Chapter ~~Lecture~~ 4 More on CAR

4.1) Graded Algebras, the graded tensor product

A $*$-algebra A is said to be ($\mathbb{Z}/2\mathbb{Z}$-) graded if it possesses an action by $*$ automorphisms of $\mathbb{Z}/2\mathbb{Z}$, called the grading. We then have $\underset{\deg 0}{A = A_0} \oplus \underset{\deg 1}{A_1}$, the eigenspace decomposition of A w.r.t. $\mathbb{Z}/2\mathbb{Z}$

And $A_0^2 \subseteq A_0$, $A_0^* = A_0$, $A_0 A_1 \subseteq A_1$, $A_1 A_1 \subseteq A_0$, $A_1^* = A_1$. Such a decomposition is equivalent to a $\mathbb{Z}/2\mathbb{Z}$ action by sending a_1 to $-a_1$ for $a_1 \in A_1$, or on homogeneous a, b, $ab \in A_{\deg a + \deg b}$

Example $CAR(\mathcal{H})$ is graded by the automorphism coming from -1 on $\mathcal{H}$. Thus an element $a(f_1)a(f_2)\ldots a(f_n)a(g_1)^* a(g_2)^* \ldots a(g_m)^*$ has degree 0 or 1 if $m+n = 0$ or $1 \bmod 2$ respectively.

There is a notion of graded tensor product $A \hat{\otimes} B$ of graded $*$ algebras A and B. As a vector space $A \otimes B$ is the vector space tensor product but multiplication and $*$ are modified

On homogeneous elements a_i and b_i we have

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} a_1 a_2 \otimes b_1 b_2$$

$$(a_1 \otimes b_1)^* = (-1)^{\deg a_1 \deg b_1} a_1^* \otimes b_1^*$$

~~and~~ $\deg(a \otimes b) = \deg a + \deg b$

Associativity etc can be checked, but don't bother if you know about crossed products, just wait.

Proposition $CAR(\mathcal{H} \oplus K) \cong CAR\,\mathcal{H} \,\hat{\otimes}\, CAR\,K$

Proof By the definition of $CAR(\mathcal{H} \oplus K)$ it suffices to define elements $A(f \oplus g)$ in $CAR\,\mathcal{H} \,\hat{\otimes}\, CAR\,K$, which generate it, and appeal to the simplicity of CAR algebras, or construct an inverse

set $A(f \oplus g) = a(f) \otimes 1 + 1 \otimes a(g)$

C^* algebras are a little more complicated because of the meaning of "$A \otimes B$". A way to handle this is the following observation, initially in the purely algebraic setting.

~~Form the~~ If $\alpha \in \mathrm{Aut}\,A$ and $\beta \in \mathrm{Aut}\,B$ are the grading automorphisms, form $A \otimes B \rtimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. Let U be the unitary implementing α, and V that implementing β. Let $\hat\alpha$ and $\hat\beta$ be the dual actions $\hat\alpha(U) = -U$, $\hat\alpha(V) = V$, $\hat\beta(U) = U$, $\hat\beta(V) = -V$.

Proposition There is a $*$ isomorphism between $A \hat\otimes B$ and the fixed point algebra $(A \otimes B \rtimes \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ where $\mathrm{Ad}\,V\hat\alpha$ and $\hat\beta$ define the action on the crossed product. The grading on $A \hat\otimes B$ corresponds to $\mathrm{Ad}\,U\,\hat\alpha$.

Proof Send $A_0 \otimes B_0$ to itself in the crossed product, $A_0 \otimes B_1$ to $(A_0 \otimes B_1)U$, $A_1 \otimes B_0$ to itself and $A_1 \otimes B_1$ to $A_1 \otimes B_1 U$. It is easy to check that the subspace spanned by these spaces is indeed the fixed point algebra, e.g. $\mathrm{Ad}\,V\hat\alpha((a_1 \otimes b_1)u) = -(-(a_1 \otimes b_1)u)$, $\hat\beta(a_1 \otimes b_1 u) = a_1 \otimes b_1 u$, and to check that multiplication, and $*$, mimic those of $A \hat\otimes B$, e.g. $(a_0 \otimes b_1)u(x_1 \otimes y_0) = -a_0 x_1 \otimes b_1 y_0 u$.

This trick can be used to define a C^* version of the graded tensor product, using whatever notion of C^* tensor product.

We also see that if φ and ψ are states on A & B respectively, there is a state $\varphi \hat\otimes \psi$ on $A \hat\otimes B$ defined by extending $\varphi \otimes \psi$ on $A \otimes B$ to the crossed product, then restricting. (Curiously, if φ is not α-invariant, it could be non zero on $A_1 \otimes B_0$)

The following lemma will be useful.

Lemma Suppose B is a graded algebra with grading automorphism β. Suppose there is a unitary u, $u^2=1$, $\alpha(u)=u$ in B, with $\beta=\mathrm{Ad}\,u$. Then if A is any graded algebra $A=A_0+A_1$, then

$A_0\otimes 1 + A_1\otimes u$ is a copy of A in $A\hat{\otimes} B$, commuting with $1\otimes B$.

Proof. Obviously $A_0\otimes 1$ commutes with $1\otimes B$, and $1\otimes B_0$ commutes with $A_1\otimes u$ so let $a_1\in A_1$, $b_1\in B_1$ and consider

$$(a_1\otimes u)(1\otimes b_1) = -a_1\otimes b_1 u$$
$$(1\otimes b_1)(a_1\otimes u) = -a_1\otimes b_1 u.$$

Moreover $(A_0\otimes 1)(A_1\otimes u)\subseteq A_1\otimes u$ and $(A_1\otimes u)(A_1\otimes u)\subseteq A_0\otimes 1$.

4.2 The structure of $CAR(\mathcal{H})$

It is now very easy to deduce the structure of $CAR(\mathcal{H})$ as a C^*-algebra. First choose an orthonormal basis $\{v_i\}$. Note that $a(v_i)a(v_i)^*$ & $a(v_i)^*a(v_i)$ are projections adding up to 1 so that $CAR(\mathbb{C})=M_2(\mathbb{C})$ and $a(v_i)a(v_i)^*-a(v_i)^*a(v_i)$ is a self adjoint unitary implementing the grading.

By induction and the previous lemma $CAR([v_1,v_2\ldots v_n])\cong M_{2^n}(\mathbb{C})=M_2(\mathbb{C})\otimes M_2(\mathbb{C})\otimes\ldots\otimes M_2(\mathbb{C})$

Hence we may form the UHF C^*-algebra $\overset{\infty}{\otimes} M_2(\mathbb{C})$ (unique norm) and the closure of the $a(f)$'s for f in the linear span of the v_i's is $\cong\mathcal{H}$ ($\|a(f)\|=\|f\|$). Thus the UHF algebra is generated by $a(f)$'s satisfying the CAR relations. On the other hand $CAR(\mathcal{H})$ is simple: if some linear combination of words on $a(f_i)$'s & $a(g_j)$'s is in an ideal, consider $CAR(V)$, V being the linear span of the ~~a(f)~~ f_i's & g_j's. This is simple by what we have shown, so $1\in$ ideal.

QED

4.3 Quasi free states.

Theorem. If $\mathcal{H}$ is a Hilbert space and $a \in \mathcal{B}(\mathcal{H})$, $0 \le a \le 1$, there is a state φ_a on CAR($\mathcal{H}$) uniquely defined by

$$\varphi_a\left(a(g_m)^* a(g_{m-1})^* \dots a(g_1)^* a(f_1) a(f_2) \dots a(f_n)\right) = \delta_{m,n} \det(\langle a f_i, g_j \rangle)$$

Proof. We start with the special case where a is a projection p. First, in finite dimensions, choose an o.n.b η_i for $(1-p)\mathcal{H}$ and let $v = \eta_1 \wedge \eta_2 \dots \wedge \eta_k \in \Lambda\mathcal{H}$. Then, on $\Lambda\mathcal{H}$, consider the vector state $\langle \cdot v, v \rangle$. By multilinearity, to prove the required formula for $\varphi_p = \langle \cdot v, v \rangle$ we may assume that all the g_i's and f_i's are in $p\mathcal{H}$ or $(1-p)\mathcal{H}$. If any of the f_i's, or g_i's is in $(1-p)\mathcal{H}$ both "φ_p" and $\langle \cdot v, v \rangle$ are zero. If all the f's and g's are in $p\mathcal{H}$ then

$$\begin{aligned}\langle a(f_1) a(f_2) \dots a(f_n) v, a(g_1) a(g_2) \dots a(g_m) v \rangle &= \langle f_1 \wedge f_2 \dots \wedge f_n \wedge \eta_1 \dots \wedge \eta_k, g_1 \wedge \dots \wedge g_m \wedge \eta_1 \dots \wedge \eta_k \rangle \\ &= \delta_{m,n} \det \begin{pmatrix} \langle f_i, g_j \rangle & 0 \\ 0 & \langle \eta_i, \eta_j \rangle \end{pmatrix} \\ &= \det(\langle p f_i, g_j \rangle)\end{aligned}$$

Now if $\mathcal{H}$ is infinite dimensional we may no longer form $\eta_1 \wedge \eta_2 \dots \wedge \eta_k$. On the other hand we may choose finite dimensional subspaces $V_n \subseteq p\mathcal{H}$, $W_n \subseteq (1-p)\mathcal{H}$ with $\bigcup_n (V_n + W_n)$ dense in $\mathcal{H}$ and define ψ_n on $CAR(V_n + W_n)$ by the above. Clearly $\psi_n = \psi_{n+1}|_{CAR(V_n+W_n)}$ so ψ_n define a state on CAR($\mathcal{H}$) satisfying the formula for φ_a.

Finally, if $0 \le a \le 1$ is arbitrary we form the projection $p = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ and observe that the restriction of φ_p to $CAR(\mathcal{H} \oplus 0)$ satisfies the formula for φ_a.

QED

<u>Proposition</u> If p is a projection, φ_p is pure.

<u>Proof</u> Inductive limit of pure states is pure. QED

<u>Exercise</u> If $a^2 \neq a$, can φ_a be pure?

Let us look at a few special cases.

(i) $p = 1$. Then φ_p is the Fock space vacuum state

(ii) $p = 0$. Then $\varphi_p = 0$ on all ~~words~~ monomials $a(\vec{g})^* a(\vec{f})$ (with obvious notation) But $\varphi(1) = 1$... This is the antifock state since $\varphi(a(\vec{f}) a(\vec{g})^*) = \delta_{mn} \det\langle f_i, g_j \rangle$

(iii) $\dim \mathcal{H} = 1$ then $a = \mu$, $0 \le \mu \le 1$ in $\mathbb{R}$ and

$$\varphi_a(a(1)^* a(1)) = \mu \qquad \varphi_a(a(1)) = 0 = \varphi_a(a(1)^*)$$

$$\varphi_a(a(1) a(1)^*) = 1 - \mu$$

i.e. $\varphi(x) = \operatorname{trace}\left(\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} x\right)$ in the iso $CAR(\mathbb{C}) \cong M_2(\mathbb{C})$.

Now ~~in the~~ quasi-free states are <u>even</u> (0 on odd elements) so we see that in our iso ~~$A \hat{\otimes} CAR(\mathbb{C})$ with φ & φ_μ~~, ~~if φ is even, $\varphi \otimes \varphi_\mu$ is φ~~ $CAR(\mathcal{H}) \cong \bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$, if a is diagonalisable with eigenvalues μ_i, then

$$\varphi_a = \bigotimes_{i=1}^{\infty} \operatorname{trace}\left(\begin{pmatrix} \mu_i & 0 \\ 0 & 1-\mu_i \end{pmatrix} \cdot\right).$$

Thus one may create hyperfinite factors of all types, III_λ, $\lambda\in[0,1]$ by varying the eigenvalue sequence.

Note that in finite dimensions we used the vector $\eta_1\wedge\cdots\wedge\eta_k$ in the usual Fock space. This can be thought of as a state with fermionic particles $\eta_1\ldots\eta_k$. Observe that $a(\eta_i)^*(\eta_1\wedge\cdots\wedge\eta_k)=(-1)^i\eta_1\wedge\wedge\hat{\eta_i}\cdots\wedge\eta_k$.

Thus $a(\eta_i)^*$ can be thought of as creating a hole — a missing η_i. In infinite dimensions one can use the same notation for the vacuum vector Ω (the image of $1\in CAR(\mathcal{H})$ in GNS_{φ_p}) — choose a basis $\eta_1,\eta_2,\ldots$ of $(1-p)\mathcal{H}$ and write $\Omega=\eta_1\wedge\eta_2\wedge\eta_3\wedge\ldots$.

One may then write, e.g. $a(\eta_2)^*\Omega$ as $-\eta_1\wedge\eta_3\wedge\eta_4\wedge\ldots$. This is the "Dirac sea" with all the $(1-p)\mathcal{H}$ states filled and the $a(\eta_i)^*$ creating "holes" or antiparticles.

In any case it is clear that the GNS Hilbert space $\mathcal{F}_p$ from the state φ_p on $CAR(\mathcal{H})$ ~~has~~ is spanned (densely) by Ω and elements of the form $a(f_n)a(f_{n-1})\ldots a(f_1)a(g_1)^*a(g_2)^*a(g_3)^*\ldots a(g_m)^*\Omega$ with $f_i\in p\mathcal{H}$ and $g_i\in(1-p)\mathcal{H}$ (in an arbitrary such expression, with f's and g's in either $p\mathcal{H}$ or $(1-p)\mathcal{H}$, take the last occurence of a g not in $(1-p)\mathcal{H}$ and anticommute it across to the vacuum, which it kills, then do the same to the f's not in $p\mathcal{H}$).

The next result is very useful.

<u>Proposition</u> In $\mathcal{F}_p$, Ω is, up to a scalar multiple, the unique vector killed by all the $a(g)$'s for $g\in(1-p)\mathcal{H}$ and $a(f)^*$'s for $f\in p\mathcal{H}$.

Proof First note that for $f \in (1-p)\mathcal{H}$, $\|a(f)\Omega\|^2 = \langle a(f)^* a(f)\Omega, \Omega\rangle$ ⓷⓪
$= \varphi(a(f)^* a(f)) = \langle pf, f\rangle = 0$, similarly for $f \in p\mathcal{H}$,
$\|a(f)^*\Omega\|^2 = \varphi(a(f)a(f)^*) = \langle f, f\rangle - \varphi(a(f)^* a(f)) = \langle f, f\rangle - \langle pf, f\rangle = 0$.

To see the converse, observe that $\Omega^\perp$ is, by the above, spanned (densely) by vectors of the form $a(f_1)\dots a(f_n) a(g_1)^* \dots a(g_m)^*\Omega$, with at least one f_i or g_i present. A vector ξ satisfying the hypoteses of the proposition is obviously orthogonal to such vectors. QED

Before examining equivalence of quasi-free states let us point out a remarkable "physical application" of the passage from $\mathcal{H}$ to $\mathcal{F}_p$.

a) If $u \in \mathcal{U}(\mathcal{H})$ is a unitary commuting with p, then ~~if~~ we define ~~$\Gamma(u)$~~ $\alpha_u \in \mathrm{Aut}(CAR)$ by functoriality, α_u is "canonically implemented" on $\mathcal{F}_p$ ~~with~~ This is simply because α_u preserves φ so by C^*-algebra general nonsense one may define the unitary Γ on ~~$\mathcal{H}$~~ $\mathcal{F}_\varphi$ by $\Gamma\, \pi_\varphi(x)\Omega = \pi_\varphi(\alpha_u(x))\Omega$.
(In particular $\Gamma a(f)\Gamma^* =$ ~~$a(uf)$~~ $\alpha_u(a(f)) = a(uf)$.

b) Now suppose we have a candidate "Hamiltonian" on $\mathcal{H}$, ie. a self adjoint operator $h: \mathcal{H} \to \mathcal{H}$. Suppose for simplicity that h has a basis ξ_i of eigenvectors with $i \in \mathbb{Z}$, $\lambda_i = \langle h\xi_i, \xi_i\rangle \geq 0$ for $i \geq 0$. Physics abhors negative energy so h doesn't pass muster as a Hamiltonian. However consider p = projection onto the closed span of the ξ_i's, $i \geq 0$ (the spectral projection of h for the interval $[0, \infty)$ in general)

The troublesome "time evolution" on $\mathcal{H}$ would be $u_t = e^{ith}$, ~~But to~~ which is canonically implemented by $\mathcal{U}_t$ on $\mathcal{F}_p$. But look what happens to the negative spectrum of h:

all vectors of the form $a(\xi_{i_1}) a(\xi_{i_2}) \dots a(\xi_{i_k}) a(\xi_{j_1})^* \dots a(\xi_{j_\ell})^* \Omega$ are eigenvectors (in fact an o.n.b. of eigenvectors) for the 1-parameter group $\mathcal{U}_t$. But for $j<0$

$$\begin{aligned}
U_t(a(\xi_j)^*)\Omega &= \alpha_t(a(\xi_j)^*)\Omega \\
&= a(u_t \xi_j)^* \Omega \\
&= a(e^{iht}\xi_j)^* \Omega \\
&= a(e^{i\lambda_j t}\xi_j)^* \Omega \\
&= e^{-i\lambda_j t}(a(\xi_j)^*)\Omega
\end{aligned}$$

So if $e^{iHt} = \mathcal{U}t$ then all the pesky negative eigenvectors of h have been converted to positive ones for H!!