, we are particularly interested in characterizing planar silent sources with compact support.

Unidirectional magnetization distributions, by which we mean magnetizations that have fixed direction but variable nonnegative magnitude, are of significant interest. These distributions occur naturally owing to the process of remanence acquisition in rocks formed in the presence of a uniform external magnetic field. It is not uncommon, however, to find a second component superimposed on a unidirectional magnetization component as a result of partial remagnetization by secondary processes like weathering or application of hand magnets. We term a magnetization *bidirectional* if it can be expressed as the sum of two unidirectional components.

More generally, we will investigate *unidimensional* and *bidimensional* magnetizations. By the former, we mean a magnetization of the form $Q(\mathbf{x})\mathbf{u}$, where $\mathbf{u} = (u_1, u_2, u_3)$ is a fixed vector in $\mathbf{R}^3$ and Q is a scalar valued function (possibly taking positive as well as negative values) for $\mathbf{x}$ in the support of the magnetization. Similarly, we call a magnetization bidimensional if it can be expressed as the sum of two unidimensional components. Clearly, any unidirectional (resp. bidirectional) magnetization is a unidimensional (resp. bidimensional) magnetization.

The outline of this paper is as follows. After the introduction of the basic formulas in Section 2.1, we focus in the remainder of Section 2 on the case of planar magnetizations whose components lie in $L^p(\mathbf{R}^2)$, $1 < p < \infty$. In Theorem 2.1, the operator relating a planar magnetization distribution to its magnetic potential is expressed in terms of Poisson and Riesz transforms from which we deduce the limiting values of the potential from above and below the source plane. The Hardy-Hodge decomposition for $(L^p(\mathbf{R}^2))^3$ is presented in Theorem 2.2 and is the main tool for characterizing the kernel of the planar magnetization operator (cf. Theorem 2.3); thereby we describe all equivalent magnetizations. In particular, we show that a thin plate magnetization is "silent from both sides" (i.e., the magnetic field vanishes above and below the sample) if its normal component is zero and its tangential component is divergence-free; moreover, the same conclusion holds for compactly supported magnetizations that are "silent from above" (i.e., observed to produce no field above the sample). In Corollary 2.5, we show that a unidimensional nontrivial magnetization that is compactly supported cannot be silent. The same is true for bidimensional magnetizations that are nontangential (cf. Corollary 2.7). Furthermore, we show that any magnetization is equivalent to a unidimensional one (cf. Theorem 2.6). For $p = 2$, the orthogonality of the Hardy-Hodge decomposition provides the equivalent magnetization with minimal $L^2(\mathbf{R}^2)$ (cf. Theorem 2.3 (i)).

In Section 3, we prove analogs of the results in Section 2 for other relevant spaces of magnetization distributions. The boundary cases $p = 1$ and $p = \infty$ of $L^p(\mathbf{R}^2)$ lead us to the spaces $\mathfrak{H}^1(\mathbf{R}^2)$ and $BMO(\mathbf{R}^2)$. To extend our analysis to all compactly supported distributions we consider the spaces $W^{-\infty,p}(\mathbf{R}^2)$, $1 < p < \infty$, and $BMO^{-\infty}$. These spaces allow us to consider all compactly supported distributional silent sources as well as certain distributions with unbounded support. In Corollaries 3.4 and 3.5 we characterize unidimensional and bidimensional silent sources in these spaces. In the case of $BMO^{-\infty}$, we find that unidimensional (and nontangential bidimensional) silent sources must be certain 'ridge' functions.

Finally, in Section 4, we briefly discuss Fourier-based inversion techniques and present some examples of source recovery. In a companion paper [15], we provide further details concerning such inversions.

2. Magnetic Potentials and Magnetizations

2.1. **Basic Relations.** We first informally review the case of a quasi-static $\mathbf{R}^3$-valued magnetization $\mathbf{M}$ supported on some subset of $\mathbf{R}^3$. In this case, the constitutive relation between the magnetic-flux density $\mathbf{B}$ and the magnetic field $\mathbf{H}$ (cf. [10, Section 5.9.C]) is given by

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7}\text{Hm}^{-1}$ is the *magnetic constant* (or *vacuum permeability*). Maxwell's equations, in the absence of any external current density $\mathbf{J}$, give $\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{0}$ and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Since $\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{0}$, the *Helmholtz-Hodge decomposition* (see Section 2.3) implies that $\mathbf{H} = -\boldsymbol{\nabla}\phi$ for some scalar function ϕ (called the *magnetic scalar potential* for $\mathbf{H}$), which gives

$$\mathbf{B} = \mu_0(-\boldsymbol{\nabla}\phi + \mathbf{M}), \tag{2}$$

and on taking the divergence we obtain the Poisson equation

$$\Delta\phi = \boldsymbol{\nabla}\cdot\mathbf{M}, \tag{3}$$

where $\Delta := \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the usual Laplacian. In the most general setting, we shall consider (3) in the distributional sense, where the components of $\mathbf{M}$ lie in $\mathcal{D}'(\mathbf{R}^3)$; that is, the dual of the space $\mathcal{D}(\mathbf{R}^3)$ of compactly supported, infinitely differentiable functions on $\mathbf{R}^3$.

Recalling that the *Coulomb potential* $1/(4\pi|\mathbf{r}|)$ is a fundamental solution of $-\Delta$ (i.e., a Green's function), where $\mathbf{r}$ is the position vector in $\mathbf{R}^3$, we infer that ϕ is, up to an additive constant, the Riesz potential in $\mathbf{R}^3$ of $-\boldsymbol{\nabla}\cdot\mathbf{M}$. We assume without loss of generality that this constant is zero; that is,

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\iiint \frac{(\boldsymbol{\nabla}\cdot\mathbf{M})(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}'. \tag{4}$$

If $\mathbf{M}$ is compactly supported or decays sufficiently fast at infinity, then (using $\boldsymbol{\nabla}(1/|\mathbf{r}|) = -\mathbf{r}/|\mathbf{r}|^3$) we may write $\phi(\mathbf{r}) = \Gamma(\mathbf{M})(\mathbf{r})$, where

$$\Gamma(\mathbf{M})(\mathbf{r}) := \frac{1}{4\pi}\iiint \frac{\mathbf{M}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,d\mathbf{r}', \tag{5}$$

since the right-hand sides of (4) and (5) are well-defined and agree for any $\mathbf{r}$ not in the support of $\mathbf{M}$. Hereafter, we shall consider only spaces of distributions $\mathbf{M}$ for which (5) is well-defined for all $\mathbf{r}$ not in the support of $\mathbf{M}$.

We shall single out the third component of $\mathbf{r}\in\mathbf{R}^3$ by writing $\mathbf{r} = (\mathbf{x}, z)$, where $\mathbf{x}\in\mathbf{R}^2$. Hereafter we assume that the support of the magnetization is contained in the $z = 0$ plane, and refer to this as the *thin plate* case. More precisely, we assume that $\mathbf{M}$ is a tensor product distribution in $(\mathcal{D}'(\mathbf{R}^3))^3$ of the form

$$\begin{aligned}\mathbf{M}(\mathbf{x}, z) &= (m_1(\mathbf{x}), m_2(\mathbf{x}), m_3(\mathbf{x}))\otimes\delta_0(z)\\ &=: \mathbf{m}(\mathbf{x})\otimes\delta_0(z) =: (\mathbf{m}_T(\mathbf{x}), m_3(\mathbf{x}))\otimes\delta_0(z),\end{aligned} \tag{6}$$

where $\mathbf{m}_T = (m_1, m_2)$ and m_3 are distributions corresponding, respectively, to the tangential and normal components of $\mathbf{m}$. Since everything will now depend on the distribution $\mathbf{m}\in\left(\mathcal{D}'(\mathbf{R}^2)\right)^3$, we put

$$\Lambda(\mathbf{m}) := \Gamma(\mathbf{M}), \tag{7}$$

where $\mathbf{M}$ and $\mathbf{m}$ are related as in (6). Then, computing the integral by Fubini's rule for distributions, (5) becomes

$$\Lambda(\mathbf{m})(\mathbf{x}, z) = \frac{1}{4\pi}\iint\left(\frac{\mathbf{m}_T(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')}{(|\mathbf{x}-\mathbf{x}'|^2+z^2)^{3/2}} + \frac{m_3(\mathbf{x}')z}{(|\mathbf{x}-\mathbf{x}'|^2+z^2)^{3/2}}\right)d\mathbf{x}', \tag{8}$$

for all $(\mathbf{x}, z)$ such that either $z \neq 0$ or $\mathbf{x}$ is not in the support of $\mathbf{m}$.

2.2. **Potentials as Poisson integrals.** In this section, our first goal is to describe the behavior of $\Lambda(\mathbf{m})(\mathbf{x}, z)$, as a function of $\mathbf{x}$, when $z\to 0^+$ (the case when $z\to 0^-$ is similar). Using a representation for this limit that involves Riesz transforms, we then determine necessary and sufficient conditions for a magnetization $\mathbf{m}$ to be a silent source; that is, for $\Lambda(\mathbf{m})(\mathbf{x}, z)$ to be identically zero for z above and below the thin plate (see Theorem 2.3).

Throughout this section, we consider the model case where $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$ for $p \in (1,\infty)$; that is, each component of $\mathbf{m}$ belongs to the familiar Lebesgue space of real-valued measurable functions f with norm

$$\|f\|_{L^p(\mathbf{R}^2)} := \left(\iint |f(\mathbf{x})|^p \, d\mathbf{x}\right)^{1/p} < \infty.$$

In Section 3 we extend our analysis from $(L^p(\mathbf{R}^2))^3$ to more general classes of distributions $\mathbf{m}$ which include all compactly supported distributions.

We begin with an analysis of the right-hand side of (8) as $z \to 0^+$. Notice that the term for the second integrand is half the familiar Poisson transform $P_z * m_3(\mathbf{x})$ of m_3, where $*$ denotes convolution and the Poisson kernel at height z for the upper half-space $\mathbf{R}^2 \times \mathbf{R}_+$ is given by

$$P_z(\mathbf{x}) := \frac{1}{2\pi} \frac{z}{(|\mathbf{x}|^2 + z^2)^{3/2}}. \tag{9}$$

By well-known properties of the Poisson kernel (an approximate identity for convolution), the limit of this term is $m_3/2$, both pointwise a.e. (even nontangentially) and in L^p-norm [22, Ch. III, Thm. 1].

The term corresponding to the first integrand in (8) is half the (dot-product) convolution of $\mathbf{m}_T$ with the kernel

$$\mathbf{H}_z(\mathbf{x}) := \frac{1}{2\pi} \frac{\mathbf{x}}{(|\mathbf{x}|^2 + z^2)^{3/2}} \tag{10}$$

and its boundary behavior is not as immediately clear. To elucidate this behavior, we express $\mathbf{m}_T * \mathbf{H}_z$ in terms of the Riesz transforms of the components of $\mathbf{m}$ composed with a Poisson transform (see (12) below). Recall that for $f \in L^p(\mathbf{R}^2)$, $p \in (1,\infty)$, the *Riesz transforms of* f, denoted by $R_1(f)$ and $R_2(f)$, are defined by

$$R_j(f)(\mathbf{x}) := \lim_{\epsilon \to 0} \frac{1}{2\pi} \iint_{\mathbf{R}^2 \setminus B(\mathbf{x},\epsilon)} f(\mathbf{x}') \frac{(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\mathbf{x}', \qquad j = 1, 2, \tag{11}$$

with $B(\mathbf{x},\epsilon)$ indicating the disc with center $\mathbf{x} = (x_1, x_2)$ and radius ϵ. Then (cf. [22, Ch. II, Sec. 4.2, Thm. 3]) the limit (11) exists a.e. when $f \in L^p(\mathbf{R}^2)$, the transform R_j continuously maps $L^p(\mathbf{R}^2)$ into itself, and (cf. [22, Ch. III, Secs. 4.3-4.4])

$$\mathbf{m}_T * \mathbf{H}_z = P_z * \big(R_1(m_1) + R_2(m_2)\big). \tag{12}$$

Relation (12) could have been surmised as follows. Both sides are harmonic in the upper half-space; furthermore, $(2\pi)^{-1}\mathbf{x}/|\mathbf{x}|^3$ is the pointwise limit of $\mathbf{H}_z$ as $z \to 0^+$ and allowing an interchange of limits we see that the boundary values on the plane $z = 0$ of both sides of (12) are the same. From the above discussion we have

Theorem 2.1. *Let* $p \in (1,\infty)$ *and suppose* $\mathbf{m} = (\mathbf{m}_T, m_3) = (m_1, m_2, m_3) \in (L^p(\mathbf{R}^2))^3$. *Then the function* $\Lambda(\mathbf{m})(\mathbf{x}, z)$ *defined by* (8) *is harmonic for* $(\mathbf{x}, z) \in \mathbf{R}^3$

with $z \neq 0$. At such points it also has the following representation in terms of the Riesz and Poisson transforms:

$$\Lambda(\mathbf{m})(\mathbf{x}, z) = \frac{1}{2} P_{|z|} * \left(R_1(m_1) + R_2(m_2) + \frac{z}{|z|} m_3 \right)(\mathbf{x}). \tag{13}$$

Moreover, the limiting relation

$$\lim_{z \to 0^\pm} \Lambda(\mathbf{m})(\mathbf{x}, z) = \frac{1}{2} \left(R_1(m_1)(\mathbf{x}) + R_2(m_2)(\mathbf{x}) \pm m_3(\mathbf{x}) \right) \tag{14}$$

holds pointwise a.e. and in $L^p(\mathbf{R}^2)$-norm.

In the Fourier domain the operator R_j has multiplier $-i\kappa_j/|\boldsymbol{\kappa}|$ (cf. [8, Prop. 4.1.14]); that is,

$$\widehat{R_j f}(\boldsymbol{\kappa}) = -i \frac{\kappa_j}{|\boldsymbol{\kappa}|} \hat{f}(\boldsymbol{\kappa}) \qquad \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbf{R}^2, \tag{15}$$

whenever $f \in \mathcal{D}(\mathbf{R}^2)$ where

$$\hat{f}(\boldsymbol{\kappa}) := \iint f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\kappa}} \, d\mathbf{x}, \tag{16}$$

is the Fourier transform of f.

We recall the following basic identities for the Riesz transforms $R_i : L^p(\mathbf{R}^2) \to L^p(\mathbf{R}^2)$:

$$R_1 R_2 = R_2 R_1 \text{ and } R_1^2 + R_2^2 = -\mathrm{Id}, \tag{17}$$

where $R_i R_j$ denotes the composition of R_i and R_j, R_j^2 is the composition of R_j with itself, and Id denotes the identity operator on $L^p(\mathbf{R}^2)$. It follows immediately from (15) that the identities hold when restricted to $\mathcal{D}(\mathbf{R}^2)$ and so must hold on all of $L^p(\mathbf{R}^2)$ by density and the continuity of the Riesz transforms.

2.3. **The Hardy-Hodge decomposition.** We say that two magnetizations in $(L^p(\mathbf{R}^2))^3$ are *equivalent from above* (resp. *below*) if they produce the same potential in the upper (resp. lower) half-space *via* (8). We say that a magnetization is *silent from above* (resp. *below*) if it is equivalent from above (resp. below) to the null magnetization. Since the Poisson transform has a trivial kernel in $L^p(\mathbf{R}^2)$, Theorem 2.1 implies that $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$ is silent from above if and only if $\Lambda(\mathbf{m})(\cdot, z) = 0$ for some $z > 0$ if and only if $R_1(m_1) + R_2(m_2) + m_3 = 0$ and silent from below if and only if $R_1(m_1) + R_2(m_2) - m_3 = 0$. Hence, $\mathbf{m}$ is silent if and only if $R_1(m_1) + R_2(m_2) = 0$ and $m_3 = 0$. We introduce a refinement of the classical Helmholtz-Hodge decomposition that allows us to write a 3-dimensional vector field on $\mathbf{R}^2$ uniquely as a sum of three terms that are, respectively, silent from above, silent from below, and silent. We call this decomposition the *Hardy-Hodge decomposition* and remark that it appears not to have been previously considered. The Hardy-Hodge decomposition works more generally for $\mathbf{R}^{n+1}$-valued vector fields on $\mathbf{R}^n$, but we shall stick to $n = 2$.

Let us first recall the classical *Helmholtz-Hodge decomposition* of two-dimensional vector fields on $\mathbf{R}^2$. If $\mathbf{h} = (h_1, h_2) \in \left(\mathcal{D}'(\mathbf{R}^2)\right)^2$ is a 2-dimensional vector of distributions on $\mathbf{R}^2$, then $\boldsymbol{\nabla} \cdot \mathbf{h} := \partial_{x_1} h_1 + \partial_{x_2} h_2$ and $\boldsymbol{\nabla} \times \mathbf{h} := \partial_{x_1} h_2 - \partial_{x_2} h_1$. For $p \in (1, \infty)$, let

$$\mathrm{Sole}(L^p(\mathbf{R}^2)) := \{\mathbf{f} = (f_1, f_2) : \ \mathbf{f} \in (L^p(\mathbf{R}^2))^2, \ \boldsymbol{\nabla} \cdot \mathbf{f} = 0\}$$

and

$$\mathrm{Irrt}(L^p(\mathbf{R}^2)) := \{\mathbf{g} = (g_1, g_2) : \ \mathbf{g} \in (L^p(\mathbf{R}^2))^2, \ \boldsymbol{\nabla} \times \mathbf{g} = 0\}.$$

That is, $\mathrm{Sole}(L^p(\mathbf{R}^2))$ is comprised of "solenoidal" vector fields (with components in $L^p(\mathbf{R}^2)$), while $\mathrm{Irrt}(L^p(\mathbf{R}^2))$ consists of "irrotational" vector fields.

Every $\mathbf{g} \in \mathrm{Irrt}(L^q(\mathbf{R}^2))$ is the gradient of some $\mathbf{R}$-valued distribution, that is, there exists $\mathbf{d} \in \mathcal{D}(\mathbf{R}^2)$ such that $\mathbf{g} = (\partial_x \mathbf{d}, \partial_y \mathbf{d})$ [21, Ch. II, Sec. 6, Thm. VI]. By construction $\mathbf{d}$ has first partial derivatives in $L^p(\mathbf{R}^2)$, hence its restriction to any bounded open set Ω lies in the Sobolev space $W^{1,p}(\Omega)$ comprised of functions in $L^p(\Omega)$ whose first distributional derivatives again lie in $L^p(\Omega)$ (this follows by regularization from the Poincaré inequality). However, $\mathbf{d}$ may not be in $W^{1,p}(\mathbf{R}^2)$ because it needs not lie in $L^p(\mathbf{R}^2)$ (although its derivatives do). Such $\mathbf{d}$ form the so-called *homogeneous Sobolev space of exponent* p, denoted by $\dot{W}^{1,p}(\mathbf{R}^2)$. We refer the reader to [2], [27, Ch. 2-3] and [22, Ch. V-VI] for standard facts on Sobolev spaces.

Next, for any two-dimensional vector field $g = (g_1, g_2)$, we let $J((g_1, g_2)) := (-g_2, g_1)$. The map J is an isometry from $\mathrm{Irrt}(L^q(\mathbf{R}^2))$ onto $\mathrm{Sole}(L^p(\mathbf{R}^2))$ such that $J^2 = -\mathrm{id}$, and so each $\mathbf{f} \in \mathrm{Sole}(L^p(\mathbf{R}^2))$ is of the form $(-\partial_y \mathbf{d}, \partial_x \mathbf{d})$ for some $\mathbf{d} \in \dot{W}^{1,p}(\mathbf{R}^2)$.

Now, the Helmholtz-Hodge decomposition states that, for $p \in (1, \infty)$,

$$(L^p(\mathbf{R}^2))^2 = \mathrm{Sole}(L^p(\mathbf{R}^2)) \oplus \mathrm{Irrt}(L^p(\mathbf{R}^2)), \tag{18}$$

is a topological direct sum as we next briefly review (cf. [9, Sec. 10.6]). Given $\mathbf{h} = (h_1, h_2) \in \left(L^p(\mathbf{R}^2)\right)^2$, let $\mathbf{g}$ and $\mathbf{f}$ be given by

$$\mathbf{g} := -\left(R_1(h), R_2(h)\right) \quad \text{with} \quad h := \sum_{j=1}^{2} R_j(h_j), \quad \text{and} \quad \mathbf{f} := \mathbf{h} - \mathbf{g}. \tag{19}$$

Then $\mathbf{h} = \mathbf{f} + \mathbf{g}$ by construction, and using (15) it is easily checked by density that $\mathbf{g} \in \mathrm{Irrt}(L^p(\mathbf{R}^2))$ and $\mathbf{f} \in \mathrm{Sole}(L^p(\mathbf{R}^2))$. The sum in (18) is direct, for if $\mathbf{h} \in \mathrm{Sole}(L^p(\mathbf{R}^2)) \cap \mathrm{Irrt}(L^p(\mathbf{R}^2))$ then it is the gradient of a harmonic distribution (thus in fact of a harmonic function) with L^p-summable derivatives, which must therefore be constant. The sum is also topological, because the projections $\mathbf{h} \to \mathbf{f}$ and $\mathbf{h} \to \mathbf{g}$ are continuous by (19) and the L^p-boundedness of Riesz transforms.

We further recall that when p, q are conjugate exponents, the spaces $\mathrm{Sole}(L^p(\mathbf{R}^2))$ and $\mathrm{Irrt}(L^q(\mathbf{R}^2))$ are orthogonal under the pairing

$$\langle \mathbf{g}, \mathbf{f} \rangle := \iint \mathbf{g}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \, d\mathbf{x};$$

in particular (18) is an orthogonal sum when $p = 2$.

We now state and prove an analog of the Helmholtz-Hodge decomposition for functions $\mathbf{f} \in (L^p(\mathbf{R}^2)))^3$. For this purpose we define

$$
\begin{aligned}
H_p^+ &= H^+\left(L^p(\mathbf{R}^2)\right) := \{(R_1(f), R_2(f), f) : \ f \in L^p(\mathbf{R}^2)\}, \\
H_p^- &= H^-\left(L^p(\mathbf{R}^2)\right) := \{(-R_1(f), -R_2(f), f) : \ f \in L^p(\mathbf{R}^2)\}, \\
\mathrm{S}_p &= \mathrm{S}\left(L^p(\mathbf{R}^2)\right) := \{(s_1, s_2, 0) : \ (s_1, s_2) \in \mathrm{Sole}(L^p(\mathbf{R}^2))\}.
\end{aligned} \tag{20}
$$

Theorem 2.2 (Hardy-Hodge Decomposition)**.** *Let $p \in (1, \infty)$. Then we have the following topological direct sum:*

$$(L^p(\mathbf{R}^2))^3 = H_p^+ \oplus H_p^- \oplus \mathrm{S}_p. \tag{21}$$

More specifically, each $\mathbf{f} = (f_1, f_2, f_3) \in (L^p(\mathbf{R}^2))^3$ can be written as

$$\mathbf{f} = P_{H_p^+}(\mathbf{f}) + P_{H_p^-}(\mathbf{f}) + P_{\mathrm{S}_p}(\mathbf{f}), \tag{22}$$

where

$$P_{H_p^+}(\mathbf{f}) = \Big(R_1(f^+), R_2(f^+), f^+\Big), \qquad f^+ := \frac{-R_1(f_1) - R_2(f_2) + f_3}{2}, \tag{23}$$

$$P_{H_p^-}(\mathbf{f}) = \Big(-R_1(f^-), -R_2(f^-), f^-\Big), \qquad f^- := \frac{R_1(f_1) + R_2(f_2) + f_3}{2}, \tag{24}$$

$$P_{\mathrm{S}_p}(\mathbf{f}) = \Big(-R_2(d), R_1(d), 0\Big), \qquad d := R_2(f_1) - R_1(f_2). \tag{25}$$

Proof. As mentioned below Equation (19), using (15) it is easily checked that $P_{\mathrm{S}_p}(\mathbf{f}) \in \mathrm{S}_p$ for any $\mathbf{f} \in \mathcal{D}(\mathbf{R}^2)$ and, by density, for any $\mathbf{f} \in (L^p(\mathbf{R}^2)))^3$.

Let $\mathbf{f} \in (L^p(\mathbf{R}^2)))^3$ be fixed. Using (17) one may readily verify that (22) holds with $P_{H_p^+}(\mathbf{f})$, $P_{H_p^-}(\mathbf{f})$, and $P_{\mathrm{S}_p}(\mathbf{f})$ given as in (23),(24), and (25) showing that the decomposition in (21) exists as a sum. On the other hand, we obtain formula (23) by observing in view of (17) that the map $(v_1, v_2, v_3) \mapsto -R_1(v_1) - R_2(v_2) + v_3$ annihilates H_p^- and S_p while giving twice the third component of H_p^+. Formula (24) follows similarly by considering the map $(v_1, v_2, v_3) \mapsto R_1(v_1) + R_2(v_2) + v_3$. Formula (25) then follows from a short computation. This establishes uniqueness of the decomposition, and the latter is topological because the coordinate projections are continuous by (23),(24), and (25). □

Let us stress that H_p^+ is the limit as $z \to 0^+$, in the sense of distributions, of a sequence of vector fields $\mathbf{x} \mapsto \nabla U(\mathbf{x}, z)$, where U is harmonic in the upper half-space. In fact, relation (12), applied with $\mathbf{m}_T = (f, 0)$ and then $\mathbf{m}_T = (0, f)$, shows for $z > 0$ that $P_z * (R_1(f), R_2(f))^t$ is the gradient with respect to $\mathbf{x}$ at $(\mathbf{x}, z)$ of minus the renormalized Riesz potential of f:

$$J_f(\mathbf{x}, z) := \frac{1}{2\pi} \iint f(\mathbf{x}') \left(\frac{1}{(|\mathbf{x} - \mathbf{x}'|^2 + z^2)^{1/2}} - \frac{1}{(|\mathbf{x}'|^2 + 1)^{1/2}} \right) d\mathbf{x}', \tag{26}$$

where the $O(|\mathbf{x}'|^{-2})$-behavior of the kernel for large $|\mathbf{x}'|$ ensures that the integral has a meaning for $f \in L^p(\mathbf{R}^2)$. Since the derivative of $-J_f$ with respect to z is $P_z * f$, we get that $P_z * (R_1(f), R_2(f), f)^t$ is the full gradient of $-J_f$ at $(\mathbf{x}, z)$, as desired. Likewise, $H^-(\mathcal{E})$ is the limit as $z \to 0^-$, in the sense of distributions, of a sequence of vector fields $\mathbf{x} \mapsto \nabla W(\mathbf{x}, z)$, where W is harmonic in the lower half-space.

By analogy with dimension one and holomorphic Hardy spaces, we call $P_{H_p^+}(\mathbf{f})$ the *projection of* $\mathbf{f}$ *onto harmonic gradients*, and $P_{H_p^-}(\mathbf{f})$ the *projection of* $\mathbf{f}$ *onto anti-harmonic gradients*. The term $P_{S_p}(\mathbf{f})$, which has no analog in dimension one where every function is a gradient, is the projection of $\mathbf{f}$ on divergence-free tangential vector fields.

Observe that (21) is an orthogonal decomposition if $p = 2$. Indeed, we know by orthogonality of the Helmholtz-Hodge decomposition that S_2 is orthogonal to both H_2^+ and H_2^-. That the latter are orthogonal to each other is immediately seen from (17) and (15).

2.4. **Equivalent magnetizations and silent source characterization.**

The Hardy-Hodge decomposition is particularly useful when analyzing the kernel of the operator $\mathbf{m} \mapsto \phi$, which is of fundamental importance to the inverse magnetization problem.

2.4.1. *Equivalent magnetizations.*

Theorem 2.3. *Let $p \in (1, \infty)$ and $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$.*

i) *The magnetization $P_{H_p^-}(\mathbf{m})$ (resp. $P_{H_p^+}(\mathbf{m})$) is equivalent to $\mathbf{m}$ from above (resp. below); in the case $p = 2$, then $P_{H_p^-}(\mathbf{m})$ (resp. $P_{H_p^+}(\mathbf{m})$) is the magnetization of minimal $(L^2(\mathbf{R}^2))^3$-norm that is equivalent to $\mathbf{m}$ from above (resp. below).*

ii) *The magnetization $\mathbf{m}$ is silent from above (resp. below) if and only if $P_{H_p^-}(\mathbf{m}) = 0$ (resp. $P_{H_p^+}(\mathbf{m}) = 0$).*

iii) *The magnetization $\mathbf{m}$ is silent from above and below if and only if it belongs to S_p; that is, if and only if $\mathbf{m}_T$ is divergence-free and $m_3 = 0$.*

iv) *If $\operatorname{supp} \mathbf{m} \neq \mathbf{R}^2$, then $\mathbf{m}$ is silent from above if and only if it is silent from below; that is, if and only if $\mathbf{m}_T$ is divergence-free and $m_3 = 0$ (i.e.,$\mathbf{m} \in \mathrm{S}_p$).*

Remarks:

- From (6), it follows that $\boldsymbol{\nabla} \cdot \mathbf{M}(\mathbf{x}, z) = \boldsymbol{\nabla} \cdot \mathbf{m}_T(\mathbf{x}) \otimes \delta_0(z) + m_3(\mathbf{x}) \otimes \delta_0'(z)$, and thus one direction of assertion (iii) is apparent. What is not apparent is that *every* silent source must have a divergence free tangential component and a vanishing normal component.
- Assertion (iv) implies in particular that if $f \in C^2(\mathbf{R}^2)$ and has compact support, then $\mathbf{m} := (\partial_y f, -\partial_x f, 0)$ is a silent source. In Figure 1 we provide an example with $f(x, y) = \phi(x)\phi(y)$, where $\phi(t) := (1/2)(1 - \cos(2\pi t))$ for $t \in [0, 1]$ and is zero otherwise.

Proof. Items (i)–(iii) follow from Theorems 2.1 and 2.2, and the orthogonality of the Hardy-Hodge decomposition in L^2.

To prove (iv), assume at first that $\mathbf{m} = (m_1, m_2, m_3)$ is silent from above. Then $P_{H_p^-}(\mathbf{m}) = 0$ by assertion (ii); that is, $R_1(m_1) + R_2(m_2) + m_3 = 0$ in view of (24). Consequently, using (23), the Hardy-Hodge decomposition reduces to

$$\mathbf{m} = (R_1(m_3), R_2(m_3), m_3) + (d_1, d_2, 0), \tag{27}$$

where $\mathbf{d} := (d_1, d_2)$ is divergence free. As remarked after the proof of Theorem 2.2, the Poisson extension $P_z * (R_1(h), R_2(h), h)(\mathbf{x})$ is minus the gradient of the renormalized Riesz potential J_{m_3} at $(\mathbf{x}, z)$. Now, by our hypothesis there is a nonempty open set $U \subset \mathbf{R}^2$ disjoint from $\operatorname{supp} \mathbf{m}$. By inspection on (26) the function $-J_{m_3}$, which is harmonic in the upper half-space by construction, extends harmonically across U to the lower half-space. Moreover, by (27), $\mathbf{d}$ is a gradient on U and since it is divergence free it must be the gradient of a harmonic function of two variables, say $W(\mathbf{x})$ there. Putting $\widetilde{W}(\mathbf{x}, z) := W(\mathbf{x})$, we thus define a harmonic function on the cylinder $\mathcal{C} := U \times \mathbf{R}$, and the function $H := -J_{m_3} + \widetilde{W}$ is harmonic on $\mathcal{C}$. On U, the gradient ∇H is identically zero because it is equal to $\mathbf{m}$ by (27). Clearly, the tangential derivatives of H of all orders also vanish on U, and since H is harmonic on $\mathcal{C}$ it then follows that the second normal derivative is zero on U. Replacing H by $\partial H/\partial z$ we obtain inductively that the normal derivatives of H of all orders vanish on U. Since H is harmonic on $\mathcal{C}$ it is also real-analytic there; hence it must be identically zero on $\mathcal{C}$.

Because $\partial \widetilde{W}/\partial z = 0$ by construction, it follows that $\partial H/\partial z = -\partial J_{m_3}/\partial z = 0$ on $\mathcal{C}$. But for $z > 0$ the latter quantity is just $P_z * m_3$. Thus, the Poisson integral of m_3 is zero on $\mathcal{C} \cap \{z > 0\}$; hence it is identically zero in the upper half-space by real analyticity. Consequently $m_3 = \lim_{z \to 0^+} P_z * m_3$ is the zero distribution.

From assertion (iii) and (27), we now conclude that $\mathbf{m}$ is silent from below, $m_3 = 0$, and $\mathbf{m}_T$ is divergence free. □

Projection onto harmonic or anti-harmonic gradients is a nonlocal operator as it involves Riesz transforms. This fact, which accounts for much of the complexity of inverse magnetization problems in the thin plate case, is conveniently expressed in the following form.

Corollary 2.4. *Let $p \in (1, \infty)$ and $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$. If $\mathbf{m} \notin \mathrm{S}_p$, then*

$$\operatorname{supp} \mathbf{m} \cup \operatorname{supp} P_{H_p^-}(\mathbf{m}) = \mathbf{R}^2. \tag{28}$$

Proof. We can assume that $\operatorname{supp} \mathbf{m} \neq \mathbf{R}^2$. By Theorem 2.3 part i), we know that $\mathbf{n} := \mathbf{m} - P_{H_p^-}(\mathbf{m})$ is silent from above. If (28) does not hold, then the support of $\mathbf{n}$ is a strict subset of $\mathbf{R}^2$ and so $\mathbf{n} \in \mathrm{S}_p$ by 2.3 part (iv); that is, $P_{H_p^+}(\mathbf{m}) = 0$ in the Hardy-Hodge decomposition. Thus $\mathbf{m}$ is silent from below by Theorem 2.3 part ii) and since $\operatorname{supp} \mathbf{m} \neq \mathbf{R}^2$ we get from Theorem 2.3 part iv) that in fact $\mathbf{m} \in \mathrm{S}_p$, a contradiction. □

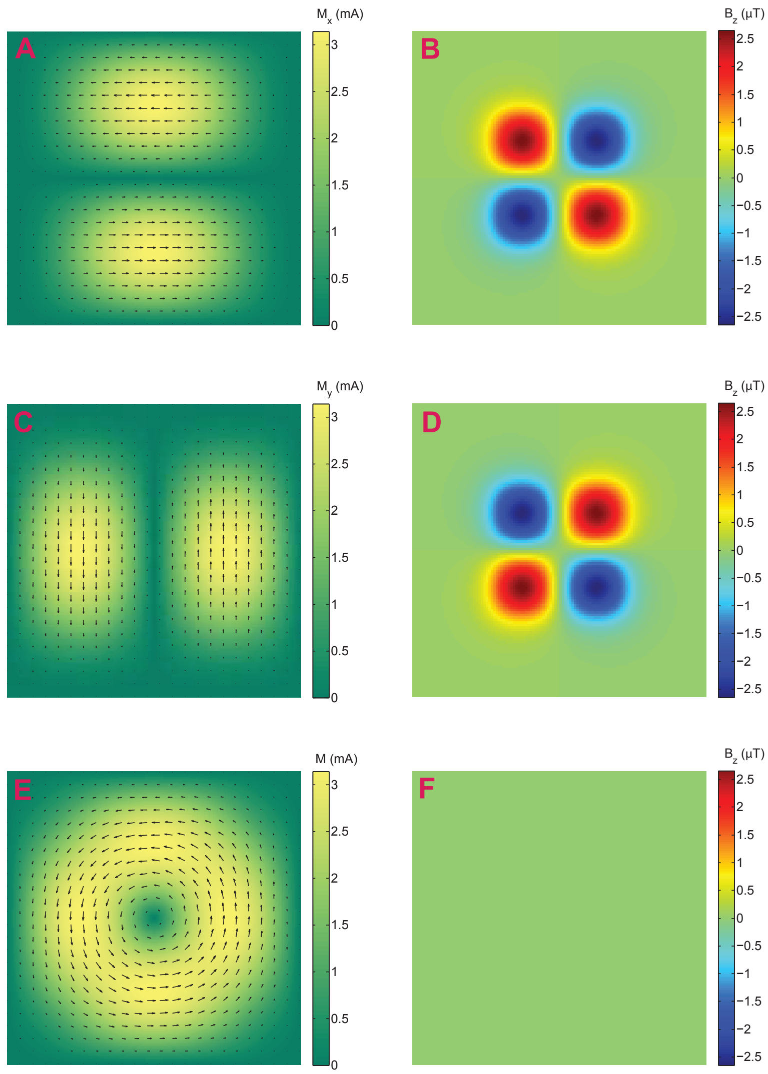

FIGURE 1. Example of a silent source magnetization defined by $\mathbf{m}(x, y) = (\psi(x)\psi'(y), -\psi'(x)\psi(y), 0)$ where $\psi(t) := (1/2)(1 - \cos(2\pi t))$ for $t \in [0, 1]$ and is zero otherwise. Parts A and B show the magnetization $\mathbf{m}_1(x, y) = (\psi(x)\psi'(y), 0, 0)$ and resulting vertical component of the magnetic field measured at height $z = 0.1$ mm. Parts C and D show the magnetization $\mathbf{m}_2(x, y) = (0, -\psi'(x)\psi(y), 0)$ and resulting vertical component of the magnetic field measured at height $z = 0.1$ mm. Parts E and F illustrate the silent source magnetization $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ and resulting null vertical component of the magnetic field measured at height $z = 0.1$ mm. Each image corresponds to an area of 1 mm × 1 mm.

2.4.2. *Unidimensional and bidimensional magnetizations.* We call a magnetization $\mathbf{m}$ *unidimensional* if $\mathbf{m} = Q\mathbf{u}$ for some fixed $\mathbf{u} \in \mathbf{R}^3$ and some scalar valued distribution Q. The sum of two unidimensional magnetizations we call *bidimensional*. As mentioned in the introduction, such magnetizations occur naturally for materials formed in a uniform external magnetic field. In such cases, Q will typically be assumed to be positive. However, in this paper we do not address issues related to such a positivity assumption.

A unidimensional magnetization with components in L^p is determined uniquely by its direction and the field it generates on one side of the $\{z = 0\}$ plane. More precisely, we have the following result which is also valid for any compactly supported unidimensional magnetization, see Corollary 3.7.

Corollary 2.5. *Suppose $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u}$, where $\mathbf{u} = (u_1, u_2, u_3)$ is a nonzero vector in $\mathbf{R}^3$ and Q is in $L^p(\mathbf{R}^2)$ for some $1 < p < \infty$. Then $\mathbf{m}$ is silent from above (resp. below) if and only if $Q = 0$ (and therefore $\mathbf{m} = \mathbf{0}$).*

Proof. Put $\mathbf{u} = (u_1, u_2, u_3)^t$ and assume $\mathbf{m}$ is silent from above. By Theorem 2.3 point (ii) and relation (24), this means $u_1 R_1(Q) + u_2 R_2(Q) + u_3 Q = 0$. The same is then true of its Poisson transform, and since for $z > 0$ we saw that $P_z * (R_1(Q), R_2(Q), Q)^t = -\nabla J_Q$ (*cf.* (26)), we deduce that the harmonic function J_Q in the upper half-pane is constant on lines parallel to u.

If $u_3 \neq 0$ these lines are transversal to $\{z = 0\}$, in which case it is immediate by continuation along them that J_Q extends to a harmonic function on the whole of $\mathbf{R}^3$. Moreover, choosing coordinates $(\tilde{x}_1, \tilde{y}_2, \tilde{x}_3)$ in $\mathbf{R}^3$ such that u points in the vertical direction, we get a harmonic function $\tilde{J}_Q(\tilde{x}_1, \tilde{x}_2)$ of two variables only, whose gradient lies in $L^p(\mathbf{R}^2)$. By Lemma 5.1 in Appendix such a function is constant, and so is J_Q. We conclude that $\nabla J_Q = 0$ hence $Q = 0$.

Assume now that $u_3 = 0$, i.e., that $u = (u_1, u_2, 0)$ is parallel to the plane $\{z = 0\}$. Then J_Q is constant along horizontal lines parallel to u and so is its normal derivative $P_z * Q$. Passing to the limit when $z \to 0^+$, we find that the distributional derivative of Q in the direction (u_1, u_2) is zero. In this case, assuming without loss of generality that $u_1 = 1$ and $u_2 = 0$ (performing if necessary a rotation in the plane $\{z = 0\}$ and a suitable renormalization of Q), we find that Q as a distribution must be of the form $\mathbf{1}_{x_1} \otimes r(x_2)$ for some distribution r on $\mathbf{R}$, [21, Ch. IV, Sec. 5]. However, such a distribution cannot lie in L^p unless it is identically zero. $\square$

It is remarkable that *any* magnetization is equivalent from one side to a unidimensional magnetization whose direction may be chosen almost arbitrarily. This is asserted in Theorem 2.6 below which should be compared with the discussion in [4] (for the case of planar distributions).

Theorem 2.6. *Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$ be such that $u_3 \neq 0$. For any magnetization $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$, $1 < p < \infty$, there is a unique $Q \in L^p(\mathbf{R}^2)$ such that $Q\mathbf{u}$ is equivalent to $\mathbf{m}$ from above.*

Of course, a similar statement holds regarding equivalence from below.

Proof. By Theorem 2.3 part (i), $Q\mathbf{u}$ is equivalent to $\mathbf{m} = (m_1, m_2, m_3)$ from above if and only if $P_{H_p^-}(Q\mathbf{u}) = P_{H_p^-}(\mathbf{m})$, that is, if and only if

$$u_1 R_1(Q) + u_2 R_2(Q) + u_3 Q = R_1(m_1) + R_2(m_2) + m_3 =: h. \tag{29}$$

We first consider the case $p = 2$. Taking Fourier transforms in (29) and using (15), it follows that (29) holds if and only if

$$\hat{Q}(\boldsymbol{\kappa}) = \frac{\hat{h}(\boldsymbol{\kappa})}{u_3 - i\mathbf{u}_T \cdot \boldsymbol{\kappa}/|\boldsymbol{\kappa}|}, \tag{30}$$

where we remark that the denominator has modulus at least $|u_3|$, showing the right-hand side of the above equation is in $L^2(\mathbf{R}^2)$. Hence there is Q solving (29).

Let now $p \in (1, \infty)$. Being smooth away from the origin, bounded, and homogeneous of degree 0, the function $1/(u_3 - i\mathbf{u}_T \cdot \boldsymbol{\kappa}/|\boldsymbol{\kappa}|)$ is a multiplier of L^p by Hörmander's theorem [22, Ch. IV, Sec. 3.2, Cor. to Thm. 3.2]. This means that the map $h \mapsto Q$, initially defined by (30) on $L^2(\mathbf{R}^2) \cap L^p(\mathbf{R}^2)$ *via* Plancherel's theorem, extends by density to a continuous map from $L^p(\mathbf{R}^2)$ into itself. Therefore, by continuity of Riesz transforms, a solution Q to (29) exists in this case too.

Uniqueness of Q follows from Corollary 2.5. $\square$

Unlike unidimensional magnetizations, bidimensional magnetizations are not determined by their two directions and the field they generate on one side of the $\{z = 0\}$ plane. This follows easily from Theorem 2.6 as applied to unidimensional $\mathbf{m}$. Still, as the next corollary shows, a nontangential bidimensional magnetization with components in L^p is determined by its directions and the field it generates from above *and* below. (This result extends to bidimensional compactly supported magnetizations as shown in Corollary 3.7.)

Corollary 2.7. *Suppose* $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u} + R(\mathbf{x})\mathbf{v}$ *where* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are nonzero vectors in* $\mathbf{R}^3$ *while* Q, R *are in* $L^p(\mathbf{R}^2)$ *for some* $1 < p < \infty$.

(a) *If* u_3 *or* v_3 *is nonzero, then* $\Lambda(\mathbf{m}) \equiv 0$ *(i.e.,* $\mathbf{m}$ *is silent) if and only if* $\mathbf{m} = \mathbf{0}$.

(b) *If* $u_3 = v_3 = 0$*, then* $\Lambda(\mathbf{m}) \equiv 0$ *if and only if* $\mathbf{m}_T(\mathbf{x}) = Q(\mathbf{x})(u_1, u_2) + R(\mathbf{x})(v_1, v_2)$ *is divergence free.*

Proof. If either $u_3 = 0$ and $v_3 \neq 0$ or $u_3 \neq 0$ and $v_3 = 0$, we get from Theorem 2.3 that either Q or R is zero and we are back to the situation of Corollary 2.5. If u_3, v_3 are both nonzero, we may assume they are equal (by possibly renormalizing Q) and then $Q = -R$ by Theorem 2.3, so we are back to the situation of Corollary 2.5 upon replacing $\mathbf{u}$ by $\mathbf{u} - \mathbf{v}$. This proves (a). Assertion (b) rephrases Theorem 2.3 (*iii*). $\square$

Remark: case (*b*) of Corollary 2.7 further splits as follows. Either (u_1, u_2) and (v_1, v_2) are linearly dependent, in which case $\mathbf{m}$ is unidimensional and so $\mathbf{m} = 0$ by Corollary 2.5, or else we may introduce new coordinates $(\tilde{x}, \tilde{y})$ in $\mathbf{R}^2$ such that $x = u_1\tilde{x} + v_1\tilde{y}$, $y = u_2\tilde{x} + v_2\tilde{y}$, and put $\tilde{Q}(\tilde{x}, \tilde{y}) = Q(x, y)$, $\tilde{R}(\tilde{x}, \tilde{y}) = R(x, y)$. Then $Q(\mathbf{x})(u_1, u_2) + R(\mathbf{x})(v_1, v_2)$ is divergence free if and only if $\tilde{Q}_{\tilde{x}} = -\tilde{R}_{\tilde{y}}$, that is, if and only if $(-\tilde{R}, \tilde{Q})$ is the gradient of some $\mathbf{d} \in \dot{W}^{1,p}(\mathbf{R}^2)$. Conversely, any bidimensional silent tangential magnetization arises in this manner from some $\mathbf{d} \in \dot{W}^{1,p}(\mathbf{R}^2)$ *via* a linear change of variables in the plane.

2.5. **Compactly supported magnetizations.** Although it is useful, and in fact necessary for the purpose of analysis, to consider magnetizations with arbitrary support, compactly supported magnetizations are of special importance as they are those arising in physical applications.

Since their support is not the whole of $\mathbf{R}^2$ by definition, Theorem 2.3 point (iv) implies that "equivalent from above" is the same as "equivalent from below" among compactly supported magnetizations. This has a number of practical implications on identifiability. For instance, we have the following result that should be held in contrast with Theorem 2.6.

Proposition 2.8. *Let $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u}$ be a compactly supported unidimensional magnetization with $Q \in L^p(\mathbf{R}^2)$, $1 < p < \infty$, and $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$, $u_3 \neq 0$. Then, $\mathbf{m}$ is equivalent (from above or below) to no other compactly supported unidimensional magnetization.*

Proof. Let $\mathbf{m}'$ be a compactly supported unidimensional magnetization which is equivalent to $\mathbf{m}$. Then $\mathbf{m}-\mathbf{m}'$ is a compactly supported bidimensional magnetization which is silent. Since $u_3 \neq 0$, Corollary 2.7 implies it is the zero magnetization. □

Note that Proposition 2.8 is no longer true if $u_3 \neq 0$, as follows from the remark after Corollary 2.7.

Below we characterize equivalent magnetizations with fixed compact support. It is useful at this point to remember from (20) the definition of S_p.

Proposition 2.9. *Let $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$ be supported on a compact set $K \subset \mathbf{R}^2$, with $1 < p < \infty$. The space of all magnetizations supported on K which are equivalent to $\mathbf{m}$ (either from above or below) is comprised of all sums $\mathbf{m} + \mathbf{s}$, where $\mathbf{s} \in \mathrm{S}_p$ is supported on K. Such magnetizations are in fact equivalent to $\mathbf{m}$ from above* and *below.*

Proof. Theorem 2.3 part (iv) characterizes silent magnetizations from above or below, whose support is a strict subset of $\mathbf{R}^2$, as being tangential and divergence free. Thus, $\mathbf{m}'$ is equivalent to $\mathbf{m}$ (from above or below) and supported on K if and only if $\mathbf{m}'-\mathbf{m}$ is tangential, divergence free, and supported on K. □

Proposition 2.9 calls for a more detailed description of those $\mathbf{s} \in \mathrm{S}_p$ that are supported on K. In particular, it will be of interest to determine $\mathbf{s} \in \mathrm{S}_p$ so that $\mathbf{m} + \mathbf{s}$ has minimal L^2 norm. This topic will treated in a sequel to this paper.

In the next section we consider magnetizations $\mathbf{m}$ in larger spaces of distributions than $(L^p(\mathbf{R}^2))^3$ in order to characterize silent sources in as general a sense as possible. Specifically, we want to include compactly supported distributions such as Dirac deltas (and hence magnetic dipoles) and also distributions with unbounded support since these can be associated with artifacts arising in the practical computations. The main technical issues to resolve are concerned with the fact that the Riesz transforms involve singular kernels and that the kernels for the Poisson and Riesz transforms have unbounded support and cannot be applied to arbitrary distributions.

3. The spaces $(W^{-\infty,p})^3$, $1 < p < \infty$, $(\mathfrak{H}^{\infty,1})^3$, and $(BMO^{-\infty})^3$

As seen in the previous section, the occurrence of Riesz transforms in the equations relating magnetization to the magnetic field naturally leads one to consider function spaces that are stable under such transforms, and L^p is the prototype of such a space when $1 < p < \infty$. Unfortunately, the L^p theory developed in Section 2 does not extend to magnetizations in L^1 and L^∞ (i.e., integrable and bounded functions, respectively), but it does to classical technical substitutes for these spaces; i.e., the real Hardy space $\mathfrak{H}^1$ and the space BMO of functions with bounded mean oscillation (see definitions below). In fact $\mathfrak{H}^1$ is smaller than L^1 and BMO is bigger than L^∞, hence we may view $\mathfrak{H}^1$ as an approximation of L^1 from below and BMO as an approximation of L^∞ from above. We cannot ignore BMO and work exclusively in L^∞ because if the initial magnetization lies in L^∞, almost all the quantities we introduce to compute relevant features thereof (like silence, equivalence, and so on) will generally lie in BMO and no longer in L^∞. There is firm ground to consider bounded magnetizations with unbounded support; e.g., the need to handle ridge distributions that appear as artifacts in Fourier based approaches to recovery. In contrast, it is tempting to forget about $\mathfrak{H}^1$. However, since BMO is dual to the latter, it is technically very cumbersome if not impossible to work with BMO and not introduce $\mathfrak{H}^1$.

Reasons to introduce the space $W^{-\infty,p}$ are different; they stem from the fact that a magnetization as simple as a dipole is mathematically not a function but a distribution, and therefore falls outside the scope of L^p theory. To remedy this, it is natural to include in our considerations magnetizations that are (distributional) derivatives of L^p functions to obtain a wider class that contains all distributions with compact support, in particular any finite collection of dipoles. The resulting class of magnetizations will be exactly $W^{-\infty,p}$. Likewise, the space $\mathrm{BMO}^{-\infty}$ is considered to include magnetizations that are derivatives of bounded functions.

To extend the above analysis for $L^p(\mathbf{R}^2)$ to a space of distributions $\mathcal{E} \subset \mathcal{D}'(\mathbf{R}^2)$ we need that $\mathcal{E}$ admits Poisson and Riesz transforms that can be convolved with H_z for $z \neq 0$ and for which equation (12) holds. In addition, we want such a space to contain all compactly supported distributions (or mean zero compactly supported distributions). By duality, this is tantamount to finding a space $\mathcal{F}$ of C^∞-smooth test functions densely containing $\mathcal{D}(\mathbf{R}^2)$ and meeting similar requirements.

3.1. **The spaces $(W^{-\infty,p})^3$ for $1 < p < \infty$.** For $p \in (1,\infty)$ let $q \in (1,\infty)$ be the conjugate to p and let $W^{\infty,q} = W^{\infty,q}(\mathbf{R}^2)$ denote the Sobolev class comprised of functions in $L^q(\mathbf{R}^2)$ whose distributional derivatives of any order also belong to $L^q(\mathbf{R}^2)$. It follows from Sobolev's embedding theorem [22, Ch. V, Sec. 2.2., Thm. 2] that $W^{\infty,q}$ consists of C^∞-smooth functions, and if $W^{\infty,q}$ is endowed with the topology of L^q-convergence of functions and all their partial derivatives it becomes a locally convex complete topological vector space with countable basis [21, Ch. VI, Sec. 8]. The natural injection $\mathcal{D}(\mathbf{R}^2) \to W^{\infty,q}$ is continuous with dense image, hence the dual $W^{-\infty,p}$ of $W^{\infty,q}$ is a space of distributions. Actually $W^{-\infty,p}$ contains all distributions with compact support, as every element of $W^{-\infty,p}$ is a finite sum of distributional derivatives of L^p-functions [1] [21, Ch. VI, Sec. 8, Thm. XXV].

We next establish that $W^{\infty,q}$ has the required properties with respect to Riesz and Poisson transforms. First note that $R_j(f)$ is C^∞-smooth when $f \in \mathcal{S}(\mathbf{R}^2)$, the space of Schwartz functions (i.e., C^∞-smooth functions that decays faster than the reciprocal of any polynomial as well as their derivatives of all orders), as follows from (15) upon differentiating the Fourier inversion formula under the integral sign (recall Schwartz functions are mapped into Schwartz functions by Fourier transform). Further, R_j preserves C^∞-smooth functions in L^q, $1 < q < \infty$, for if g is such a function we can write $g = g_1 + g_2$ where g_1 vanishes in a neighborhood of $x_0 \in \mathbf{R}^2$ while $g_2 \in \mathcal{D}(\mathbf{R}^2)$, and near x_0 smoothness of $R_j g$ is equivalent to smoothness of $R_j g_2$ as follows from (11) by inspection.

Secondly,

$$\iint R_i(f)g = -\iint f R_i(g), \qquad f \in L^q, \quad g \in L^p, \tag{31}$$

in other words the adjoint of R_j is $-R_j$. Indeed, from (15) and the isometric character of the Fourier transform, we see that it is true if $f, g \in L^2$ and the case of arbitrary $p, q \in (1, \infty)$ follows by density.

Thirdly, if $f \in L^q$ is C^∞ smooth with first partial derivatives in L^q, $1 < q < \infty$, then R_j commutes with taking these partial derivatives. To see this, observe from (15) that it holds for Schwartz functions, and if f is as indicated pick $\varphi \in \mathcal{D}(\mathbf{R}^2)$ to write

$$\iint R_i(\partial_{x_j} f)(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x} = -\iint \partial_{x_j} f(\mathbf{x})\, R_i(\varphi)(\mathbf{x})\, d\mathbf{x} = \iint f(\mathbf{x})\, \partial_{x_j}(R_i(\varphi))\,(\mathbf{x})\, d\mathbf{x} \tag{32}$$

$$= \iint f(\mathbf{x})\, R_i(\partial_{x_j}\varphi)(\mathbf{x})\, d\mathbf{x} = -\iint R_i(f)(\mathbf{x})\, \partial_{x_j}\varphi(\mathbf{x})\, d\mathbf{x} = \iint \partial_{x_j}(R_i(f))\,(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x}, \tag{33}$$

where integration by parts is possible in (32) because $f R_i(\varphi) \in L^1$ and in (33) because φ has compact support. Since $\varphi \in \mathcal{D}(\mathbf{R}^2)$ was arbitrary, we get that $R_i(\partial_{x_j} f) = \partial_{x_j}(R_i(f))$, as desired.

It is clear from what precedes that R_j maps $W^{\infty,q}$ continuously into itself. We may thus define for $m \in W^{-\infty,p}$ its Riesz transform $R_j(m)$ as the distribution in $W^{-\infty,p}$ given by

$$\langle R_j(m), f\rangle := -\langle m, R_j(f)\rangle, \qquad f \in W^{\infty,q}.$$

[1] Recall that distributions with compact support are finite sums of derivatives of compactly supported continuous functions [21, Ch. III, Sec. 7, Thm. XXVI].

We remark that this is the unique linear extension of R_j from L^p to $W^{-\infty,p}$ that commutes with differentiation.

In addition P_h* acts on $W^{\infty,q}$ since it commutes with differentiation, hence we can define the Poisson transform $P_h * m$ of $m \in W^{-\infty,p}$ in the usual manner by

$$\langle P_h * m, f\rangle := \langle m, \check{P}_h * f\rangle = \langle m, P_h * f\rangle, \qquad f \in W^{\infty,q},$$

where $\check{g}(\mathbf{x}) := g(-\mathbf{x})$ and we used that P_h is even. More generally, if $m \in W^{-\infty,p}$ and r is a positive number such that $1/p + 1/r - 1 \geq 0$, convolution of m with a member of $W^{\infty,r}$ is well-defined in W^{∞,q_1} whenever $1/q_1 := 1/p + 1/r - 1$ [21, Ch. VI, Sec. 8, Eqn. (VI.8.4)]. In particular we see that in fact $P_h * m \in W^{\infty,p}$ and that $\mathbf{d} * \mathbf{H}_h \in W^{\infty,q_1}$ for all $q_1 \in (p,\infty)$ if $\mathbf{d} \in (W^{-\infty,p})^2$, where $\mathbf{H}_h$ was defined in (10).

It is not difficult to show that $W^{\infty,p}$-functions are bounded [21, Ch. VI, Sec. 8], hence $W^{\infty,q} \subset W^{\infty,q_1}$ if $q \leq q_1$. In particular both sides of (12) exist in W^{∞,q_1} when $\mathbf{m}_T \in (W^{-\infty,p})^2$, at least if $p < q_1 < \infty$. We prove that they coincide (so that in fact both sides belong to $W^{\infty,p}$) by verifying that they define the same distribution. Indeed, pick $\varphi = (\varphi_1, \varphi_2) \in (\mathcal{D}(\mathbf{R}^2))^2$ and observe from the definitions, since H_h is odd, that

$$\langle H_h * \mathbf{m}_T, \varphi\rangle = -\langle \mathbf{m}_T, H_h * \varphi\rangle, \qquad \langle P_h * ((R_1,R_2)\cdot \mathbf{m}_T), \varphi\rangle = -\langle \mathbf{m}_T, (R_1,R_2)(P_h * \varphi)\rangle,$$

where $(R_1,R_2)\cdot(\psi_1,\psi_2)$ is understood to be $R_1(\psi_1) + R_2(\psi_2)$. Since Poisson and Riesz transforms commute on $\mathcal{D}(\mathbf{R}^2)$ by (15) and the fact that P_h* also arises from a multiplier in the Fourier domain (this multiplier is $e^{-2\pi|\kappa|h}$ [22, Ch. II, Sec. 2.1]), we are left to show that (12) holds with φ instead of $\mathbf{m}_T$. But, as previously pointed out, this holds at the L^p-level already, thereby establishing (12) when $\mathbf{m}_T \in (W^{-\infty,p})^2$.

The arguments that led us to (13) now apply again to show that the latter holds if $\mathbf{m} \in (W^{-\infty,p})^3$, and (14) likewise holds when the limit is understood in $W^{-\infty,p}$.

Equation (13) entails that $\Lambda(\mathbf{m})$ is a harmonic function of $(\mathbf{x}, h)$ in the upper half space, since members of $W^{-\infty,p}$ are finite sums of derivatives of L^p-functions and all partial derivatives of $P_h(\mathbf{x})$ are harmonic there.

3.2. **The spaces $\mathfrak{H}^{\infty,1}$ and $BMO^{-\infty}$.** Although $\cup_{1<p<\infty}(W^{-\infty,p})^3$ is a fairly large class already, it does not contain all bounded magnetizations, not even constant ones (whose potential should be zero in view of (4)). To include them requires $p = \infty$ and thus $q = 1$ in the above analysis. There is no difficulty in defining $W^{\infty,1}$ and $W^{-\infty,\infty}$ the same way as $W^{\infty,q}$ and $W^{-\infty,p}$ [21, Ch. VI, Sec. 8], and all the properties related to Poisson transforms that we need are still valid. However, we face the problem that R_j, which is still well defined on $W^{\infty,1}$ (the latter is included in L^q for all $q > 1$), no longer maps this space into itself. In fact, for $f \in L^1$, the Riesz transforms $R_j(f)$ exists a.e. [22, Ch. II, Sec. 4.5, Thm. 4] but may not lie in L^1. To circumvent this problem, we shall shrink the space of test functions and obtain a quotient space of distributions modulo constants as new framework.

Recall that the subspace $\mathfrak{H}^1 = \mathfrak{H}^1(\mathbf{R}^2)$ of L^1 comprised of functions whose Riesz transforms again belong to L^1 is a Banach space with norm

$$\|f\|_{\mathfrak{H}^1} := \|f\|_{L^1} + \|R_1(f)\|_{L^1} + \|R_2(f)\|_{L^1},$$

known as the real Hardy space of index 1 [22, Ch. VII, Sec. 3.2, Cor. 1], and that R_j continuously maps $\mathfrak{H}^1$ into itself [22, Ch. VII, Sec. 3.4, Thm. 9]. Also useful is the so-called maximal function characterization of $\mathfrak{H}^1$ [8, Ch. 6, Cor. 6.4.8], asserting that if ψ is a Schwartz function with nonzero mean and if for each $t > 0$ we set $\psi_t(\mathbf{x}) := \psi(\mathbf{x}/t)/t^2$, then there are constants C_1, C_2 depending on ψ such that

$$C_1\|f\|_{\mathfrak{H}^1} \leq \left\|\sup_{t>0} |\psi_t * f|\right\|_{L^1} \leq C_2\|f\|_{\mathfrak{H}^1}. \tag{34}$$

The space $\mathfrak{H}^1$ densely contains bounded functions with zero mean (these are particular "atoms" [23, Ch. III, Sec. 2.1.1]), in particular it contains the subspace $\mathcal{D}_0(\mathbf{R}^2) \subset \mathcal{D}(\mathbf{R}^2)$ of C^∞-smooth compactly supported functions with zero mean.

Since R_j commutes with translations, it is easily checked that Poisson transforms continuously map $\mathfrak{H}^1$ into itself, and if $f \in \mathfrak{H}^1$ then $P_h * f$ tends to f both in $\mathfrak{H}^1$ and pointwise a.e. as $h \to 0^+$. Moreover Poisson transforms still commute with Riesz transforms on $\mathfrak{H}^1$ because we know it is so on the dense subspace of bounded compactly supported functions with zero mean (those lie in L^p for $p > 1$).

The left hand side of (12) still makes sense when $\mathbf{m}_T \in (\mathfrak{h}^1)^2$, for we can write for each $A > 0$

$$(\mathbf{m}_T * \mathbf{H}_z)(\mathbf{x}) = \iint_{|\mathbf{x}'|<A} \mathbf{m}_T(\mathbf{x}-\mathbf{x}')\mathbf{H}_z(\mathbf{x}')\,d\mathbf{x}' + \iint_{|\mathbf{x}'|\geq A} \mathbf{m}_T(\mathbf{x}-\mathbf{x}')\mathbf{H}_z(\mathbf{x}')\,d\mathbf{x}' \tag{35}$$

where the first integral is the convolution of two L^1 functions while the second is the integral of the L^1-function $\mathbf{x}' \mapsto \mathbf{m}_T(\mathbf{x} - \mathbf{x}')$ against a bounded function. Since translation of the argument is uniformly continuous in $\mathfrak{h}^1$ (for it is uniformly continuous in L^1 and it commutes with Riesz transforms), we deduce from (35) by density of compactly supported functions with zero mean in $\mathfrak{h}^1$ that (12) holds good when $\mathbf{m}_T \in (\mathfrak{h}^1)^2$ too.

We also recall $BMO = BMO(\mathbf{R}^2)$, the space of functions with bounded mean oscillation consisting of locally integrable h such that

$$\|h\|_{BMO} := \sup_{B\subset\mathbf{R}^2} \frac{1}{|B|}\iint_B |h(\mathbf{x}) - m_B(h)|\,d\mathbf{x} < +\infty, \quad m_B(h) := \frac{1}{|B|}\iint_B h(\mathbf{x})\,dx, \tag{36}$$

where the supremum is taken over all balls $B \subset \mathbf{R}^2$ and $|B|$ indicates the volume of B. It is easy to see that $\|h\|_{BMO} = 0$ if and only if h is constant, and that $\|\ \|_{BMO}$ is a complete norm on the quotient space $BMO/\mathbf{R}$. Clearly $L^\infty \subset BMO$.

We now define as a new test space the Hardy Sobolev class $\mathfrak{H}^{\infty,1} = \mathfrak{H}^{\infty,1}(\mathbf{R}^2) \subset W^{\infty,1}$ consisting of functions lying in $\mathfrak{H}^1$ together with their partial derivatives of any order. We endow $\mathfrak{H}^{\infty,1}$ with the topology of $\mathfrak{H}^1$ convergence of functions and all their derivatives. Clearly $\mathcal{D}_0 \subset \mathfrak{H}^{\infty,1}$.

By what we said before Poisson transforms act continuously on $\mathfrak{H}^{\infty,1}$, and so do the Riesz tranforms since they preserve C^∞-smoothness in $\mathfrak{H}^1$ for the same reason they do in L^q, $q>1$. Moreover (12) holds for $\mathbf{m}_T\in(\mathfrak{H}^{\infty,1})^2$ because we know it holds in $(\mathfrak{h}^1)^2$ already.

Let $\mathcal{S}=\mathcal{S}(\mathbf{R}^2)$ denote the space of Schwartz functions and $\mathcal{S}_0\subset\mathcal{S}$ the subspace of functions with zero mean. It follows from Lemma 5.3 in Appendix that $\mathcal{S}_0\subset\mathfrak{h}^1$, hence also $\mathcal{S}_0\subset\mathfrak{h}^{\infty,1}$ (derivatives trivially have zero mean). In [22, Ch.VII, Secs. 3.3.1 &3.3.3] it is shown that each $f\in\mathfrak{H}^1$ can be approximated there by a sequence $\{f_k\}\subset\mathcal{S}_0$, and examination of the proof reveals that the partial derivatives of f_k also approximate the partial derivatives of f in $\mathfrak{H}^1$ when the latter belong to that space. Moreover, if we equip $\mathcal{S}$ with its usual topology defined by the seminorms

$$N_{n,m}(f):=\sup_{\alpha_1+\alpha_2\leq n}\ \sup_{\mathbf{x}\in\mathbf{R2}}\left|(1+|\mathbf{x}|)^m\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}f(\mathbf{x})\right|, \tag{37}$$

it can be proved using (34) (see Lemma 5.3) that it is finer than the topology induced on $\mathcal{S}_0$ by $\mathfrak{H}^{\infty,1}$. Altogether we get that the natural injection $\mathcal{S}_0\to\mathfrak{H}^{\infty,1}$ is continuous with dense image. As the natural injection $\mathcal{D}(\mathbf{R^2})\to\mathcal{S}$ is itself continuous with dense image [21, Ch. VII, Sec. 3 Thm. III & Sec. 4] hence also the natural injection $\mathcal{D}_0(\mathbf{R^2})\to\mathcal{S}_0$, we conclude that the natural injection $\mathcal{D}_0(\mathbf{R}^2)\to\mathfrak{H}^{\infty,1}$ is in turn continuous with dense image.

Since $\mathcal{D}_0(\mathbf{R}^2)$ has codimension 1 in $\mathcal{D}(\mathbf{R}^2)$ and is annihilated by constant distributions, we deduce from what precedes that the dual of $\mathfrak{H}^{\infty,1}$ is a quotient space of distributions by the constants.

To identify the latter, recall [8, thm. 7.2.2] that the quotient space $BMO/\mathbf{R}$ is dual to $\mathfrak{H}^1$ under the pairing

$$\langle h,g\rangle=\iint h(\mathbf{x})g(\mathbf{x})\,d\mathbf{x},\quad h\in BMO,\ g\in\mathfrak{H}^1. \tag{38}$$

More precisely, for fixed h, the integral in the right hand side of (38) converges absolutely when g is bounded and compactly supported with zero mean, and the linear form thus obtained has norm comparable to $\|h\|_{BMO}$ hence it extends to the whole of $\mathfrak{H}^1$ by continuity. Note that (38) indeed only depends on the coset of h in $BMO/\mathbf{R}$, since $\mathfrak{H}^1$-functions have zero mean.

The $\mathfrak{H}^1$-BMO duality allows one to naturally define the Riesz transforms on $BMO/\mathbf{R}$ (thus also on BMO) by the formula

$$\langle R_j(h),f\rangle:=-\langle h,R_j(f)\rangle,\qquad f\in\mathfrak{H}^1,\quad h\in BMO. \tag{39}$$

A more concrete definition of $R_j(h)$ for $h\in BMO$ may in fact be obtained upon additively renormalizing (11), replacing for instance the right hand side with

$$\lim_{\varepsilon\to0}\frac{1}{2\pi}\iint_{\mathbf{x}'\in\mathbf{R}^2,\,|\mathbf{x}-\mathbf{x}'|>\varepsilon}h(\mathbf{x}')\left(\frac{x_j-x_j'}{|\mathbf{x}-\mathbf{x}'|^3}+\frac{x_j'}{(1+|\mathbf{x}'|^2)^{3/2}}\right)d\mathbf{x}',\qquad j=1,2. \tag{40}$$

The existence of the above integral for fixed ε depends on the fact that if $h \in BMO$, then to each $\delta, r > 0$, there is a constant $C(\delta)$ such that [8, prop. 7.1.5]

$$r^\delta \iint \frac{|h(\mathbf{x}) - m_{B(x_0,r)}(h)|}{(r + |\mathbf{x} - \mathbf{x}_0|)^{2+\delta}}\, d\mathbf{x} \leq C(\delta)\, \|h\|_{BMO}, \quad \mathbf{x}_0 \in \mathbf{R}^2, \tag{41}$$

where $B(\mathbf{x}_0, r)$ indicates the ball of center $\mathbf{x}_0$ with radius r.

Now, if we let $W_0^{\infty,1} \subset W^{\infty,1}$ indicate the subspace of functions with zero mean, the map $J(f) := (f, R_1(f), R_2(f))$ identifies $\mathfrak{H}^{\infty,1}$ with a closed subspace of $\left(W_0^{\infty,1}\right)^3$. Therefore, by the Hahn-Banach theorem, each continuous linear form Ψ on $\mathfrak{H}^{\infty,1}$ is of the form $\Psi(f) = \langle G, J(f)\rangle$ for some $G \in (W^{-\infty,\infty})^3$. Because each component of G is a finite sum of derivatives of L^∞-functions [21, Ch. VI, Sec. 8, Thm. XXV] and the latter space is mapped into BMO by R_j, we conclude from (39) that Ψ is a finite sum of derivatives of BMO-functions. Conversely any such sum defines a continuous linear form on $\mathfrak{H}^{\infty,1}$ by $\mathfrak{H}^1$-BMO duality hence the dual of $\mathfrak{H}^{\infty,1}$, that we denote by $BMO^{-\infty} = BMO^{-\infty}(\mathbf{R}^2)$ consists of finite sums of derivatives of BMO-functions modulo constants.

The discussion after Theorem 2.2 requires some adjustement. For $f \in \mathfrak{h}^1$, it is still true that $P_z * (R_1(f), R_2(f), f)$ is the gradient of a harmonic function in the upper half-space, but to describe it we no longer normalize the Riesz potential as in (26). Instead, we use the same splitting as in (35) to show that the ordinary Riesz potential

$$L_f(\mathbf{x}, z) := \frac{1}{2\pi} \iint f(\mathbf{x} - \mathbf{x}') \frac{1}{(|\mathbf{x}'|^2 + z^2)^{1/2}}\, d\mathbf{x}' p \tag{42}$$

exists for fixed $\mathbf{x}$. Next, we recall that $\mathcal{S}_0$ is dense in $\mathfrak{h}^1$, and when $g \in \mathcal{S}_0$ we know that L_g is harmonic in $\{z > 0\}$ with gradient $-P_z * (R_1(g), R_2(g), g)$. Since translation of the argument is uniformly continuous in $\mathfrak{h}^1$, we conclude that L_g converges to L_f locally uniformly in $\{z > 0\}$ if g tends to f in $\mathfrak{h}^1$. In particular L_f is harmonic and ∇L_f is the limit of ∇L_g, namely $-P_z * (R_1(f), R_2(f), f)$ as desired.

The case where $f \in BMO$ rests on a different normalization: this time we set

$$T(\mathbf{x}, \mathbf{t}, z) := \frac{1}{(|\mathbf{x} - \mathbf{t}|^2 + z^2)^{1/2}} - \frac{1}{(|\mathbf{t}|^2 + 1)^{1/2}} - \frac{\mathbf{x} \cdot \mathbf{t}}{(|\mathbf{t}|^2 + 1)^{3/2}},$$

and subsequently we let

$$K_f(\mathbf{x}, z) := \frac{1}{2\pi} \iint f(\mathbf{t}) T(\mathbf{x}, \mathbf{t}, z)\, d\mathbf{t}. \tag{43}$$

Observe that the $O(1/|\mathbf{t}|^3)$ behaviour of T for large $|\mathbf{t}|$ and (41) together imply that K_f is well defined. Differentiating under the integral sign, we infer that K_f is harmonic in $\{z > 0\}$ with gradient

$$\nabla K_f(\mathbf{x}, z) = \frac{1}{2\pi} \iint f(\mathbf{t}) \left(-\frac{\mathbf{x} - \mathbf{t}}{(|\mathbf{x} - \mathbf{t}|^2 + z^2)^{3/2}} - \frac{\mathbf{t}}{(|\mathbf{t}|^2 + 1)^{3/2}}\ ,\ -\frac{z}{(|\mathbf{x} - \mathbf{t}|^2 + z^2)^{3/2}} \right)^t d\mathbf{t}.$$

The third component of ∇K_f is $-P_z * f$. To evaluate the other two, observe from (41) that

$$\iint |f(t)| \left| \frac{\mathbf{x}-\mathbf{t}}{(|\mathbf{x}-\mathbf{t}|^2+z^2)^{3/2}} + \frac{\mathbf{t}}{(|\mathbf{t}|^2+1)^{3/2}} \right| d\mathbf{t} < +\infty$$

locally uniformly with respect to $\mathbf{x}$. Therefore, if we integrate for fixed $z > 0$ the $\mathbf{R}^2$-valued function

$$\psi(\mathbf{x}) := \frac{1}{2\pi} \iint f(t) \left(-\frac{\mathbf{x}-\mathbf{t}}{(|\mathbf{x}-\mathbf{t}|^2+z^2)^{3/2}} - \frac{\mathbf{t}}{(|\mathbf{t}|^2+1)^{3/2}} \right) d\mathbf{t}$$

against some $\varphi \in \mathcal{D}_0$, we may use Fubini's theorem to obtain

$$\langle \psi, \varphi \rangle = \langle f, H_z * \varphi \rangle = \langle f, P_z * (R_1(\varphi), R_2(\varphi)) \rangle = -\langle P_z * (R_1(f), R_2(f)), \varphi \rangle,$$

where we used (12) for $\mathbf{m}_T \in (\mathfrak{H}^1)^2$ and (39) together with the fact that Poisson and Riesz transforms commute on $\mathfrak{h}^1$, hence also on BMO by duality.

Thus, by density of $\mathcal{D}_0$ in $\mathfrak{h}^1$, we get $\psi = -P_z * (R_1(f), R_2(f), f) + (C_1, C_2)$ for some constants C_1, C_2. By inspection $C_j = P_1 * R_j(f)(0)$, hence $-P_z * (R_1(f), R_2(f), f)$ is indeed the gradient of the harmonic function

$$H(\mathbf{x}, z) := K_f - x_1 P_1 * R_1(f)(0) - x_2 P_1 * R_2(f)(0), \qquad z > 0. \tag{44}$$

When $f \in BMO^{-\infty}$, it is a finite sum of derivatives of BMO-functions (modulo constants):

$$f = \sum_{j=1}^{N} \frac{\partial^{n_j+m_j}}{\partial x_1^{n_j} \partial x_2^{m_j}} f_j, \qquad f_j \in BMO.$$

Subsequently, using that Poisson tranforms commute with differentiation, we find that $-P_z * (R_1(f), R_2(f), f)$ is the gradient of the harmonic function

$$H_f(\mathbf{x}, z) := \sum_{j=1}^{N} \frac{\partial^{n_j+m_j}}{\partial x_1^{n_j} \partial x_2^{m_j}} H_{f_j}(\mathbf{x}, z), \qquad z > 0. \tag{45}$$

It is now straightforward if $\mathbf{m} \in BMO^{-\infty}$ to obtain (13), as well as (14) in the distributional sense, following the steps we used when $\mathbf{m} \in W^{-\infty,p}$, $1 < p < \infty$.

3.3. Results for magnetizations $\mathbf{m}$ in $(W^{-\infty,p})^3$, $1 < p < \infty$, $(\mathfrak{H}^1)^3$, or $(BMO^{-\infty})^3$.

From the above discussions, we obtain the following theorem generalizing Theorem 2.1.

Theorem 3.1. *Let $\mathcal{E}$ be either $W^{-\infty,p}$, $\mathfrak{H}^1$, or $BMO^{-\infty}$ and suppose $\mathbf{m} = (\mathbf{m}_T, m_3) = (m_1, m_2, m_3) \in (\mathcal{E})^3$. Then the function $\Lambda(\mathbf{m})(\mathbf{x}, z)$ defined by (8) is harmonic for $(\mathbf{x}, z) \in \mathbf{R}^3$ with $z \neq 0$. At such points it also has the following representation in terms of the Riesz and Poisson transforms:*

$$\Lambda(\mathbf{m})(\mathbf{x}, z) = \frac{1}{2} P_{|z|} * \left(R_1(m_1) + R_2(m_2) + \frac{z}{|z|} m_3 \right)(\mathbf{x}). \tag{46}$$

Moreover, the limiting relation

$$\lim_{z\to 0^\pm} \Lambda(\mathbf{m})(\mathbf{x},z) = \frac{1}{2}\left(R_1(m_1)(\mathbf{x}) + R_2(m_2)(\mathbf{x}) \pm m_3(\mathbf{x})\right) \tag{47}$$

holds in $\mathcal{E}$.

Remark: The convergence in (47) will of course be stronger for smoother $\mathbf{m}$. For instance if $\mathbf{m} \in L^q$ for some $q \in (1,\infty)$, or if $\mathbf{m} \in \mathfrak{H}^1$, then the convergence holds both pointwise a.e. (even nontangentially) and in norm [22, Ch. VII, Sec. 3.2], while if $\mathbf{m}$ belongs to BMO we get both pointwise a.e. and weak-* convergence.

The invariance of $W^{\infty,q}(\mathbf{R}^2)$ under R_j now induces of a Hodge subdecomposition:

$$W^{\infty,q} \times W^{\infty,q} = \mathrm{Sole}(W^{\infty,q}) \oplus \mathrm{Irrt}(W^{\infty,q}) \tag{48}$$

where $\mathrm{Sole}(W^{\infty,q}) := W^{\infty,q} \cap \nabla(L^q)$ and $\mathrm{Irrt}(W^{\infty,q}) := W^{\infty,q} \cap \mathrm{Irrt}(L^q)$.

From (19) we see that $\mathbf{g} = (g_1, g_2)$ belongs to $\mathrm{Irrt}(W^{\infty,q})$ if, and only if it is of the form $(R_1(h), R_2(h))$ for some $h \in (W^{\infty,q})$. In particular, by continuity of Riesz transforms, the subspace of those pairs $(R_1(\varphi), R_2(\varphi))$ with $\varphi \in \mathcal{D}(\mathbf{R}^2)$ is dense in $\mathrm{Irrt}(W^{\infty,q})$. If we set

$$I_\varphi(\mathbf{x}) := \frac{1}{2\pi}\iint \frac{\varphi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}',$$

we get from the Hardy-Littlewood-Sobolev theorem on fractional integration [22, Ch. V, Sec. 1.2, Thm. 1] that $I_\varphi \in L^\alpha$ for each $\alpha \in (2,\infty)$. Moreover it follows from [22, Ch. V, Sec. 2.2] that

$$-(R_1(\varphi), R_2(\varphi)) = (\partial_{x_1} I_\varphi, \partial_{x_2} I_\varphi)$$

in the sense of distributions, hence also in the strong sense since all derivatives of I_φ are smooth. Let $\psi_n \in \mathcal{D}(\mathbf{R}^2)$ be a sequence of nonnegative functions with uniformly bounded derivatives such that $\psi_n(\mathbf{x}) = 1$ for $|\mathbf{x}| \le n$ and $\psi_n(\mathbf{x}) = 0$ for $|\mathbf{x}| \ge n+1$. Using that $I_\varphi = O(1/|\mathbf{x}|)$ for large $|\mathbf{x}|$ (because φ has compact support), it is easy to check from the Leibnitz rule and Hölder's inequality that each partial derivative $\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}(\psi_n I_\varphi)$ with $n_1 + n_2 \ge 1$ converges in L^q to the corresponding partial derivative of I_φ as $n \to \infty$. Hence the space of pairs $(\partial_{x_1}(\psi), \partial_{x_2}(\psi))$, with $\psi \in \mathcal{D}(\mathbf{R}^2)$, is in turn dense in $\mathrm{Irrt}(W^{\infty,q})$.

Now, if we put

$$\mathrm{Sole}(W^{-\infty,p}) := \{\mathbf{f} = (f_1, f_2) :\ f_j \in W^{-\infty,p},\ \boldsymbol{\nabla}\cdot\mathbf{f} = 0\},$$

it follows by definition that $\mathbf{f} \in W^{-\infty,p}$ lies in $\mathrm{Sole}(W^{-\infty,p})$ if, and only if

$$0 = -\langle \nabla\cdot\mathbf{f}, \psi\rangle = \langle f_1, \partial_{x_1}\psi\rangle + \langle f_2, \partial_{x_2}\psi\rangle = \langle \mathbf{f}\cdot\nabla\psi\rangle, \qquad \psi \in \mathcal{D}(\mathbf{R}^2),$$

and by what precedes this is if and only if $\mathbf{f}$ annihilates $\mathrm{Irrt}(W^{\infty,q})$.

Next, upon rewriting the second half of (19) with the help of (17) as

$$\mathbf{g}_\mathbf{f} = \Big(R_2\big(R_1(h_2) - R_2(h_1)\big), -R_1\big(R_1(h_2) - R_2(h_1)\big)\Big),$$

we find reasoning as before that pairs of the form $(\partial_{x_2}\psi, -\partial_{x_1}\psi)$ with $\psi \in \mathcal{D}(\mathbf{R}^2)$ are dense in $\mathrm{Sole}(W^{\infty,q})$. Thus if we let

$$\mathrm{Irrt}(W^{-\infty,p}) := \{\mathbf{g} = (g_1, g_2) : \ g_j \in W^{-\infty,p}, \ \boldsymbol{\nabla} \times \mathbf{g} = 0\},$$

we find by definition that $\mathbf{g} \in W^{-\infty,p}$ lies in $\mathrm{Irrt}(W^{-\infty,p})$ if, and only if

$$0 = -\langle \nabla \times \mathbf{g}, \psi\rangle = \langle g_2, \partial_{x_1}\psi\rangle - \langle g_1, \partial_{x_2}\psi\rangle = \langle \mathbf{g} \cdot (-\partial_{x_2}\psi, \partial_{x_1}\psi)\rangle, \qquad \psi \in \mathcal{D}(\mathbf{R}^2),$$

which is if and only if $\mathbf{g}$ annihilates $\mathrm{Sole}(W^{\infty,q})$.

By duality, (48) now gives us a Hodge decomposition:

$$W^{-\infty,p} \times W^{-\infty,p} = \mathrm{Sole}(W^{-\infty,p}) \oplus \mathrm{Irrt}(W^{-\infty,p}), \tag{49}$$

where the first (resp. second) summand in the right hand side of (49) is the annihilator of the second (resp. first) summand in the right hand side of (48). In particular the sum in (49) is direct, for an element in the intersection of the two summands annihilates every member of $W^{\infty,q}(\mathbf{R}^2) \times W^{\infty,q}(\mathbf{R}^2)$ by (48), therefore it is the zero distribution.

The same reasoning yields Hodge decompositions:

$$\mathfrak{H}^1(\mathbf{R}^2) \times \mathfrak{H}^1(\mathbf{R}^2) = \mathrm{Sole}(\mathfrak{H}^1) \oplus \mathrm{Irrt}(\mathfrak{H}^1), \tag{50}$$

$$\mathfrak{H}^{\infty,1}(\mathbf{R}^2) \times \mathfrak{H}^{\infty,1}(\mathbf{R}^2) = \mathrm{Sole}(\mathfrak{H}^{\infty,1}) \oplus \mathrm{Irrt}(\mathfrak{H}^{\infty,1}), \tag{51}$$

and

$$BMO^{-\infty}(\mathbf{R}^2) \times BMO^{-\infty}(\mathbf{R}^2) = \mathrm{Sole}(BMO^{-\infty}) \oplus \mathrm{Irrt}(BMO^{-\infty}), \tag{52}$$

where the notations are self-explanatory; the only modification to the previous reasoning is that, in order to show pairs of the form $(\partial_{x_2}\psi, -\partial_{x_1}\psi)$ with $\psi \in \mathcal{D}_0(\mathbf{R}^2)$ are dense in $\mathrm{Sole}(\mathfrak{H}^{\infty,1})$, we use the $\mathfrak{H}^1$-extension of the Hardy-Littlewood-Sobolev theorem [24, Sec. 6, Thm G] and the fact that $I_\varphi(\mathbf{x})$ is $O(1/|\mathbf{x}|^2)$ for large $|\mathbf{x}|$ when $\varphi \in \mathcal{D}_0(\mathbf{R}^2)$.

We next generalize the Hardy-Hodge decomposition presented in Theorem 2.2. If $\mathcal{E}$ is any of the spaces L^q $(1 < q < \infty)$, $W^{\infty,q}$, $W^{-\infty,p}$ $(1 < p < \infty)$, $\mathfrak{H}^1$, $\mathfrak{H}^{\infty,1}$, or $BMO^{-\infty}$, then we define

$$H^+(\mathcal{E}) := \{\mathbf{f} = (R_1(f), R_2(f), f) : \ f \in \mathcal{E}\}$$
$$H^-(\mathcal{E}) := \{\mathbf{f} = (-R_1(f), -R_2(f), f) : \ f \in \mathcal{E}\}$$

and

$$\mathrm{Sole}^*(\mathcal{E}) := \{\mathbf{g} = (g_1, g_2, 0) : \ (g_1, g_2) \in \mathrm{Sole}(\mathcal{E})\}.$$

From the above discussion it now follows that the arguments leading to Theorems 2.2 and 2.3, Corollary 2.4 and Proposition 2.9 hold when $L^p(\mathbf{R}^2)$ is replaced by one of the spaces $(W^{-\infty,p})^3$, $1 < p < \infty$, $\mathfrak{h}^1$, or $(BMO^{-\infty})^3$. Thus, we have:

Theorem 3.2. *Let $\mathcal{E}$ be one of the spaces $\mathcal{E} = W^{-\infty,p}$, $1 < p < \infty$, $\mathcal{E} = \mathfrak{H}^1$, or $\mathcal{E} = BMO^{-\infty}$. Then Theorems 2.2, 2.3, Corollary 2.4 and Proposition 2.9, hold with $L^p(\mathbf{R}^2)$ replaced by $\mathcal{E}$.*

We remark that in the case $\mathcal{E} = BMO^{-\infty}$ in Theorem 3.2, equality is in the sense of $BMO^{-\infty}$, that is, equality of distributions up to a constant. For example, in part (iii) of the analog of Theorem 2.3 for the case $\mathcal{E} = BMO^{-\infty}$, the condition "$m_3 = 0$" now means "$m_3$ is constant" (when viewed as a distribution).

We next consider $\mathcal{E}$-analogs of Corollaries 2.5 and 2.7 characterizing unidimensional and bidimensional silent sources. The cases where $\mathcal{E} = \mathfrak{h}^1$ and $\mathcal{E} = BMO/\mathbf{R} \subset BMO^{-\infty}$ complement our treatment of L^p, $1 < p < \infty$, given in Section 2. Indeed, $\mathfrak{h}^1$ appears as a substitute for L^1 in the present context while BMO is a substitute for L^∞. In fact, these spaces are the closest substitutes since we need stability under Riesz transforms.

Corollary 3.3. *Corollaries 2.5 and 2.7 remain valid if $Q, R \in \mathfrak{h}^1$.*

Proof. The proofs are the same except that we use L_Q from (42) instead of J_Q. □

The situation $Q, R \in BMO$ is different and illustrates well the theory just developed. It shows in particular that nonzero bounded silent-from-above unidimensional magnetizations exist, but they assume a very special (yet classical) form:

Corollary 3.4. *Suppose $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u}$, where $\mathbf{u} = (u_1, u_2, u_3)$ is a nonzero vector in $\mathbf{R}^3$ and Q is in $BMO(\mathbf{R}^2)$. Then $\mathbf{m}$ is silent from above (resp. below) if and only if either $\mathbf{m}$ is constant (i.e., Q is constant) or $u_3 = 0$ and $\mathbf{m}$ is a "unidimensional ridge" function of the form $\mathbf{m}(\mathbf{x}) = \mathbf{u}h(\mathbf{x} \cdot \mathbf{v})$, where $\mathbf{v} \in \mathbf{R}^2$ is orthogonal to (u_1, u_2) and $h \in BMO(\mathbf{R})$. In such a case, $\mathbf{m}$ is silent both from above and below.*

Proof. That a constant or ridge magnetization as indicated is silent (from above and below) follows from Theorem 2.3 point *iii*).

The proof of the converse proceeds along the lines of of Corollary 2.5. The case where $u_3 \neq 0$ is argued the same way, replacing J_Q by H_Q defined in (44) and using Lemma 5.2 instead of Lemma 5.1, to the effect that H_Q is affine. Therefore $\nabla H_Q = P_z * (R_1(Q), R_2(Q), Q)$ is constant, in particular Q is constant and so is $\mathbf{m}$.

When $u_3 = 0$, we assume again without loss of generality that $\mathbf{u} = (1, 0, 0)$ and we conclude in the same way that $Q = \mathbf{1}_{x_1} \otimes r(x_2)$ for some distribution r on $\mathbf{R}$. The latter is easily seen to be a BMO function, say h. Hence $\mathbf{m} = (1, 0, 0)^t h(x_2)$ is a ridge function as announced. □

Corollary 3.5. *Suppose $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u} + R(\mathbf{x})\mathbf{v}$ where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are nonzero vectors in $\mathbf{R}^3$ while Q and R are in $BMO(\mathbf{R}^2)$.*

(a) *If u_3 or v_3 is nonzero, then $\Lambda(\mathbf{m}) \equiv 0$ (i.e., $\mathbf{m}$ is silent) if and only if $\mathbf{m}$ is a unidimensional ridge function as defined in Corollary 3.4.*
(b) *If $u_3 = v_3 = 0$, then $\Lambda(\mathbf{m}) \equiv 0$ if and only if $\mathbf{m}_T(\mathbf{x}) = Q(\mathbf{x})(u_1, u_2) + R(\mathbf{x})(v_1, v_2)$ is divergence free.*

comprised

Proof. The proof is similar to that of Corollary 2.7, granted Corollary 3.4. □

We turn to generalizations of Theorem 2.6.

Theorem 3.6. *Theorem 2.6 holds with $L^p(\mathbf{R}^2)$ replaced by $\mathfrak{h}^1(\mathbf{R}^2)$. The existence part continues to hold in $W^{-\infty,p}$, $1<p<\infty$, and $BMO^{-\infty}$.*

Proof. The solvability of equation (29) for $Q \in L^p(\mathbf{R}^2)$ when $\mathbf{m} \in (L^p(\mathbf{R}^2))^3$ entails that it is solvable for $Q \in W^{\infty,q}(\mathbf{R}^2)$ when $\mathbf{m} \in (W^{\infty,q}(\mathbf{R}^2))^3$, since transformations arising from a Fourier multiplier commute with derivations. Subsequently, by duality, (29) is still solvable for $Q \in W^{-\infty,p}(\mathbf{R}^2)$ when $\mathbf{m} \in (W^{-\infty,p}(\mathbf{R}^2))^3$. Moreover, using the $\mathfrak{h}^1$-version of Hörmander's theorem [22, Ch. VII, Thm. 9], the proof of Theorem 2.6 shows that equation (29) is solvable for $Q \in \mathfrak{H}^1(\mathbf{R}^2)$ when $\mathbf{m} \in (\mathfrak{h}^1(\mathbf{R}^2))^3$, and argueing as before this remains true when $\mathfrak{h}^1$ gets replaced by $\mathfrak{h}^{\infty,1}$ and $BMO^{-\infty}$. Uniqueness in the case of $\mathfrak{h}^1$ comes from Corollary 3.3. $\square$

Note that, in view of Corollary 3.4, neither Corollary 2.5 nor the uniqueness part of Theorem 2.6 can hold when $\mathbf{m} \in BMO^{-\infty}$. When Q, R lie in $W^{-\infty,p}$, $1<p<\infty$, a proof of Corollaries 2.5 and 2.7 would require generalizing Lemmas 5.1 and 5.2 which is beyond the scope of this paper. Thus, the study of magnetizations in these classes that are silent from above will be left for future investigations. However, the weak version below is of interest because of the practical importance of compactly supported magnetizations.

Corollary 3.7. *Corollaries 2.5, 2.7 and Proposition 2.8 remain valid if Q, R are arbitrary distributions with compact support.*

Proof. A distribution with compact support lies in $W^{-\infty,p}$ for any $p \in (1,\infty)$. Hence by Theorem 3.2, such a distribution is silent from above if and only if it is silent, that is, if and only if it lies in Sole*. In particular $Qu_3 = 0$ if $Q\mathbf{u}$ is silent, thus in the proof of Corollary 2.5 only the case $u_3 = 0$ needs to be analyzed further. The result follows then from the fact that no distribution of the form $\mathbf{1}_{x_1} \otimes r(x_2)$ can have compact support.

The proofs of Corollary 2.7 and Proposition 2.8 are unchanged. $\square$

4. Fourier Transform Reconstructions

Recall from (16) our convention for the Fourier transform $\hat{f}$ of a function f defined on $\mathbf{R}^2$:

$$\hat{f}(\boldsymbol{\kappa}) := \iint f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\kappa}}\, d\mathbf{x}, \qquad \mathbf{x} = (x_1, x_2),\ \boldsymbol{\kappa} = (\kappa_1, \kappa_2).$$

The integral is absolutely convergent for $f \in L^1(\mathbf{R}^2)$, and if $f \in L^p(\mathbf{R}^2)$ for some $p \in (1,2]$ then it may be interpreted as the limit in $L^p(\mathbf{R}^2)$ of the integral over the ball $B(0,r) \subset \mathbf{R}^2$ as $r \to \infty$.

Because of the convolution structure of Λ, it is natural to recast (8) in the Fourier domain (cf. [6], [16], and [20]). In particular, (53) below corresponds to [6, Eq. (11)]. In the following, we shall consider the Fourier transform of functions $g(\mathbf{x}, z)$ defined on $\mathbf{R}^2 \times R$ with respect to $\mathbf{x}$. Such a transform shall be denoted by $\hat{g}(\boldsymbol{\kappa}, z)$ for fixed $z \in \mathbf{R}$ and Fourier variable $\boldsymbol{\kappa} \in \mathbf{R}^2$.

Proposition 4.1. *Suppose that* $\mathbf{m} \in L^p(\mathbf{R}^2)$ *for some* $p \in (1, 2]$ *or that* $\mathbf{m}_T \in (\mathfrak{H}^1)^2$ *and* $m_3 \in L^1(\mathbf{R}^2)$*. Then for* $z \neq 0$ *and letting* $\phi := \Lambda(\mathbf{m})$*, we have*

$$\hat{\phi}(\boldsymbol{\kappa}, z) = \frac{e^{-2\pi|z||\boldsymbol{\kappa}|}}{2}\left(-\frac{z}{|z|}\hat{m}_3(\boldsymbol{\kappa}) + i\left(\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \hat{\mathbf{m}}_T(\boldsymbol{\kappa})\right)\right), \tag{53}$$

where (53) *holds for almost every* $\boldsymbol{\kappa} \in \mathbf{R}^2$ *if* $\mathbf{m} \in L^p(\mathbf{R}^2)$*,* $1 < p \leq 2$*, and for every* $\boldsymbol{\kappa} \in \mathbf{R}^2$ *in case* $\mathbf{m}_T \in (\mathfrak{H}^1)^2$ *and* $m_3 \in L^1(\mathbf{R}^2)$*.*

Proof. Equation follows at once from (13), (15), the remark after Theorem 3.1, and the fact that P_z* arises from the multiplier $e^{-2\pi|\boldsymbol{\kappa}|z}$ in the Fourier domain [22, Ch. II, Sec. 2.1] (note that $\hat{\mathbf{m}}_T(0) = 0$ if $\mathbf{m}_T \in (\mathfrak{H}^1)^2$ since $\mathfrak{H}^1$-functions have zero mean). $\square$

In a typical scanning microscope setup, the normal component B_3 of the magnetic field $\mathbf{B}$ is measured in a horizontal plane $z = h$ for some $h \neq 0$. From (2) we have $\mathbf{B}(\mathbf{x}, z) = -\mu_0 \boldsymbol{\nabla}\phi(\mathbf{x}, z)$ for $z \neq 0$. Writing $\phi = \Lambda(\mathbf{m})$ and taking the Fourier transform, we have (with ϕ_z denoting $\partial\phi/\partial z$)

$$\begin{aligned} \hat{\mathbf{B}}(\boldsymbol{\kappa}, h) &= \mu_0\left[(2\pi i\boldsymbol{\kappa})\hat{\phi}(\boldsymbol{\kappa}, h) - \widehat{\phi_z}(\boldsymbol{\kappa}, h)\mathbf{k}\right] \\ &= (2\pi i)\mu_0\left[\boldsymbol{\kappa} - i\frac{h}{|h|}|\boldsymbol{\kappa}|\mathbf{k}\right]\hat{\phi}(\boldsymbol{\kappa}, h), \end{aligned} \tag{54}$$

where $\mathbf{k}$ denotes unit vector in the z-direction and the second equality follows by interchanging differentiation with respect to z with the Fourier transform with respect to (x, y) in (53). Thus, $\phi(\mathbf{x}, h)$ can be obtained from $B_3(\mathbf{x}, h)$ using

$$\hat{\phi}(\boldsymbol{\kappa}, h) = (2\pi\mu_0|\boldsymbol{\kappa}|)^{-1}\frac{h}{|h|}\hat{B}_3(\boldsymbol{\kappa}, h). \tag{55}$$

We divide the reconstruction of $\mathbf{m}$ from B_3 into the following steps: (a) estimate ϕ on the plane $z = h$ from samples of B_3 on this plane using (55); (b) estimate the boundary values $\phi(\cdot, 0^+)$ and/or $\phi(\cdot, 0^-)$ through downward continuation; and (c) estimate $\mathbf{m}$ from the boundary values $\phi(\cdot, 0^+)$ and/or $\phi(\cdot, 0^-)$. Steps (a) and (b) for determining $\phi(\cdot, 0^+)$ from $B_3(\cdot, z)$ lead to the equation

$$\hat{B}_3(\boldsymbol{\kappa}, z) = 2\pi\mu_0|\boldsymbol{\kappa}|\hat{\phi}(\boldsymbol{\kappa}, z) = 2\pi\mu_0|\boldsymbol{\kappa}|e^{-2\pi|z||\boldsymbol{\kappa}|}\hat{\phi}(\boldsymbol{\kappa}, 0^+) = V(\boldsymbol{\kappa})\hat{\phi}(\boldsymbol{\kappa}, 0^+), \tag{56}$$

where $V(\boldsymbol{\kappa}) := 2\pi\mu_0|\boldsymbol{\kappa}|e^{-2\pi|z||\boldsymbol{\kappa}|}$, which allows the determination of $\hat{\phi}(\boldsymbol{\kappa}, 0^+)$ from $\hat{B}_3(\boldsymbol{\kappa}, z)$ provided $\boldsymbol{\kappa} \neq 0$ although not in a stable manner, i.e., the inversion of (56) is ill-posed. Without special assumptions such as unidimensionality, step (c) can only solve for $\hat{m}_3(\boldsymbol{\kappa})$ and $\boldsymbol{\kappa} \cdot \hat{\mathbf{m}}_T(\boldsymbol{\kappa})$ given $\phi(\boldsymbol{\kappa}, 0^+) = \widehat{\Lambda(\mathbf{m})}(\boldsymbol{\kappa}, 0^+)$ and $\phi(\boldsymbol{\kappa}, 0^-) = \widehat{\Lambda(\mathbf{m})}(\boldsymbol{\kappa}, 0^-)$. However, in certain cases, such as unidimensional magnetizations, we can perform the inversion of step (c) as we discuss in the next section.

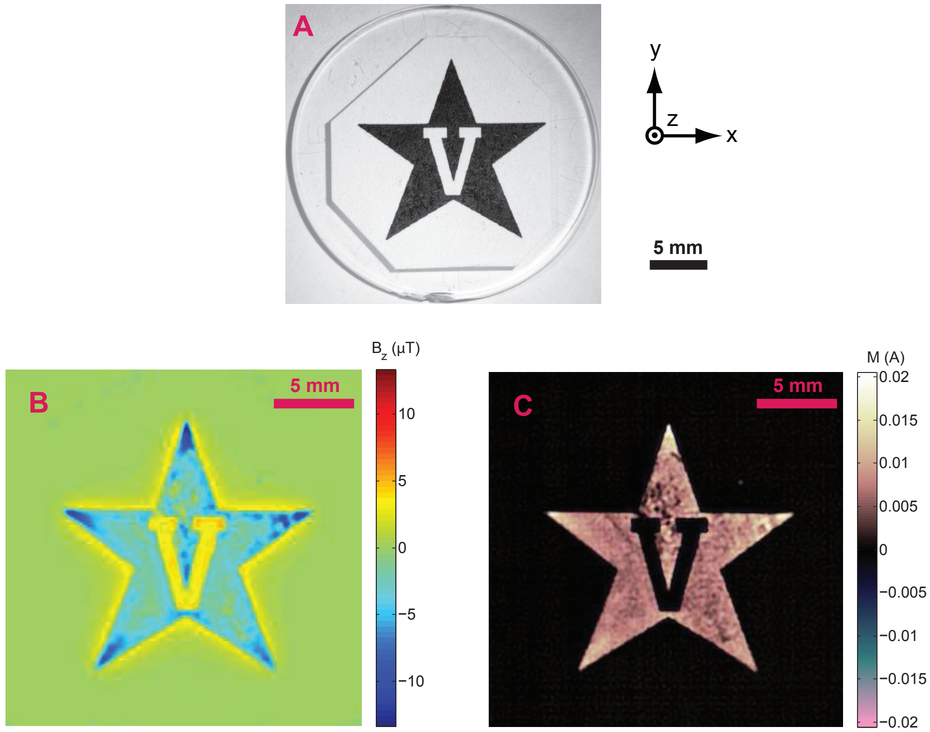

FIGURE 2. Inversion of experimental magnetic data from a synthetic sample measured with SQUID microscope. (A) Optical photograph of the synthetic sample comprised of a piece of paper with Vanderbilt University's 'Star V' logo printed on it. (B) Map of the z component of the remanent magnetic field produced by the sample, which was magnetized in the -z direction. (C) Estimated unidimensional magnetization distribution, which is essentially unidirectional, matching principal characteristics of the true magnetization.

4.1. **Reconstructing unidimensional magnetizations.** Here we consider the problem of reconstructing $\mathbf{m} \in (L^2(\mathbf{R}^2))^3$ in the case of a unidimensional magnetization when its direction $\mathbf{u}$ is known. That is, $\mathbf{m}$ is of the form

$$\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u} = Q(\mathbf{x})(\mathbf{u}_T, u_3), \tag{57}$$

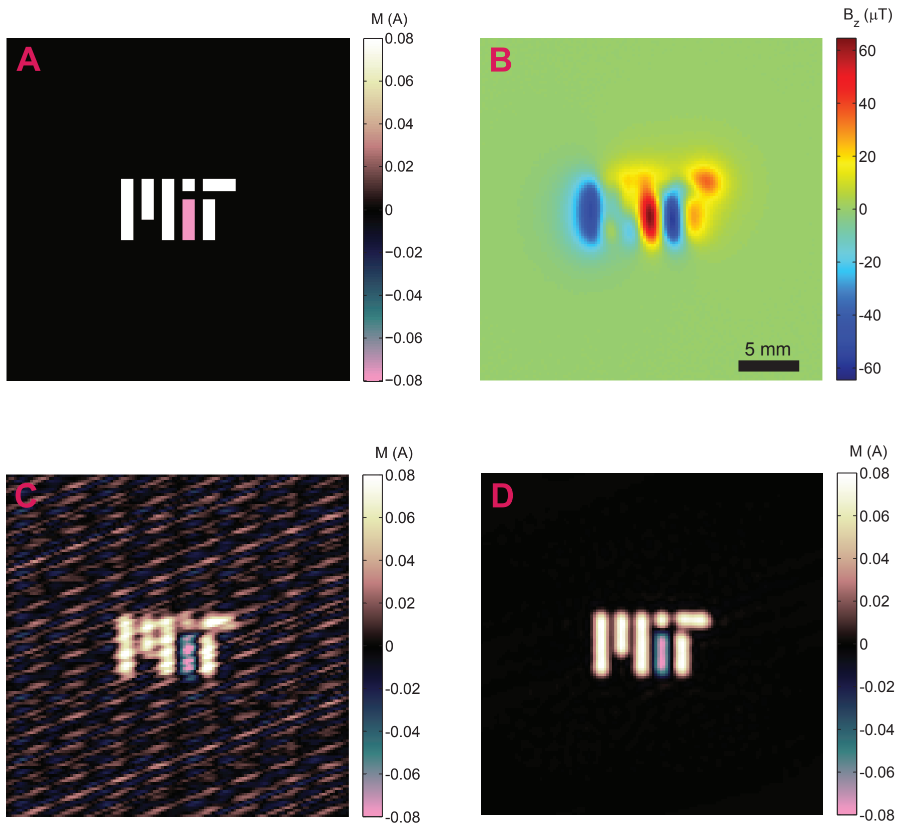

FIGURE 3. Inversion of the magnetic field produced by a simulated piecewise-continuous magnetization distribution in the shape of the Massachusetts Institute of Technology's logo. (A) Intensity plot of the synthetic magnetization distribution. (B) Simulated map of the z component of the magnetic field. (C) Estimated unidimensional magnetization distribution. (D) Solution obtained by means of an improved Wiener deconvolution algorithm, with only a minor impact on accuracy and spatial resolution.

for some $\mathbf{u} \in \mathbf{R}^3$ and $Q \in L^2(\mathbf{R}^2)$. From (57), we have $\mathbf{m}_T = Q\mathbf{u}_T$ and $m_3 = Qu_3$, and so Proposition 4.1 gives for $\boldsymbol{\kappa} \neq (0,0)$,

$$\begin{aligned}\hat{\phi}(\boldsymbol{\kappa}, 0^+) &= \lim_{z\to 0^+}\left[\frac{e^{-2\pi|z||\boldsymbol{\kappa}|}}{2}\left(-\frac{z}{|z|}u_3\hat{Q}(\boldsymbol{\kappa}) + i\left(\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}\cdot\mathbf{u}_T\hat{Q}(\boldsymbol{\kappa})\right)\right)\right]\\ &= \frac{1}{2}\left[-u_3 + i\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}\cdot\mathbf{u}_T\right]\hat{Q}(\boldsymbol{\kappa}) = C_{\mathbf{u}}(\boldsymbol{\kappa})\hat{Q}(\boldsymbol{\kappa}),\end{aligned}$$

where

$$C_{\mathbf{u}}(\boldsymbol{\kappa}) := \frac{1}{2}\left[-u_3 + i\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}\cdot\mathbf{u}_T\right].$$

Since

$$|C_{\mathbf{u}}(\boldsymbol{\kappa})|^2 = (1/4)\left[u_3^2 + \frac{(\boldsymbol{\kappa}\cdot\mathbf{u}_T)^2}{|\boldsymbol{\kappa}|^2}\right] \le (1/4)\left[u_3^2 + \|\mathbf{u}_T\|^2\right],$$

we have

$$(1/2)|u_3| \le |C_{\mathbf{u}}(\boldsymbol{\kappa})| \le (1/2)\|\mathbf{u}\|. \tag{58}$$

If $u_3 \neq 0$, then we can stably solve for Q using

$$\hat{Q}(\boldsymbol{\kappa}) = C_{\mathbf{u}}(\boldsymbol{\kappa})^{-1}\hat{\phi}(\boldsymbol{\kappa}, 0^+). \tag{59}$$

We next provide bounds on the error in a unidimensional reconstruction resulting from errors in the assumed direction and/or from errors in the downward continued potential $\phi(\cdot, 0^+)$.

Lemma 4.2. *Suppose $Q, P \in L^2(\mathbf{R}^2)$, $\mathbf{u} = (\mathbf{u}_T, u_3)$, and $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_T, \tilde{u}_3)$ be fixed vectors in $\mathbf{R}^3$. Let $\mathbf{m} := Q\mathbf{u}$, $\tilde{\mathbf{m}} := P\tilde{\mathbf{u}}$, $\phi_0 := \Lambda(\mathbf{m}, 0^+)$ and $\psi_0 := \Lambda(\tilde{\mathbf{m}}, 0^+)$.*

Then

$$\|Q - P\|_2 \le \frac{\|\mathbf{u} - \tilde{\mathbf{u}}\|}{|\tilde{u}_3|}\|Q\|_2 + \frac{2}{|\tilde{u}_3|}\|\phi_0 - \psi_0\|_2. \tag{60}$$

Proof. For $\boldsymbol{\kappa} \neq \mathbf{0}$ we have

$$\begin{aligned}
\hat{Q}(\boldsymbol{\kappa}) - \hat{P}(\boldsymbol{\kappa}) &= C_{\mathbf{u}}(\boldsymbol{\kappa})^{-1}\hat{\phi}_0(\boldsymbol{\kappa}) - C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})^{-1}\hat{\psi}_0(\boldsymbol{\kappa}) \\
&= \left(C_{\mathbf{u}}(\boldsymbol{\kappa})^{-1} - C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})^{-1}\right)\hat{\phi}_0(\boldsymbol{\kappa}) + C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})^{-1}\left(\hat{\phi}_0(\boldsymbol{\kappa}) - \hat{\psi}_0(\boldsymbol{\kappa})\right) \\
&= \frac{C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa}) - C_{\mathbf{u}}(\boldsymbol{\kappa})}{C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})}C_{\mathbf{u}}(\boldsymbol{\kappa})^{-1}\hat{\phi}_0(\boldsymbol{\kappa}) + C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})^{-1}\left(\hat{\phi}_0(\boldsymbol{\kappa}) - \hat{\psi}_0(\boldsymbol{\kappa})\right) \\
&= C_{\tilde{\mathbf{u}}}(\boldsymbol{\kappa})^{-1}\left(C_{(\mathbf{u}-\tilde{\mathbf{u}})}(\boldsymbol{\kappa})\hat{Q}(\boldsymbol{\kappa}) + \left(\hat{\phi}_0(\boldsymbol{\kappa}) - \hat{\psi}_0(\boldsymbol{\kappa})\right)\right).
\end{aligned}$$

Then, using the above and (58), we obtain

$$\|\hat{Q} - \hat{P}\|_2 \le (1/\tilde{u}_3)(\|\tilde{\mathbf{u}} - \mathbf{u}\|\|\hat{Q}\|_2 + 2\|\hat{\phi}_0 - \hat{\psi}_0\|_2),$$

and the result follows from Parseval's identity. □

Combining steps (a), (b), and (c) as described in the preceding section, we get for the case of unidimensional magnetizations:

$$\hat{Q}(\boldsymbol{\kappa}) = C_{\mathbf{u}}(\boldsymbol{\kappa})^{-1}\hat{\phi}(\boldsymbol{\kappa}, 0^+) = \left(C_{\mathbf{u}}(\boldsymbol{\kappa})V(\boldsymbol{\kappa})\right)^{-1}\hat{B}_3(\boldsymbol{\kappa}, z), \tag{61}$$

which is ill-posed and thus requires a regularization procedure. In the companion paper [15], we explore various regularization schemes as well as issues arising from the fact that the measurements are taken on a finite grid of points. In the following examples, we consider the regularized inversion

$$\hat{\phi}(\boldsymbol{\kappa}, 0^+) \approx \frac{\overline{C_{\mathbf{u}}(\boldsymbol{\kappa})} V(\boldsymbol{\kappa})}{|C_{\mathbf{u}}(\boldsymbol{\kappa}) V(\boldsymbol{\kappa})|^2 + \sigma(\boldsymbol{\kappa})^2} \hat{B}_3(\boldsymbol{\kappa}, z), \tag{62}$$

which is a 'Wiener' deconvolution in the case of additive noise and signal-to-noise power spectrum $\sigma^2(\boldsymbol{\kappa})$.

Figure 2 illustrates an inversion based on (62) for a physical sample comprised of a piece of paper with Vanderbilt University's 'Star V' logo printed on it (see image (A)) and magnetized in the direction $\mathbf{u} = (0, 0, -1)$. The paper was glued to a nonmagnetic quartz disc to ensure flatness and facilitate scanning. The sample was magnetized prior to mapping by applying a field pulse of 0.9 T. The z-component of the magnetic field produced by the remanent magnetization of the sample, illustrated in image (B), was measured from above by a SQUID microscope in a grid of 294×294 positions with step size of 0.075 mm, covering an area of 22×22 mm^2. The sample-to-sensor distance was approximately 0.27 mm. Image (C) shows an inversion of these data based on (62) where $\sigma^2(\boldsymbol{\kappa})$ is chosen of the form $\gamma\rho^{-3}(|\boldsymbol{\kappa}|^2 + \rho^2)^{3/2}$ for positive parameters γ and ρ chosen experimentally (cf. [15]).

In the case of a tangential unidimensional magnetization (i.e., $u_3 = 0$), the third step is no longer stable as is clear in the Fourier domain since $C_{\mathbf{u}}(\boldsymbol{\kappa})$ vanishes along the line $\mathbf{u}_T \cdot \boldsymbol{\kappa} = 0$. To illustrate this point, in Figure 3 we show the reconstruction of such a tangential magnetization for a simulated piecewise-continuous magnetization distribution in the shape of the Massachusetts Institute of Technology's logo. The synthetic distribution is comprised of a set of rectangular slabs uniformly magnetized in the horizontal plane ($u_1 = \cos 20°$, $u_2 = \sin 20°$, $u_3 = 0$). The bottom part of the letter 'I' is magnetized in the antipodal direction (i.e., $u_1 = -\cos 20°$, $u_2 = -\sin 20°$, $u_3 = 0$). All slabs have the same magnetization strength of 0.08 A. Image (B) shows the simulated z component of the magnetic field produced by this distribution computed on a 128 x 128 square grid of positions (0.022 mm step size) at a sample-to-sensor distance of 0.15 mm. Gaussian white noise was added to simulate instrument noise, yielding a signal-to-noise ratio of 100:1 or 40 dB. Image (C) shows the estimated magnetization distribution obtained by inversion in the Fourier domain of the magnetic data, using the regularization from (62) with $\sigma^2(\boldsymbol{\kappa})$ again chosen of the form $\gamma\rho^{-3}(|\boldsymbol{\kappa}|^2 + \rho^2)^{3/2}$. Notice the artifacts along the magnetization direction associated with noncompactly ('ridge-like') supported silent sources as described in Corollary 3.4. Image (D) shows an inversion obtained by means of an improved Wiener deconvolution algorithm which tames these artifacts by digitally filtering the magnetic data prior to inversion in order to minimize finite mapping area effects. (Essentially, this filtering is implemented using spectral windows with better response characteristics than the rectangular (boxcar) one. See [15] for details.)

We next briefly consider the problem of recovering a unidimensional magnetization when the direction is unknown but both $\phi(\cdot, 0^+)$ and $\phi(\cdot, 0^-)$ are available. In this case, we have

$$u_3\hat{Q}(\boldsymbol{\kappa}) = d(\boldsymbol{\kappa}) := \hat{\phi}(\boldsymbol{\kappa}, 0^+) - \hat{\phi}(\boldsymbol{\kappa}, 0^-) \tag{63}$$

and

$$i\left(\mathbf{u}_T \cdot \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}\right)\hat{Q}(\boldsymbol{\kappa}) = s(\boldsymbol{\kappa}) := \hat{\phi}(\boldsymbol{\kappa}, 0^+) + \hat{\phi}(\boldsymbol{\kappa}, 0^-). \tag{64}$$

If d is not identically zero, then $u_3 \neq 0$ and we may recover Q from (63) and then recover $\mathbf{u}_T/u_3$ from (64) by, for example in the case that $Q \in L^1(\mathbf{R}^2)$ with no noise, by evaluating (64) for a set of $\boldsymbol{\kappa}$ such $\hat{Q}(\boldsymbol{\kappa}) \neq 0$. If $d \equiv 0$, then $u_3 = 0$. Assuming that $\hat{Q}(\boldsymbol{\kappa})$ does not vanish along a line through the origin, i.e.,$\mathbf{a} \cdot \boldsymbol{\kappa} = 0$ for some fixed $\mathbf{a}$, then we may recover $\mathbf{u}_T$ since the only line that $s(\boldsymbol{\kappa})$ vanishes on will be given by $\mathbf{a} = \mathbf{u}_T$. Note that if Q is compactly supported and nonzero, then $\hat{Q}$ cannot vanish along a line. Finally, we remark that the bidimensional case with a *known* normal direction $\mathbf{n}$ (i.e., $\mathbf{m}(\mathbf{x}) \cdot \mathbf{n} = 0$) that is not vertical (i.e., $\mathbf{n}_T = (n_1, n_2) \neq \mathbf{0}$) can be solved similarly. Without loss of generality, we may choose vectors $\mathbf{u}$ and $\mathbf{v}$ perpendicular to $\mathbf{n}$ such that $u_3 = 1$ and $v_3 = 0$. If $\mathbf{m}(\mathbf{x}) = Q(\mathbf{x})\mathbf{u} + R(\mathbf{x})\mathbf{v}$, then from (63) we have $\hat{Q} = d$ and from (64) we obtain

$$\hat{R}(\boldsymbol{\kappa}) = \left(\frac{-i|\boldsymbol{\kappa}|}{\boldsymbol{\kappa} \cdot \mathbf{u}_T}\right) s(\boldsymbol{\kappa}) - \left(\frac{\boldsymbol{\kappa} \cdot \mathbf{v}_T}{\boldsymbol{\kappa} \cdot \mathbf{u}_T}\right) d(\boldsymbol{\kappa}), \qquad (\boldsymbol{\kappa} \cdot \mathbf{u}_T \neq 0).$$

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5. Appendix

In the proof of Corollaries 2.5 and 3.4, we appealed to the following basic facts on harmonic functions.

Lemma 5.1. *A harmonic function on $\mathbf{R}^2$ whose gradient lies in $(L^p)^2$ for some $p \in [1, \infty)$ is a constant.*

Proof. Let g be harmonic with L^p gradient. Then the partial derivatives $\partial_{x_1} g$, $\partial_{x_2} g$ are harmonic and lie in L^p. Let h denote any of them. For $\mathbf{x} \in \mathbf{R}^2$, put $B(\mathbf{x}, r)$ for the disk centered at $\mathbf{x}$ of radius r. By the mean value theorem

$$h(\mathbf{x}) = \frac{1}{\pi r^2} \iint_{B(\mathbf{x},r)} h(\mathbf{t})\, d\mathbf{t};$$

hence by Hölder's inequality

$$|h(\mathbf{x})| \leq \frac{1}{\pi r^2} \|h\|_{L^p(\mathbf{R}^2)} (\pi r^2)^{1-1/p}.$$

Letting $r \to \infty$ we get that $h = 0$ if $p > 1$. If $p = 1$, we get that h is bounded, therefore a constant, and since it is summable $h = 0$ again, as desired. $\square$

Lemma 5.2. *A harmonic function on $\mathbf{R}^2$ whose gradient lies in $(BMO)^2$ is affine.*

Proof. Considering the partial derivatives, this is tantamount to showing that a harmonic function $g \in BMO$ must be constant.

To prove this, let $B(\xi, r)$ indicate the disk centered at ξ of radius r and pick $\mathbf{x} \in \mathbf{R}^2$ with $|\mathbf{x}| = 1$. For each $a > 1$, we get from the mean value property and standard estimates on BMO-functions [8, ex. 7.1.6] that

$$|g(ax) - g(0)| = |m_{B(ax,1)}(g) - m_{B(0,1)}(g)| \leq C \log(a+1) \|g\|_{BMO} \tag{65}$$

for some absolute constant C. Thus $|g(t)| = o(|t|)$ as $|t| \to \infty$, hence g is constant. $\square$

The following lemma was used in section 3.2. Recall from (37) the topology of the Schwartz space $\mathcal{S}$ and the notation $\mathcal{S}_0$ for the subspace of functions with zero mean.

Lemma 5.3. *If $f \in \mathcal{S}_0$, then*

$$\|f\|_{\mathfrak{h}^1} \leq C(N_{0,0}(f) + N_{0,4}(f))$$

where C is independent of f.

Proof. Let $\psi \in \mathcal{D}$ be nonnegative with support in $\{\mathbf{x} : |\mathbf{x}| < R\}$. In view of (34), we must bound the L^1-norm of the function

$$\Phi(\mathbf{x}) = \sup_{t>0} \left| \int_{\mathbf{R}^2} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2} \right|, \qquad \mathbf{x} \in \mathbf{R}^2. \tag{66}$$

Note that the integral on the right-hand side of (66) bears on the ball $B(\mathbf{x}, tR) = \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| < tR\}$ whose measure is $\pi t^2 R^2$). We consider three cases:

Case 1 If $|\mathbf{x}| < 1$, then trivially

$$\left| \int_{\mathbf{R}^2} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2} \right| \leq \pi R^2 \|\psi\|_{L^\infty} N_{0,0}(f).$$

Case 2 If $|\mathbf{x}| \geq 1$ and $t < |\mathbf{x}|/2R$, $B(\mathbf{x}, tR)$ contains only $\mathbf{y}$ with $|\mathbf{y}| > |\mathbf{x}|/2$. Thus

$$\left| \int_{\mathbf{R}^2} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2} \right| \leq 8\pi R^2 \|\psi\|_{L^\infty} \frac{N_{0,3}(f)}{(2+|\mathbf{x}|)^3}.$$

Case 3 If $|\mathbf{x}| \geq 1$ and $t \geq |\mathbf{x}|/2R$, we write since f has zero mean

$$\int_{\mathbf{R}^2} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2} = \int_{\mathbf{R}^2} \left(\psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) - \psi\left(\frac{\mathbf{x}}{t}\right)\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2}$$

and using the mean value theorem we get

$$\begin{aligned}
\left| \int_{\mathbf{R}^2} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) f(\mathbf{y}) \, \frac{d\mathbf{y}}{t^2} \right| &\leq \|\nabla\psi\|_{L^\infty} \int_{\mathbf{R}^2} |\mathbf{y} f(\mathbf{y})| \, \frac{d\mathbf{y}}{t^3} \\
&\leq \frac{8R^3}{|\mathbf{x}|^3} \|\nabla\psi\|_{L^\infty} N_{0,4}(f) \int \frac{d\mathbf{y}}{(1+|\mathbf{y}|)^3}.
\end{aligned}$$

Altogether, $|\Phi(\mathbf{x})| \leq C_1 N_{0,0}(f)$ if $|\mathbf{x}| < 1$ and $|\Phi(\mathbf{x})| \leq C_2 N_{0,4}(f)/|\mathbf{x}|^3$ if $|\mathbf{x}| \geq 1$, from which the desired result follows. $\square$

L. Baratchart: Projet APICS, INRIA, 2004 route des Lucioles, BP 93, Sophia-Antipolis, 06902 Cedex, FRANCE

E-mail address: Laurent.Baratchart@inria.fr

D. P. Hardin and E. B. Saff: Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: Doug.Hardin@Vanderbilt.Edu

E-mail address: Edward.B.Saff@Vanderbilt.Edu

E. A. Lima and B. P. Weiss: Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail address: limaea@MIT.edu

E-mail address: bpweiss@MIT.edu