Partition Congruences and the Localization Method

Nicolas Allen Smoot

Vanderbilt Number Theory Seminar

8 September, 2020

Doctoral Program Computational Mathematics Numerical Analysis and Symbolic Computation

RISC Research Institute for Symbolic Computation

JKU Johannes Kepler University Linz

FWF Der Wissenschaftsfonds.

What is a Partition?

Ramanujan's Conjectures

Let $n \geq 0$, $\alpha \geq 1$, and $24\lambda_{\ell,\alpha} \equiv 1 \pmod{\ell^\alpha}$.

Conjecture (Ramanujan, 1919)

$$p\left(5^\alpha n + \lambda_{5,\alpha}\right) \equiv 0 \pmod{5^\alpha},$$
$$p\left(7^\alpha n + \lambda_{7,\alpha}\right) \equiv 0 \pmod{7^\alpha},$$
$$p\left(11^\alpha n + \lambda_{11,\alpha}\right) \equiv 0 \pmod{11^\alpha}.$$

Ramanujan's Conjectures

Let $n \geq 0$, $\alpha \geq 1$, and $24\lambda_{\ell,\alpha} \equiv 1 \pmod{\ell^{\alpha}}$.

Theorem (Ramanujan, Watson, Atkin)

$$p\left(5^{\alpha}n+\lambda_{5,\alpha}\right) \equiv 0 \pmod{5^{\alpha}},$$
$$p\left(7^{\alpha}n+\lambda_{7,\alpha}\right) \equiv 0 \pmod{7^{\left\lfloor \frac{\alpha}{2} \right\rfloor+1}},$$
$$p\left(11^{\alpha}n+\lambda_{11,\alpha}\right) \equiv 0 \pmod{11^{\alpha}}.$$

$\alpha = 1$

Theorem (Ramanujan)

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

Rewriting, we have

$$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{n+1} = 5q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}.$$

$\alpha = 1, 2$

For $\alpha = 1$: $$(q^5; q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{n+1} = 5 \cdot q \frac{(q^5; q^5)_\infty^6}{(q; q)_\infty^6}.$$

$\alpha = 1, 2$

For $\alpha = 1$: $$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{n+1} = 5 \cdot q \frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}.$$

For $\alpha = 2$: $$(q;q)_\infty \sum_{n=0}^{\infty} p(5^2 n+24)q^{n+1}$$
$$= 5^{12} \cdot q^5 \frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^{30}} + 5^{10} \cdot 6 \cdot q^4 \frac{(q^5;q^5)_\infty^{24}}{(q;q)_\infty^{24}}$$
$$+ 5^7 \cdot 63 \cdot q^3 \frac{(q^5;q^5)_\infty^{18}}{(q;q)_\infty^{18}} + 5^5 \cdot 52 \cdot q^2 \frac{(q^5;q^5)_\infty^{12}}{(q;q)_\infty^{12}}$$
$$+ 5^2 \cdot 63 \cdot q \frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^{30}}.$$

$\alpha = 1, 2$

$$\text{Let } t = q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}, \text{ and } q = e^{2\pi i\tau}, \text{ with } \tau \in \mathbb{H}.$$

Theorem

$$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{n+1} = 5t,$$

$$(q;q)_\infty \sum_{n=0}^{\infty} p(5^2n+24)q^{n+1} = 5^{12}\cdot t^5 + 5^{10}\cdot 6\cdot t^4 + 5^7\cdot 63\cdot t^3 + 5^5\cdot 52\cdot t^2 + 5^2\cdot 63\cdot t.$$

5^α

Theorem

Let λ_α *be the smallest positive solution to* $24x \equiv 1 \pmod{5^\alpha}$*. Then*

$$L_{2\alpha-1} = (q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5^{2\alpha-1}n + \lambda_{2\alpha-1})q^{n+1} \in \mathbb{Z}[t],$$

$$L_{2\alpha} = (q;q)_\infty \sum_{n=0}^{\infty} p(5^{2\alpha}n + \lambda_{2\alpha})q^{n+1} \in \mathbb{Z}[t].$$

We write $L_{\alpha+1} = U^{(\alpha)}(L_\alpha)$, where $U^{(\alpha)}$ are *linear* operators.

5^α

We have

- $L_\alpha \in \mathbb{Z}[t]$,
- $L_{\alpha+1} = U^{(\alpha)}\left(L_\alpha\right)$,
- $U^{(\alpha)}\left(5^k \cdot f\right) = 5^k \cdot U^{(\alpha)}\left(f\right)$,
- Also, $L_1 = 5t$.

Theorem

For every $\alpha \in \mathbb{Z}_{\geq 1}$,

$$U^{(\alpha)}\left(\frac{L_\alpha}{5^\alpha}\right) \in 5 \cdot \mathbb{Z}[t].$$

So going from L_α to $L_{\alpha+1}$, we pick up an extra power of 5.

5^α

We have

- $L_\alpha \in \mathbb{Z}[t]$,
- $L_{\alpha+1} = U^{(\alpha)}(L_\alpha)$,
- $U^{(\alpha)}\left(5^k \cdot f\right) = 5^k \cdot U^{(\alpha)}(f)$,
- Also, $L_1 = 5t$.

Theorem

For every $\alpha \in \mathbb{Z}_{\geq 1}$,

$$U^{(\alpha)}\left(\frac{L_\alpha}{5^\alpha}\right) \in 5 \cdot \mathbb{Z}[t].$$

So going from L_α to $L_{\alpha+1}$, we pick up an extra power of 5. □

7^α, 11^α

The same technique works for

$$p\left(7^\alpha n+\lambda_{7,\alpha}\right)\equiv 0 \pmod{7^{\left\lfloor\frac{\alpha}{2}\right\rfloor+1}},$$

It does *not* work for

$$p\left(11^\alpha n+\lambda_{11,\alpha}\right)\equiv 0 \pmod{11^\alpha}.$$

Modular Group Action

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : N | c \right\}.$$

Let $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. We define a group action

$$\Gamma_0(N) \times \hat{\mathbb{H}} \longrightarrow \hat{\mathbb{H}},$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Define the orbits $[\tau]_N := \{\gamma\tau : \gamma \in \Gamma_0(N)\}$.

Modular Curves

Definition

For any $N \in \mathbb{Z}_{\geq 1}$, we define the classical modular curve of level N as the set of all orbits of $\Gamma_0(N)$ applied to $\hat{\mathbb{H}}$:

$$\mathrm{X}_0(N) := \left\{[\tau]_N : \tau \in \hat{\mathbb{H}}\right\}$$

Definition

For each $N \geq 1$ there exists some $d \geq 1$ and orbits $[r_k]_N$, $0 \leq k \leq d-1$, such that

$$\mathbb{Q} \cup \{\infty\} = \bigsqcup_{k=0}^{d-1} [r_k]_N.$$

The orbits $[r_k]_N$ are the cusps of $\mathrm{X}_0(N)$.

Modular Functions

Let $q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.

Definition

A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is modular over $\Gamma_0(N)$ if

- For any $\tau_1, \tau_2 \in \mathbb{H}$ such that $\tau_1 \in [\tau_2]_N$, $f(\tau_1) = f(\tau_2)$,
- For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n \geq n_\gamma} \alpha_\gamma(n) q^{n \cdot \gcd(c^2, N)/N},$$

with $n_\gamma \in \mathbb{Z}$, $\alpha_\gamma(n_\gamma) \neq 0$.

Modular Curves

For each partition congruence family, we can associate a compact Riemann surface.

$$p\left(5^{\alpha}n+\lambda_{5,\alpha}\right)\equiv 0 \pmod{5^{\alpha}} \longrightarrow \mathrm{X}_0(5),$$
$$p\left(7^{\alpha}n+\lambda_{7,\alpha}\right)\equiv 0 \pmod{7^{\lfloor\frac{\alpha}{2}\rfloor+1}} \longrightarrow \mathrm{X}_0(7),$$
$$p\left(11^{\alpha}n+\lambda_{11,\alpha}\right)\equiv 0 \pmod{11^{\alpha}} \longrightarrow \mathrm{X}_0(11).$$

These are the classical modular curves of level 5, 7, 11 (resp.).

Genus

The genus of a Riemann surface X, denoted $g(\mathrm{X})$, is the number of holes in the surface.

- $g(\mathrm{X}_0(1)) = 0$,
- $g(\mathrm{X}_0(5)) = 0$,
- $g(\mathrm{X}_0(7)) = 0$,
- $g(\mathrm{X}_0(11)) = 1$,
- $g(\mathrm{X}_0(20)) = 1$.

Why is this important?

Weierstrass Gap Theorem

Theorem

Let X *be a compact Riemann surface, and let*

$$f : \mathrm{X} \to \mathbb{C}$$

be holomorphic over X*, except for a pole at a point* $p \in \mathrm{X}$*. Then the order of* f *at* p *can assume any negative integer, with exactly* $g(\mathrm{X})$ *exceptions.*

Weierstrass Gap Theorem

Example: Let $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.

$$\frac{1}{t} = \frac{1}{q}\prod_{m=1}^{\infty}\left(\frac{1-q^m}{1-q^{5m}}\right)^6 = \frac{1}{q} + c(0) + c(1)q + \ldots$$

is holomorphic, except for $q = 0$ $(\tau = i\infty)$. And t induces

$$\begin{aligned}\hat{t} &: \mathrm{X}_0(5) \longrightarrow \mathbb{C}\\ &: [\tau]_5 \longmapsto t(\tau).\end{aligned}$$

$$g\left(\mathrm{X}_0(5)\right) = 0.$$

Weierstrass Gap Theorem

Example:

$$t = \frac{1}{q^2}\prod_{m=1}^{\infty}\left(\frac{1-q^{4m}}{1-q^{20m}}\right)^4\left(\frac{1-q^{10m}}{1-q^{2m}}\right)^2,$$

$$\rho = \frac{1}{q^3}\prod_{m=1}^{\infty}\left(\frac{1-q^{4m}}{1-q^{m}}\right)\left(\frac{1-q^{5m}}{1-q^{20m}}\right)^5$$

are holomorphic over $\mathrm{X}_0(20)$, except for $q=0$, with orders $-2, -3$. There is no such function with order -1.

$$g\left(\mathrm{X}_0(20)\right) = 1.$$

Corollaries

Let $\mathcal{M}^c\left(\Gamma_0(N)\right)$ be the space of modular functions over $\Gamma_0(N)$ with a pole at only one cusp $[c]_N$.

Corollary

If $g\left(\mathrm{X}_0(N)\right) = 0$, then there exists a function t such that $\mathcal{M}^c\left(\Gamma_0(N)\right) = \mathbb{C}[t]$.

Corollary

If $g\left(\mathrm{X}_0(N)\right) = 1$, then there exist functions t, ρ such that $\mathcal{M}^c\left(\Gamma_0(N)\right) = \mathbb{C}[t] \oplus \rho\mathbb{C}[t]$.

$c(n)$

Let $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$. Define $E_2(\tau)$ by

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Definition

Define $c(n)$ with the generating function

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{(2\cdot E_2(2\tau) - E_2(\tau))}{(q^2;q^2)_\infty}.$$

Congruences on $c(n)$

Theorem (Wang, Yang)

Let $n \geq 0$, $\alpha \geq 1$, *and* $12\delta_\alpha \equiv 1 \pmod{5^\alpha}$. *Then*

$$c\left(5^\alpha n + \delta_\alpha\right) \equiv 0 \pmod{5^\alpha},$$

Congruences on $\mathrm{spt}_\omega(n)$

Theorem (Wang, Yang)

Let $n \geq 0$, $\alpha \geq 1$, and $12\delta_\alpha \equiv 1 \pmod{5^\alpha}$. Then

$$\mathrm{spt}_\omega\left(2 \cdot 5^\alpha n + \delta_\alpha\right) \equiv 0 \pmod{5^\alpha},$$

with $\omega(q)$ defined as Ramanujan's third order mock theta function.

Proof in Terms of L_α

Let

$$L_{2\alpha-1} := (q^{10}; q^{10})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha-1} n + \delta_{2\alpha-1}\right) q^{n+1},$$

$$L_{2\alpha} := (q^{2}; q^{2})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha} n + \delta_{2\alpha}\right) q^{n+1}.$$

Proof in Terms of L_α

Let

$$L_{2\alpha-1} := (q^{10}; q^{10})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha-1} n + \delta_{2\alpha-1}\right) q^{n+1},$$

$$L_{2\alpha} := (q^2; q^2)_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha} n + \delta_{2\alpha}\right) q^{n+1}.$$

For example,

$$L_1 = (q^{10}; q^{10})_\infty \sum_{n=0}^{\infty} c\left(5n+3\right) q^{n+1}.$$

Proof in Terms of L_α

Let

$$L_{2\alpha-1} := (q^{10};q^{10})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha-1}n+\delta_{2\alpha-1}\right)q^{n+1},$$

$$L_{2\alpha} := (q^{2};q^{2})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha}n+\delta_{2\alpha}\right)q^{n+1}.$$

For example,

$$L_1 = (q^{10};q^{10})_\infty \sum_{n=0}^{\infty} c\left(5n+3\right)q^{n+1}.$$

The game is to show that $L_\alpha \equiv 0 \pmod{5^\alpha}$.

Technique by Wang and Yang

To begin with, examine L_1. In Wang and Yang's form:

Technique by Wang and Yang

To begin with, examine L_1. In Wang and Yang's form:

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

Technique by Wang and Yang

To begin with, examine L_1. In Wang and Yang's form:

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

with t, ρ eta quotients with integer power expansions, and F a modular form with constant term 1, over $\Gamma_0(10)$.

Technique by Wang and Yang

To begin with, examine L_1. In Wang and Yang's form:

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

with t, ρ eta quotients with integer power expansions, and F a modular form with constant term 1, over $\Gamma_0(10)$.

$$\frac{L_\alpha}{5^\alpha \cdot F} \in \mathbb{Z}[t] \oplus \rho\mathbb{Z}[t].$$

This is characteristic of the Paule–Radu method for proving congruences when the associated modular curve has genus 1.

Technique by Wang and Yang

To begin with, examine L_1. In Wang and Yang's form:

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

with t, ρ eta quotients with integer power expansions, and F a modular form with constant term 1, over $\Gamma_0(10)$.

$$\frac{L_\alpha}{5^\alpha \cdot F} \in \mathbb{Z}[t] \oplus \rho\mathbb{Z}[t].$$

This is characteristic of the Paule–Radu method for proving congruences when the associated modular curve has genus 1. However, the genus of $\mathrm{X}_0(10)$ is 0.

Our First Attempt

$$\text{Let } x := \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5(1-q^{5m})}{(1-q^m)^5(1-q^{10m})}.$$

$$\text{Then } \mathcal{M}^0\left(\Gamma_0(10)\right) = \mathbb{C}[x].$$

Theorem

$$x^3 \cdot \frac{L_1}{F} \in \mathbb{C}[x].$$

Corollary

$$\text{For all } \alpha \geq 1, \frac{L_\alpha}{F} \in \mathbb{C}[x, x^{-1}].$$

First Attempt

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right).$$

In terms of x:

First Attempt

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right).$$

In terms of x:

$$L_1 = F \cdot \left(-\frac{624}{625x^3} - \frac{2487}{625x^2} + \frac{801}{625x} - \frac{422}{125} - \frac{3148x}{125} + \frac{19904x^2}{625} + \frac{512x^3}{625} - \frac{256x^4}{625} \right).$$

Second Attempt

$$x = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5(1-q^{5m})}{(1-q^m)^5(1-q^{10m})}$$

Lemma

$$x \equiv 1 \pmod{5}.$$

Let $x = 1 + 5y$. Interestingly,

$$y = q \prod_{m=1}^{\infty} \frac{(1-q^{2m})(1-q^{10m})^3}{(1-q^m)^3(1-q^{5m})}.$$

Problem

We have a problem:

$$\frac{L_1}{F} \notin \mathbb{C}[y, y^{-1}].$$

But we *do* have

$$\frac{L_1}{F} \in \mathbb{C}[y, x^{-1}]...$$

Comparison of Expressions for L_1

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

In our form:

Comparison of Expressions for L_1

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

In our form:

$$L_1 = \frac{F}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right),$$

Comparison of Expressions for L_1

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

In our form:

$$L_1 = \frac{F}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right),$$

Important! $F, y, \frac{1}{1+5y}$ have integer power series expansions.

Comparison of Expressions for L_1

$$L_1 = F \cdot \left(245 \cdot t + 3750 \cdot t^2 + 15625 \cdot t^3 - \rho \cdot \left(125 \cdot t + 3125 \cdot t^2\right)\right),$$

In our form:

$$L_1 = \frac{F}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right),$$

Important! $F, y, \frac{1}{1+5y}$ have integer power series expansions.
Similar identities hold for L_2, L_3, etc.

Weak Result

We let $\alpha \geq 1$,

$$\mathcal{S} := \left\{ (1+5y)^n : n \in \mathbb{Z}_{\geq 0} \right\},$$
$$\mathbb{Z}[y]_{\mathcal{S}} := \text{ the localization of } \mathbb{Z}[y] \text{ at } \mathcal{S}.$$

Then we have the following:

Theorem (Me!)

$$\frac{1}{5^{\alpha} F} \cdot L_{\alpha} \in \mathbb{Z}[y]_{\mathcal{S}}.$$

Strong Result

We let $\alpha \geq 1$,

$$\psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{12} \right\rfloor + 1.$$

Then we have the following:

Theorem (Me!)

$$\frac{(1+5y)^{\psi(\alpha)}}{5^{\alpha} F} \cdot L_{\alpha} \in \mathbb{Z}[y].$$

L_α

$$L_1 = \frac{F}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right).$$

We will prove that

$$\frac{1}{5^\alpha F} \cdot L_\alpha = \sum_{m\geq 1} s(m) \cdot 5^{\mu(m)} \cdot \frac{y^m}{(1+5y)^n},$$

with $n \in \mathbb{Z}_{\geq 1}$ fixed, s, μ integer-valued functions, and s discrete.

U Operator

$$U_5\left(L_{2\alpha-1}\right) = L_{2\alpha},$$
$$U_5\left(Z \cdot L_{2\alpha}\right) = L_{2\alpha+1},$$

for a certain eta quotient Z. We define

$$U^{(i)}\left(f\right) := \frac{1}{F} \cdot U_5\left(F \cdot Z^{1-i} \cdot f\right).$$

Then

$$\frac{L_{\alpha+1}}{F} = U^{(i)}\left(\frac{L_\alpha}{F}\right),$$

for $i \equiv \alpha \pmod{2}$.

U Operator

$$\frac{1}{5^{\alpha}F} \cdot L_{\alpha} = \sum_{m\geq 1} s(m) \cdot 5^{\mu(m)} \cdot \frac{y^m}{(1+5y)^n},$$

We study

$$U^{(i)}\left(\frac{y^m}{(1+5y)^n}\right).$$

General Relation

Theorem

There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ *and functions* $\pi_i : \mathbb{Z}^2_{\geq 1} \to \mathbb{Z}_{\geq 0}$ *such that*

$$U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right) = \frac{1}{(1+5y)^{5n-4}} \sum_{r\geq\lceil m/5\rceil} h_1(m,n,r)\cdot 5^{\pi_1(m,r)}\cdot y^r,$$

$$U^{(0)}\left(\frac{y^m}{(1+5y)^n}\right) = \frac{1}{(1+5y)^{5n-2}} \sum_{r\geq\lceil (m+2)/5\rceil} h_0(m,n,r)\cdot 5^{\pi_0(m,r)}\cdot y^r.$$

General Relation

$$\pi_1(m,r) := \begin{cases} 0, & 1 \leq m \leq 2 \text{ and } r = 1 \\ 3, & 1 \leq m \leq 2 \text{ and } r = 3 \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & 1 \leq m \leq 2 \text{ and } r \neq 1, 3 \\ 2, & m = 3 \text{ and } r = 2 \\ \left\lfloor \frac{5r-2}{6} \right\rfloor, & m = 3 \text{ and } r \neq 2 \\ \left\lfloor \frac{5r-m+1}{6} \right\rfloor, & m \geq 4, \end{cases}$$

$$\pi_0(m,r) := \begin{cases} \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 1 \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 2 \text{ and } r \neq 3, 4, 5 \\ \left\lfloor \frac{5r-5}{6} \right\rfloor, & m = 2 \text{ and } 3 \leq r \leq 5 \\ \left\lfloor \frac{5r-m-2}{6} \right\rfloor, & m \geq 3. \end{cases}$$

Proof Strategy

$$\mathcal{Z}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta(m)} \cdot y^m : s \text{ is discreet} \right\},$$

$$\mathcal{V}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\phi(m)} \cdot y^m : s \text{ is discreet} \right\}.$$

$$\theta(m) := \begin{cases} \left\lfloor \frac{5m-5}{6} \right\rfloor, & 1 \leq m \leq 2 \\ \left\lfloor \frac{5m-5}{6} \right\rfloor - 1, & m \geq 3, \end{cases}$$

$$\phi(m) := \begin{cases} \left\lfloor \frac{5m-5}{6} \right\rfloor, & 1 \leq m \leq 3 \\ \left\lfloor \frac{5m-5}{6} \right\rfloor - 1, & m \geq 4. \end{cases}$$

Proof Strategy

$$\mathcal{Z}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta(m)}\cdot y^m : s \text{ is discreet} \right\},$$

$$\mathcal{V}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\phi(m)}\cdot y^m : s \text{ is discreet} \right\}.$$

Show that $\frac{1}{5F}L_1 \in \mathcal{Z}_3$,

Show that for any $f\in\mathcal{Z}_n$, $\frac{1}{5}U^{(1)}(f)\in\mathcal{V}_{5n-4}$,

Show that for any $f\in\mathcal{V}_n$, $\frac{1}{5}U^{(0)}(f)\in\mathcal{Z}_{5n-2}$.

Even-to-Odd Index

Let $f \in \mathcal{V}_n$. Then

$$\begin{aligned}
U^{(0)}(f) =& U^{(0)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\phi(m)}\cdot y^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 5^{\phi(m)}\cdot U^{(0)}\left(\frac{y^m}{(1+5y)^n}\right)\\
=& \frac{1}{(1+5y)^{5n-2}}\sum_{m\geq 1}\sum_{r\geq\lceil (m+2)/5\rceil} s(m)\cdot h_0(m,n,r)\cdot 5^{\phi(m)+\pi_0(m,r)}\cdot y^r\\
=& \frac{1}{(1+5y)^{5n-2}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)\cdot 5^{\phi(m)+\pi_0(m,r)}\cdot y^r
\end{aligned}$$

We want to show that

$$\phi(m)+\pi_0(m,r)\geq \theta(r)+1 \text{ for all } r\geq 1,$$
$$\text{so that } \frac{1}{5}U^{(0)}(f)\in \mathcal{Z}_{5n-2}.$$

Odd-to-Even Index

Let $f \in \mathcal{Z}_n$. Then

$$\begin{aligned} U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta(m)}\cdot y^m\right)\\ =& \sum_{m\geq 1} s(m)\cdot 5^{\theta(m)}\cdot U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right)\\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{m\geq 1}\sum_{r\geq \lceil m/5\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta(m)+\pi_1(m,r)}\cdot y^r\\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta(m)+\pi_1(m,r)}\cdot y^r \end{aligned}$$

We want to show that

$$\theta(m)+\pi_1(m,r) \geq \phi(r)+1 \text{ for all } r\geq 1,$$
$$\text{so that } \frac{1}{5}U^{(1)}(f)\in \mathcal{V}_{5n-4}.$$

5-adic Irregularity

We are going to prove that

$$\phi(m) + \pi_0(m, r) \geq \theta(r) + 1 \text{ for all } r \geq 1,$$
$$\theta(m) + \pi_1(m, r) \geq \phi(r) + 1 \text{ for all } r \geq 1.$$

5-adic Irregularity

We are going to prove that

$$\phi(m) + \pi_0(m, r) \geq \theta(r) + 1 \text{ for all } r \geq 1,$$
$$\theta(m) + \pi_1(m, r) \geq \phi(r) + 1 \text{ for all } r \geq 1.$$

No we aren't.

5-adic Irregularity

We are going to prove that

$$\phi(m) + \pi_0(m, r) \geq \theta(r) + 1 \text{ for all } r \geq 1,$$
$$\theta(m) + \pi_1(m, r) \geq \phi(r) + 1 \text{ for all } r \geq 1.$$

No we aren't.

$$\phi(m) + \pi_0(m, r) \geq \theta(r) + 1 \text{ for all } r \geq 1 \text{ is true.}$$
$$\theta(m) + \pi_1(m, r) \geq \phi(r) + 1, \text{ on the other hand...}$$

5-adic Irregularity

Let $f \in \mathcal{Z}_n$. Then

$$\begin{aligned}
U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta(m)}\cdot y^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 5^{\theta(m)}\cdot U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right)\\
=& \frac{1}{(1+5y)^{5n-4}}\sum_{m\geq 1}\sum_{r\geq\lceil m/5\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta(m)+\pi_1(m,r)}\cdot y^r\\
=& \frac{1}{(1+5y)^{5n-4}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta(m)+\pi_1(m,r)}\cdot y^r
\end{aligned}$$

The coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ is

$$\sum_{m=1}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta(m)+\pi_1(m,1)}.$$

5-adic Irregularity

The coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ has the form

$$= \sum_{m=1}^{5} s(m) \cdot h_1(m,n,1) \cdot 5^{\theta(m)+\pi_1(m,1)}$$

$$= \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) + s(4) \cdot h_1(4,n,1) \cdot 5 + s(5) \cdot h_1(5,n,1) \cdot 5^2$$

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 5.$$

5-adic Irregularity

Lemma

For all m, n such that $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$\begin{aligned}
h_0(1, n, 1) &\equiv 1 \pmod{5}, \\
h_0(2, 5n-4, 1) &\equiv 0 \pmod{5}, \\
h_0(3, n, 1) &\equiv 1 \pmod{5}, \\
h_0(1, n, 2) &\equiv 4 \pmod{5}, \\
h_0(2, 5n-4, 2) &\equiv 4 \pmod{5}, \\
h_0(3, n, 2) &\equiv 4 \pmod{5}, \\
h_0(2, 5n-4, 3) &\equiv 1 \pmod{5}, \\
h_1(m, n, 1) &\equiv 1 \pmod{5}.
\end{aligned}$$

5-adic Irregularity

Our coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ for $U^{(1)}(f)$ is

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 5$$

$$\equiv \sum_{m=1}^{3} s(m) \pmod 5.$$

5-adic Irregularity

Our coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ for $U^{(1)}(f)$ is

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 5$$
$$\equiv \sum_{m=1}^{3} s(m) \pmod 5.$$

Examine L_1:

$$L_1 = \frac{5 \cdot F}{(1+5y)^3} \cdot \left(24y + 361y^2 + 2410y^3 + 7900y^4 + 10000y^5\right).$$

5-adic Irregularity

Definition

$$\mathcal{W}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta(m)} \cdot y^m : \sum_{m=1}^{3} s(m) \equiv 0 \bmod 5 \right\},$$

$$\mathcal{V}_n := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m) \cdot 5^{\phi(m)} \cdot y^m \right\}.$$

Here s again represents a discrete integer-valued function.

Resolving 5-adic Irregularity

Theorem

Suppose $f \in \mathcal{W}_n$. Then

$$\frac{1}{5} \cdot U^{(1)}(f) \in \mathcal{V}_{5n-4},$$
$$\frac{1}{5^2} \cdot U^{(0)} \circ U^{(1)}(f) \in \mathcal{W}_{25n-22}.$$

Sketch

Let $f \in \mathcal{W}_n$. Then

$$\frac{1}{5^2} \cdot \left(U^{(0)} \circ U^{(1)} (f) \right) = \frac{1}{(1+5y)^{25n-22}} \sum_{w \geq 1} t(w) \cdot 5^{\theta(w)} y^w,$$

$$t(w) = \sum_{r=1}^{5w-2} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, w)$$
$$\times 5^{\theta(m)+\pi_1(m,r)+\pi_0(r,w)-\theta(w)-2}.$$

Sketch

$$t(1) = \sum_{r=1}^{3} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 1) \cdot 5^{\lambda(m,r,1)},$$

$$t(2) = \sum_{r=1}^{8} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 2) \cdot 5^{\lambda(m,r,2)},$$

$$t(3) = \sum_{r=1}^{13} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 3) \cdot 5^{\lambda(m,r,3)},$$

$$\lambda(m,r,w) := \theta(m) + \pi_1(m,r) + \pi_0(r,w) - 2.$$

We want to show that $t(1), t(2), t(3) \in \mathbb{Z}$, and that $t(1) + t(2) + t(3) \equiv 0 \pmod 5$.

Sketch

$$
\begin{aligned}
t(1)+t(2)+t(3) \equiv & \frac{1}{5}\cdot\left(\sum_{j=1}^{2} h_0(1,5n-4,j)\right)\cdot\left(\sum_{m=1}^{3} s(m)\cdot h_1(m,n,1)\right)\\
&+h_0(1,5n-4,3)\cdot\left(\sum_{m=1}^{3} s(m)\cdot h_1(m,n,1)\right)\\
&+\left(\sum_{j=1}^{2} h_0(1,5n-4,j)\right)\cdot s(4)\cdot h_1(4,n,1)\\
&+\left(\sum_{j=1}^{3} h_0(2,5n-4,j)\right)\cdot\sum_{m=1}^{2} s(m)\cdot h_1(m,n,2)\\
&+\left(\sum_{j=1}^{2} h_0(3,5n-4,j)\right)\cdot s(3)\cdot h_1(3,n,3) \pmod 5.
\end{aligned}
$$

Sketch

It's That Lemma Again

For all m, n such that $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$\begin{aligned}
h_0(1, n, 1) &\equiv 1 \pmod{5},\\
h_0(2, 5n-4, 1) &\equiv 0 \pmod{5},\\
h_0(3, n, 1) &\equiv 1 \pmod{5},\\
h_0(1, n, 2) &\equiv 4 \pmod{5},\\
h_0(2, 5n-4, 2) &\equiv 4 \pmod{5},\\
h_0(3, n, 2) &\equiv 4 \pmod{5},\\
h_0(2, 5n-4, 3) &\equiv 1 \pmod{5},\\
h_1(m, n, 1) &\equiv 1 \pmod{5}.
\end{aligned}$$

Therefore, $t(1) + t(2) + t(3) \equiv 0 \pmod{5}$.

Sketch

It's That Lemma Again

For all m, n such that $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$\begin{aligned}
h_0(1, n, 1) &\equiv 1 \pmod{5}, \\
h_0(2, 5n-4, 1) &\equiv 0 \pmod{5}, \\
h_0(3, n, 1) &\equiv 1 \pmod{5}, \\
h_0(1, n, 2) &\equiv 4 \pmod{5}, \\
h_0(2, 5n-4, 2) &\equiv 4 \pmod{5}, \\
h_0(3, n, 2) &\equiv 4 \pmod{5}, \\
h_0(2, 5n-4, 3) &\equiv 1 \pmod{5}, \\
h_1(m, n, 1) &\equiv 1 \pmod{5}.
\end{aligned}$$

Therefore, $t(1) + t(2) + t(3) \equiv 0 \pmod{5}$. □

Proof of our Strong Result

Proof (I)

$$\frac{1}{5F} \cdot L_1 \in \mathcal{W}_3.$$

Suppose that for some $\alpha \in \mathbb{Z}_{\geq 1}$, there exists some $n \in \mathbb{Z}_{\geq 1}$ such that

$\frac{1}{5^{2\alpha-1}F} \cdot L_{2\alpha-1} \in \mathcal{W}_n$. Then

$L_{2\alpha-1} = 5^{2\alpha-1}F \cdot f_{2\alpha-1}$, for $f_{2\alpha-1} \in \mathcal{W}_n$. Now,

$L_{2\alpha} = U_5\left(L_{2\alpha-1}\right) = U_5\left(5^{2\alpha-1}F \cdot f_{2\alpha-1}\right) = 5^{2\alpha-1}F \cdot U^{(1)}(f_{2\alpha-1})$.

There exists some $f_{2\alpha} \in \mathcal{V}_{5n-4}$ such that $U^{(1)}(f_{2\alpha-1}) = 5 \cdot f_{2\alpha}$. Therefore,

$$L_{2\alpha} = 5^{2\alpha}F \cdot f_{2\alpha}, \text{ and } \frac{1}{5^{2\alpha}F} \cdot L_{2\alpha} \in \mathcal{V}_{5n-4}.$$

Proof of our Strong Result

Proof (II)

$$L_{2\alpha+1} = U_5\left(Z \cdot L_{2\alpha}\right) = U_5\left(5^{2\alpha} F \cdot Z \cdot f_{2\alpha}\right) = 5^{2\alpha} F \cdot U^{(0)}(f_{2\alpha}).$$

There exists some $f_{2\alpha+1} \in \mathcal{W}_{25n-22}$ such that $U^{(0)}(f_{2\alpha}) = 5 \cdot f_{2\alpha+1}$. Therefore,

$$L_{2\alpha+1} = 5^{2\alpha+1} F \cdot f_{2\alpha+1}, \text{ and } \frac{1}{5^{2\alpha+1} F} \cdot L_{2\alpha+1} \in \mathcal{W}_{25n-22}.$$

Proof of our Strong Result

Proof (III)

$$\psi(\alpha) = \left\lfloor \frac{5^{\alpha+1}}{12} \right\rfloor + 1.$$

Establishing that $\psi(\alpha)$ give the appropriate indices for $\mathcal{V}_n, \mathcal{W}_n$ is an elementary exercise in number theory. Prove that

$$\psi(1) = 3,$$
$$5\psi(2\alpha - 1) - 4 = \psi(2\alpha),$$
$$5\psi(2\alpha) - 2 = \psi(2\alpha + 1).$$

□

Computational Considerations

We have a degree 5 modular equation for y (and for $1+5y$).
So for any pattern, we would expect 25 initial relations for each value of i to prove by induction—50 relations, total.
Let $x = 1+5y$. Then

$$
\begin{aligned}
U^{(i)}\left(\frac{y^m}{(1+5y)^n}\right) &= \frac{1}{5^m}\cdot U^{(i)}\left(\frac{(x-1)^m}{x^n}\right)\\
&= \frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left(x^{r-n}\right)\\
&= \frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left((1+5y)^{r-n}\right).
\end{aligned}
$$

Computational Considerations

If $n \geq 0$, then

$$U^{(i)}\left((1+5y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} \cdot 5^k \cdot U^{(i)}\left(y^k\right).$$

So all we really need are ten relations—five for each i—for $U^{(i)}\left(y^k\right)$. Then we can confirm the initial relations for any patern on $U^{(i)}\left(\frac{y^m}{(1+5y)^n}\right)$.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.
- We require 50 initial cases.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.
- We require 50 initial cases.
- However, these are algebraically dependent: we only need to directly prove 10 initial cases...

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.
- We require 50 initial cases.
- However, these are algebraically dependent: we only need to directly prove 10 initial cases...
- (in contrast to the 20 that Wang and Yang needed)

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.
- We require 50 initial cases.
- However, these are algebraically dependent: we only need to directly prove 10 initial cases...
- (in contrast to the 20 that Wang and Yang needed)
- Our proof reveals some interesting algebraic structure in the form of the localized polynomial ring.

Summary

- Wang and Yang's proof utilized techniques for handling congruences with an associated Riemann surface of genus 1.
- However, the congruences for this problem are associated with a genus 0 Riemann surface.
- Our new approach uses one modular function instead of two.
- We require 50 initial cases.
- However, these are algebraically dependent: we only need to directly prove 10 initial cases...
- (in contrast to the 20 that Wang and Yang needed)
- Our proof reveals some interesting algebraic structure in the form of the localized polynomial ring.
- Finally, there are some extremely difficult steps in showing that going from L_α to $L_{\alpha+1}$ *always* picks up an extra power of 5.

Localization Method

- Let $\mathcal{L} := (L_\alpha)_{\alpha\geq 1}$ be a sequence of modular functions over some $\Gamma_0(N)$, such that

Localization Method

- Let $\mathcal{L} := (L_\alpha)_{\alpha\geq 1}$ be a sequence of modular functions over some $\Gamma_0(N)$, such that
- $g\left(\mathrm{X}_0(N)\right) = 0$

Localization Method

- Let $\mathcal{L} := (L_\alpha)_{\alpha \geq 1}$ be a sequence of modular functions over some $\Gamma_0(N)$, such that
- $g\left(\mathrm{X}_0(N)\right) = 0$
- Let y be a chosen so that $\mathcal{M}^{a/c}\left(\Gamma_0(N)\right) = \mathbb{C}[y]$.

Localization Method

- Let $\mathcal{L} := (L_\alpha)_{\alpha\geq 1}$ be a sequence of modular functions over some $\Gamma_0(N)$, such that
- $g\left(\mathrm{X}_0(N)\right) = 0$
- Let y be a chosen so that $\mathcal{M}^{a/c}\left(\Gamma_0(N)\right) = \mathbb{C}[y]$.

There exists some $p \in \mathbb{C}[X]$ and some nonnegative integer sequence $\{\psi(\alpha)\}_{\alpha\geq 1}$ such that $p(y) \in \mathcal{E}^{a/c}\left(\Gamma_0(N)\right)$ and

$$p(y)^{\psi(\alpha)} \cdot L_\alpha \in \mathbb{C}[y].$$

Localization Method

$$\text{If } L_\alpha = \sum_{m\geq 1} s(m)\cdot \ell^{\nu_\alpha(m)}\cdot \frac{y^m}{p(y)^{\psi(\alpha)}} \in \mathbb{Z}[y]_{\mathcal{S}},$$

with $\mathcal{S} := \{p(y)^n : n \in \mathbb{Z}_{\geq 0}\}$, and

$$U^{(\alpha)}\left(L_\alpha\right) = L_{\alpha+1}$$

for some linear operator sequence $\left(U^{(\alpha)}\right)_{\alpha\geq 1}$, then we want to understand

$$U^{(\alpha)}\left(\frac{y^m}{p(y)^{\psi(\alpha)}}\right).$$

Future Work

- This result has enormous potential!

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.
- The result relates arithmetic, algebra, analysis, topology, computational methods, and experimental math.

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.
- The result relates arithmetic, algebra, analysis, topology, computational methods, and experimental math.
- There are theoretical reasons to believe that this modified technique may be used to prove *all* congruence families over a genus 0 surface.

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.
- The result relates arithmetic, algebra, analysis, topology, computational methods, and experimental math.
- There are theoretical reasons to believe that this modified technique may be used to prove *all* congruence families over a genus 0 surface.
- But we need more examples!

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.
- The result relates arithmetic, algebra, analysis, topology, computational methods, and experimental math.
- There are theoretical reasons to believe that this modified technique may be used to prove *all* congruence families over a genus 0 surface.
- But we need more examples!
- I'm looking through the literature now for any pathological congruences that are resistant to classical techniques.

Future Work

- This result has enormous potential!
- This is an extension of the original technique by Ramanujan and Watson.
- The localization structure has never been studied before in this subject.
- The result relates arithmetic, algebra, analysis, topology, computational methods, and experimental math.
- There are theoretical reasons to believe that this modified technique may be used to prove *all* congruence families over a genus 0 surface.
- But we need more examples!
- I'm looking through the literature now for any pathological congruences that are resistant to classical techniques.
- I'm working on a research proposal to study this method further (Hint, *Hint*).

References

- F. Diamond, J. Shurman, *A First Course in Modular Forms*, 4th Printing., Springer Publishing (2016).
- M. Knopp, *Modular Functions in Analytic Number Theory*, 2nd Ed., AMS Chelsea Publishing, 1993.
- P. Paule, S. Radu, "The Andrews–Sellers Family of Partition Congruences," *Advances in Mathematics* 230, pp. 819-838 (2012).
- S. Ramanujan, G.H. Hardy, P.V. Seshu Aiyar, B.M. Wilson, Bruce C. Berndt, *Collected Papers of Srinivasa Ramanujan*, Cambridge University Press, 1927; Reissued AMS–Chelsea, 2000.
- N. Smoot, "A Single-Variable Proof of the Omega SPT Congruence Family Over Powers of 5," (Submitted) Available at `https://risc.jku.at/m/nicolas-smoot/` (2020).
- L. Wang, "New Congruences for Partitions Related to Mock Theta Functions," *Journal of Number Theory* 175, pp. 51-65 (2017).
- L. Wang, Y. Yang, "The Smallest Parts Function Associated with $\omega(q)$," (Submitted), Available at `https://arxiv.org/pdf/1812.00379.pdf` (2018).