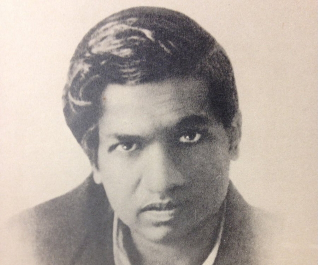
VARIANTS OF LEHMER'S CONJECTURE

J. Balakrishnan, W. Craig, K. Ono, and W.-L. Tsai

"ON CERTAIN ARITHMETICAL FUNCTIONS" (1916)



Srinivasa Ramanujan

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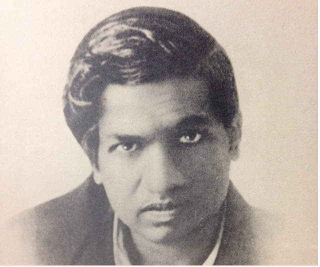

Srinivasa Ramanujan

Ramanujan defined the tau-function with the **infinite product**

$$\sum_{n=1}^{\infty} \tau(n) q^n := q \left((1-q^1)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots\right)^{24}$$

$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - \ldots.$$

THE PROTOTYPE

FACT

The function $\Delta(z) := \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$
is a **weight 12 modular (cusp) form for** $\mathrm{SL}_2(\mathbb{Z})$.

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For $\mathrm{Im}(z) > 0$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *this means that*

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12} \Delta(z).$$

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UBIQUITY OF FUNCTIONS LIKE $\Delta(z)$

- *Arithmetic Geometry: Elliptic curves, BSD Conjecture,...*
- *Number Theory: Partitions, Quad. forms, ...*
- *Mathematical Physics: Mirror symmetry,...*
- *Representation Theory: Moonshine, symmetric groups,...*

TESTING GROUND (HECKE OPERATORS)

THEOREM (MORDELL (1917))

The following are true:

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- *(30s) Theory of Hecke operators (linear endomorphisms)*

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- *(30s) Theory of Hecke operators (linear endomorphisms)*
- *(70s) Atkin-Lehner Theory of* **newforms** *(i.e. eigenforms)*

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DAWN OF GALOIS REPRESENTATIONS

- (Serre & Deligne, 70s) *Reformulated using representations*
$$\rho_{\Delta,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{F}_\ell).$$

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- (Wiles, 90s) *Used to prove Fermat's Last Theorem.*

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  Proof of the Weil Conjectures $\Longrightarrow$ *Ramanujan's Conjecture.*

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- (Ramanujan-Petersson)
  Generalized to newforms and generic automorphic forms.

LEHMER'S CONJECTURE

D. H. Lehmer

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CONJECTURE (LEHMER (1947))

For every $n \geq 1$ *we have* $\tau(n) \neq 0$.

RESULTS ON LEHMER'S CONJECTURE

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If $\tau(n) = 0$, *then* n *is prime.*

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We have that

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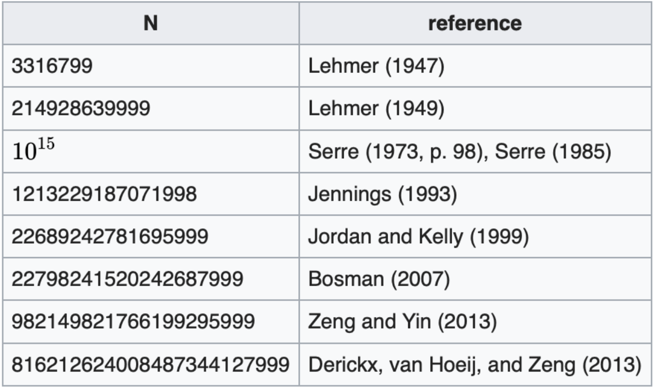
$$\#\{\text{prime } p \leq X \ : \ \tau(p) = 0\} \ll \pi(X) \cdot \frac{(\log\log X)^2}{\log(X)}.$$

Namely, the set of p *for which* $\tau(p) = 0$ *has* **density zero***.*

NUMERICAL INVESTIGATIONS



N	reference
3316799	Lehmer (1947)
214928639999	Lehmer (1949)
10^{15}	Serre (1973, p. 98), Serre (1985)
1213229187071998	Jennings (1993)
22689242781695999	Jordan and Kelly (1999)
22798241520242687999	Bosman (2007)
982149821766199295999	Zeng and Yin (2013)
816212624008487344127999	Derickx, van Hoeij, and Zeng (2013)

Lehmer's Conjecture confirmed for $n \leq N$

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At most finitely many non-CM level N newforms

$$f = q + \sum_{n=2}^{\infty} a_f(n) q^n$$

have $a_f(p) = 0$.

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(3) Classifying soln's to $\tau(n) = \alpha$ not done in any other cases.

CAN $|\tau(n)| = \ell^m$, A POWER OF AN ODD PRIME?

THEOREM (B-C-O-T)

If $|\tau(n)| = \ell^m$, *then* $n = p^{d-1}$, *with* p *and* $d \mid \ell(\ell^2 - 1)$ *are odd primes.*

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1. *List the finitely many odd primes* $d \mid \ell(\ell^2-1)$.
2. *For each* d, **simply** *solve* $\tau(p^{d-1}) = \pm\ell^m$ *for primes* p.

A SATISFYING RESULT

THEOREM (B-C-O-T + UVA REU)

For $n > 1$ we have

$$\tau(n) \notin \{\pm 1, \pm 691\} \cup \{\pm \ell \;:\; 3 \leq \ell < 100 \text{ prime}\}.$$

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REMARK (UVA REU)

These results have been extended to $|\tau(n)| = \alpha$ odd.

GENERAL RESULTS

OUR SETTING

Let $f \in S_{2k}(N)$ *be a level* N *weight* $2k$ **newform** *with*

$$f(z) = q + \sum_{n=2}^{\infty} a_f(n) q^n \cap \mathbb{Z}[[q]] \qquad (q := e^{2\pi i z})$$

and **trivial mod 2 residual Galois representation**.

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- *Elliptic curves* $E/\mathbb{Q}$ *with a rational 2-torsion point.*
- *All forms of level* $2^a M$ *with* $a \geq 0$ *and* $M \in \{1, 3, 5, 15, 17\}$.

GENERAL RESULTS (ℓ AN ODD PRIME)

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Suppose that $2k \geq 4$ *and* $a_f(2)$ *is even.*
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Assuming GRH, we have

$$a_f(n) \notin \{\pm 1\} \cup \{\pm \ell \ : \ 3 \leq \ell \leq 97 \text{ prime with } \ell \neq 37\} \cup \{-37\}.$$

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$$a_f(3^2) = 37, \quad a_f(3^2) = -11, \quad a_f(3^2) = -23,$$
$$a_f(3^4) = 19, \quad a_f(5^2) = 19, a_f(7^2) = -19,$$
$$a_f(7^4) = 11, \quad a_f(17^2) = -13, \quad a_f(43^2) = 17.$$

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3. UVA REU *will study odd wt, Nebentypus, and general* α.

EXAMPLE: THE WEIGHT 16 HECKE EIGENFORM

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The Hecke eigenform $E_4\Delta$

$$E_4(z)\Delta(z) := \left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\cdot\Delta(z)$$

has no coefficients with absolute value $3 \leq \ell \leq 37$ (GRH $\Longrightarrow \ell \leq 97$.)

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EXAMPLE

We have $M^{\pm}(3, m) = 2m + \sqrt{m} \cdot 10^{32}$.

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REMARK

In 2013 Lygeros and Rozier found further prime values of $\tau(n)$.

NUMBER OF PRIME DIVISORS OF $\tau(n)$

NOTATION

$\Omega(n)$:= number of prime divisors of n **with multiplicity**

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If $n > 1$ is an integer, then

$$\Omega(\tau(n)) \geq \sum_{\substack{p \mid n \\ prime}} (\sigma_0(\mathrm{ord}_p(n)+1)-1) \geq \omega(n).$$

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2. *A generalization exists for newforms with integer coefficients and trivial residual mod 2 Galois representation.*

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$\Longrightarrow \quad \{1 = \tau(p^0), \tau(p), \tau(p^2), \tau(p^3), \dots\}$ is **periodic** modulo ℓ.

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$$\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-2}).$$

$$\implies \quad \{1 = \tau(p^0), \tau(p), \tau(p^2), \tau(p^3), \dots\} \text{ is } \textbf{periodic} \text{ modulo } \ell.$$

(3) The **first time** $\ell \mid \tau(p^{d-1})$ has $d \mid \ell(\ell^2 - 1)$.

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(7) Any soln gives an integer point on a genus $g \geq 1$ algebraic curve, which by Siegel has **finitely many** (if any) integer points.

PRIMITIVE PRIME DIVISORS

DEFINITION

A term $a(n)$ in an integer sequence $\{a(1), a(2), \ldots\}$ has a **primitive prime divisor** if there is a prime ℓ for which TFAT:

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EXAMPLE (CARMICHAEL 1913)

The Fibonacci numbers in red are defective:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$$

$F_{12} = 144$ is **the last** defective one!

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Their **Lucas numbers** $\{u_n(\alpha, \beta)\} = \{u_1 = 1, u_2 = \alpha + \beta, \dots\}$ are:

$$u_n(\alpha, \beta) := \frac{\alpha^n - \beta^n}{\alpha - \beta} \in \mathbb{Z}.$$

PRIMITIVE PRIME DIVISORS

THEOREM (BILU, HANROT, VOUTIER (2001))

Lucas numbers $u_n(\alpha, \beta)$, *with* $n > 30$, *have primitive prime divisors.*

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A classification of defective Lucas numbers is obtained:

- *Finitely many* **sporadic** *sequences*
- *Explicit* **parameterized infinite families.**

RELEVANT LUCAS SEQUENCES

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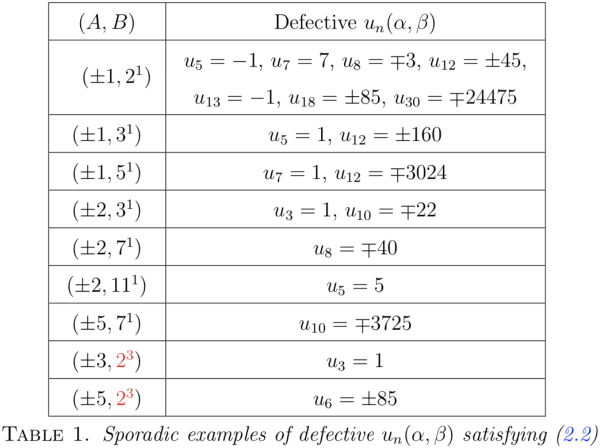
COROLLARY (BRUTE FORCE)

The potentially modular defective Lucas numbers have been classified.



(A, B)	Defective $u_n(\alpha, \beta)$
$(\pm 1, 2^1)$	$u_5 = -1,\ u_7 = 7,\ u_8 = \mp 3,\ u_{12} = \pm 45,$ $u_{13} = -1,\ u_{18} = \pm 85,\ u_{30} = \mp 24475$
$(\pm 1, 3^1)$	$u_5 = 1,\ u_{12} = \pm 160$
$(\pm 1, 5^1)$	$u_7 = 1,\ u_{12} = \mp 3024$
$(\pm 2, 3^1)$	$u_3 = 1,\ u_{10} = \mp 22$
$(\pm 2, 7^1)$	$u_8 = \mp 40$
$(\pm 2, 11^1)$	$u_5 = 5$
$(\pm 5, 7^1)$	$u_{10} = \mp 3725$
$(\pm 3, 2^3)$	$u_3 = 1$
$(\pm 5, 2^3)$	$u_6 = \pm 85$

TABLE 1. *Sporadic examples of defective $u_n(\alpha, \beta)$ satisfying (2.2)*

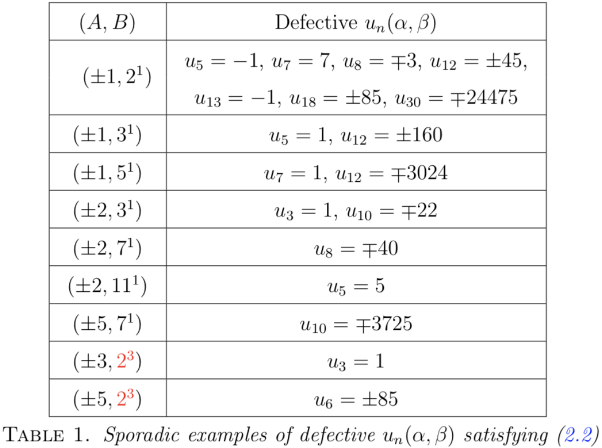


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TABLE 1. *Sporadic examples of defective $u_n(\alpha, \beta)$ satisfying (2.2)*

REMARK

Since $(A, B) = (A, p^{2k-1})$, there are only two with weight $2k \geq 4$.

$$B_{1,k}^{r,\pm} : Y^2 = X^{2k-1} \pm 3^r, \quad B_{2,k} : Y^2 = 2X^{2k-1} - 1, \quad B_{3,k}^{\pm} : Y^2 = 2X^{2k-1} \pm 2,$$
$$B_{4,k}^{r} : Y^2 = 3X^{2k-1} + (-2)^{r+2}, \quad B_{5,k}^{\pm} : Y^2 = 3X^{2k-1} \pm 3, \quad B_{6,k}^{r,\pm} : Y^2 = 3X^{2k-1} \pm 3 \cdot 2^r.$$

(A, B)	Defective $u_n(\alpha, \beta)$	Constraints on parameters
$(\pm m, p)$	$u_3 = -1$	$m > 1$ and $p = m^2 + 1$
$(\pm m, p^{2k-1})$	$u_3 = \varepsilon 3^r$	$(p, \pm m) \in B_{1,k}^{r,\varepsilon}$ with $3 \nmid m$, $(\varepsilon, r, m) \neq (1, 1, 2)$, and $m^2 \geq 4\varepsilon 3^{r-1}$
$(\pm m, p^{2k-1})$	$u_4 = \mp m$	$(p, \pm m) \in B_{2,k}$ with $m > 1$ odd
$(\pm m, p^{2k-1})$	$u_4 = \pm 2\varepsilon m$	$(p, \pm m) \in B_{3,k}^{\varepsilon}$ with $(\varepsilon, m) \neq (1, 2)$ and $m > 2$ even
$(\pm m, p^{2k-1})$	$u_6 = \pm(-2)^r m(2m^2 + (-2)^r)/3$	$(p, \pm m) \in B_{4,k}^{r}$ with $\gcd(m, 6) = 1$, $(r, m) \neq (1, 1)$, and $m^2 \geq (-2)^{r+2}$
$(\pm m, p^{2k-1})$	$u_6 = \pm \varepsilon m(2m^2 + 3\varepsilon)$	$(p, \pm m) \in B_{5,k}^{\varepsilon}$ with $3 \mid m$ and $m > 3$
$(\pm m, p^{2k-1})$	$u_6 = \pm 2^{r+1} \varepsilon m(m^2 + 3\varepsilon \cdot 2^{r-1})$	$(p, \pm m) \in B_{6,k}^{r,\varepsilon}$ with $m \equiv 3 \bmod 6$ and $m^2 \geq 3\varepsilon \cdot 2^{r+2}$

TABLE 2. Parameterized families of defective $u_n(\alpha, \beta)$ satisfying (2.2)
Notation: $m, k, r \in \mathbb{Z}^+$, $\varepsilon = \pm 1$, p *is a prime number.*

KEY LEMMAS

LEMMA (RELATIVE DIVISIBILITY)

If $d \mid n$, *then* $u_d(\alpha, \beta) \mid u_n(\alpha, \beta)$.

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We let $m_\ell(\alpha, \beta)$ *be the* **smallest** $n \geq 2$ *for which* $\ell \mid u_n(\alpha, \beta)$.
If $\ell \nmid \alpha\beta$ *is an odd prime with* $m_\ell(\alpha, \beta) > 2$, *then* $m_\ell(\alpha, \beta) \mid \ell(\ell^2 - 1)$.

PROPERTIES OF NEWFORMS

THEOREM (ATKIN-LEHNER, DELIGNE)

If $f(z) = q + \sum_{n=2}^{\infty} a_f(n)q^n \in S_{2k}(N) \cap \mathbb{Z}[[q]]$ *is a newform, then TFAT.*

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(7) For each $d \mid \ell(\ell^2 - 1)$ classify integer points for the “curve”

$$a_f(p^{d-1}) = \pm\ell.$$

☐

FORMULAS FOR $a_f(p^2)$ AND $a_f(p^4)$

LEMMA

TFAT.

1. *If* $a_f(p^2) = \alpha$, *then* $(p, a_f(p))$ *is an integer point on*

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FORMULAS FOR $a_f(p^{2m})$ FOR $m \geq 3$

DEFINITION

In terms of the generating function

$$\frac{1}{1-\sqrt{Y}T+XT^2} =: \sum_{m=0}^{\infty} F_m(X,Y)\cdot T^m = 1+\sqrt{Y}\cdot T+\ldots$$

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we have the **special cyclotomic Thue polynomials**

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LEMMA

If f is a newform, then

$$a_f(p^{2m}) = F_{2m}(p^{2k-1}, a_f(p)^2).$$

EXPLICIT EXAMPLE

THEOREM (B-C-O-T + UVA REU)

For $n > 1$ we have

$$\tau(n) \notin \{\pm 1, \pm 691\} \cup \{\pm \ell \ : \ 3 \leq \ell < 100 \text{ prime}\}.$$

Sketch of the Proof

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4. Use Galois rep'ns + Mordell-Weil + Chabauty-Coleman + facts about Thue eqns to rule these out (**a lot here**).

☐

SUMMARY: NUMBER OF PRIME DIVISORS

THEOREM (B-C-O-T)

If $n > 1$ is an integer, then

$$\Omega(\tau(n)) \geq \sum_{\substack{p \mid n \\ prime}} (\sigma_0(\mathrm{ord}_p(n)+1)-1) \geq \omega(n).$$

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REMARKS

1. *This lower bound is sharp.*
2. *"Same" result when the mod 2 Galois rep'n is trivial.*

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2. *If* $\gcd(3 \cdot 5, 2k-1) \neq 1$ *and* $n > 1$, *then*

$$a_f(n) \notin \{\pm 1\} \cup \{\pm \ell \;:\; 3 \leq \ell < 37\} \cup \{-37\}.$$

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Assuming GRH, we have

$$a_f(n) \notin \{\pm 1\} \cup \{\pm \ell \ : \ 3 \leq \ell \leq 97 \text{ prime with } \ell \neq 37\} \cup \{-37\}.$$

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If $2k \geq 4$ *and* $a_f(2)$ *is even, then TFAT:*

1. *If* $|a_f(n)| = \ell^m$*, then* $n = p^{d-1}$*, with odd primes* $d \mid \ell(\ell^2-1)$ *and* p.

2. *If* $\gcd(3 \cdot 5, 2k-1) \neq 1$ *and* $n > 1$*, then*

$$a_f(n) \notin \{\pm 1\} \cup \{\pm \ell \ : \ 3 \leq \ell < 37\} \cup \{-37\}.$$

Assuming GRH, we have

$$a_f(n) \notin \{\pm 1\} \cup \{\pm \ell \ : \ 3 \leq \ell \leq 97 \text{ prime with } \ell \neq 37\} \cup \{-37\}.$$

THEOREM (B-C-O-T)

For prime powers ℓ^m*, if* f *has weight* $2k > M^{\pm}(\ell, m) = O_\ell(m)$*, then*

$$a_f(n) \neq \pm \ell^m.$$