$19\frac{1}{2}$

Note on relative tensor product:

WARNING: Given $\xi \in H^\circ$, $\eta \in K^\circ$, $x \in M$, we ~~[illegible]~~ do not have $\xi x \otimes_M \eta = \xi \otimes_M x\eta$. Instead, we have $\xi x \otimes_M \eta = \xi \otimes_M \sigma^{\psi}_{-i/2}(x)\eta$.

To prove this, we compute: (assuming ψ a state)

Formally, For $x, y \in M$,

$$
\begin{aligned}
L_{\xi x}(Jy^*\Omega) &= \xi x y \\
&= L_\xi (J(xy)^*\Omega) \\
&= L_\xi JSxy\Omega \\
&= L_\xi \Delta^{1/2} xy\Omega \\
&= L_\xi (\Delta^{1/2} x \Delta^{-1/2})\Delta^{1/2} y\Omega \\
&= L_\xi \sigma_{-i/2}(x)\, JSy\Omega \\
&= L_\xi \sigma_{-i/2}(x)(Jy^*\Omega)
\end{aligned}
$$

Hence
$$
\begin{aligned}
\langle \xi_1 x \otimes_M \eta_1, \xi_2 \otimes_M \eta_2\rangle &= \langle \overbrace{L_{\xi_2}^* L_{\xi_1 x}}^{\in M} \eta_1, \eta_2\rangle_K \\
&= \langle L_{\xi_2}^* L_{\xi_1} \sigma^{\psi}_{-i/2}(x)\eta_1, \eta_2\rangle_K \\
&= \langle \xi_1 \otimes_M \sigma^{\psi}_{-i/2}(x)\eta_1, \xi_2 \otimes \eta_2\rangle
\end{aligned}
$$

Fermions + CAR Quantum physics lore says the states of a system are given by lines in a Hilbert space H. If H and K are state spaces for two (non-interacting) systems, then the joint system has state space $H \otimes_{\mathbb{C}} K$. To describe a system of arbitrarily many identical particles, we are led to the Hilbert space completion of the tensor algebra of H (i.e. $T(H) = \bigoplus_{n=0}^{\infty} \otimes^n H$).

We take $H^{\otimes 0} = \mathbb{C}\Omega$, $\|\Omega\| = 1$ "the vacuum vector."

Fermions are characterized by the Pauli exclusion principle that no two particles can lie in the same state. Thus their states ~~with the~~ are described by the antisymmetric subspace

$$\Lambda H = \bigoplus_{n=0}^{\infty} \Lambda^n H \subseteq T(H).$$

Fermionic Fock space:

Define the antisymmetrization projection $p : T(H) \circlearrowleft$ by

$$p(\xi_1 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

(Exercise: check this is a projection).

Let $\Lambda^n H = p(H^{\otimes n})$ and $\Lambda H = pT(H) = \bigoplus_{n=0}^{\infty} \Lambda^n H$.

Define $\xi_1 \wedge \cdots \wedge \xi_n = \sqrt{n!}\, p(\xi_1 \otimes \cdots \otimes \xi_n)$.

inner product ΛH inherits from $T(H)$

Exercises: i) $\langle \xi_1 \wedge \cdots \wedge \xi_n, \eta_1 \wedge \cdots \wedge \eta_m \rangle = \delta_{n,m} \det\left((\langle \xi_i, \eta_j \rangle_H)_{i,j=1}^n \right)$

ii) $\xi_{\sigma(1)} \wedge \cdots \wedge \xi_{\sigma(n)} = (-1)^\sigma \xi_1 \wedge \cdots \wedge \xi_n$

iii) If $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis for H, then $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : i_1 < i_2 < \cdots < i_n\}$ is an orthonormal basis for ΛH.

Remark: One can also define ΛH as a quotient of $T(H)$, rather than a subspace, or directly using (i)-(iii).

<u>Creation operators</u>: For $f \in H$, define $a(f) : \Lambda^n H \to \Lambda^{n+1} H$

by $a(f)(\xi_1 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \cdots \wedge \xi_n$ (Exercise: this is well defined).

We ~~thus~~ can define $a(f) : \underset{\text{algebraic}}{\bigoplus} \Lambda^n H \to \bigoplus \Lambda^n H$, and

check that $\|a(f)\| = \|f\|_H$, (Pf: Exercise), so that $a(f)$ extends to a bounded operator on ΛH.

The map $f \mapsto a(f)$ is an isometric embedding of Banach Spaces $H \to B(\Lambda H)$.

Proposition (Exercises):

i) $a(f)^*(\xi_1 \wedge \dots \wedge \xi_n) = \sum_{k=1}^{n} (-1)^{k+1} \langle \xi_k, f \rangle \, \xi_1 \wedge \dots \wedge \hat{\xi}_k \wedge \dots \wedge \xi_n$ $\quad$ $\hat{}$ means omitted

ii) CAR relations $\quad \{a(f), a(g)\} = 0 \;, \quad \{a(f), a(g)^*\} = \langle f, g \rangle \mathbb{1}$

anticommutator $\{x, y\} = xy + yx$

iii) The $*$-algebra generated the $a(f)$'s acts irreducibly on ΛH (Hint: Ω is the unique vector, up to scaling, that satisfies $a(f)^*\Omega = 0 \;\; \forall f \in H$)

<u>Def</u> We define $CAR(H)$ to be the universal $*$-algebra with generators $a(f)$ for $f \in H$, subject to the relations: $f \mapsto a(f)$ linear, $\{a(f), a(g)\} = 0$, $\{a(f), a(g)^*\} = \langle f, g \rangle \mathbb{1}$.

By (ii) above, the anihilation/creation operators on ΛH give a representation of $CAR(H)$. We will eventually
"wave representation"
be interested in the universal C^*-algebra generated by $CAR(H)$, and we will see that this is the C^*-algebra generated by the wave representation on ΛH. (In particular, the wave rep. is faithful).