The Stochastic Implications of Autonomous Creation and Destruction

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August, 2022

Abstract

A model of stochastic, autonomous creative destruction is developed to study the impact of a change in the volatility of inter-firm productivity shocks. The model gives rise to a distribution of firm productivities, as well as a distribution of exit dates for firms. It is shown that the observed increase in the variance of firm-specific technology shocks can account for the slowdown in growth in recent decades, as well as the reductions in firm exit and entry. This increased volatility also has a complicated effect on income inequality. The equilibrium does not necessarily yield an optimal degree of exit, and so it may be welfare-improving to tax workers in order to pay firms to cease operating early. The value of firms is then calculated, and this yields a novel asset pricing formula that involves a survival function that reflects the expected random, productivity-dependent lifetime of the firm. It is shown that asset returns, and the volatility of these returns have a predictable component, which is related to firm productivity, that is not present in typical models of asset pricing, and also possess a hazard function that embodies the risk associated with firm mortality. The variance of asset prices (i.e. firm values) becomes arbitrarily large as the firm's productivity falls, even though the variance of the underlying productivity shocks is fixed.

JEL codes: E0, G1, O3, O4, H2

Keywords: Economic Growth, Creative Destruction, Innovation, Tax Policy, Asset Pricing and Returns

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†I have benefited from the comments of Erzo Luttmer, and the assistance of Jae Cho.

1 Introduction

In this paper a model of stochastic, autonomous creative destruction is studied that is useful for investigating various important questions. The framework is used to study how an increase in the variability of inter-firm productivity shocks could affect economic growth, as well as outcomes such as welfare, inequality, and the rate of business creation and destruction. Such a study is of interest because there is some evidence that the variability of these productivity shocks has indeed increased in recent decades. The use of the model with autonomous creative destruction is of interest because this framework has different agents engaged in the optimal creation (i.e. innovation) and destruction (i.e. exit) decisions. The model also has interesting implications for asset pricing, and is useful for studying the linkage between the productivity of a firm, and the rate of return on its shares. A unique asset pricing formula is derived, which has a survival function that characterizes the horizon over which payoffs will be forthcoming. This formula implies that there is an inverse, monotonic relationship between the productivity of the firm and the volatility of the firm's share price. This helps enhance our understanding of the "size puzzle" in financial economics.

There has been substantial research into models of growth that exhibit the "creative destruction" feature. Early efforts in this area are those of Aghion and Howitt [3], Grossman and Helpman [22]. More recently, there have been a few papers that have sought to link this creative destruction concept with the accompanying implications for financial markets (see Kogan, Papanikolaou, and Stoffman [31], as well as Kung and Schmid [32]).[1] One seemingly universal problem with this literature is that there is there are no independent creation and destruction mechanisms in these models. That is, when one firm successfully innovates this means that some other firm must exit. For example, in the baseline model of Aghion and Howitt, there is a monopolist with a "lifetime patent" that produces until someone else innovates, at which time the incumbent exits and the new monopolist takes over. This is expedient as a modeling technique, but this does not seem to capture the true nature of how actual markets work. Typically firms exit because the change in a factor or product prices reduces firm productivity, not because there is some fixed capacity of firms in any market. Furthermore, in such a model if there were to be too little innovation, there must necessarily be too little destruction as well.

The analysis of Huffman [23] rectifies this problem by employing a "span of control" type of model that has separate agents (or firms) making the creation and destruction decision. Although these decisions interact, or influence each other in a multitude of ways, they are made by separate agents. This means that it is possible to influence these autonomous decisions separately, with different policy tools. In this paper, *stochastic* productivity shocks are introduced, which is useful for answering some fundamental questions. The model builds not just on the work of Huffman [23], in that a complete characterization of the decision rules and the distribution of firms, measured either in terms of employment or productivity, is derived.

While the model studied here has some features in common with the work of Luttmer

[1] There are some papers at the intersection of this growth and finance literature, that feature firm exit as part of the growth process (for example, see Gomes, and Schmid [17]), although many of these papers have the exit at a constant, exogenously specified rate (Gomes, Kogan and Zhang [16], Corhay, Kung and Schmid [10].

[35], [36], there are plenty of points of departure as well. In Luttmer's model firm owners are also workers, and they will operate their firms operate until their profit drops to zero because they cannot cover their continual fixed costs. In the current paper, an agent is *either* a firm owner *or* a worker, but never both simultaneously. Workers can also engage in an innovation activity that, when successful, will produce a newer technology that can be used when they start up their firm. Furthermore, here firms will cease operating when market prices make the further operation of the firm less attractive than the outside option which is being a worker/innovator. Since a firm owner can always choose to become a worker, the wage influences the opportunity cost of shutting down the firm. Since the two papers have different factors that influence the decision to shut down a firm, the reasons for these decisions will be distinctive in the two models.[2] This heterogeneity in occupations can then be used to study measures of income inequality, which is not a topic pursued in Luttmer's work. In Luttmer's model, the effort devoted by innovators to discovering a new technology is fixed, whereas here the workers can influence the likelihood of successfully innovating. Since this effort is related to the potential reward, the growth rate will be tied to the potential rewards to innovation.[3] Additionally, it is possible to use the current framework to study issues such as how government policies, such as those involving redistribution, would influence the growth rate. Furthermore, the topics of interest or points of emphasis in the papers are considerably different. Lastly, Luttmer does not focus on the financial or asset pricing implications of the model, which is an important insight that is pursued in this model.

The model studied here is used to explore the implications of a change in the volatility of inter-firm productivity shocks. This is a question of considerable importance because there is some evidence that there is an increase in this volatility in recent years. This is documented in Decker, Haltiwanger, Jarmin, and Miranda [14], Foster, Grim, Haltiwanger, and Wolf [15] and Herskovic, Kelly, Lustig, and Van Nieuwerburgh [21]. Much of this literature focuses on how a change in the exogenous volatility of shocks could have implications for movements of factors of production, including labor. However, it is then also of interest to also see what the broader repercussions of such a change would be on other outcomes such as the growth rate as well as the rate of entry and exit of firms. It is not at all clear what the effect of, say, a "mean-preserving spread" in this distribution should have on the economy, as well as the rate of entry and exit of firms. First, since the average of the distribution is unchanged, perhaps the effect negligible. Secondly, if, as seems likely, there is exit by firms with lower productivity, the average productivity of surviving firms could be higher, which could increase growth. But third, if the average productivity of surviving firms is higher, then this could raise factor prices, which would reduce the profitability of all firms, which could lower growth because of the reduced incentive to innovate. Similarly, it is also not clear how more variability would affect the rate of entry and exit of firms. It will be shown below that these rates seem to have fallen in recent decades.

This paper, and the literature just cited, is tangentially related to the work of Arellano, Bai, and Kehoe [5] as well as Bloom, Floetto, Jaimovich, Saporta-Eksten and Terry [7]. These papers suggest that an increase in the dispersion of firm-level productivity shocks

[2] One emphasis here is that the exit decision in the model may not be made in a socially optimal manner, so it may be proper for the government to further distort this decision.

[3] Also, Luttmer has a growing population, while there is no need for this in the present model.

can induce a recession.[4] However, the models they use, and the issues that are studied are quite different from those analyzed in this paper.[5]

The model also has implications for how a change in inter-firm productivity can influence the degree of innovation activity in an economy. This is of interest because, as will be shown below, concomitant with the diminution of growth in recent decades, there also seems to be a reduction in the other measures of economic vitality, such as the rates of birth and death of firms.

Because the model studied here characterizes the explicit nature of the decisions made by different agents, it is also useful for studying many other issues as well. For example, it is possible to investigate how an increase in the volatility of inter-firm productivity will influence the average age or lifetime of a firm. Also, it is possible to investigate how various measures of inequality are affected.

One innovation of this paper is that it will produce growth, as well as a non-degenerate distribution of income and firm sizes and productivities, even though the underlying population is ex-ante *homogeneous*. In many other models, this is not the case. For example, in Jaimovich and Rebelo [27] the heterogeneity is assumed, in that the agents have different entrepreneurial abilities, which then leads to different productivities of firms. But by employing identical agents in the model ensures that any consequent inequality can only be attributable to the decisions of agents, and the resulting economic outcomes.

Still another novel feature of the approach is a unique characterization of asset prices, where the asset in question is the discounted value of the firm. This price can be characterized in a tractable manner and can give a unique perspective on the reason for the potential excess volatility of such prices. This asset-pricing formula from the model can then be used to suggest that there may be a unique source of asset-price volatility, that seems to be largely ignored until now. This volatility enters through a productivity-dependent *survival function* inherent in the price, that captures the mortality of the firm. Even though agents are risk-neutral, and the standard deviation of innovations to productivity is the same for all firms, some of these asset prices can display a high elasticity (or volatility) with respect to a change in dividends. This effect is especially important for firms that have relatively low productivity. This will then provide a linkage between the literature on asset pricing, and that of the creative-destruction, and growth literature. The rates of return on these assets are also analyzed, and these returns also have a new source of volatility or risk. This volatility enters through a *productivity-dependent hazard function*, which would seem to be a new feature not present in the existing literature.

This is an important issue because it seems natural to think that there should be a close relationship between the behavior of firm productivity, and the behavior of its stock or asset price. Why wouldn't the asset prices merely mirror the productivity shocks? Some work has taken place on this issue. For example, İmrohoroğlu and Tüzel [26] document the differences in financial variables across firms of different sizes and levels of productivity.[6]

[4] Christiano, Motto, and Rostagno [12] also study the implications of changes in the volatility of shocks to the quality of capital.

[5] The growth feature of this model also shares some insights with that of Kogan, Papanikolaou, Seru, and Stoffman [30]. They develop an innovative method of quantifying some factors that can influence the degree of creative destruction. They find that the number of citations that a patent receives can be an important factor.

[6] See also a related study in Bloom et al. [7].

They find that rates of return are considerably more volatile for smaller, or low-productivity firms. This would seem to be related to the size puzzle in financial economics (see, for example, Banz [6] and Reinganum [38]). But since asset prices reflect movements in firm-level productivity, then this suggests that smaller firms have a more volatile distribution of productivity shocks, which is then reflected in their asset prices and rates of return. But there does not seem to be any evidence that this latter point is true: there is no reason to believe that the distribution of productivity shocks depends on the size of the firm. It is then an open question as to how these observations can be reconciled.

It is shown that the creative destruction feature of the model studied here serves to *magnify the volatility* of asset prices for smaller firms. There is an inverse, and non-linear relationship between firm size, or productivity, and the volatility of asset prices. This then raises another question: Why aren't asset prices more volatile? Could it be that some asset prices are not sufficiently volatile?

2 Observations

There has been a slowdown in growth that has taken place in the past few decades across a wide range of economies. Figure 1 shows the growth rate of per-capita GDP in the US since 1948. The smoothed line is the trend that results from HP filtering the data. The average (unfiltered) growth rate from 1948 until 1989 was 3.57%. The average from 1992 until 2019 was 2.5%.

A similar pattern is seen in growth for Canada, which is shown in figure 2. The average per-capita growth rate from 1961-1988 was 2.65%, but this fell to 1.54% after 1991. A similar pattern can be seen in growth rates for the European Union as well as for Japan.

At the same time as this growth slowdown has been taking place, there has been another notable phenomenon. It has been documented by Decker, Haltiwanger, Jarmin and Miranda [14], as well as by Foster, Grim, Haltiwanger and Wolf [15] that there has been an increase in the variability of inter-firm productivity over the past two and one half, to three decades.[7] This increase in dispersion seems rather sharp, and on the order of 25%.

Over a similar period, there seems to have been a change in the level of innovation activity, as measured by the entry and exit of establishments. Figure 3 shows the rate of entry and exit of establishments in the US has evolved from 1978 until 2019. This figure clearly shows a trend in that both of these rates have fallen over this period. Alternatively, the data for the rate of entry of small, young firms also shows a similar downward trend over this period.[8] If one looks instead at the amount of employment attributed to small, young firms, as a fraction of total employment, the graph looks very similar. To the extent that these young firms are an indication of the level of innovative activity in an economy,

[7] See also Herskovic, Bernard, Bryan Kelly, Hanno Lustig, and Sijn Van Nieuwerburgh [21]. Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry [7] also describe how this increase in the dispersion of productivity shocks was more pronounced during the most recent recession.

[8] This is the number of firms that are less than one year old, that have less than 5 employees, divided by the number of existing firms in that year. These small firms account for an overwhelming number of young firms in each year. By focusing on firms with few employees, this eliminates older, larger firms that may have broken up into several small entities. The source for this data is the Business Dynamics Statistics (BDS) Time Series, from the *US Census Bureau*.

this suggests that this innovation has waned in recent decades, and would likely result in reduced growth.

This is not just a phenomenon experienced in the US. Figure 4 shows the quarterly rate of entry and exit of new private sector firms in Canada, from 2000 until 2019.[9] Once again, this figure shows that these rates are distinctively declining. It is interesting to note that if one looks at the rates of job creation and destruction, the pattern looks very similar to that in these figures.

Taken together, figures 3 and 4 show remarkably similar trends, in what might be termed a reduction in dynamism of the economy. Once again, one is led to wonder if this behavior is a reflection of reduced innovation, which then results in lower economic growth, as is seen in figures 1 and 2.

3 Description of the Model

Time is assumed to be continuous, and there is no aggregate uncertainty. In this model, there will be a dynamic evolution of agents from workers to business (or firm) owners, and this movement will accompany, and be related to the growth rate. It will be assumed that all individuals are risk-neutral, and so merely wish to consume their income. Their preferences are assumed to be a function of the discounted stream of consumption $(c_t,\, t \geq 0)$:

$$\int_0^\infty e^{-rt} \left[c_t - h\left(x_t, Z_t\right)\right] dt.$$

where r is the rate of time preference. The function $h\left(x_t, Z_t\right)$ denotes the flow of disutility associated with using some time to engage in research, an activity undertaken by workers only. At any date, there are two types of individuals. There are workers, who supply their unit of labor inelastically which means that they earn the market wage, which is the consumed $c_t = w_t$. Additionally, there are firm owners who use all their time to manage their firm. These firms hire labor at the market wage, in order to maximize profit. They own the technology for the firm, which is denoted z_t, which produces output $(y_t = z_t\left(n_t^\alpha\right))$. Their compensation is merely the profit, which is output minus wage payments $(\pi_t = y_t - w_t n_t)$. The firm owner has proprietary ownership over his technology (z_t), and so owners of inferior technologies cannot costlessly upgrade or steal superior technologies. They can, however, when they have the opportunity, develop or invent an upgraded technology. The firm owner consumes his profits each period $(c_t = \pi_t)$.

There are several useful reasons for employing linear preferences. First, it will be the case that policies that entail redistributions of consumption or income will not directly influence overall welfare. Additionally, the use of linear preferences ensures that the study of changes in the volatility of productivity shocks will not operate through risk aversion, or the intertemporal elasticity of substitution.

Workers are also permitted to use some time (x_t) to attempt to discover a new technology, which may eventually permit them to become a firm-owner, or manager.[10] This

[9] The entry (exit) rate is obtained by dividing the number of entries (exits) by the average total number of enterprises in the previous and current quarters. This data has also been annualized. The source for this is *Statistics Canada*.

[10] Perhaps the most intuitive way to think of this arrangement is that workers use some of their leisure time

activity is successful with some probability, and is characterized by a Poisson process with rate $\mu(x_t)$. However, this activity also has disutility measured as $h(x_t, Z_t)$, which is increasing in both arguments. Here Z_t is the productivity of new firms at that date, and so the cost of innovation is related to the magnitude of the subsequent benefit. Existing firm owners cannot engage in this activity, and so for them $x = 0$ (and $h(0, Z_t) = 0$).

It will be assumed that firm owners spend all their efforts to run their firm, and do not have any work/leisure decision of their own. The technology of their firm can change at some stochastic rate. Firm owners always have the option of disposing of their technology (i.e. shutting down their firm) and becoming a worker at the market wage. However, only workers are assumed to have the opportunity to develop or invent a new technology. This requires effort or disutility. When new technologies or firms are developed, this raises the cost or compensation of labor, which therefore increases the costs and reduces the profits of existing firms. At some point, an owner of an older firm will find his profit to be less than the market wage. At this time, he will elect to shut down the firm, and to become a laborer. At this point he can seek to obtain a new technology, which will give rise to a new firm in the future. There will then be a churning of workers and firms as this economy grows.

4 A Stochastic Characterization of Technology

At any date $t \geq 0$, all new firms begin with a productivity of Z_t, and this evolves in the following deterministic manner

$$Z_t = Z_0 e^{\kappa t} > 0.$$

The growth rate (κ) will be taken as a parameter, beyond the control of all agents. However, later a description of the determinants of this growth rate will be given. Firms in each cohort are identical initially, but not for long! Following entry, the productivity of a firm that was a date-t entrant, but which is now age a, evolves according to the following:

$$z_{t,a} = Z_t e^{(\kappa_1 a + \sigma W_a)},$$

where W_a is a standard Brownian motion that is independent across firms. This approach is useful because at any date t, all operational technologies at that date will be measured against the singular value of that of new entrants (Z_t), and this will serve as a unique reference point. Note that this assumption implies that an older firm benefits from technological change that has happened since it began operations., because $z_{t-a,a}$ is linked to Z_t. Once a firm ceases operations, $z_{t,a}$ becomes zero forever. So here κ represents the growth rate of the technology for new entrants, while κ_1 will represent the growth rate of technology for incumbents.[11] This also implies that

$$\frac{z_{t,a}}{Z_{t+a}} = \left(e^{-\kappa a}\right) e^{(\kappa_1 a + \sigma W_a)}. \tag{1}$$

to engage in experimentation or innovation. This could be thought of as non-market or informal employment which eventually may have a payoff. This feature is similar to the assumption, and intuitive idea, in Kogan, Papanikolaou and Stoffman [31] that market for ideas, and especially new ideas, is incomplete. Also, see footnote (12) below.

[11] In this way, there will be innovation, or productivity growth, by both incumbents and entrants, as is documented by Decker, Haltiwanger, Jarmin and Miranda [13]. However, only the productivity growth by entrants is determined by the decisions of agents.

For a firm with existing technology $z_{t,a}$, we will let

$$d\left(\frac{z_{t,a}}{Z_{t+a}}\right)/\left(\frac{z_{t,a}}{Z_{t+a}}\right) = (\kappa_1 - \kappa)\, da + \sigma dW_a,$$

where $\frac{z_{t,a}}{Z_{t+a}}$ denotes the *relative* technology for that firm at date $t+a$. That is, this expression will be the relative technology of a firm in the sense that it is measured relative to the level of technology of new entrants (Z_{t+a}) at that date.

Note that in characterizing the distribution of firm productivities, the age of each firm is essentially irrelevant, and instead what is important is the state of the firm's technology relative to the value of that of new entrants. That is, for two firms that are of different ages $a \neq a'$, if it turns out that $z_{t-a,a} = z_{t-a',a'}$, then these firms will be treated identically at that moment. Therefore the history of any firm is immaterial and it will only be necessary to know the distribution of firm's technologies at that particular moment.

4.1 The (static) problem of the firm

The owner of a firm of age a at date t has a technology denoted by $z_{t-a,a}$. For brevity, let us temporarily just write this as z. This technology is fixed for that particular firm, and does not change over time. The firm owner can hire labor in a competitive market at a price of w_t, and this price *will* change over time. The owners of a firm maximize profits, which are written as follows:

$$\pi_t = \max_{n_t} \left\{ z\left(n_t^{\alpha}\right) - w_t n_t \right\}. \tag{2}$$

Here $w_t n_t$ represents the wage bill. The profit-maximizing demand for labor is

$$n_t = \left(\frac{z\alpha}{w_t}\right)^{\frac{1}{1-\alpha}}. \tag{3}$$

Of course, since $z_{t,a}$ is fixed, but w_t rises if there is growth in the wage, this means that employment in the firm will diminish over time. Therefore, the profit of the firm is then

$$\pi_t = (z)^{\frac{1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}} (w_t)^{\frac{\alpha}{\alpha-1}} (1-\alpha). \tag{4}$$

It will be shown below that the market wage will be proportional to the technology index Z_t, so let us use the following expression: $w_t = A_w Z_t$. This then yields the following representation for profit

$$\pi_t = A_\pi \left(e^s\right) Z_t \tag{5}$$

where A_π is a constant to be determined below, and for a firm with productivity $z_{t-a,a}$ at date t, when the technology of new entrants is Z_t, then

$$e^s \equiv \left(\frac{z_{t-a,a}}{Z_t}\right)^{\left(\frac{1}{1-\alpha}\right)}. \tag{6}$$

Here s is interpreted as the "scaled-up" measure of the productivity of a firm with technology parameter $z_{t-a,a}$, relative to the productivity of a new firm at date t, which is Z_t.

An alternative interpretation is that it is the percentage difference in output of a profit-maximizing firm with technology parameter $z_{t-a,a}$, relative to the productivity of a new firm, at that same date. If the old firm has $s = 0$, then it will have productivity identical to that of a new firm at that date, and it is isomorphic to a new firm. Henceforth, it will be convenient to suppose that e^s will be an index of the relative technology for a firm with technology parameter $z_{t-a,a}$.

Let $\underline{s}$ denote the time-invariant value of the relative technology such that firms will voluntarily *choose* to exit the market, or cease operations if their productivity reaches that level. The distribution of s will be determined below, but the value of $\underline{s}$ will act as an endogenously-determined absorbing barrier in this distribution. Hence, operational firms will have technology indexes $s \in (\underline{s}, \infty)$.

The value of $(\underline{s})$ will be determined by the equilibrium conditions. If $\underline{s} = -\infty$, this implies that firms will never cease operating, which would seem incompatible with an equilibrium of a growing economy. Conversely, if $\underline{s} \geq 0$, then firms will cease operating the moment they are created. In other words, there will be no creation of new firms or technologies, and hence no growth. Given the assumptions above, where $(\kappa_1 - \kappa) < 0$, and $\sigma > 0$, firms will eventually exit when they reach $\underline{s} < 0$. In the absence of any entry of new firms, this would imply a falling stock of firms.

The above equation also implies that for a specific firm,

$$ds = [\kappa_s dt + \sigma_s dW]$$

where

$$\kappa_s = \left(\frac{\kappa_1 - \kappa}{1 - \alpha}\right), \qquad \sigma_s = \left(\frac{\sigma}{1 - \alpha}\right). \tag{7}$$

Note that for a particular firm, the de-trended profit function is given by equation (4). This profit has a stochastic growth rate, that has a trend composed that of technology $(z_{t+a,a})$ and that of wages (w_t). The expected growth of profit for such a firm can be written as which can be written as:

$$\kappa_s + \frac{\sigma_s^2}{2} + \kappa$$

The expected growth rate of profit for a firm less the growth rate of wages will then be

$$\kappa_s + \frac{\sigma_s^2}{2}. \tag{8}$$

In what follows it will generally be assumed that this quantity is negative, since will insure that the distribution of "s" shifts to the left, and is therefore the distribution will be well-defined. But more importantly, this assumption ensures that over time the technology of older firms will, eventually erode or get worse *relative to newer firms*, even if the absolute productivity of firms improves over time. If this were not the case, then the distribution of firms would be dominated by very old firms that were very productive, and then a few young, relatively unproductive firms.

Hence, lower values of κ_s or higher values of σ_s will raise the expected growth rate of profit relative to that of wages. This will be important in influencing the expected lifetime of a firm, and the decision of when to shut down a firm.

4.2 Equilibrium Condition for Labor

It must be that the quantity of labor available equals the quantity demanded. Let N denote the amount of labor available. Then let $f(z)$ temporarily denote the distribution of operational technologies in period t. Using equation (3), the equilibrium condition must be

$$N = \int \left(\frac{z\alpha}{w_t}\right)^{\frac{1}{1-\alpha}} f(z)\,dz \tag{9}$$

or

$$w_t^{\frac{1}{1-\alpha}} = \frac{1}{N}\int (z\alpha)^{\frac{1}{1-\alpha}} f(z)\,dz. \tag{10}$$

From equation (6) we can re-write this equation in the following manner

$$w_t = A_w Z_t$$

where the productivity-adjusted wage is written as follows:

$$A_w = \alpha (N)^{\alpha-1}\left[\int e^s f_s(s)\,ds\right]^{1-\alpha}. \tag{11}$$

Here we are abusing notation by positing some distribution $f_s(s)$, but this will be determined endogenously below. It is important to note that this distribution is determined endogenously by the decisions of agents.

As can be seen this expression is decreasing in the number of workers N. The last term in square brackets is the ratio of the number of operational firms to the number of workers, which is analogous to a capital-labor ratio.

From equation (5) this implies that the productivity-adjusted profit function can be written as follows:

$$A_\pi = (1-\alpha)\left[\frac{\int e^s f_s(s)\,ds}{N}\right]^{-\alpha}. \tag{12}$$

Once, again term in square brackets is the ratio of the number of operational firms, to the number of workers, and so it makes sense that profit should be decreasing in this term.

4.3 The Dynamic Programming Problem for the Owner of a Firm

Let V_t the date-t value function for a firm-holder who has access to a technology $z_{t-a,a}$, when the market wage is w_t. At each instant the owner of a firm, with technology λ_t, receives a flow of profit of π_t. Additionally, if he wishes to stay as a non-laborer and run the firm, he gets the *increased value of the firm* ($\dot{V}(\cdot)$), but otherwise he can shut down the firm, and become a worker, with value function W_t. The Hamilton-Jacobi-Bellman for a firm-owner is then written as follows:

$$rV_t = \max\left\{\pi_t + \dot{V}_t, rW_t\right\}. \tag{13}$$

This means that an agent owning an operational firm will have a value equal to the profit received at that instant, plus the discounted value of the firm next period if it is operational, or the value switching to being a worker, whichever is greater. Note that if W_t

is homogenous of degree 1 in Z_t, then V_t can inherit this property as well. Also, it may result that typically $\dot{V} < 0$, which is to say that the value of a particular firm will fall over time as wages rise. It should be clear that the firm owner will shut down the firm, and become a worker when the following condition holds:

$$V_t = W_t \tag{14}$$

Using the following notation with productivity-adjusted value functions (i.e. $V_t = Z_t V(s_t)$), the solution to equation (13) can be written as follows

$$V(s_t) = \max_T E_t \left[\int_t^T e^{-r(v-t)} A_\pi (e^{s_v}) \, dv + e^{-r(T-t)} W_T \right]. \tag{15}$$

Here T is the *optimally chosen* shutdown date for the firm. This is a random variable, since it depends on the value of s. The Hamilton-Jacobi-Bellman equation for this problem is written as follows:

$$rV(s) = A_\pi (e^s) + \kappa V(s) + \mathcal{A}V(s) \tag{16}$$

where

$$\mathcal{A}V(s) = \kappa_s V'(s) + \sigma_s^2 V''(s)/2. \tag{17}$$

These equations have the following interpretation. At any instant, the firm owner receives the dividend of $A_\pi(e^s)$, plus the capital gain. The latter consists of two parts. First, the term is the capital gain portion of the return ($\kappa V(s)$) that is present because $V_t = Z_t V(s)$, and the aggregate technology variable (Z_t) is growing over time at the rate if κ. The other portion of the capital gain (equation (17)) is the change in the value of the firm because the relative technology (s) is changing.

Then there is the value-matching, or boundary condition implied by equation (15)

$$V_T = V(\underline{s}) = W_T. \tag{18}$$

There is also be the important smooth-pasting condition that ensures that the firm-owner ceases at the optimal date

$$\left. \frac{\partial V(s)}{\partial s} \right|_{s=\underline{s}} = 0. \tag{19}$$

It is then straightforward to verify that the value function takes the following form:

$$V(s) = B_1 (e^s) + B_2 e^{s\beta}, \tag{20}$$

where

$$B_1 = \frac{A_\pi}{(r-\kappa) - \kappa_s - \left(\frac{\sigma_s^2}{2}\right)} \tag{21}$$

$$B_2 = e^{-\underline{s}\beta} \left[W - B_1 (e^{\underline{s}}) \right]$$

$$\beta = \frac{-\kappa_s}{\sigma_s^2} - \sqrt{\left(\frac{\kappa_s}{\sigma_s^2}\right)^2 + \frac{2(r-\kappa)}{\sigma_s^2}} < 0. \tag{22}$$

The meaning of these equations is instructive. The term $B_1(e^s)$ is the expected utility that the firm-owner would get if he owned the firm, which has an index value of relative technology equal to s, *and operating this firm forever*. Obviously this term is increasing in s, and so a better technology leads to a higher value. This can be seen because B_1 is the discounted profit for such a firm where $s = 0$. But it will not be optimal for the firm-owner to operate it forever, and so the second term in equation (20) is the extra utility the firm-owner will receive when $s = \underline{s}$. At that time he will receive the extra utility from being a worker, in excess of that of being a firm owner. $(W - B_1(e^{\underline{s}}))$. By writing this value function in this manner, it naturally satisfies equation (18).

Equation (22) is the characteristic root of equation (16). Note that since $\beta < 0$, this means that as the relative technology of a firm (s) falls, the firm will get closer to the point at which it shuts down.[12]

Equation (19) can be seen to imply that the following condition is satisfied [13]

$$\underline{s} = \ln\left(\frac{-\beta W}{(1-\beta) B_1}\right).$$

This expression determines the appropriate level of relative productivity for a firm that is ceasing operations. This expression has the intuitive property that the threshold (s) is increasing in the relative payoff (W/B_1), or that the expected lifetime of a firm will be inversely related to this ratio.

4.4 The Dynamic Programming Problem for a Worker

The worker receives a wage w_t at time t. All workers are identical, no matter how profitable their firms used to be, nor how long they have been unemployed. However, the worker can also expend time trying to obtain an idea or technology (z_t) which might become productive immediately. The effort (x) that they expend in discovering a new technology is not observable by others, and therefore not contractible, so that agents can not engage in contracts based on the outcome of research efforts.

Workers have discoveries that arrive according to a Poisson distribution. Let μ be the probability of locating such a technology. At each instant, the flow of utility for a worker is the wage (w_t) net of the disutility of research effort expended ($h(x, Z_t)$).[14] Additionally, he receives the increased value of the job ($\dot{W}$), plus with some probability (μ) he acquires a new technology so that he switches to running a firm, instead of being a worker. The typical worker takes the wage w_t, or the technology (Z_t) as given while expecting to receive a new technology (Z_t) for himself. Therefore, the dynamic programming problem of a worker is then written as follows:

$$rW_t = \max_x \left\{ w_t - h(x, Z_t) + \dot{W}_t + \mu(x)[V_t - W_t] \right\} \tag{23}$$

[12] Here β has the interpretation of acting like a discount rate. But rather than being a discount rate with respect to *time*, in the usual sense, it is a discount rate with respect to *firm productivity*.

[13] This can be derived by taking the derivative of equation (20) with respect to s, evaluating the result at $s = \underline{s}$, and setting this to zero.

[14] This function should be increasing in both arguments. The reason for the inclusion of Z_t is that as the technological frontier "moves out" or progresses, it requires more effort by any potential innovator to be successful.

The optimization condition is written as follows:

$$h_1(x, Z_t) = \mu'(x)[V_t - W_t] > 0 \tag{24}$$

This last equation is noteworthy because it says that as the value functions of the workers and firm-owners get close to each other, innovation, and therefore growth, will be reduced. To insure that the economy will have a balanced growth path, with positive growth, it will be assumed below that $h(x_t, Z_t)$ will be homogeneous of degree one in Z_t. Since the wage (w_t) also has this property, the value functions (V_t and W_t) will also have this property.[15]

In general, it is also a possibility that equation (24) will hold with a strict inequality ($>$). In this case, no workers will devote effort to innovate, and hence there will not be any new firms created. For there to be balanced growth path it would then seem necessary that $\underline{s} < 0$. Otherwise if $\underline{s} \geq 0$ the number of firms would be falling over time.

Assuming that this homogeneity holds, then the productivity-adjusted value function, equation (23), can then be written as follows:

$$rWZ_t = \{A_w Z_t - h(x^*) Z_t + \kappa W Z_t + \mu(x^*)[V(0) Z_t - W Z_t]\}$$

where x^* is the solution to equation (24).

It then follows that this equation can be written in the "normalized manner" as follows:

$$rW = A_w - h(x^*) + \kappa W + \mu(x^*)[V(0) - W],$$

which can be written as follows:

$$W = \frac{A_w - h(x^*) + \mu(x^*) V(0)}{[r - \kappa + \mu(x^*)]}.$$

Here, as a more general interpretation, the function $(W())$ actually represents an amalgam of all the factors, external to the firm, that can affect the opportunity cost of operating the firm, and therefore influence the firm's existence. Here the market forces that influence both the entry and exit decision are embedded in the wage (w_t), which in turn influences the value function $(W())$. In a more detailed or complicated model, there could be many other market factors that could influence the opportunity cost of operating a firm, or producing a product $(W())$, and also for innovation $(V(s))$. The point here is that factors that influence the incentives for innovation and the opportunity cost of operating a firm are intertwined (as can be seen by equations (15) and (23)), and policies or at parameters do not just influence one of these in isolation from the other. [16]

[15] An alternative, but roughly equivalent formulation, is to assume that the individual gets to consume his wage, less some fraction (x) of this wage income that is spent on research. Consumption of the individual is then $w_t(1-x)$.

[16] An alternative formulation that captures these features is contained in Huffman [25]. In this case, there is a pool of agents who will always be workers, and another separate group of agents who are either firm owners or potential innovators, who will ultimately innovate and then operate a firm. The cost of labor affects the profitability of operating a firm and also of innovating. The cost of shutting down an existing firm is the lost profit, while the benefit is the expected future profit after developing an innovation.

4.4.1 Determination of the Growth Rate

It must be recognized that this version of the model is not well suited for studying the situation in which $\kappa_1 \geq \kappa > 0$. The reason is that in this case growth is determined strictly by the incumbents, and these incumbents rarely exit. In this instance, there is no negative growth for the productivity of firms, relative to that of new entrants. But such a process does not have a stationary distribution unless there is negative growth. Hence, it is not possible to characterize a steady-state balanced growth path.

At any date t there will be a flow of new firms created equal to the amount $\mu(x)N$. It is assumed that the growth rate will then be related to the amount of research activity, which is the number of people engaged in this activity (N) multiplied by the successful per-person outcome from this activity $\mu(x)$. These firms will all enter with technology parameter Z_t, or with relative technology $s = 0$. Hence it is assumed that [17]

$$\kappa - \kappa_1 = \mu(x)N.$$

This simple relationship relates the amount of successful research activity to the growth rate. Of course, there is then an intertemporal spillover in that new firms are able to make use of, or benefit from, the most recent innovation.[18] This is the form of the growth equation used in "idea-based growth models", such as that of Bloom, Jones, Van Reenen, and Webb[8].

5 The Steady-State Distribution of Technologies or Firms

In a steady-state, the distribution of technologies $z_{t,a}$, $a \geq 0$, is distributed over some interval of $(0, \infty)$. However, the lower limit will be strictly greater than 0 because some firms will wish to shut down operations when their technology becomes sufficiently poor. Of course, this distribution (as well as its support) will be shifting to the right because of the trend in Z_t. However, it will be convenient to characterize the technology as relative to Z_t. Since, for a specific firm let $s = \left(\frac{1}{1-\alpha}\right) \ln\left(\frac{z_{t-a,a}}{Z_t}\right)$, is such a measure or index of relative technology, it will be convenient to suppose that there is some stationary distribution of $f()$ that is time-invariant. This distribution will be over some interval $(\underline{s}, \infty)$. Let $f(s)$ denote this stationary distribution.

To do this we must study the Kolmogorov forward equation, which is written as follows:

$$\frac{\partial f(s)}{\partial t} = -\kappa_s \frac{\partial f(s)}{\partial s} + \left(\frac{\sigma_s^2}{2}\right) \frac{\partial^2 f(s)}{\partial s^2}, \text{ for } x \in (\underline{s}, 0) \cup (0, \infty). \tag{25}$$

Since the distribution is assumed to be stationary, the left side of this expression will be

[17] Another way to think about this is to recognize that at date t new entrants will have productivity slightly in excess of (i.e. to the right of) Z_t. And over some time interval τ the number of these entrants is $\tau\mu(x)N$. Therefore we have the following: $(Z_{t+\tau} - Z_t) = \tau\mu(x)N + \tau\kappa_1$. A more detailed presentation of the decision-making process and assumptions underlying this determination of the growth rate is presented in the technical appendix.

[18] This constrasts with the setup of Luttmer [36] wherein the new entrants are able to copy and improve on the technology of a firm that is just exiting the market.

zero, It will also be the case that the flow of firms exiting at the boundary will be

$$\frac{\sigma_s^2}{2} \left. \frac{\partial f(s)}{\partial s} \right|_{s=\underline{s}} . \tag{26}$$

Using the fact that, since the distribution is stationary, the left side of equation (25) is zero, and integrating this expression yields

$$0 = \frac{\sigma_s^2}{2} \left[\left. \frac{\partial f(s)}{\partial s} \right|_{s \nearrow 0} - \left. \frac{\partial f(s)}{\partial s} \right|_{s=\underline{s}} - \left. \frac{\partial f(s)}{\partial s} \right|_{s \searrow 0} \right] . \tag{27}$$

Using equation (26) this implies that the number of firms entering at $s = 0$ satisfies

$$\frac{\sigma_s^2}{2} \left[\left. \frac{\partial f(s)}{\partial s} \right|_{s \nearrow 0} - \left. \frac{\partial f(s)}{\partial s} \right|_{s \searrow 0} \right] . \tag{28}$$

Hence, equalizing the inflow and outflow of firms must mean that equation (26) must equal equation (28).[19]

With the determination of the constants of the distribution, such as the fact that the distribution must sum to $(1 - N)$, it can now be re-written or characterized as follows:

$$f(s) = \left\{ \begin{array}{ll} \left(\frac{1-N}{-\underline{s}}\right) \left(1 - e^{\delta(s-\underline{s})}\right) & \text{for } s \in (\underline{s}, 0) \\ \left(\frac{1-N}{-\underline{s}}\right) \left(1 - e^{-\delta\underline{s}}\right) e^{\delta s} & \text{for } s \in (0, \infty) \end{array} \right. . \tag{29}$$

Equation (27) can be used to show that the root of this equation is the following

$$\delta = \frac{2\kappa_s}{\sigma_s^2} < 0.$$

Figure 5 shows two different distributions, which have different values for the variance.[20]

Luttmer [35] also shows that another attractive property of this formulation is that this distribution can closely match the actual size distribution of firms in the US.

6 Summary of Equations of the Model

With the use of the formulae for the distribution $(f(\cdot))$, the equations above can then be summarized as follows. Equation (12), which determines the productivity-adjusted profit can be written as

$$A_\pi = (1 - \alpha)(N)^\alpha \left[\left(\frac{1 - N}{-\underline{s}} \right) [1 - e^{\underline{s}}] \left(\frac{2\kappa_s}{\sigma_s^2 + 2\kappa_s} \right) \right]^{-\alpha} .$$

[19] Luttmer [36],[35] studies and characterizes distributions with many of these properties, and this analysis builds from his work.

[20] Both distributions have $\kappa_s = -.01$, and $N = 0.80$, so that they have the same mass. One distribution has $\sigma_s = .05$, $\underline{s} = -.40$, while the other has $\sigma_s = .10$, $\underline{s} = -.60$.

Similarly, the equation (11) which defines the productivity-adjusted wage can be written as follows:

$$A_w = \alpha \left(N^{\alpha-1}\right) \left[\left(\frac{1-N}{-\underline{s}}\right) \left[1 - e^{\underline{s}}\right] \left(\frac{2\kappa_s}{\sigma_s^2 + 2\kappa_s}\right) \right]^{1-\alpha}. \tag{30}$$

This expression looks a little unusual for a wage equation. Usually such an equation for have something like a capital-labor ratio, but this equation has terms reflecting growth rates and dispersion measures. In fact, the term in square brackets is really proportional to the productivity levels of all firms. The reason for the terms involving κ_s and σ_s is because these characterize the steady-state distribution of firms, as shown in equation (29).

The equation determining the value function of the firm-owner, with access to relative technology index s:[21]

$$V(s) = B_1 \left(e^s\right) + B_2 e^{s\beta},$$

where

$$B_1 = \frac{A_\pi}{(r-\kappa) - \kappa_s - \left(\frac{\sigma_s^2}{2}\right)}$$

$$\beta = \frac{-\kappa_s}{\sigma_s^2} - \sqrt{\left(\frac{\kappa_s}{\sigma_s^2}\right)^2 + \frac{2(r-\kappa)}{\sigma_s^2}}.$$

and

$$B_2 = e^{-\beta(\underline{s})} \left[W - B_1 e^{(\underline{s})}\right] \tag{31}$$

The equation determining the value function of a worker:

$$W = \frac{A_w - h(x^*) + \mu(x^*) V(0)}{\left[r - \kappa + \mu(x^*)\right]}$$

The optimal research decision is determined by the following:

$$h'(x) = \mu'(x) \left[V(0) - W\right] > 0$$

The equation determining the optimal exit decision:

$$\underline{s} = \ln\left(\frac{-\beta W}{(1-\beta) B_1}\right) \tag{32}$$

The growth rate is assumed to be determined from the following expression:

$$\kappa - \kappa_1 = \mu(x) N.$$

The flow of agents in and out of the two sectors must be equal in a steady-state. This means that the flow of agents who obtain new technologies, and become firm owners, must

[21] An alternative way to write this expression is as follows: $V(z/Z)Z = B_1(z) + B_2\left(z^\beta\right)\left(Z^{1-\beta}\right)$. Despite the fact that $\beta < 0$, it can be shown that this expression is increasing in z, but it will cease to be *strictly* increasing when $z = Ze^{\underline{s}}$.

equal the number of agents exiting production and shutting down their firms:[22]

$$\begin{aligned}\mu(x) N &= \frac{\sigma_s^2}{2}\left[\left(\frac{-2\kappa_s}{\sigma_s^2}\right)\frac{1-N}{-\underline{s}}\right] \\ &= (-\kappa_s)\left(\frac{1-N}{\underline{s}}\right) \qquad (33)\end{aligned}$$

Alternatively, using equation (7) this can be re-written as $\underline{s} = -\left(\frac{1-N}{1-\alpha}\right)$.

These equations will determine the following unknowns: $\kappa, N, x, \underline{s}, W, \tilde{V}(0), A_w, A_\pi$. We then have to fix a bunch of parameters,such as α,r, σ, κ_1.

6.1 Measures of Inequality in this Economy

In this model, the agents are ex-ante identical in terms of preferences and abilities. Nevertheless, income and consumption levels can be quite different. Therefore, it is of interest to characterize inequality, and how this is influenced by various parameters.[23] First, since the distribution of firm productivities has no upper bound, workers will not be the richest agents in the economy. The question is whether they will be the poorest. Therefore it is of interest to study the following:

$$\frac{A_\pi}{A_w} = \frac{N(-\underline{s})\left(\sigma_s^2 + 2\kappa_s\right)}{(1-N)\left(1-e^{\underline{s}}\right)2\kappa_s}.$$

This is the ratio of the income of an owner of a new firm, to that of a worker. If one were to think that workers would be paid much less than the profit from a new firm, then this ratio should substantially exceed unity.

There are a few things to note about this expression. First, the term $\left(\frac{\sigma_s^2+2\kappa_s}{2\kappa_s}\right) > 1$ is positive, because the growth rate κ_s is also assumed to be sufficiently negative. Next, any change in a policy or a parameter that leads to an increase in aggregate growth κ, will further reduce κ_s and therefore reduce inequality as measured by this ratio. However, this will also likely mean a change in N and $\underline{s}$ as well. Third, the ratio is increasing in σ_s. In general, The larger is $\left(\frac{\sigma_s^2+2\kappa_s}{2\kappa_s}\right)$ the more concentrated is the distribution, and the steeper is the density of the distribution. This concentration is offset by the larger size of the drift.

Fourth, this ratio is increasing in the number of workers (N). This is because as N increases, the ratio of workers to firms rises, and this lowers wages and raises the profit of firms. Lastly, this ratio is increasing in $\underline{s}$. For the same reasons, as $\underline{s}$ rises the ratio of workers to firms rises, and this lowers wages and raises the profit of firms.

An alternative measure of inequality is the ratio of the profit from a marginal firm, to the wage of a worker. That is, compare the profit from a firm that is just about to shut down because the owner is indifferent to receiving that profit, or alternatively becoming a

[22] This means using equation (29) in either equation (26) or (28).

[23] It is possible to compute the Gini coefficient for such an economy, but it is much more cumbersome to assess how various factors would influence this measure of inequality since so many different agents are involved in such a measure. The focus here will be on a couple of specific measures of inequality.

worker. This ratio is given by the following:

$$\frac{A_\pi e^{\underline{s}}}{A_w} = \frac{\left(e^{\underline{s}}\right) N\left(-\underline{s}\right)\left(\sigma_s^2 + 2\kappa_s\right)}{\left(1-N\right)\left(1-e^{\underline{s}}\right) 2\kappa_s}.$$

Since the agent is indifferent between these two statuses, one might naturally think that this ratio should be close to unity. It will be shown that this is not always the case. This measure is increasing in $\underline{s}$ because a lower value means a lower level of productivity of the firm, and hence lower profit.

Of course, in this model, over time firm-owners wander around within the distribution of income or consumption, since their income is subject to constant shocks. However, all workers earn the same income and therefore occupy the same position within the distribution.

7 Calculating the Rate of Destruction

It would seem important to calculate the rate of creation or destruction in such a model. It is possible to show that the expected lifetime of a firm (or the expected time to transit from $s = 0$ to $s = \underline{s}$) can be measured as

$$E\left(T\right) = \frac{\underline{s}}{\kappa_s}. \tag{34}$$

The standard deviation of this first passage time can then be shown to be

$$\frac{-\underline{s}\sigma_s}{\sqrt{2\left|\kappa_s\right|}}. \tag{35}$$

Therefore, an increase in σ_s by itself, which is studied below, will not affect the average lifespan of a firm, but will increase the dispersion of these durations.

However, equation (34) is instrumental for another reason. Although σ_s does not appear explicitly in this expression, a change in σ_s can influence the equilibrium level of $\underline{s}$, and κ_s. If this results in fewer entrants of new firms, and slower growth, then this would show up as a longer expected lifetime for each firm. This would then shed some light on the phenomena illustrated in figures 5 and 6.

One might then think of the rate of business destruction as being characterized as

$$\frac{1-N}{E\left(T\right)} = \frac{\left(1-N\right)\kappa_s}{\underline{s}} = \frac{\sigma_s^2}{2}. \tag{36}$$

Equations (34) and (36) show rather clearly that there will then be a negative relationship between the rate of entry of new firms, and the average age of firms. [24]

[24] Decker et al. [13] study the relationship between the changing age distribution of firms, and the declining startup rates, and how this is related to the changing startu rates.

7.1 The Distribution of Exit Times

Since there are a continuum of firms with different productivities, there must be a distribution of exit times for these firms. It can be shown that, contingent on having state s, the distribution of exit times (t) (at which $s = \underline{s}$).can be written as follows[25]

$$g(t \mid (s-\underline{s})) = \frac{(s-\underline{s})}{\sigma_s\sqrt{2\pi t^3}} \exp\left\{-\frac{((s-\underline{s})+\kappa_s t)^2}{2\sigma_s^2 t}\right\}. \tag{37}$$

Several examples of such a distribution are shown in figure 6, for various values of $(s-\underline{s})$, which are measured in percent. This figure certainly illustrates the higher likelihood of a firm exiting soon if it has a relatively low productivity.

8 On the Possibility of an Equilibrium with Zero Growth

One might inquire if it is possible for there to be zero growth. For this to be possible it would mean that there is zero innovation ($x = 0$). This would imply that there is no entry of new firms, and hence no exit, which would mean that $\underline{s} = -\infty$. This would imply that $V(s) \searrow 0$, as $s \to -\infty$. But since firm-owners can always become workers this implies that $W \leq 0$. For this to happen it would mean that

$$A_w - h(0) \leq 0.$$

Additionally for $x = 0$ to hold it must be that

$$h'(0) > \mu'(0)\tilde{V}(0) = \mu'(0)B_1.$$

Hence certain boundary conditions must apply for there to be strictly zero growth. The wages must be so low that the owner of even the worst firm would decline to become a worker, and the costs of innovation, relative to the benefits are too low to produce any innovation.

9 A Note on the Intertemporal Spillover

Many models of endogenous growth have an intertemporal externality or spillover which is justified by the plausible observation that current innovation benefits from discoveries of the past. Such a feature is present here, but the magnitude of it is far from clear. In this framework, the technology of new entrants (Z_t) is rising at a rate that is determined by the number of innovators. That is, more aggregate innovation means higher growth. However, there are also reasons why more innovation would also reduce the incentive to innovate. First, note that there is no direct research externality here. A single individual's probability of innovating is determined by $\mu(x)$, and so it is the agent's effort (x) alone that determines success. If other workers were to change their effort, this would not directly affect other agents.

[25] See Cox and Miller [11].

Secondly, higher growth raises the growth rate of wages, which in turn raises the *cost* of labor to incumbent firms and lowers the value of these firms ($V(s)$). This lowers the value in equations (20) and (21). This in turn lowers the incentive to innovate.

Next, consider a worker who is attempting to innovate, and has a decision described by equation (24). Now consider what would happen if a positive measure of other workers were to increase their innovation level (x). This effect of this would be a marginal increase in the growth rate (κ) of wages, which would raise the value of being a worker. Although the value of obtaining a new firm ($V(0)$) would change, for reasons just described, this effect could be rather small. The point is that this increase in the level of innovation of others can lower the value of ($V_t - W_t$) in equation (24). Again, this would lower the incentive to innovate.

Lastly, with these factors in mind, it would seem that the collective effect of innovation can have complicated repercussions for present and future decisions in this economy. While there appears to be an intertemporal spillover in the model, it would seem to be much smaller than in other models.

10 Analysis of the Model

10.1 Parameterization

In the following analysis, the model will be used to study how the model behaves as various parameters are changed. Therefore, a particular parameterization has to be chosen for the benchmark of the economy. Therefore, using the following functional form will be useful:

$$h(x) = \frac{(x)^{1+\omega}}{1+\omega}, \quad \omega \geq 0.$$

Also it will be assumed that $\mu(x) = \mu \cdot x$. Parameters, such as ω, will be chosen so that in the benchmark economy the growth rate will be 3%.

Again as a benchmark, it will be assumed that $\kappa_1 = 0$, and aggregate growth will driven by innovation of new entrants. Additionally, in the benchmark model the following parameter values will be used:

$$r = .05, \sigma = .05, \alpha = .65.$$

10.2 The Effect of a Change in σ

The following analysis will proceed by comparing the steady-state growth paths for this economy under alternative parameter values. First, we will consider an increase in σ, with is the log of the relative technology shock. The results from raising this parameter are shown in figures 7 and 8. As can be seen in figure 7A, an increase in σ tends to reduce the growth rate. Figure 7B shows that this result is due to the fact that there is less innovation activity (x), which is partly offset by the increase in the number of researchers (or workers N). Note also that since N rises, $\underline{s}$ also rises and so the threshold technology for exit is now higher.

It then seems possible that the increase in the variance of firm-level productivity shocks can be at least partially responsible for the slowdown in growth observed in figures 1 and 2.

Figures 8A shows that the welfare of both workers and firm-owners is increasing in σ. The reason is that the rise in σ actually raises the growth in the technology shocks z, and thereby raises the prospects for future profit. This is offset to some extent by the reduction in growth. However, the reduced growth, resulting from the lower value of x is a consequence of the fact that $(V - W)$ falls as σ rises.

Figure 8B shows that as σ rises, the average lifetime of a firm initially rises. This is in spite of the fact that the threshold relative productivity of an exiting firm has risen. As indicated above, the inverse of this time could be interpreted as a measure of the rate of business destruction. This panel then indicates that an increase in σ results in a lower rate of destruction of businesses or firms. The reason for this is that although the threshold level of $\underline{s}$ rises, this is overwhelmed by the fall in the growth rate.

Figure 8C shows that as the σ rises, the productivity-adjusted wage (A_w) rises. This happens even though the number of workers is also rising. Normally an increase in the number of workers would reduce wages. The key to understanding this result is found in equation (30). Here it is apparent that the rise in N and $\underline{s}$, in conjunction with the increase in σ should all cause A_w to fall. But it does not fall, and the reason for this is the growth rate (κ) also falls. This effect helps to make the distribution "spread out" and therefore raise the average productivity of these firms. An example of this is shown in figure 3. The effect of the rise in σ is to censor the distribution by causing some of the low-productivity firms to drop out through the absorbing barrier, which is also somewhat higher. This change in κ_s increases the productivity-adjusted wages (equation (30)). If the ratio (A_π/A_w) is viewed as a measure of income inequality, then this suggests that a greater variance of the economic shocks σ results in a *lower* level of inequality.

Figure 8C shows that the productivity-adjusted profit is falling in σ for the same reason that wages are rising. However, as σ rises sufficiently, A_w eventually exceeds A_π. If the wage is rising in σ because of the higher productivity of firms, the higher resulting wage will then lower the profit of remaining firms. This may be a little surprising. This means that workers may eventually give up their wage and job for a payoff (i.e. profit) that is *lower* than their current wage. The reason that the worker will give up this wage for a lower profit is because there is a higher prospect of profit growth due to the higher value of σ.

Figure 8D shows the productivity-adjusted wage (A_w) as well as the productivity-adjusted profit if a marginal firm $(A_\pi e^{\underline{s}})$, which is a firm whose owner is just indifferent between operating the firm, and shutting it down to become a worker. It is evident that as σ increases the disparity between A_w and $A_\pi e^{\underline{s}}$ increases, but in the diagram, it is possible that $A_w > A_\pi e^{\underline{s}}$. In this case, for these firms, the workers are getting paid more than some firm-owners! The reason for this would seem to be that the firm owner with a very low profit can still potentially benefit from elevated future profit due to the possibility that this profit will rise in the future because the productivity of the firm will rise. This feature can also be seen equation (8). Increasing the value of σ_s raises expected profit relative to the contemporaneous wage, and so makes the firm owner delay shutting down his firm. In this case, the agent earning the lowest income in the economy would not be a worker, but would be an owner of a firm that is just about to cease operating.

It is then useful to summarize how an increase in σ affects income inequality. For low values of σ, the income of a worker is roughly equivalent to that of the marginal (lowest productivity) firm ($A_\pi e^{\underline{s}}$), and significantly lower than the income of a new firm owner (A_π). As σ rises, the relative distribution of income "spreads out", as there are wealthier firm-owners who earn higher profits. Although $\underline{s}$ rises, the relative income of the marginal firm falls because A_π falls. Also, the income of a worker (A_w) moves well inside the distribution of income, until it can be greater than A_π. As σ rises, income inequality rises in the sense that there is a wider distribution of overall incomes. However, there are complicated effects, in that the relative income of workers certainly rises.

Incidentally, one can see a linkage between figures 8B and 8D. Figure 8B shows that as σ rises, the average lifetime of a firm rises. Figure 8D shows that the firm-owner seems to wait even longer before shutting down the firm.[26]

10.3 An Increase in σ and a Corresponding Reduction in κ_1

Now consider an increase in the standard deviation of σ which is compensated for my decrease in κ_1 so that the growth component, measured as $\left(\kappa_1 + \frac{\sigma^2}{2}\right)$ is held constant. The results of this experiment are shown in figures 9 and 10. Other than the increase in σ and reduction in κ_1, the same parameter values are employed for both this and the previous analysis. The results are very similar to those of the previous experiment, shown in figures 7 and 8. In this new experiment, the fall in the growth rate (figure 9A) is much more pronounced, since the upward trend in technology shocks is removed. As is shown in figure 9B, this fall in growth is entirely due to the reduction in innovation (x), since the labor force (N) rises, as is shown in figure 9C. It should be emphasized that this reduction in growth is not attributed to some assumed change in costs or benefits of innovation. To the extent that these costs are benefits have changed, say due to the rising wages, these are endogenous responses to the change of σ.[27] The increased dispersion of firms results in many more "high-productivity" firms. This drives up wages, which reduces the profitability of firms, and reduces the incentive to innovate. That is, the increase in σ does not by itself insure that the cost of innovation must increase.

Figure 9D shows that the threshold level of $\underline{s}$ rises in this experiment so that an increase in σ results in firms exiting at a relatively higher level of productivity.

Figure 10B shows that the increase in the age of a firm is much more evident in this case than in figure 8B where there is a simple change in σ. This may be a little surprising because figure 9D shows that the shutdown threshold value $\underline{s}$ rises, and this should *lower* the average age of a firm. However, this effect is more than offset by the fact that the growth rate has fallen, and consequently firms will have a much longer lifetime because of this. Once again, the inverse of this measure can be interpreted as the rate of business destruction. This experiment again indicates that an *increase in the volatility of technology shocks translates into a reduced rate of business destruction.*

[26] Here the notion of "waiting longer" means that the firm owner will wait until their profit is even further below the market wage before shutting down the firm.

[27] This is in contrast with some papers, such as Aghion, Bergeaud et al. [2] They employ a much more complex model in which it is assumed that rising overhead costs of firms alters the incentives for innovation, and leads to higher growth in the short run, but lower growth in the longer run.

Figures 9B and 10B are also jointly important for understanding some phenomena described above. The slowdown in research effort illustrated in figure 9B would mean a reduction in the rate of entry of new firms. This is exactly what seems to appear in figures 3 and 4 for the US and Canada. So it could be that the recent reduction in the growth rate and the rate of entry of young firms could be an outcome of reduced innovative activity, but the ultimate cause of this could be the increase in the variance of firm-level productivity.[28]

This illustrates an important phenomenon that can appear in these models of growth through creative destruction. One might witness a change rate of the introduction of new goods or firms, or a change in the average age of firms, and then make the leap of logic to conclude how this should influence the growth rate, not to mention to attach value judgements to all of this. But all of these observations are the outcome of a complex series of factors that interact together.

Figures 10C and 10D look a lot like their counterparts in figures 8C and 8D. Again, it seems that A_w is increasing in σ while A_π is decreasing. Figure 10C, once again, shows that if σ is sufficiently large A_π will fall below A_w. This happens for the same reason as in the previous experiment. Once again, if the ratio (A_π/A_w) is viewed as a measure of income inequality, then this suggests that a greater variance of the economic shocks σ_s results in a lower level of inequality.

And once again the disparity between A_w and $A_\pi e^{\underline{s}}$ increases as σ rises. This is shown in figure 10D. This contributes to the elevated average age of firms in this economy, as is illustrated in figure 10B.

One notable difference in the two experiments is the behavior of welfare of both agents as σ changes. Figures 10A and 8A are quite different. Whereas in the previous experiment welfare of both agents rose as σ increased, here the welfare of workers rises modestly, but the value function of a new firm owner does not. From one perspective this is a little surprising. Since agents have linear preferences, they do not care about risk or uncertainty, so the reduction in welfare cannot come from this feature. The fall in welfare is coming instead from the reduction in the growth rate. The growth rate is falling because of the reduction in innovation (x), which can only be because there is a fall in the relative payoffs $(V - W)$. The first reason is the change in payoffs shown in figure 10C.

This fall in welfare that results from an increase in exogenous volatility is surprising for another reason. In many models, such as models involving sequential search, agents derive utility from a selectively truncated portion of the distribution.[29] Frequently in these models, an increase in volatility, or a mean-preserving spread of the distribution, results in increased utility or value because it raises the value in the truncated portion of the distribution. This is not necessarily the case here, since welfare can fall as a result of this increase in σ_s. The reason is that the increase in σ_s results in more "high-productivity" firms, which raises the

[28] It may seem paradoxical that with an increase in the variance of firm-level of productivity, would lead to an increase in the average lifetime of a firm, as is shown in figure 10B. After all, increased variance of these shocks should lead to more movement of firms within the distribution, and therefore should result in greater exit. However, the key to understanding lies in equation (34). Here, it can be seen that the increased variance lowers the value of $\underline{s}$, but this is outweighed by the reduction in the growth rate, in the denominator.

[29] For example, in models in which a potential worker is searching for a job, it may be optimal to employ a reservation wage strategy. In this case, the agent's welfare will be determined only by the portion of the distribution that lies above the reservation wage - a truncated portion of the distribution.

equilibrium wage. This in turn *raises the costs to all firms*, which lowers profit to all firms. The increase in σ_s might seem beneficial to a single firm, but it has adverse consequences when it happens to all firms.

10.4 Optimal Degree of Destruction

It is natural in models with an intertemporal externality for innovation, that the equilibrium will display too little innovation. If, as is frequently the case in the existing growth literature, innovation is inexorably linked to business exit (or destruction), then it will also be the case that there will be too little business destruction. However, this linkage is not present in this model, as there are autonomous innovation and exit decisions. It is then possible to inquire as to whether there is too much, or too little business exit, or destruction alone. In other words, it is possible to influence the business destruction margin (or decision) without also influencing the innovation margin.

It is of interest to understand how various policies might influence the economic outcomes in such a model. It is natural to inquire as to the effect of having the government introduce a cost of terminating the business. Suppose that the government introduces such a shutdown cost, and uses the revenue to fund a lump-sum tax on all individuals. Since, at this point, there is no reason to believe that such a policy would be welfare-improving, we will also consider the possibility that this shutdown cost, and transfers, should be negative as well.

Figures 11A and 12A show the impact of such a policy. Introducing such a termination cost lowers growth. Essentially, this reduces the incentive to engage in innovation. Introducing a cost of closing a business is essentially a delayed tax or cost of *starting* a business. As shown in figure 11C, this cost results in *more* firms and fewer workers. Furthermore, this will result in the marginal firm having lower productivity (i.e. lower $\underline{s}$).

Figure 12A shows that the welfare of both workers and firm owners is marginally decreasing in this cost. Not only is it wrong to tax the closure of a business, but this figure shows that instead this closure should be *subsidized*! Essentially what is happening is that in the benchmark model there is too little destruction, or at least firms are shutting down too late in the benchmark model. Subsidizing this closure would help resolve this problem. Another way of viewing this issue is to recognize that subsidizing the shutdown of the business has several effects. First, this results in fewer firms, which lowers wages and thereby lowers the cost of business to all firms. But secondly, subsidizing the shutdown of a firm is actually a delayed subsidy to starting a firm, since you cannot do the former without the latter. But this subsidy raises the incentive to innovation as well.

Figure 12B shows that introducing this cost will result in the average age of a firm will rise. Again, to raise welfare it would be desirable to reduce this average age. Figure 10A and 10B together suggest that welfare is inversely related to average firm age, which suggests that welfare is positively related to the rates of firm entry and exit.

Figure 12C shows that introducing a shutdown cost raises wages very, marginally and so, on net, the welfare of workers is reduced because the growth rate falls so much.

In summary, this analysis suggests that it may be welfare-improving to tax workers and to use the proceeds to pay marginal firms to cease operating. It would be exciting to witness what fate awaits any politician who suggests such a policy.

11 Implications for Asset Pricing

11.1 Observations

It has been recognized for some time that the behavior of rates of return can be different for large and small firms, and this has become known as the *size puzzle*. For example, Banz [6] and Reinganum [38] document that average rates of return seem to differ based firm size. While a full, detailed study of this observation is beyond the scope of this paper, the model employed here can yield some insights into this issue. Furthermore, there is a related puzzle that is also of interest.[30] İmrohoroğlu and Tüzel [26] describe how the *variability* of rates of return differs for firms of different sizes. Rates of return are much more variable for smaller firms than for larger ones. Bloom et al. [7] also document that TFP shocks seem to be reflected in stock returns, but they do not explore the more subtle issues studied below.

This raises other important economic questions. Since rates of return would naturally seem to be linked to the productivity (or technology) of the firm, this observation suggests that the volatility or productivity of smaller firms should be much greater than for larger firms. However, there does not seem to be any evidence that this is the case, or that it should be the case. So what is the mechanism that makes these returns depend on firm size?

As will be seen the model of this paper will produce asset price or return volatility that is inversely related to firm size or productivity, even when firms are subject to the exact same distribution of shocks.

11.2 Asset Pricing Implications of the Model

This model has some important implications for asset pricing. The owner of a firm in such an economy can easily have the value of the firm priced, since such a value should just reflect the expected discounted value of dividends.

In similar models with linear preferences, for a firm with relative technology index "s_t" and when the technology of new entrants is Z_t, the formula for calculating the discounted value of dividends is then

$$Q(s_t)\,Z_t = E_t\left[\int_t^\infty Z_t e^{-r(v-t)} A_\pi\left(e^{s_v}\right) dv\right]. \tag{38}$$

In the event that the firm is expected to operate forever, this can be re-written as

$$Q(s_t) = \frac{A_\pi\left(e^{s_t}\right)}{(r-\kappa) - \kappa_s - \left(\frac{\sigma_s^2}{2}\right)}. \tag{39}$$

Such an expression has the following property for the semi-elasticity:

$$\frac{\partial \log\left(Q(s)\right)}{\partial s} = 1. \tag{40}$$

[30] In particular, since the barrier ($\underline{s}$) is time-invariant, and the preferences of asset holders are assumed to be linear, it is not possible to produce differences in expected returns for firms with different productivity. However, additional research has shown that if either of these two assumptions is jettisoned, then it is possible to produce rates of return that depend on the productivity of the firm.

This is an unfortunate property because since productivity (or profit) is relatively smooth, the volatility of prices will not be anywhere close to as large as it is in the data. Even if firms were to expire randomly (with some constant, time-invariant probability), equation (40) would still hold. In other words, having firms exit does not, in and of itself, does not change this last property.

However, in this model economy studied here firms do not die randomly. They cease operating because of low productivity. Hence in the model, the proper asset pricing equation is not (38), but instead is

$$P(s_t) Z_t = E_t \left[\int_t^T Z_t e^{-r(v-t)} A_\pi (e^{s_v}) \, dv \right]. \tag{41}$$

Furthermore, not only is T a random variable, but it is also *positively correlated with* s_t. That is, firms that are highly productive today (high s_t) are likely to operate longer (T) than firms with lower productivity. As will be seen, this positive relationship between T and s_t will add a significant amount of volatility to this price.

It is possible to re-write this last equation in the following manner[31]

$$P(s) = B_1 e^s \left[1 - e^{(\underline{s}-s)(1-\beta)} \right], \tag{42}$$

where

$$B_1 = \frac{A_\pi}{(r-\kappa) - \kappa_s - \left(\frac{\sigma_s^2}{2}\right)} \tag{43}$$

and

$$\beta = \frac{-\kappa_s}{\sigma_s^2} - \sqrt{\left(\frac{\kappa_s}{\sigma_s^2}\right)^2 + \frac{2(r-\kappa)}{\sigma_s^2}} < 0. \tag{44}$$

Note that this pricing formula has the property that

$$P'(s) > 0 = P(\underline{s})$$

Whether $P''(s) \lesseqgtr 0$ depends upon the size of β, as well as relative productivity s.

An alternative way to write equation (42) is as follows:

$$P(s_t) = Q(s_t) - \left(e^{(s_t-\underline{s})\beta} \right) Q(\underline{s}) \tag{45}$$

The interpretation of this is as follows. The whole expression is the value of a firm with current technology indexed by s, *contingent on the firm operating optimally* (i.e. shutting down when $s = \underline{s}$) The first term on the right side of equation (45) is the expected discounted value of the infinite stream of dividends (i.e. from operating the firm forever). However, the firm will *not* operate forever, and instead, it will shut-down when its technology index reaches $\underline{s}$. Hence, the second term on the right side of equation (45) reflects the "loss of dividends" when the firm is shut down. The value of these lost dividends, at the shutdown date, will be $Q(\underline{s})$. This date is uncertain, as it is the (random) time it will take s to reach

[31] This equation can be derived using equation (37) to integrate over the values of T.

$\underline{s}$, which will occur with probability one.[32] Lastly, the term $\left(e^{(s_t-\underline{s})\beta}\right)$ is the expected value of the (random) discount factor between the current date, and the reaching the absorbing barrier.[33]

Another way of viewing this pricing function is to note that in equation (42) the expression $B_1 e^s$ is the value of the current asset if it were to continue to pay off, or exist forever. However the term

$$\left[1 - e^{(s-\underline{s})(\beta-1)}\right] \tag{46}$$

has the form of, and indeed is, a *survival function.* This function expresses the discounted value, multiplied by the probability of future mortality of the asset, given its current payoff of q_t. If this survival function is close to unity, then it must be that the asset is expected to have a long horizon.

One very important feature of this analysis is that like most such models, the optimization conditions (16 and 17) that give rise to the asset pricing equation are "locally linear", the ultimate solution for the price (42 or 45) have important non-linearities.

It is then of interest to compare equations (39) and (42). The former satisfies the following

$$\frac{dQ(s)}{Q(s)} = \left(\kappa_s + \frac{\sigma_s^2}{2}\right) dt + \sigma_s dW. \tag{47}$$

However, the counterpart expression for equation (42) is the following

$$\begin{aligned} \frac{dP(s)}{P(s)} &= \frac{\left[\left(1-\beta\left(e^{(s-\underline{s})(\beta-1)}\right)\right)\kappa_s + \left(1-\beta^2\left(e^{(s-\underline{s})(\beta-1)}\right)\right)\left(\frac{\sigma_s^2}{2}\right)\right]}{\left[1-e^{(s-\underline{s})(\beta-1)}\right]} dt \\ &\quad + \left[\frac{1-\beta\left(e^{(s-\underline{s})(\beta-1)}\right)}{1-e^{(s-\underline{s})(\beta-1)}}\right]\sigma_s dW. \end{aligned} \tag{48}$$

Clearly, the expression in equation (48) is a non-standard Brownian motion since *both* the instantaneous *mean* and *variance* depend on the value of s. In contrast, the instantaneous mean and variance in equation (47) are constant. We frequently think that the forecastable or predictable change in asset prices should be zero, or at least a constant. But this expression shows that the expected price change is a function of the firm's level of technology. In fact, this change in price, as well as its standard deviation, can be arbitrarily large. This is true even though the standard deviation of the underlying shocks is, in fact, constant. In other words, a finite variance of the fundamental technology shocks can translate into arbitrarily large shocks of the price of the asset.

Furthermore, equation (48) has the property that the trend or growth term satisfies the following:

$$\lim_{s\to\infty} E\left[\frac{dP(s)}{P(s)}\right] = \kappa_s + \frac{\sigma_s^2}{2},$$

[32] An observer who witnessed a firm whose value was priced according to equation (39), but who did not take into consideration that the firm would shut down at a future date when $s = \underline{s}$, might conclude that the value of the firm was selling at a discount, relative to $B_1(e^s)$. Alternatively, the observer might think that the price of the firm contained a bubble component which was *negative.*

[33] This expression can also be derived using equation (37).

which is independent of productivity. Similarly, as $s \to \infty$, the instantaneous standard deviation in equation (48) is σ_s. Hence, only for firms with very high productivity does equation (48) approximate equation (47).

On the other hand, for firms with low levels of productivity

$$\lim_{s \to \underline{s}} E\left[\frac{dP(s)}{P(s)}\right] = -\infty.$$

Similarly, as $(s - \underline{s}) \to \infty$, the instantaneous standard deviation in equation (48) approaches $+\infty$, so the firm's size or productivity certainly affects the volatility of returns in a non-linear manner.

One way to view this issue is by referring back to equation (45). In this equation, if the second term were absent (because $\underline{s} = -\infty$) then a 1% change in productivity would translate into a change in the price of the same magnitude. But the second term in equation (45) introduces an additional complication. Obviously the elasticity of this term with respect to "s" is β. If β is significantly different from zero, prices can exhibit a response to a change in productivity that is much greater than a unitary elasticity. The reason is that an increase in productivity does not just signal higher future productivity (which is reflected in the first term of equation (39)), but it also indicates that the firm is going to be operational longer! The larger is β, the bigger this response is likely to be. This emphasizes the importance of the value of β.

There is yet another avenue through which this feature is important, and has to do with the effect of a change in the drift of the asset (κ_1 or κ_s). In a typical model with long-lived assets, a change in the drift parameter can have a pronounced impact on the price, because it operates through changing B_1 in equation (43). However, in this model, this growth rate also influences the resulting value of β. This can further magnify the effect of a change in the drift parameter on the price of the asset, for lower levels of productivity $(s - \underline{s})$. The reason for this is that a change in the growth rate can have a substantial impact on the *expected lifetime* of the firm, and thereby alter its value immensely. This effect becomes arbitrarily large as $(s - \underline{s}) \searrow 0$.

This feature is further illustrated in figure 13 which shows the expected value equation (48) in the vertical axis, and the relative value of the productivity index $(s - \underline{s})$ on the horizontal axis). To be clear, the vertical axis ranges from -800% up to zero. This figure shows in rather stark terms that as the productivity of the firm falls, the expected change in the price of that asset is expected to fall in rather dramatic terms. The level of productivity of the firm should have strong predictive power *at the low end of the productivity distribution*, but not as much at the high end of the distribution. As the figure shows, raising the standard deviation of productivity shocks (σ) raises the expected value of the price change, since there is now a greater likelihood of a positive shock.[34]

It should be noted that in this environment in which agents are risk-neutral, all assets will have the same expected rate of return. The other portion of the rate of return is the dividend price ratio, which is given by

$$\frac{A_\pi e^s}{P(s)}. \tag{49}$$

[34] The figure is suggestive that this process is analogous to that of a vortex from which it is very difficult to escape.

The fact that firms with relatively low productivity have large negative capital gains is offset by the fact that the dividend-price ratio is extremely high for these firms, and this ratio approaches $+\infty$ as $s \searrow \underline{s}$.

Next, consider the standard deviation term in equation (48). It is straightforward to see the following:

$$\lim_{s \to \underline{s}} \left[\frac{1 - \beta \left(e^{(s-\underline{s})(\beta-1)} \right)}{1 - e^{(s-\underline{s})(\beta-1)}} \right] = \frac{1-\beta}{0} = +\infty, \tag{50}$$

and hence the variance of the price change $\to +\infty$, as productivity falls. This implies that the volatility of asset returns should be significantly greater for firms or assets with low levels of productivity.[35] On the other hand,

$$\lim_{s \to \infty} \left[\frac{1 - \beta \left(e^{(s-\underline{s})(\beta-1)} \right)}{1 - e^{(s-\underline{s})(\beta-1)}} \right] = 1. \tag{51}$$

Once again, none of this has to do with risk-aversion. These results do not rely on a changing variance of the underlying shocks, since the variance (σ) is assumed to be constant. This shows that even if the productivity shocks have the same properties across all firms, this translates into a highly non-linear statistical properties for the asset prices of the firms.

The term in square brackets in the last two expressions is the derivative of the logarithm of the survival function in equation (46), and as such is termed a *hazard function*. This hazard function appears in the standard deviation of the price change, present in equation (48), and re-written here:

$$\left[\frac{1 - \beta \left(e^{(s-\underline{s})(\beta-1)} \right)}{1 - e^{(s-\underline{s})(\beta-1)}} \right] \sigma_s.$$

This expression highlights the fact that two separate features that contribute to the variability of asset returns in this environment. First, there is the standard uncertainty regarding future payoffs, which is captured by σ_s. But also, there is the mortality risk that is captured by the productivity-dependent hazard function, in the expression in square brackets.

This is further illustrated in figure 14 where the standard deviation term in equation (48) is plotted as a function of the level of relative productivity. The vertical axis ranges from zero to 2500%. As is shown, firms with low productivity levels should have *extremely* volatile asset prices. An increase in the value of σ raises this volatility, although it should be noted that this parameter also operates through a change in β as well.

One might question this analysis by wondering if this high volatility is present only in a small sliver of the distribution of firms, and therefore of not much consequence. After all, it is important to be sure how volatile the asset price of the typical or median firm will be.

The answer to this is illustrated in Table 1, which shows the expected growth term, given in equation (48), for different values of $(s - \underline{s})$, which is how far productivity is above the shutdown threshold, or absorbing barrier. The first column indicates where the firm's productivity is in the distribution of firms, measured from the least to the most productive. The second column indicates how much above the shutdown (or absorbing) barrier, this

[35] Zhang [40] also studies a model in which the volatility of returns can depend on firm size. However, his model is much more complicated than that studied here. Among other things, his technology employs capital accumulation along with asymmetric adjustment costs.

firm's productivity is. For example, the first row is the productivity of a firm that is greater than only 1% of other existing firms, and less than 99% of such firms. At the same time, this same firm's productivity is only 3.2% above the shutdown threshold at that moment. The third column indicates the expected instantaneous change in the price change for that firm. The last column measures the instantaneous standard deviation for the price change for this firm.

There are several things to note from this comparison. For both tables, the expected price change is substantially less than zero, but the standard deviation of the price changes is remarkably large. As can be seen, the expected change is negatively and exponentially influenced by how close the productivity is to the exit threshold. This negative growth can be quite substantial, even for firms that have higher productivity than the median firm.

Table 1

$\sigma = .05$

Percentage	$(s - \underline{s})$	$100 \cdot E\left(\frac{dP}{P}\right)$	Standard Deviation of Price Change
1%	3.2%	-249%	303%
2%	5.1%	-192%	210%
5%	9.1%	-73%	127%
10%	14.3%	-48%	85%
15%	18.8%	-38%	67%
20%	23%	-39%	56%
50%	45.8%	-17%	31%
100%	-	-6.7%	12.5%

Table 2 shows the same statistics for an identical economy in which the σ is twice as large as for the previous table. In visual terms, this results in a wider variation in the distribution of firm productivity. For a fixed point in the productivity distribution of firms, the productivity level, relative to the shutdown barrier, is higher in Table 2. However, at these same points, the expected instantaneous percentage change in the price is lower in Table 2. The reason for this is that when the standard deviation of productivity shocks (σ) is relatively high, there is a greater likelihood that the firm's productivity will recover and the firm will become highly productive in the future (see equation (35)). Therefore, there is a higher probability that the price will rise as well. However, as the tables show, for a higher value of (σ), there is then a higher standard deviation of the price change. It makes sense that as volatility of the productivity shocks increase, the volatility of the price changes should also increase.

Table 2

$\sigma = .10$

Percentage	$(s - \underline{s})$	$100 \cdot E\left(\frac{dP}{P}\right)$	Standard Deviation of Price Change
1%	7.0%	-70%	321%
2%	10.4%	-49%	227%
5%	17.5%	-30%	142%
10%	26%	-21%	100%
15%	32.9%	-17%	82%
20%	39%	-14%	71%
50%	68.2%	-9%	46%
100%	-	-4.1%	25%

Both of these tables show that for even the median firm in the distribution(s) the mean and standard deviation of the price change is very different from that of high-productivity firms. Therefore the effect of ultimate exit on asset prices, both the mean and volatility, is not something that can be dismissed as something that would only show up as influencing a small slice of the distribution of firms.

Tables 3 and 4 allow for a different comparison across economies, *relative to new entrants*, with different values of the standard deviation of the productivity shock (σ). The first column in both tables indicates where the firm lies in the distribution of productivities. In these cases we consider 3 firms: i) one that has productivity identical to that of a new firm ($s = 0$), ii) one that has a productivity 50% below that of a new firm, iii) one that has a productivity of 50% above that of a new firm.

Table 3

$\sigma = .05$

$100 \cdot (s)$	$100 \cdot E\left(\frac{dP}{P}\right)$	Standard Deviation of Price Change
-50%	-32%	58%
0	-12.1%	22%
50%	-8.7%	16%

As can be seen, fixing productivity relative to that of new entrants, a higher value of (σ), results in a higher expected fall in the price of the asset. Once again, with a higher value of (σ) there is a greater likelihood that the firm's productivity will recover and the firm will become highly productive in the future. However, a higher value of (σ) results in a higher variation of this price change as well.

Table 4

$\sigma = .10$

$100 \cdot (s)$	$100 \cdot E\left(\frac{dP}{P}\right)$	Standard Deviation of Price Change
-50%	-29%	137%
0	-8.7%	46%
50%	-5.8%	33%

It should be re-emphasized that in this environment with risk-neutral agents, all assets will have the same expected return. The fact that the expected price change in these tables is negative only means that this portion of the return is offset by a higher dividend-price ratio (equation (49)). Nevertheless, there will certainly be higher volatility of the returns to assets associated with low-productivity firms, even though the stochastic process of the productivity shocks is the same for all firms.

Finally, the following should be noted. Consider an outside observer who interprets the movements in asset prices through the lens of a traditional asset pricing formula, such as equation (41) with $T = \infty$, or equation (42) with $\underline{s} = -\infty$. This observer would look at the (excess) volatility of the assets of the low-productivity firms and conclude that they were much too volatile to be consistent with the underlying fundamentals. The observer might even conclude that the asset-holders in this economy were being irrational, or at least reacting in much too sensitive a manner to changes in the productivity of the firm. Of course, consumers in this environment are being perfectly rational, and the volatility of prices is wholly justified, once one understands the nature of the underlying risks.

11.2.1 The Effect on Asset Prices of a Change in σ_s

The analysis of section (10) focuses on how a change in σ_s would influence economic outcomes. It is then of interest to see how this would influence the prices of assets in this environment. There would seem to be four effects of this change. The first thing to note is that, within this environment, this increased variance contributes to the upward trend in productivity growth, and thereby raises asset prices. This can be seen in raising the value in equation (43). This effect will raise the asset price.

Secondly, note that from equation (44) an increase in σ_s raises the value of β (or makes it closer to zero, since it is negative). Equation (45) then can be used to show that $\partial P(s)/\partial\beta < 0$, and so the increase in σ_s will lower the asset price.

But there is then a third effect as well. This comes from the fact that, as was shown in section (10), an increase in σ_s will raise the value of the threshold value $\underline{s}$. This will then raise the value of $Q(\underline{s})$ and lower the value of the price $P(s)$ in equation (45). Once again, this is truly where the creative destruction feature has some traction. This change makes firms cease operations with a higher level of relative productivity ($\underline{s}$), which *lowers* the discounted value of the firm's payoffs.

Lastly, this increase in σ_s would lower the value of the survival function in equation (46) because this would shorten the horizon over which future payoffs (or dividends) are calculated for all assets. This would lower the value of the asset price.

It would then seem that an increase in σ_s may raise the price or value of high productivity assets, but lower the value of low productivity assets.

Note that in a similar model without the creative destruction feature, only the first effect would be present and so an increase in σ_s would raise the asset price. The last three features are present only because it is expected that assets are mortal.

11.2.2 Correlations with Consumption

In models where agents have linear, or risk-neutral preferences, it is universally the case that a change in aggregate consumption by itself should not affect asset prices, since prices are determined by the discounted expected future payoffs, discounted at a *constant* rate. However, in the model presented above this is not the case. To see this, use equations (42), together with the substitution of $e^s = (z_t/Z_t)$ to re-write the asset pricing equation (41) in the following manner.

$$P(s_t) Z_t = B_1 z_t - B_1 (z_t)^{\beta} (Z_t)^{1-\beta} e^{\underline{s}(1-\beta)}. \tag{52}$$

This is the price at date t denominated in units of consumption at that date. Once again, the first term on the right side is the value of the payoffs of the firm, if it were to *operate forever*. The second term is the expected loss of value due to the exit of the firm, conditional on it terminating operations optimally (i.e. when $s_t = \underline{s}$). Note that in this model aggregate consumption is proportional to Z_t. Next, consider a thought experiment in which, over a short period the value of z_t is roughly constant, but the value of Z_t increases. That is the payoff of the asset is unchanged, but consumption rises. Since $\beta < 0$, the price of the asset in equation (52) would fall. The reason is that the growth in consumption (or technology) has moved the technology of this firm a little closer to the exit barrier, and so lowered the value of the firm. Hence, this could translate into a negative correlation between asset prices and aggregate consumption, even though the discount factor was constant.

Additionally, the increase in aggregate consumption (through the increase in Z_t) would also result in a lower return on existing assets in the economy. Through this channel, there would then be a negative relationship between changes in consumption and asset returns, in an environment where agent's preferences were linear.

To the outside observer, who was not aware of the preferences of individuals, they might observe the contemporaneous rise in consumption and fall in asset price, and conclude that the rise in consumption has caused the (endogenous) interest rate to have risen. Of course, the discount rate is constant. The creative destruction feature of the economy can make it seem like interest rates may be changing when they are not.

A more detailed analysis of how the features of this model, and how the correlations of asset payoffs with other variables like consumption can in turn affect asset returns is conducted in Huffman [24]. This paper employs risk-averse agents, so the formulae for asset prices and returns are much more sophisticated than those derived here.

11.2.3 Diversification

A natural question to ask here is if it is possible to diversify this risk. If one pooled together the assets with similar levels of productivity (s), then one could diversify away the risk associated with changing levels of s, which is to say that this would reduce σ_s. However, this would not reduce the risk associated with reaching $\underline{s}$, because this would be common for all of those assets. Therefore, there really are two separate types of risk here. One is easier to diversify away than is the other.

One might consider how a hypothetical risk-averse might seek to hold an optimally-diversified portfolio of assets in this environment. If, as in the benchmark model, the firms all have the same, independent productivity parameters, then a well-diversified portfolio is likely to involve reducing the holdings of assets that have low or falling productivity, since these returns are likely to have higher risk.

11.2.4 Extensions

One might quibble with the fact that in this primitive model, equation (45) implies that all firms with the same relative productivity (s) will have the same level of employment and output, the same threshold productivity ($\underline{s}$), and hence the same distribution of exit times. This does not seem consistent with the fact that occasionally large firms can be in a financially precarious position. Fortunately, there is a simple modification of the model that can accommodate this observation. To rectify this, consider instead of using the production function in equation (2), that we use the following production function

$$z\left(n_t^{\alpha}\right) - z\chi.$$

Here χ is some continuous fixed cost (but it could be negative as well). Think of this as some operating cost. Since this is additively separable from the production function, this parameter will not affect the level of employment. However, this will certainly affect the profit and will affect and appear in the value matching condition which determines the exit decision. Therefore, it will be the case that the threshold level of productivity ($\underline{s}$) will be influenced by the size of χ. Now suppose that the value of this parameter varies by firm. Then it can be the case that there are some high productivity firms with much higher values of $\underline{s}$ than other low productivity firms. In other words, with this feature, it is possible to have large firms (i.e. high employment) that are much closer to exit than would be some smaller firms. This would complicate the market clearing condition for labor (equation (9)) because we would then have to integrate not just over the values of s or (z_t), but also over χ.

In general, here χ reflects the opportunity cost of keeping a firm operational. From a positive perspective, any feature that raises the marginal cost of operating the firm could be reflected in χ. For example, if a richer model one might imagine firms having various levels of debt, and that a high level of debt can increase the probability of the firm exiting. Hence one might identify debt as one of the features that can influence the value of χ.

Lastly, it was shown in section (10.4) that the equilibrium of such an economy may not display the optimal degree of business destruction. This feature would then naturally show up in asset prices as well. That is, asset prices would not be the same as in a world with optimal allocations. Any policies (such as those illustrated in section (10.4)) that were implemented to change the degree of business destruction to something closer to a socially optimal level would then also influence the asset prices. Actually, it might be possible to implement policies to change the asset prices, and thereby influence the rate of business destruction.

12 Final Remarks

The model studied here is uniquely capable of yielding insights into how changes in the variability of firm-level productivity shocks could influence economic growth, as well as other important economic outcomes. One might initially expect that higher variability in such shocks could raise growth because it would then lead to a greater likelihood of high-productivity shocks. Since low productivity firms are free to exit, this would naturally lead the remaining firms to have higher productivity and perhaps higher growth. However, the "creative destruction" feature of the economy is instrumental in showing how the growth rate could actually fall. Higher productivity shocks raise factor prices which in turn lowers the profitability of all firms, and thereby reduces the incentives for innovation.

In recent decades there has been an apparent increase in the variance of inter-firm productivity. Concomitant with this has been a slowdown in growth and a reduction in the entry of new, small firms. Until now, it was not apparent how all of these phenomena were related. However, the model studied here shows that they can be explained. Perhaps the next step is then to understand why there has been an increase in this variance.

It was shown that the creative-destruction feature also has unique implications for the pricing of assets in this environment. The resulting formula for asset prices embodies a survival function that characterizes the likelihood of the firm ceasing operations in the future. This exit or mortality risk adds another element of uncertainty to asset prices, and this uncertainty can be arbitrarily large for low-productivity firms. As a result, changes in parameters, such as the drift parameter for assets, can have some unique effects on asset prices.

Further research is being conducted into how the properties of assets, such as those studied in this model, are influenced by the introduction of risk aversion. Preliminary work reveals that these features, acting in conjunction with each other produce some unique properties for prices and returns.

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Figure 1

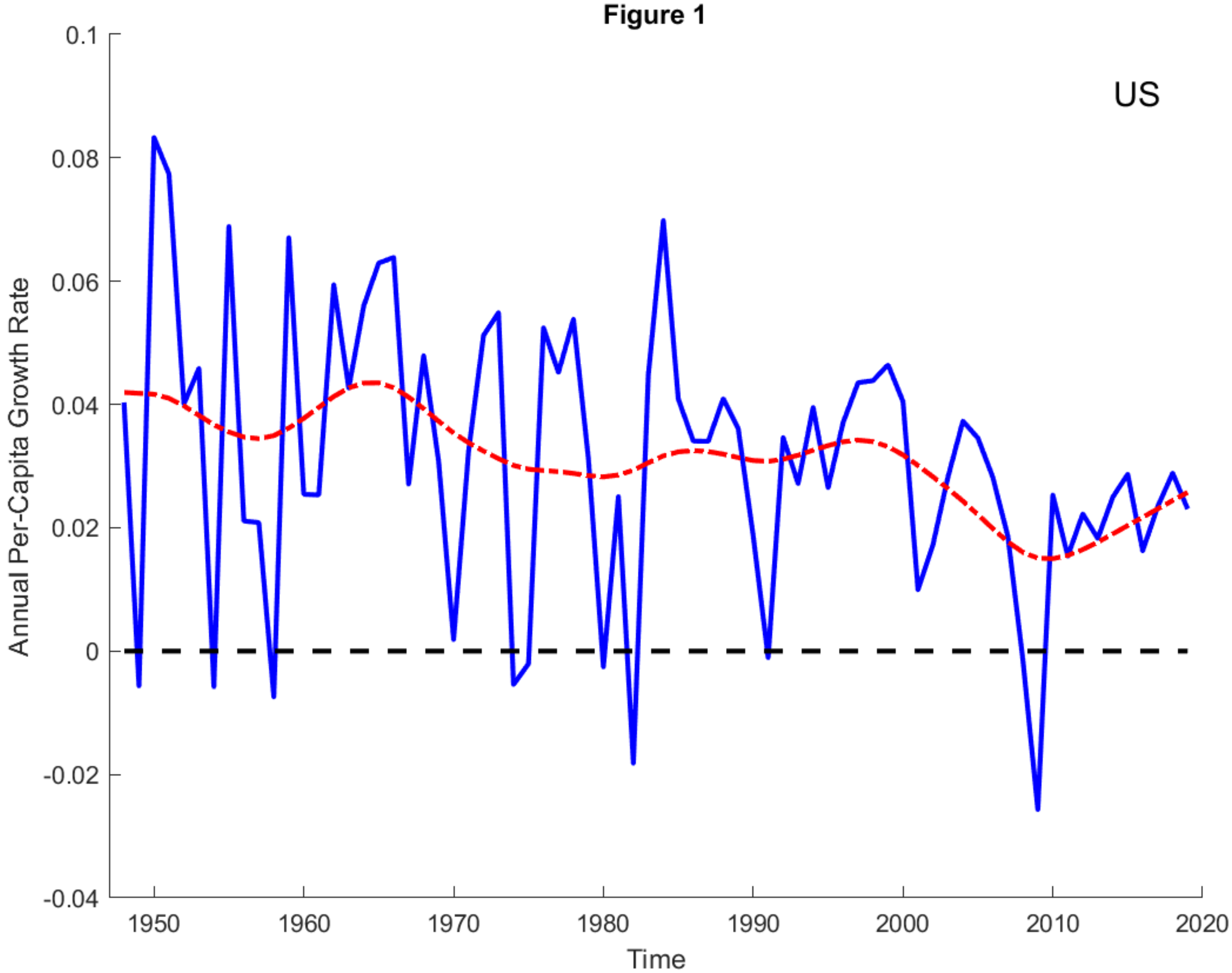
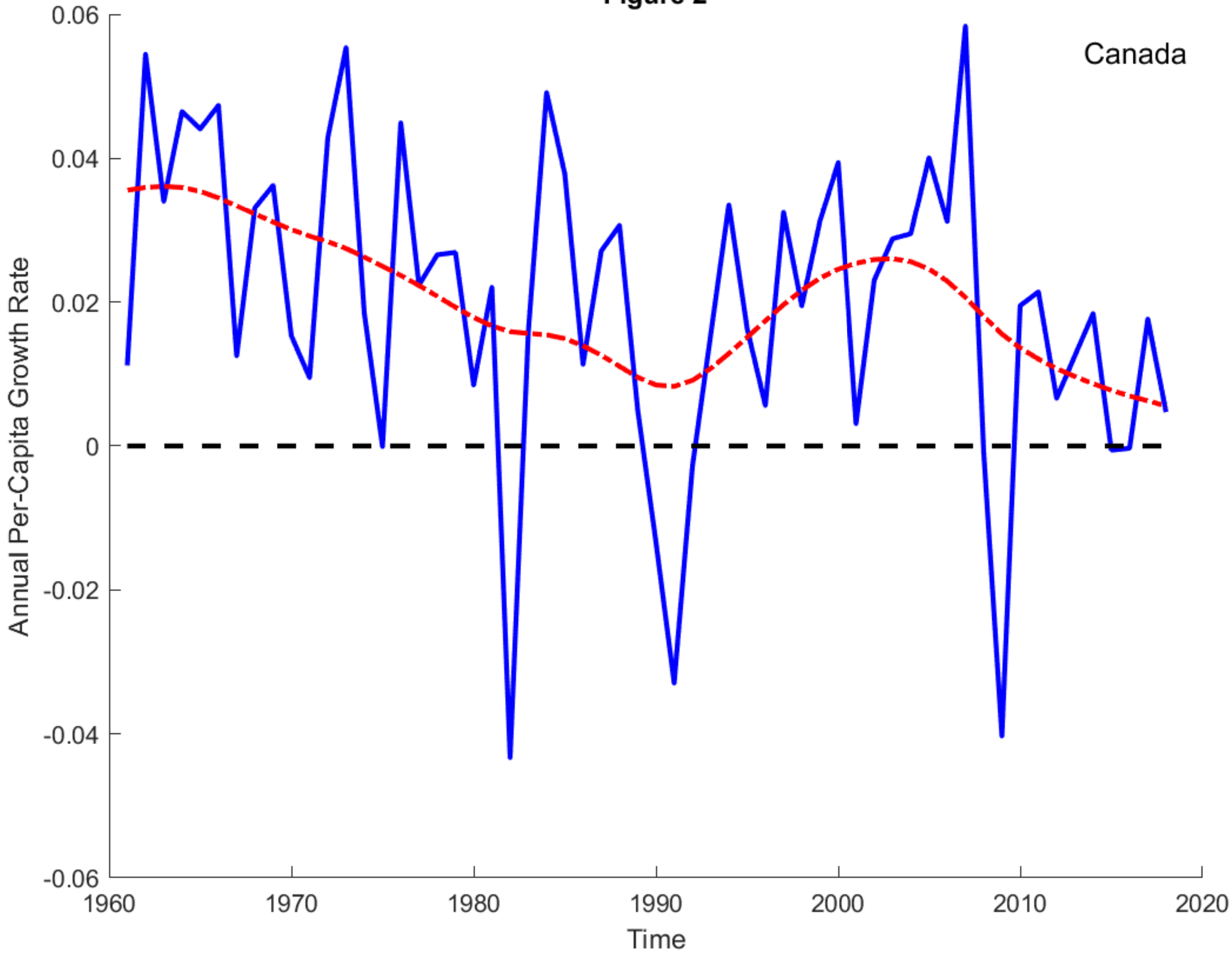

US
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
1950
1960
1970
1980
1990
2000
2010
2020
Time
Annual Per-Capita Growth Rate

Figure 2



Canada
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
Annual Per-Capita Growth Rate
1960
1970
1980
1990
2000
2010
2020
Time

Figure 3

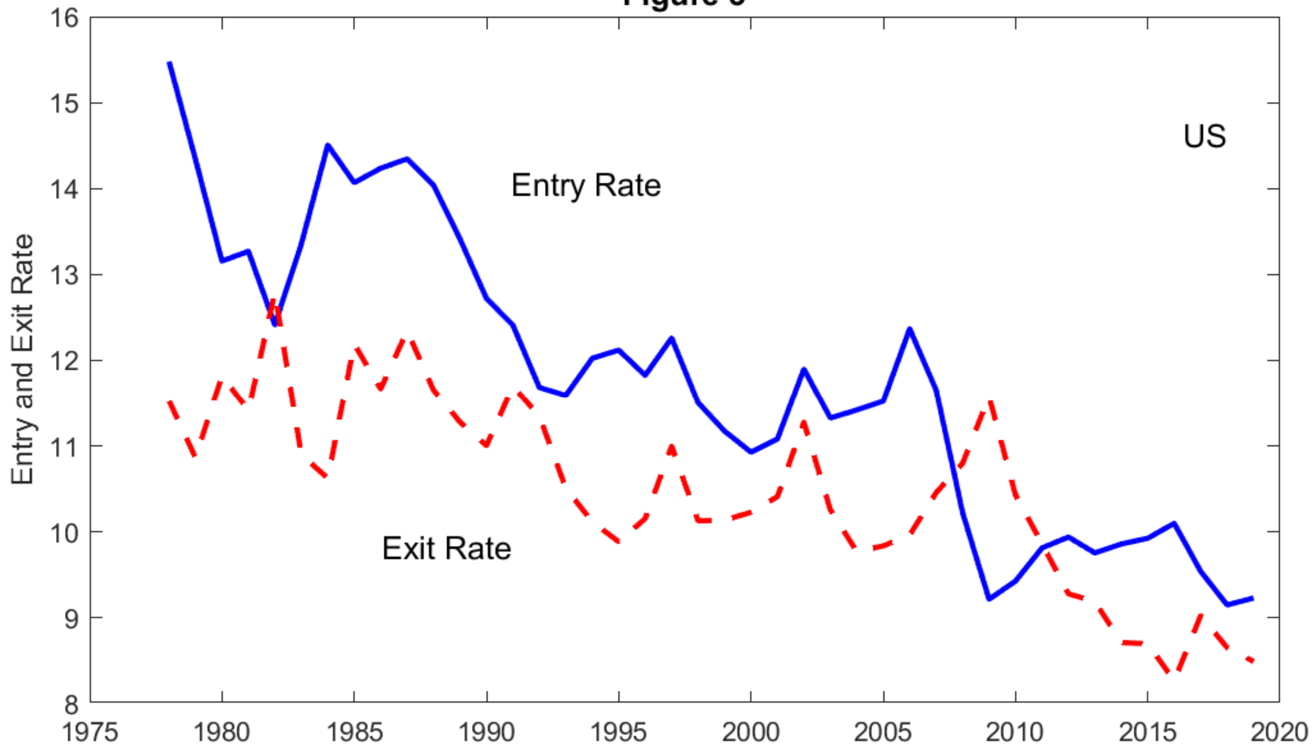
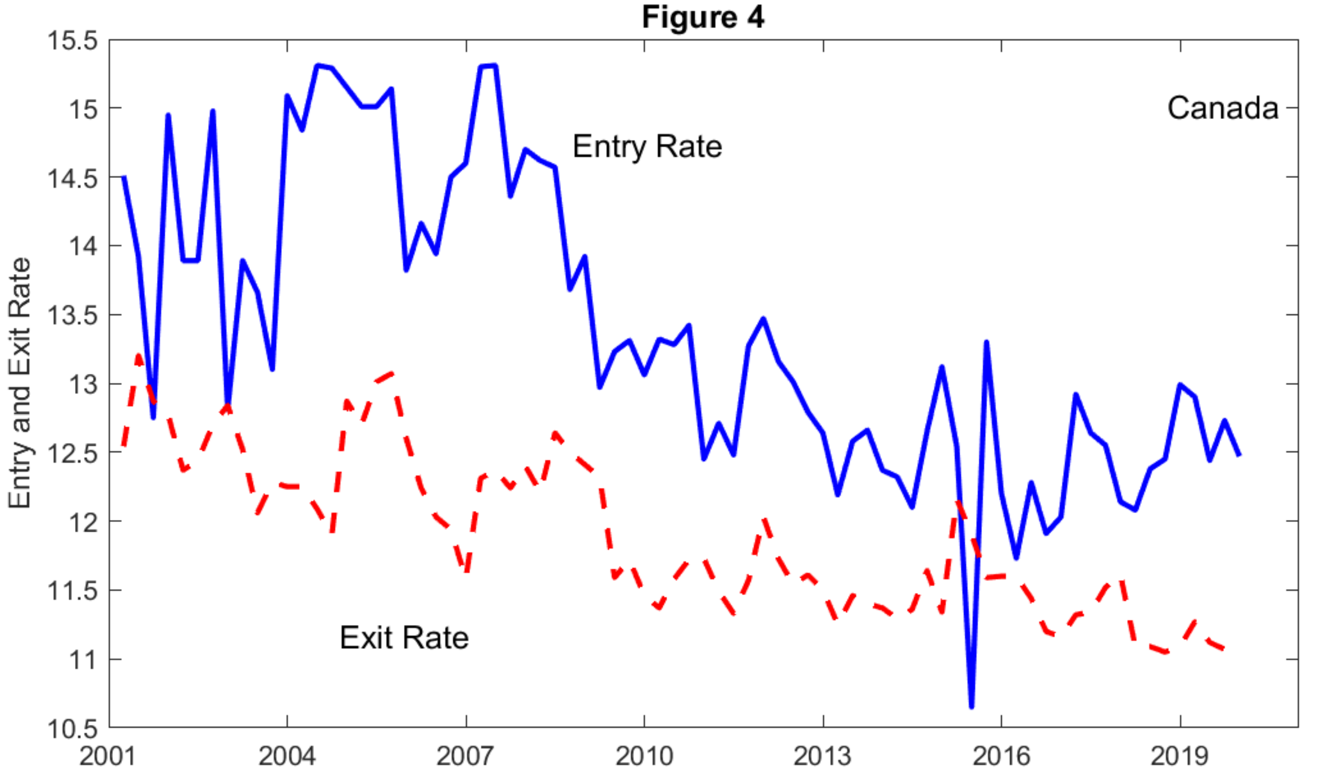


Figure 4



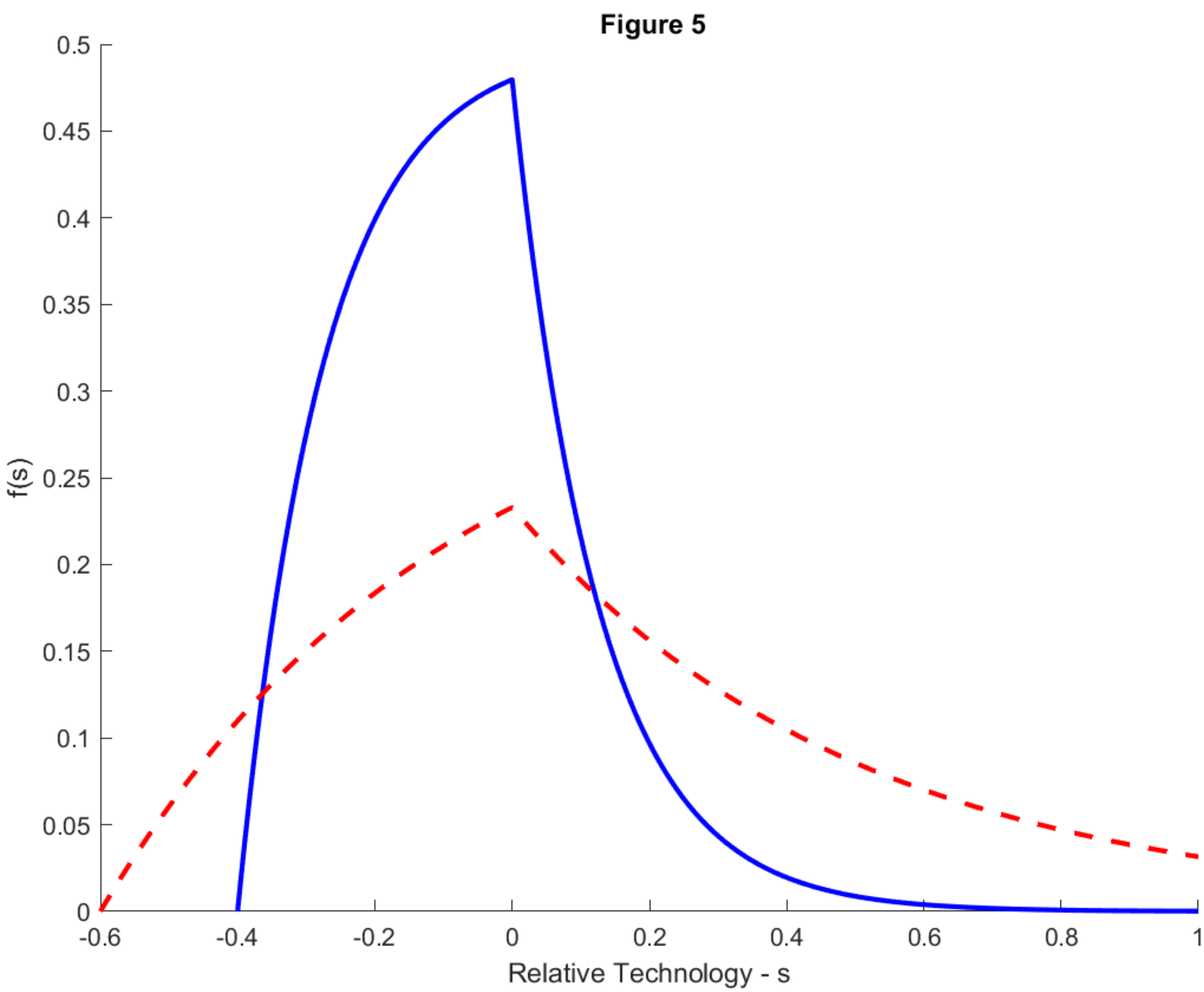
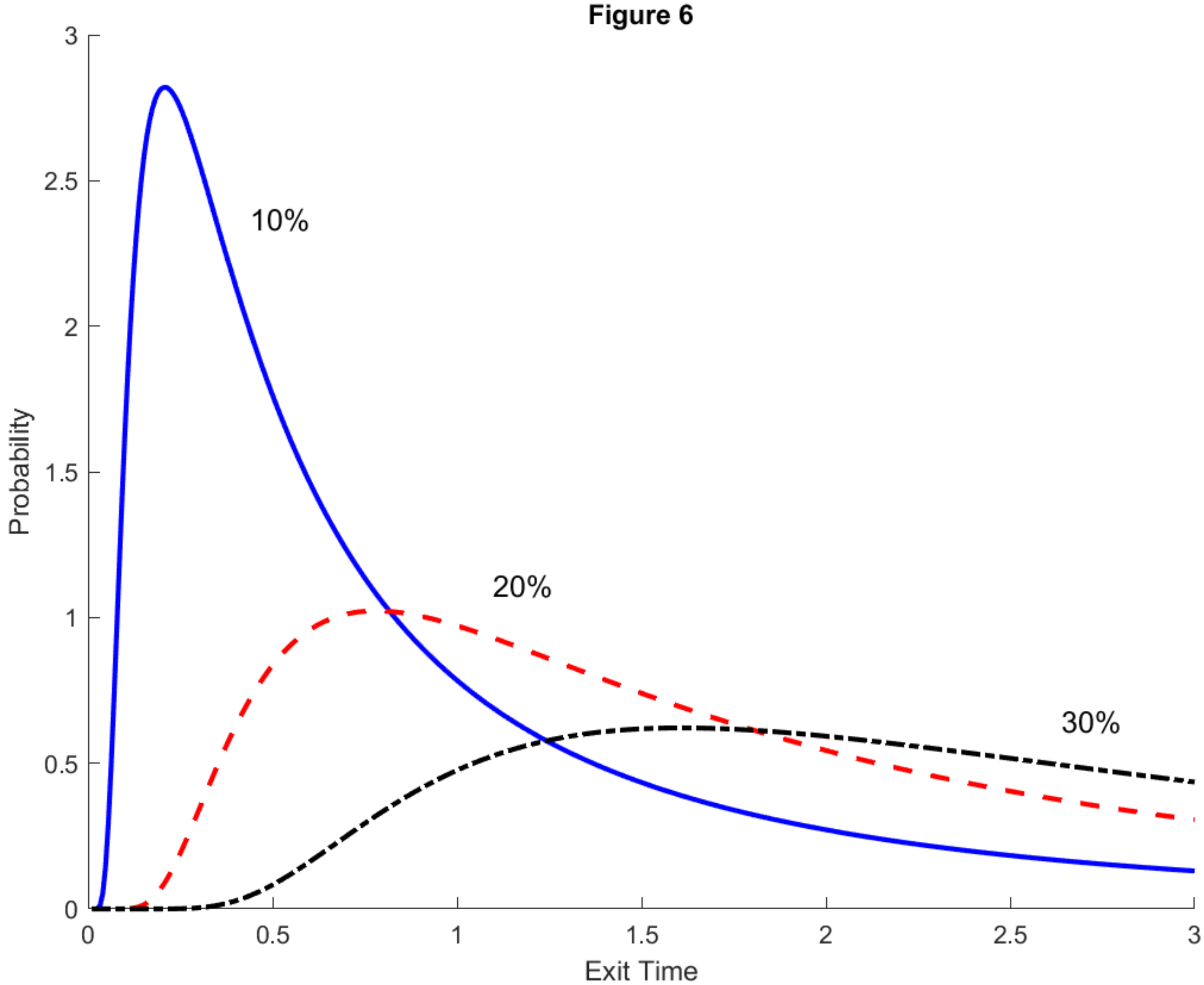


Figure 5
f(s)
Relative Technology - s
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Figure 6



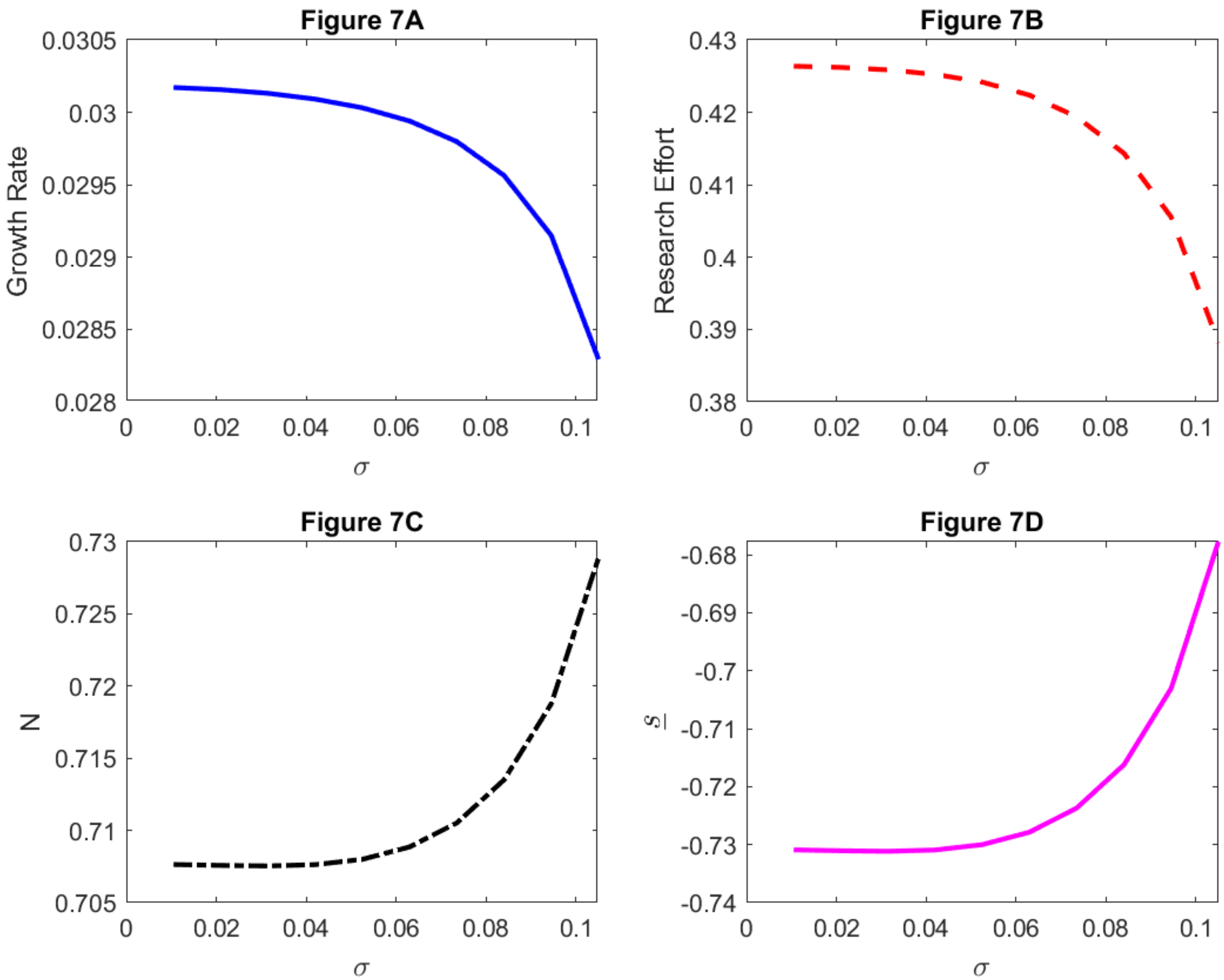


Figure 7A
Growth Rate
0.0305
0.03
0.0295
0.029
0.0285
0.028
0
0.02
0.04
0.06
0.08
0.1
σ
Figure 7B
Research Effort
0.43
0.42
0.41
0.4
0.39
0.38
0
0.02
0.04
0.06
0.08
0.1
σ
Figure 7C
N
0.73
0.725
0.72
0.715
0.71
0.705
0
0.02
0.04
0.06
0.08
0.1
σ
Figure 7D
s
-0.68
-0.69
-0.7
-0.71
-0.72
-0.73
-0.74
0
0.02
0.04
0.06
0.08
0.1
σ

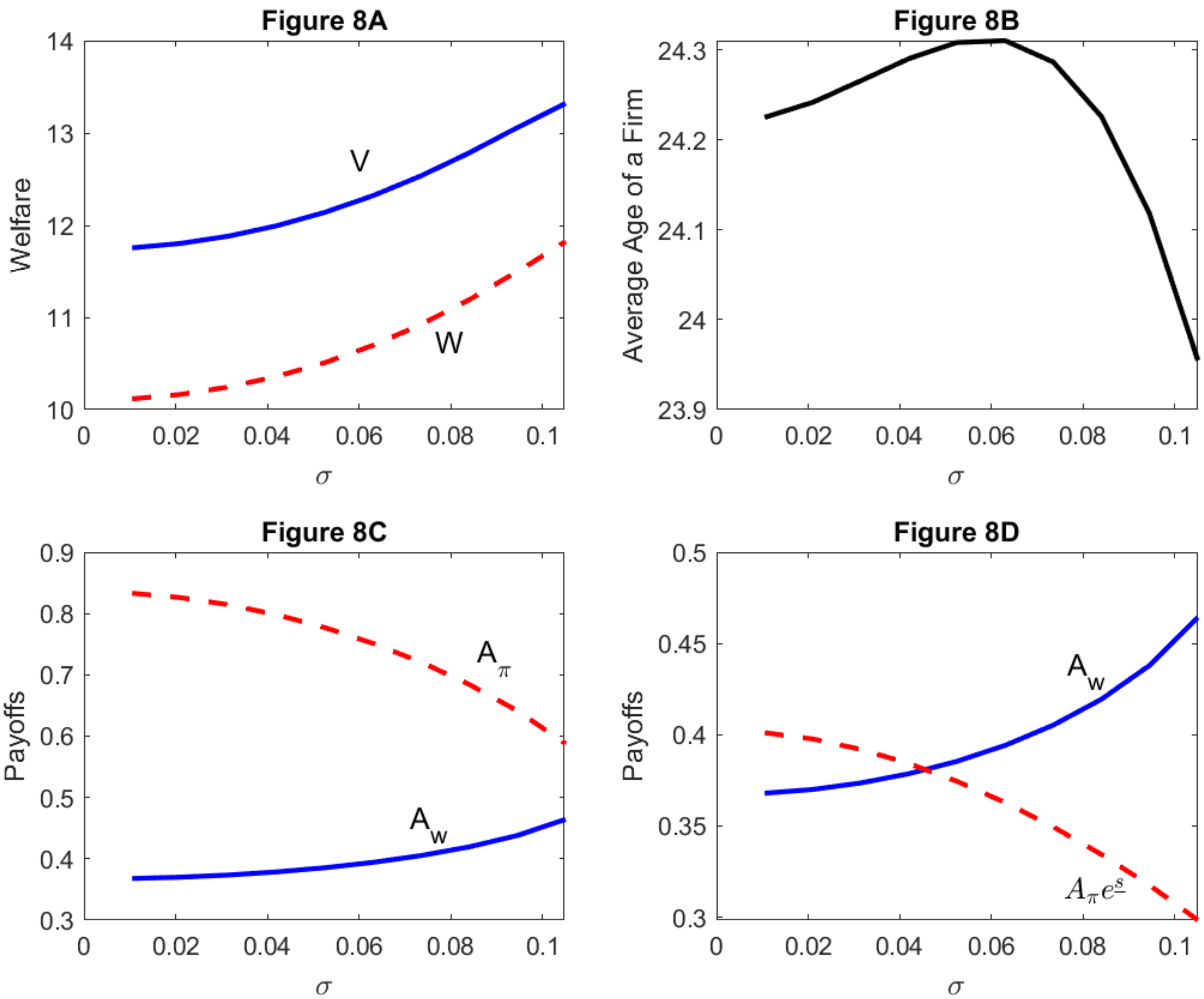


Figure 8A
Welfare
V
W
σ
Figure 8B
Average Age of a Firm
σ
Figure 8C
Payoffs
A_π
A_w
σ
Figure 8D
Payoffs
A_w
$A_\pi e^{\underline{s}}$
σ

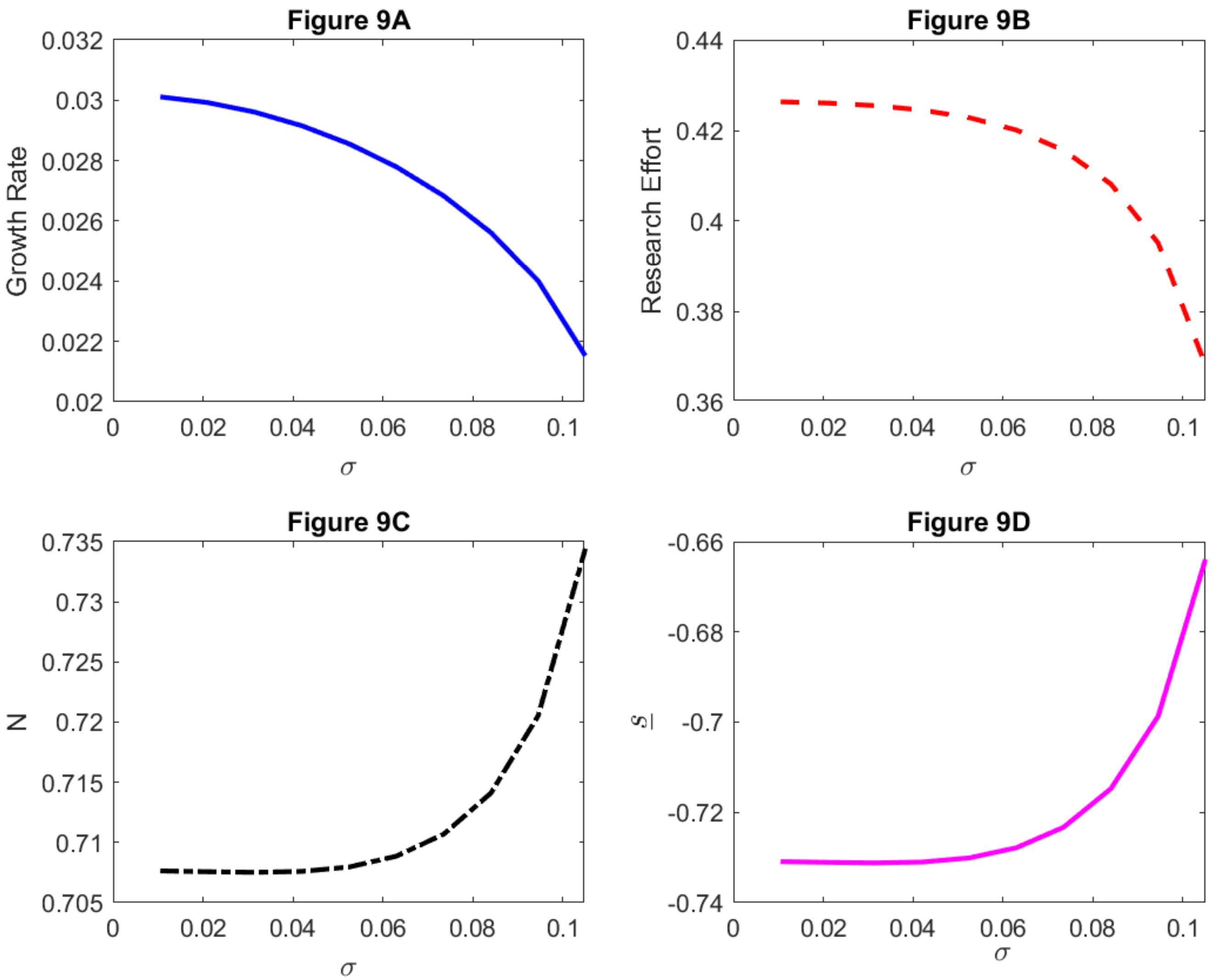


Figure 9A
Growth Rate
0.032
0.03
0.028
0.026
0.024
0.022
0.02
0
0.02
0.04
0.06
0.08
0.1
σ
Figure 9B
Research Effort
0.44
0.42
0.4
0.38
0.36
σ
Figure 9C
N
0.735
0.73
0.725
0.72
0.715
0.71
0.705
σ
Figure 9D
-0.66
-0.68
-0.7
-0.72
-0.74
σ

Figure 10A

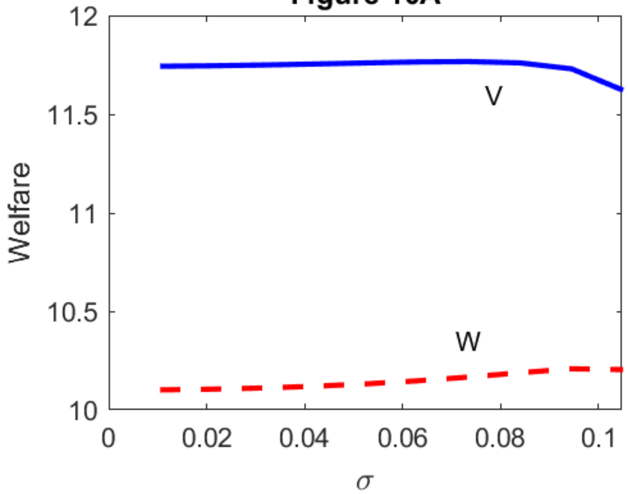


Figure 10B

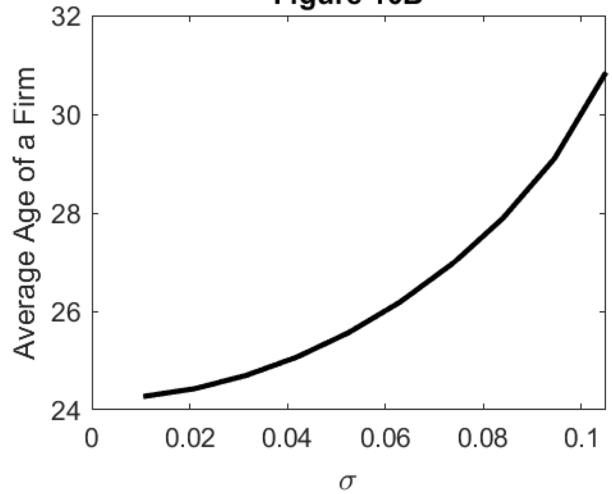


Figure 10C

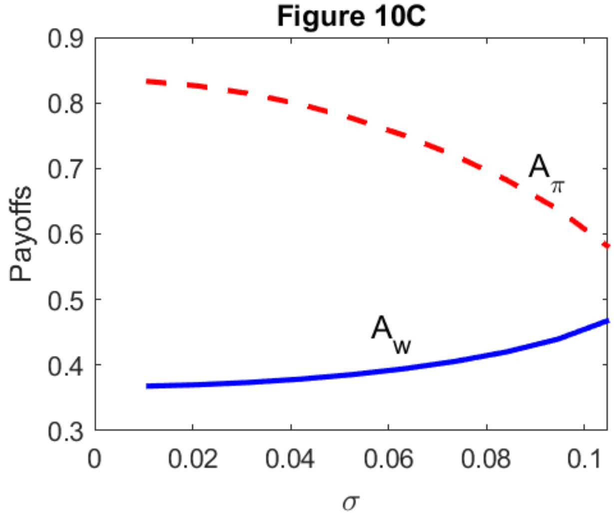
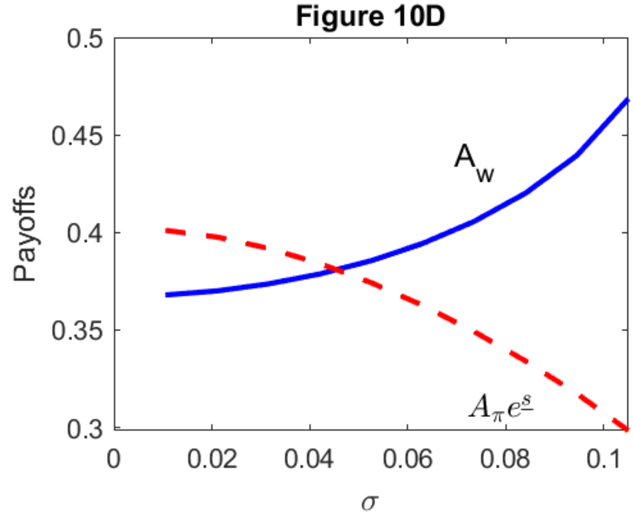


Figure 10D



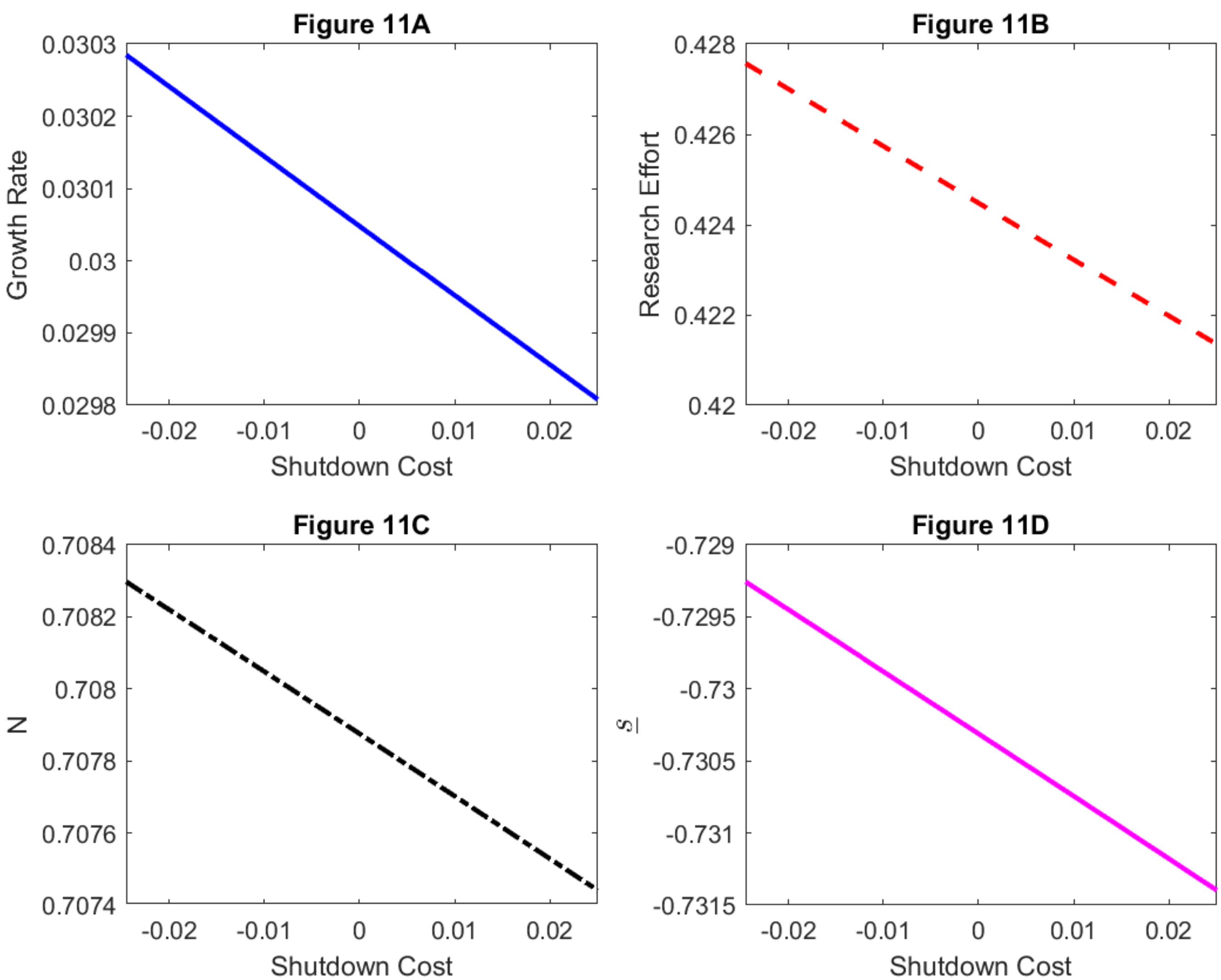


Figure 11A
Growth Rate
0.0303
0.0302
0.0301
0.03
0.0299
0.0298
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 11B
Research Effort
0.428
0.426
0.424
0.422
0.42
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 11C
N
0.7084
0.7082
0.708
0.7078
0.7076
0.7074
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 11D
-0.729
-0.7295
-0.73
-0.7305
-0.731
-0.7315
-0.02
-0.01
0
0.01
0.02
Shutdown Cost

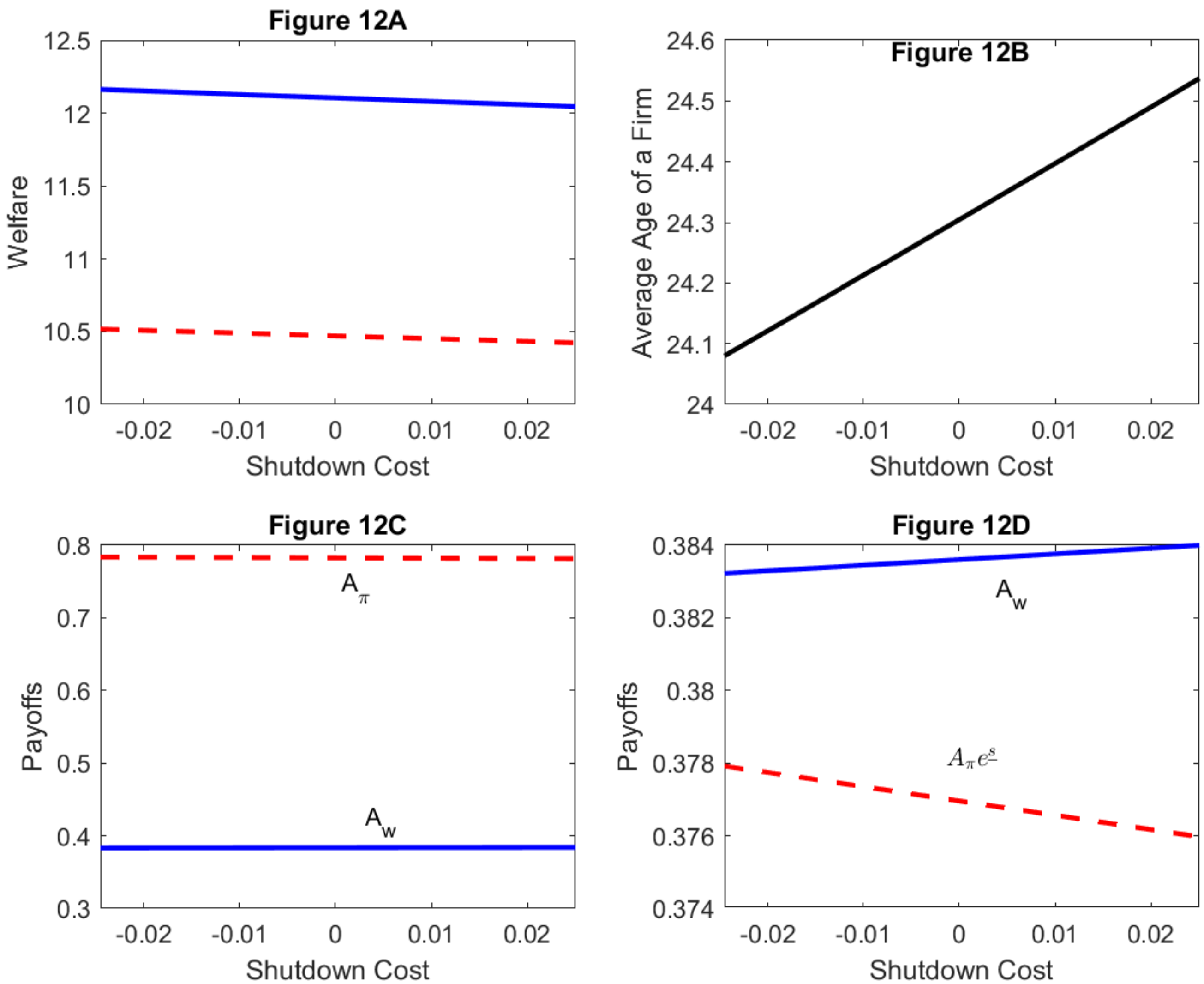
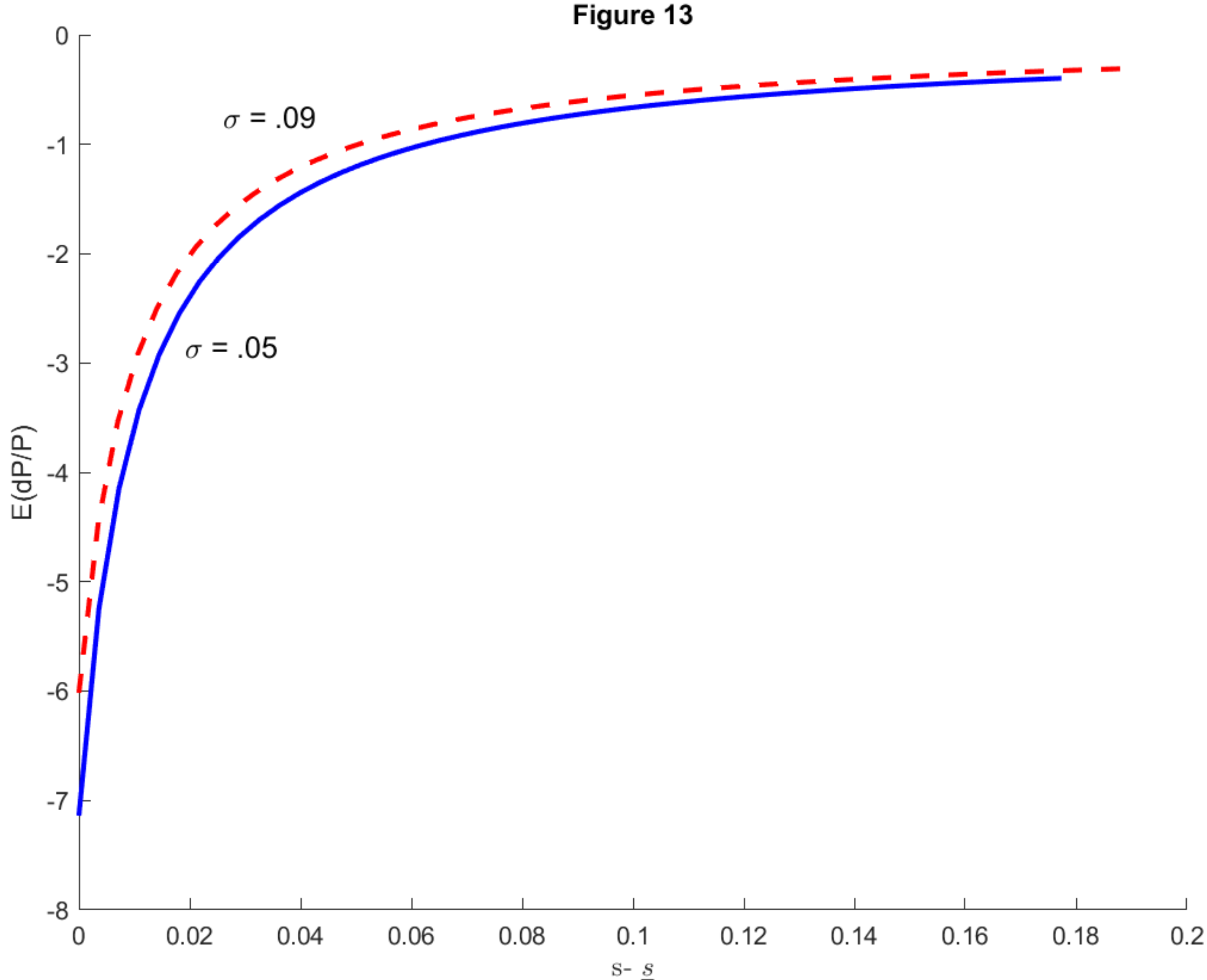


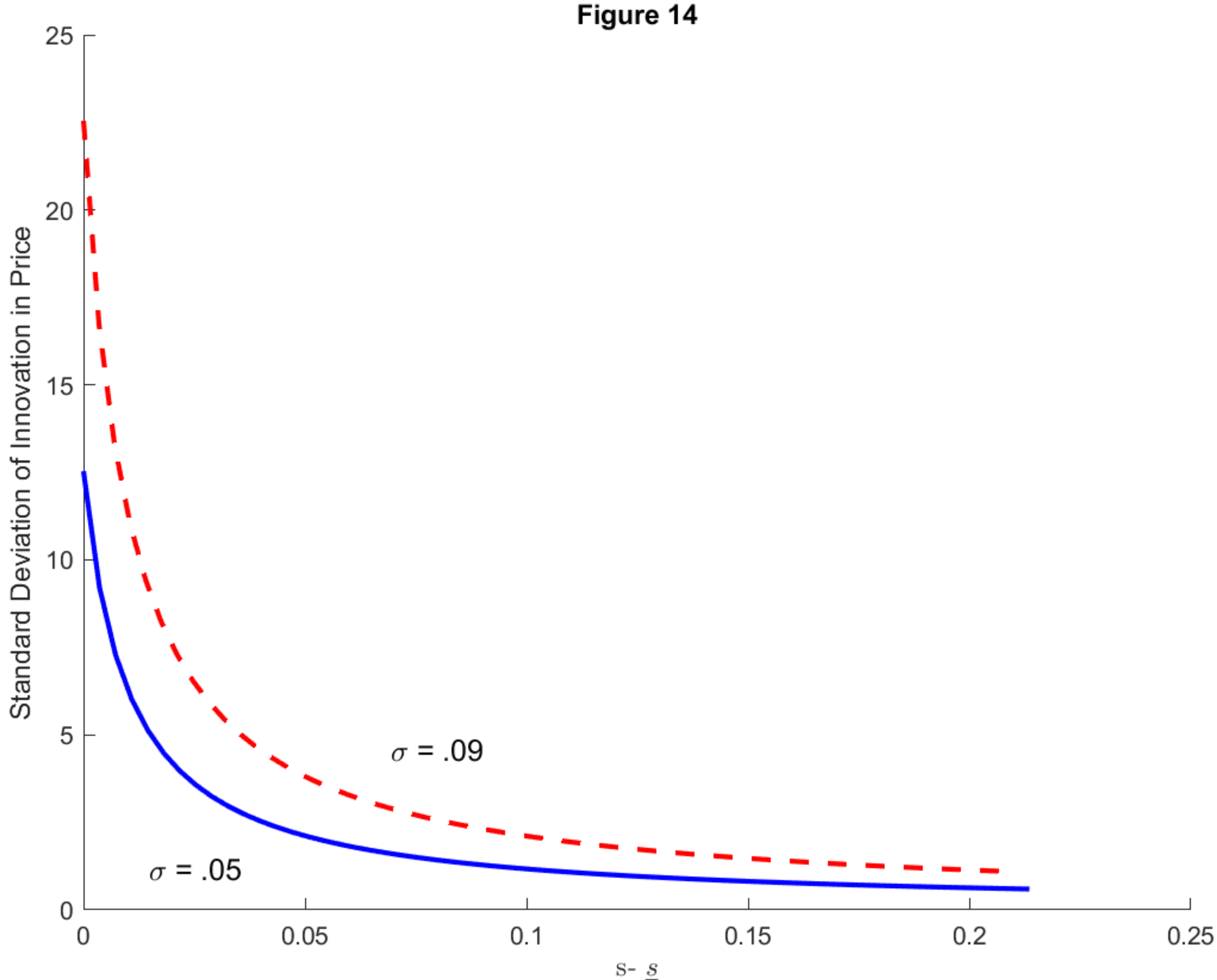
Figure 12A
Welfare
12.5
12
11.5
11
10.5
10
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 12B
Average Age of a Firm
24.6
24.5
24.4
24.3
24.2
24.1
24
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 12C
Payoffs
0.8
0.7
0.6
0.5
0.4
0.3
A_π
A_w
-0.02
-0.01
0
0.01
0.02
Shutdown Cost
Figure 12D
Payoffs
0.384
0.382
0.38
0.378
0.376
0.374
A_w
$A_\pi e^{\underline{s}}$
-0.02
-0.01
0
0.01
0.02
Shutdown Cost

Figure 13



0
-1
-2
-3
-4
-5
-6
-7
-8
E(dP/P)
σ = .09
σ = .05
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
s- s

Figure 14



Standard Deviation of Innovation in Price
25
20
15
10
5
0
0
0.05
0.1
0.15
0.2
0.25
σ = .09
σ = .05
s- $\underline{s}$

Appendix to "The Stochastic Implications of Autonomous Creation and Destruction"

Gregory W. Huffman

August 1, 2022

1 Introduction

In this appendix, a few extra details of the model will be explored. In particular, it is explained how the new innovators, who were previously workers, can obtain a new technology that is an improvement on the technology discovered by previous innovators. This in turn will influence or determine the steady-state growth rate.

2 Discrete and Continuous Time

It is well known that this type of continuous-time model can be characterized as the limiting case of a discrete-time model where the length of the time period goes to zero. But the limits have to characterized in just the right manner. To proceed with such an analysis, consider the model of the paper where the length of the period is denoted by τ. Assume that the functions below have all the necessary diffferentiability properties so that the appropriate limits can be taken. Then the HJB equation of the firm-owner can be written as follows:

$$\tau r V_t = \max \left\{ \tau \pi_t + \left(V_{t+\tau} - V_t \right), \tau r W_t \right\}. \tag{1}$$

The left side is the return to the firm owner over a period of length τ at time t. This corresponds to equation (13) in the text. Next, consider the companion problem faced by a worker, characterized as follows:

$$\tau r W_t = \max_x \left\{ \tau \left(w_t - h\left(x, Z_t\right) \right) + \left(W_{t+\tau} - W_t \right) + \tau \mu\left(x\right) \left[V_{t+\tau} - W_{t+\tau} \right] \right\} \tag{2}$$

This corresponds to equation (23) in the text. Assuming sufficient smoothness or differentiability, then letting $\tau \searrow 0$ in this last expression then yields

$$r W_t = \max_x \left\{ w_t - h\left(x, Z_t\right) + \dot{W}_t + \mu\left(x\right) \left[V_t - W_t \right] \right\}.$$

It is well known that the Brownian motion process, which is assumed for the productivity shocks, can be approximated as the limit of a discrete time Markov process, where period length approaches zero and probabilities converge in just the proper manner (see Cox and Miller [2]). This result will be employed below.

It is also known that with certain smoothness assumptions, equations (1) and (2) converge to equations (13) and (23) in the text. Next there is one other detail to nail down, which is the behavior of workers who are innovators.

3 Innovators

It is assumed that over any period of length τ that any worker has a probability of innovating which is equal to $\tau\mu(x)$. For now let us abbreviate $\tilde{\mu} = \mu(x)$ to keep the notation simple. Next, consider such a worker who innovates at date t. Now the previous generation of innovators had innovated at date $t - \tau$, and they then began to produce with technology level Z_t. However, the next generation of innovators are assumed to be able to *copy or improve* upon the previous generation's innovation. This will mean that there is essentially an intertemporal spillover or externality whereby new innovators benefit from that work of previous cohorts. These improvements take place at random time intervals.[1]

This means that the successful innovators at date t, will each inherit or receive a new technology which is denoted by $Z_{t+\tau} > Z_t$. For convenience it is assumed that

$$Z_{t+\tau} = Z_t (1 + \Delta).$$

Clearly, the size of Δ will then influence the growth rate. To make this discrete-time analysis consistent with the distributional assumptions of the continuous-time model of the paper, it is also assumed that there is a grid of points $\{k_i\}_{i=-\infty}^{i=\infty}$ where k_i represents the logarithm of a potential technology of a firm. That is, the technology of each firm moves (or jumps) discretely along this grid of points. It is assumed that

$$k_{i+1} - k_i = \ln(1 + \Delta)$$

so that the values of Z_t each fall on these grid points as well. That is, each new entrant also has a productivity that falls upon these grid points. At each date there is a distribution of productivities of firms that falls on the points $\{k_i\}_{i=-\infty}^{i=\infty}$. This is a discrete distribution that sums to the mass of $1 - N$. This discrete distribution will be the counterpart of the continuous distribution of the model in the paper.

In the continuous time version of the model the shocks of existing firms will follow a Brownian motion process. Therefore, in the discrete-time process it will be assumed that in each period, if a firm has a log productivity of k_i, then at the next date the firm will have a log productivity of k_{i+1} with probability of p, and will have log productivity of k_{i-1} with probability of $1 - p$. It will be assumed that

$$p = \frac{1}{2}\left(1 + \frac{\kappa_1\sqrt{\tau}}{\sigma}\right). \tag{3}$$

Next, let the relationship between τ and Δ be determined as follows:

$$\Delta = \sigma\sqrt{\tau} \tag{4}$$

It is shown in Cox and Miller [2] that this process for these shocks then converges to the process described in the paper as $\tau \searrow 0$.

Next, we have to describe or characterize the behavior of new entrants. Once again, it is assumed that each innovator has no control or influence over the *size* of an innovation

[1] An alternative approach would be to have the improvements take place be of a random size.

(Δ). Assume that successful new entrants don't get a certain technology of Z_t, but instead a *random* (or barely random) payoff. That is, suppose that they receive the following:

$$Z_{t+\tau} = \begin{cases} Z_t(1+\Delta) & \text{with probability } \hat{p} \\ Z_t(1-\Delta) & \text{with probability } 1-\hat{p} \end{cases}.$$

Note that here $\hat{p}$, which determines the success rate of potential innovators, is distinct from p in equation (3), which determines the rate of movement of existing firms.

Once again, the value of Δ is determined by equation (4). Next, assume that

$$\hat{p} = \frac{1}{2}\left(1 + \frac{\tilde{\kappa}\sqrt{\tau}}{\sigma}\right).$$

Here $\tilde{\kappa}$ is assumed to be a parameter, but this will be determined below.

This means that new entrants, or innovators, get a *random* improvement on the technology of recent innovators. If $\hat{p} > 1/2$, on average it will be an improvement, but it might also fall just below that of recent innovators. Then it is easy to see that

$$\begin{aligned} E\left[\frac{Z_{t+\tau} - Z_t}{Z_t}\right] &= \hat{p}\Delta + (1-\hat{p})(-\Delta) \\ &= \frac{1}{2}\left(1 + \frac{\tilde{\kappa}\sqrt{\tau}}{\sigma}\right)\Delta + \frac{1}{2}\left(-1 + \frac{\tilde{\kappa}\sqrt{\tau}}{\sigma}\right)\Delta \\ &= \left(\frac{\tilde{\kappa}\sqrt{\tau}}{\sigma}\right)\Delta \\ &= \tilde{\kappa}\tau \end{aligned} \tag{5}$$

This follows because of equation (4). This ensures that the growth rate of new entrants is indeed $\tilde{\kappa}$.

Lastly, it is assumed that the value of $\tilde{\kappa}$ depends on the aggregate level of innovation (μN) in the following manner:

$$\tilde{\kappa} = \mu N + \kappa_1.$$

As mentioned in the paper, this expression is the same equation used to characterize the determinants of growth in Bloom, Jones, Reenen, and Webb [1].

References

[1] Bloom, Nicholas, Charles I. Jones, John Van Reenen, and Michael Webb, "Are Ideas Getting Harder to Find?", *American Economic Review*, 110(4), (2020), pages 1104–1144.

[2] Cox, D. R., and H. D. Miller, *The Theory of Stochastic Processes*, Science Paperbacks, (1970).