How can cognitive science research help improve education? The case of comparing multiple strategies to improve mathematics learning and teaching

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Abstract

The current article focuses on efforts to understand how a basic learning process - comparison – can be harnessed to improve learning, especially mathematics learning in schools. To harness the power of comparison in instruction, three core questions are what, when and how to compare. Comparing different strategies for solving the same problem or easily confusable problem types is particularly effective for supporting mathematics learning. Comparing examples early in the learning process can be challenging, but delaying comparison can reduce procedural flexibility. Indeed, comparison is resource-demanding, so is more impactful when carefully supported (e.g., side-by-side visual presentation, explanation prompts). To bridge from research to practice, we communicated research findings to teachers and policy makers as well as developed curricular materials, instructional routines and professional development to help math teachers leverage these learning processes. We concluded with key open questions.

KEYWORDS: Learning processes; comparison; mathematics learning; multiple strategies

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Students too often memorize ideas without understanding the ideas or being able to flexibly apply them to new contexts. For example, only 11% of 15-year-olds from around the world could work strategically using well-developed thinking and reasoning skills to solve math problems; in the United States, only 5% could (OECD, 2016). Cognitive science research provides many insights into potential ways to improve teaching and learning in schools, but those insights infrequently make their way into classrooms (National Academies of Sciences, 2018). In the current article, we focus on our efforts to understand how one basic learning process - comparison –can be harnessed to improve learning, especially mathematics learning in schools.

We often learn through comparison. For example, we compare different brands of products, we compare one treatment option to another, and we compare new words, objects and ideas to ones we already know. A study aggregating the results of previous studies found that comparison improved learning across a range of topics, including math, science and language (Alfieri et al., 2013). Further, in mathematics education, best practices laud comparison as an important and effective instructional approach (NCTM, 2014).

Theoretically, comparison promotes analogical reasoning. When studying individual examples, people often focus on unimportant, surface features that are not relevant to the target concepts and procedures (Gick & Holyoak, 1983). Comparing two examples leads people to create an analogy between the two examples, helping them notice important, deep structural aspects of the examples, identify meaningful similarities and differences, and highlight the shared relational structure of the two examples (Gentner, 1983; Gentner et al., 2003; Schwartz & Bransford, 1998). In turn, this facilitates people's transfer of the knowledge to new situations and problems (Gick & Holyoak, 1983). However, comparison often requires substantial mental effort (Richland et al., 2016). Thus, evidence-based guidelines are needed for using comparison effectively in instruction.

Table 1

Sample Items for Assessing Procedural Knowledge, Procedural Flexibility and Conceptual Knowledge

Knowledge Type	Sample Items
1. Procedural knowledge	
1a. Similar to instructional problems	(a) $1/2\ (x + 1) = 10$ (b) $3(h + 2) + 4(h + 2) = 35$
1b. New problem types	(a) $3(m - 2)/5 = 33/5$ (b) $3(2x + 3x - 4) + 5(2x + 3x - 4) = 48$
2. Procedural flexibility	
2a. Generate multiple methods	(a) Solve this equation in two different ways: 18 = 3(x + 2). (b) Which of your ways do you think is easiest and fastest?
2b. Recognize multiple methods	(a) For the equation $2(x + 1) + 4 = 12$, identify all possible steps that could be done next. (4 choices)
2c. Evaluate nonconventional methods	(a) What step did the student use to get from the first line to the second line? $5(x + 3) + 6 = 5(x + 3) + 2x$ $6 = 2x$ (b) Do you think that this is a good way to start this problem? Circle an answer and explain your reasoning: A) A very good way; B) OK to do, but not a very good way; C) Not OK to do
3. Conceptual knowledge	(a) Which of the following is a like term to (could be combined with) $\mathbf{7(\boldsymbol{j} + 4)}$? A) $7(j + 10)$; B) $7(p + 4)$; C) j, D) $2(j + 4)$, E) A and D (b) Look at this pair of equations. Without solving the equations, decide if these equations are equivalent (have the same answer) and explain your reasoning. $98 = 21x$ $98 + 2(x + 1) = 21x + 2(x + 1)$

Note: Adapted from "The importance of prior knowledge when comparing examples: Influences of conceptual and procedural knowledge of equation solving," by B. Rittle-Johnson, J. Star, & K.

Durkin, 2009, *Journal of Educational Psychology*, *101*, p. 841. Copyright 2009 by the American Psychological Association.

Research on engaging in comparison to promote mathematics learning in the past 15 years has revealed new insights about what, when and how to compare. In this research, target learning outcomes were procedural knowledge (i.e., knowledge of what actions to take to solve problems, such as equation-solving procedures), procedural flexibility (i.e., knowledge of multiple procedures and when to use each), and conceptual knowledge (i.e., knowledge of abstract and general principles, such as equivalency) (Star et al., 2016). Examples of items we have used to assess each knowledge type are shown in Table 1. Conceptual and procedural knowledge are typically the focus of mathematics instruction and assessment, while procedural flexibility is not, despite evidence that procedural flexibility is an important component of mathematics expertise (Star, 2005).

What to Compare?

Integrating the cognitive science and mathematics education literatures highlights the importance of considering what is being compared. This is a fundamental aspect of comparison, yet one that historically had not received focused attention in cognitive science research. Comparing multiple strategies – that can both be correct or that can vary in correctness – or comparing easily confusing problem types are the most promising types of comparison for promoting mathematics learning identified to date (see Rittle-Johnson & Star, 2011).

Comparing Multiple Correct Strategies

Comparing multiple strategies for solving the same problem was rarely studied in the cognitive science literature, but it is the type of comparison most often used by expert mathematics teachers and recommended in mathematics education standards (NCTM, 2014).

Often, two correct strategies for solving the same problem are compared, such as two strategies for solving an equation.

To experimentally evaluate the impact of comparing multiple strategies on student mathematics learning, in a series of 5 studies, we redesigned 2 or 3 math lessons on a topic and implemented these lessons during mathematics classes. Students who compared multiple strategies saw the same problem solved two different ways on each page of a workbook with questions asking them to compare the two (see Figure 1). Students in the sequential condition studied one example per page with questions asking them to explain that individual strategy. Across these studies, with hundreds of students, students who compared multiple strategies gained greater procedural flexibility, often gained greater procedural knowledge, and sometimes gained greater conceptual knowledge than students who studied the same examples sequentially (Rittle-Johnson & Star, 2011; Star et al., 2016). Students' explanations during the intervention confirmed that those who compared strategies often compared the similarities and differences in solution steps across examples and evaluated their efficiency and accuracy; in turn, frequency of making explicit comparisons during the intervention was predictive of learning outcomes. Overall, comparing correct strategies helped students differentiate important characteristics of strategies and when and why one strategy was better for solving a particular problem.

We evaluated the effectiveness of this type of comparison relative to the most commonly studied form of comparison in the cognitive science literature - comparison of problems with different surface features, such as story context, but the same underlying solution strategy (i.e., isomorphic problems; Gick & Holyoak, 1983). We created two versions of comparing isomorphic problems solved with the same strategy, given mixed evidence on how similar the problems should be (see Rittle-Johnson & Star, 2009). In one condition, the isomorphic problems were the same problem type and thus had very similar surface features and solution strategy (e.g., $5(y+1) = 3(y+1) + 8$ and $10(x+3) = 6(x+3) + 16$ solved by subtracting the composite variable from both sides (e.g., subtracting (y+1)). In the other condition, the isomorphic problems had different problem features and thus had moderately similar surface features but the same underlying solution strategy (e.g., $5(y+1) = 3(y+1) + 8$ and $3(h-2) + 5(h-2) = 24$ solved using a composite variable strategy). Comparing multiple strategies led to greater conceptual knowledge and procedural flexibility than the other two comparison conditions, and procedural knowledge was similar for all conditions (Rittle-Johnson & Star, 2009). Comparing multiple strategies could be more beneficial than comparing isomorphic problems for learning some aspects of mathematics.

A. Compare Condition

Mandy's Solution:	**Erica's Solution:**
$5(y+1) = 3(y+1)+8$ $5y+5 = 3y+3+8$ *Distribute* $5y+5 = 3y+11$ *Combine* $2y+5 = 11$ *Subtract on Both* $2y = 6$ *Subtract on Both* $y = 3$ *Divide on Both*	$5(y+1) = 3(y+1)+8$ $2(y+1) = 8$ *Subtract on Both* $y+1 = 4$ *Divide on Both* $y = 3$ *Subtract on Both*

1. Mandy and Erica solved the problem differently, but they got the same answer. Why?
2. Why might you choose to use Erica's way?

B. Sequential Condition

Mandy's Solution:

$5(y+1) = 3(y+1)+8$

$5y+5 = 3y+3+8$ *Distribute*

$5y+5 = 3y+11$ *Combine*

$$\begin{aligned} 2y + 5 &= 11 && \textit{Subtract on Both} \\ 2y &= 6 && \textit{Subtract on Both} \\ y &= 3 && \textit{Divide on Both} \end{aligned}$$

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1. Would you choose to use Mandy's way to solve problems like this? Why or why not?

-----NEXT PAGE-----

Erica's Solution:

$$\begin{aligned} 10(x + 3) &= 6(x + 3) + 16 && \\ 4(x + 3) &= 16 && \textit{Subtract on Both} \\ x + 3 &= 4 && \textit{Divide on Both} \\ x &= 1 && \textit{Subtract on Both} \end{aligned}$$

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1. Check Erica's solution by substituting her answer into the equation. Did Erica get the right answer?

Figure 1. Sample pages from intervention packet for (A) compare correct strategies and (B) sequential conditions. Reprinted from "Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations," by B. Rittle-Johnson & J. Star, 2007, *Journal of Educational Psychology*, *99*, p. 564. Copyright 2007 by the American Psychological Association.

Comparing Confusable Problem Types

Rather than comparing isomorphic problems, comparing easily confusing problem types (which are not isomorphic) has promise for promoting math learning. In laboratory research with college students, participants who compared examples of algebra word problems from different categories (e.g., dilution vs. catch-up) were better able to sort new examples by problem category, rather than surface features, and to describe their structural features than students who studied examples one at a time (Cummins, 1992). Similarly, across two studies conducted in small groups, middle-school students with little prior knowledge compared examples of addition vs. multiplication of algebraic expressions (e.g., $xy + xy + xy = 3xy$ vs. $xy \bullet xy \bullet xy = x^3y^3$)or studied them sequentially, with all addition examples before multiplication examples. In the comparison condition, explicit comparison of the addition and multiplication examples and the distinction between them was supported via direct instruction from a researcher or self-explanation prompts (depending on the study). Students in the comparison condition developed stronger conceptual and procedural knowledge than students in the sequential condition in both studies (Ziegler & Stern, 2014, 2016). Comparing examples of different problem types that are easily confused can help people learn to better distinguish and understand the categories and associated solution strategies.

Comparing Correct and Incorrect Strategies

Other cognitive science research has highlighted the value of studying incorrect examples in combination with correct examples to increase depth of thinking about correct ideas and reduce use of incorrect ideas in the future (e.g., Alvermann & Hague, 1989; Booth et al., 2013; Siegler & Chen, 2008). We evaluated the impact of prompting students to compare correct and incorrect examples instead of comparing only correct examples. For fourth- and fifth-grade students learning about decimal magnitude, students who compared incorrect and correct strategies gained greater conceptual and procedural knowledge than students who only compared correct strategies (Durkin & Rittle-Johnson, 2012). Comparing correct and incorrect examples promoted greater noticing of conflicting ideas and more focused attention on the distinguishing features of the correct strategies, including the relevant concepts.

Overall, the cognitive science literature on analogical reasoning provided theoretical and empirical guidance on comparison and how it aids learning, but applying these ideas to learning mathematics highlighted new types of comparison and helped specify the impact of different types of comparison on learning.

When to Compare?

Using comparison as an instructional method also highlighted the need to decide *when* in the learning process to use comparison. According to theories of analogical reasoning, prior knowledge of one of the to-be-compared examples may be important because analogical reasoning is particularly effective when learners can make inferences about a new idea by identifying its similarities and differences with a known idea and making predictions about the new idea based on its alignment with the known idea (Gentner, 1983). Thus, should comparison be delayed until learners know one of the to-be-compared strategies? Indeed, instructional supports that help learners with prior knowledge are sometimes not effective with learners with little prior knowledge (Kalyuga et al., 2003).

Our initial evidence suggested yes. In our initial study with students with variable prior knowledge, comparing correct strategies was less effective than sequential study of the strategies for students who were not previously familiar with one of the strategies, while it was more effective for students with prior knowledge of one of the strategies (Rittle-Johnson et al., 2009). This condition by prior knowledge interaction was specific to prior knowledge of a strategy; general prior math knowledge did not interact with condition.

However, according to theories of analogical learning, people can learn from comparing two unfamiliar examples, as it can help them notice potentially relevant features via identifying similarities and differences in the examples (Gentner et al., 2003). In a follow-up study, we gave students with variable prior knowledge more time to learn a smaller amount of material. With this added support, students who compared correct strategies immediately gained greater procedural flexibility and similar conceptual and procedural knowledge relative to students who studied one strategy before exposure to additional strategies, regardless of prior knowledge (Rittle-Johnson et al., 2012). Learning from comparing two unfamiliar examples can aid learning, but it requires sufficient support to avoid overwhelming learners.

How to Support Comparison?

Comparison often requires substantial mental effort. The cognitive science literature has revealed key ways to support comparison and increase the probability that comparison will support learning (see also Richland et al., 2016). 1) *Make the examples clear and visible and present both examples simultaneously*, not one at a time. In math and some science topics, *worked examples* (a problem and step-by-step strategy for solving it) are very effective visible examples to help novices learn new procedures and related concepts (Atkinson et al., 2000). Students make better comparisons and learn more when they do not have to rely on their memory of one example while comparing it to another example (Begolli & Richland, 2015). 2) *Present examples side-by-side and use common terminology, gestures and other cues (e.g., highlight key parts in the same color) to guide attention to important similarities and differences in the examples.* Visual cues such as gesturing back and forth between similar aspects of two side-by-side examples improves appropriate transfer of the demonstrated procedure to new contexts (Richland & McDonough, 2010). Labeling two examples with the same term makes it much more likely learners will compare the examples and notice their underlying similarities (Namy & Gentner, 2002). 3) *Prompt for student explanation of key points about the comparison.* Prompts to compare and explain specific aspects of two examples guides attention and improves learning from comparison more than generic prompts to compare (Gentner et al., 2003). It is also important to ask, and allow students to answer, higher-level open-ended questions about the comparison (Star, Newton, et al., 2015). Finally, 4) *Summarize the main points of the comparison after students reflect*. Direct instruction on the key points after students compare supplements learners' comparisons and improves learning from comparison (Gick & Holyoak, 1983; Schwartz & Bransford, 1998). Providing direct instruction after students compare examples can be more effective than providing it before learners compare (Alfieri et al., 2013); comparison prepares students to learn more from direct instruction (Schwartz & Bransford, 1998).

Bridging to Practice

Cognitive-science inspired research on mathematics learning in classroom contexts provided evidence-based recommendations for improving mathematics instruction. How do we bridge from research to practice to impact classroom instruction provided by teachers? One key is to have an interdisciplinary research team of psychology and education researchers who work directly with practitioners. Another key is to disseminate findings to broad audiences, including teachers and decision makers, via practitioner-focused journals (e.g., Star et al., 2010), conferences (e.g., NCTM regional conferences) and webinars. Another is to influence consensus documents for educators on evidence-based instructional practices (Woodward et al., 2012), including investing the effort in chairing creation of these documents (Star, Caronongan, et al., 2015).

To promote high-quality adoption in classrooms, creating curriculum materials and teacher professional development (PD) are often needed. We have developed a supplemental curriculum and teacher PD entitled *Comparison and Discussion of Multiple Strategies (CDMS)* for Algebra I instruction. This required us to substantially expand the number of, types of, and curricular coverage of our materials, develop and iteratively improve pedagogical routines and teacher professional development structures and materials, and expand our assessments. Teachers in the treatment condition improved their use of comparison, supported by our materials, during the summer professional development and the school year (Newton & Star, 2013). However, in our first year-long study with Algebra I teachers, implementation of our materials was very infrequent, and students in the treatment condition did not learn more than students in the control condition (Star, Pollack, et al., 2015). In our second attempt, we revised the curriculum materials (see Figure 2 for samples) and helped teacher align them with their curriculum. We refined and specified an instructional routine, as shown in Figure 3, including a think-pair-share routine for reflecting on the comparison. First, students *think* on their own for a minute about the discussion prompt. Next, each student *pairs* with another student to discuss the prompt, summarizing their ideas in writing. Then, students share their ideas in a whole class discussion. We provided a graphic organizer for students to record their thinking in each phase. We also added professional development during the school year where we helped teachers plan and provided individualized feedback on implementation. Overall, unpublished results suggest that these efforts greatly increased the quantity and quality of implementation, and teachers were generally positive about the approach. Preliminary analyses suggest students in the treatment condition learned more than those in the control condition on researcher-designed measures. Thus, we have made substantial progress in promoting comparison of multiple strategies in mathematics classrooms, but much more work remains.

Discussion

Comparison is a powerful learning process. In problem-solving domains such as mathematics, comparing multiple strategies and comparing confusable problem types promotes conceptual knowledge, procedural knowledge and/or procedural flexibility. However, comparison requires substantial mental effort by learners, and learners can become overwhelmed by it without adequate support, especially if all of the material is unfamiliar. Supports, such as presenting examples side-by-side and using cues to guide attention to important similarities and differences in the examples, facilitate learning from comparison. To help teachers use comparison more frequently and effectively in their classrooms, curricular materials, well-specified instructional routines, and sustained professional development are likely needed.

Despite the progress that has been made, many open questions remain. Theoretically, theories and formal models of analogical learning have not systematically considered or modeled comparison of two different strategies for solving the same problem. For instance, what impact do alignable differences (differences related to commonalities) versus nonalignable differences (features in one strategy that have no corresponding feature in the other strategy) in the strategies have on what people learn from the comparisons? How can this impact be modeled in alignment-based models of similarity? Additional open questions include: (1) What are other effective ways to encourage and support comparison of multiple strategies (e.g., prompting students to generate a second way to solve a problem or to compare their own incorrect strategy to a correct strategy)? (2) Does comparing multiple strategies impact students' attitudes, such as their productive disposition towards mathematics (i.e., see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy)? (3) How can we support math teachers in learning to appropriately use and support comparison? Can they detect when their students need more support and adjust accordingly? Does developing teachers' understanding of how and why comparison promotes learning improve their implementation? (4) How do teachers' beliefs, such as their self-efficacy for helping all students learn challenging mathematics, impact how they use comparison, and do curriculum and professional development such as ours change teachers' beliefs? (5) How can we integrate insights from using comparison to promote science learning (e.g., comparing observable and modeled events, see Jee & Anggoro, 2019) with insights from mathematics learning?

More broadly, for cognitive science research to impact education, researchers must (a) make a clear connection to important educational outcomes that are valued by practitioners and clearly define and measure those outcomes and (b) conduct research that provides evidence for the what, when, and how of the instructional intervention. Conducting such research informs theory as well as practice. For example, when choosing what to compare, our decision to focus on comparing multiple strategies was driven by best practices of teachers, and this research drove the need to extend theories of analogical reasoning to a type of comparison not previously considered. Bridging between theory and practice is not a one-way street from theory to practice; rather, it is bi-directional.

A)

Which is better?

Topic 3.5

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

Tim's "substitution" way

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

I solved the second equation for x.

$x = 3y + 10$

I plugged this into the first equation.

$3(3y + 10) + 2y = 8$

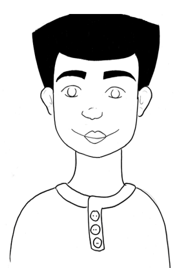
I then solved for y.

$9y + 30 + 2y = 8$
$11y + 30 = 8$
$11y = -22$
$y = -2$

I plugged y into the second equation to find x.

$x - 3(-2) = 10$
$x + 6 = 10$
$x = 4$

The solution is (4, -2)



Emma's "elimination" way

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

I multiplied the bottom equation by -3.

$3x + 2y = 8$
$-3(x - 3y = 10)$

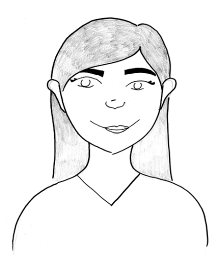
I then used elimination and solved for y.

$3x + 2y = 8$
$\underline{-3x + 9y = -30}$
$11y = -22$
$y = -2$

I plugged y into the second equation to find x.

$x - 3(-2) = 10$
$x + 6 = 10$
$x = 4$

The solution is (4, -2)



Why did Tim choose to plug y = –2 into the second equation to find x instead of the first equation?

Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

B)

Which is correct? Topic *1.6*

Emma and Layla were asked to simplify the expression 2(x + 1) + 3(x + 6)

Emma's "distribute first" way	**Layla's "combine like terms" way**
2(x + 1) + 3(x + 6)	2(x + 1) + 3(x + 6)
2x + 2 + 3x + 18	5(2x + 7)
5x + 20	*10x + 35*
First, I distributed the parentheses. *Then, I combined like terms to get 5x + 20.*	*First, I combined what was in the parentheses.* *Then, I distributed the parentheses to get 10x + 35.*

What were the like terms in Emma's "distribute first" way? What were the like terms combined in Layla's "combine like terms" way?

Which method is correct? For the incorrect method, what needs to be done to make it correct?

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Figure 2. Sample Algebra I curriculum materials for A) comparing two correct strategies (*Which is better?)* and B) comparing a correct and common incorrect strategy (*Which is correct?)*.

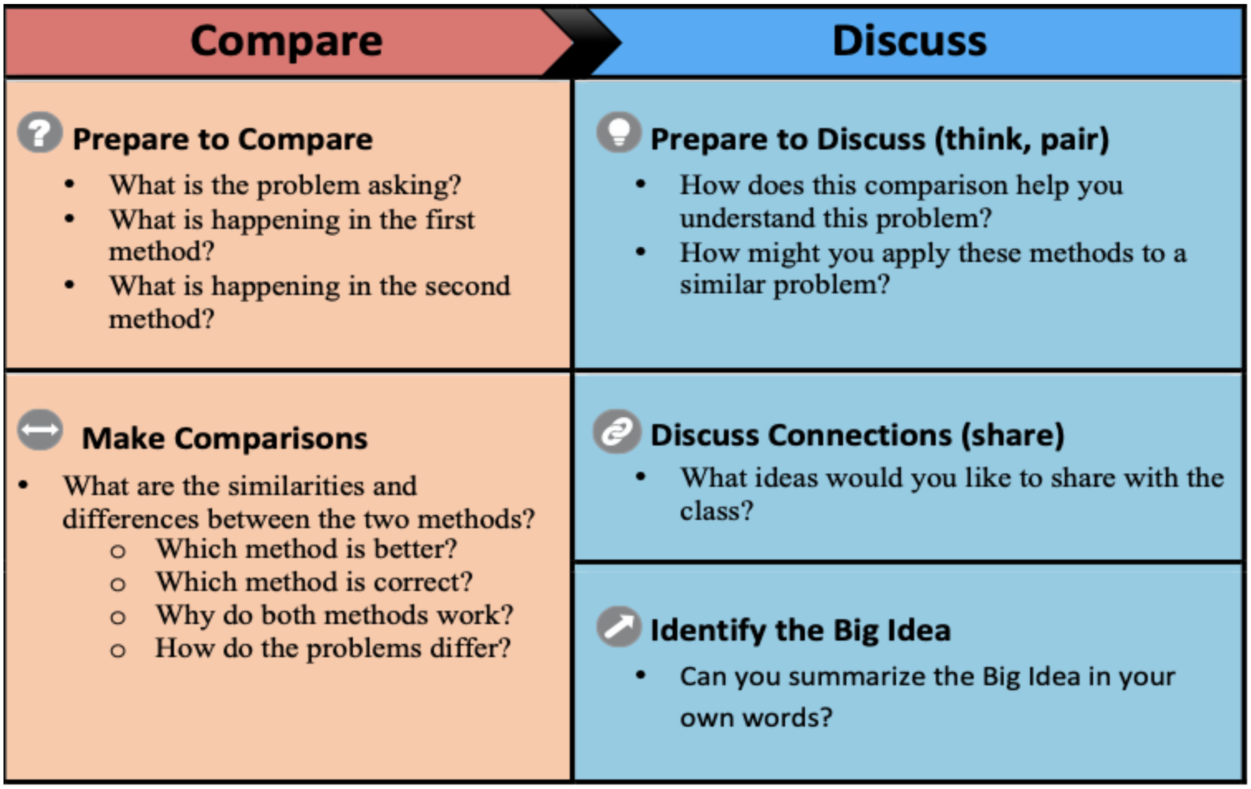

Figure 3. Instructional routine for promoting comparison and discuss of multiple strategies in the classroom. Recommended to spend about 8 minutes in compare phase and 12 minutes in discuss phase.

Acknowledgements and Endnotes

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Recommended Readings

Alfieri, L., Nokes-Malach, T. J., & Schunn, C. D. (2013). Learning through case comparison: A meta-analytic review. *Educational Psychologist, 48*(2), 87-113. doi: 10.1080/00461520.2013.775712. Aggregates the results of previous studies on comparing examples and identifies factors that may improve learning from comparison.

Durkin, K., Star, J. R., & Rittle-Johnson, B. (2017). Using Comparison of Multiple Strategies in the Mathematics Classroom: Lessons Learned and Next Steps. *ZDM, 49*, 585 - 597. doi.org/10.1007/s11858-017-0853-9. Reviews empirical research on using comparison of multiple strategies to promote mathematics learning and provides instructional recommendations in more depth than this article.

Gentner, D., Loewenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology, 95*(2), 393-405. doi:10.1037/0022-0663.95.2.393. Expands a theoretical acccount for analogical reasoning, based on structure-mapping theory, to learning from comparing two unfamiliar examples and provides evidence in support of the theory.

Richland, L. E., Begolli, K. N., Simms, N., Frausel, R. R., & Lyons, E. A. (2016). Supporting mathematical discussions: The roles of comparison and cognitive load. *Educational Psychology Review, 29*(1), 41-53. doi:10.1007/s10648-016-9382-2. Reviews research on the cognitive challenges for students when comparing solution strategies and on teaching practices for helping students engage in comparison successfully.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations.

Journal of Educational Psychology, 99(3), 561-574. doi:10.1037/0022-0663.99.3.561.

The authors' first study on using comparison to support mathematics learning, which provides details on the method and results.

Figure Captions

(also embedded in text with Figures)

Figure 1. Sample pages from intervention packet for (A) compare and (B) sequential conditions from Rittle-Johnson, B. & Star, J. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*. *99*(3), 561-574.

Figure 2. Sample Algebra I curriculum materials for A) comparing two correct strategies (*Which is better?)* and B) comparing a correct and common incorrect strategy (*Which is correct?)*.

Figure 3. Instructional routine for promoting comparison and discuss of multiple strategies in the classroom. Recommended to spend about 8 minutes in compare phase and 12 minutes in discuss phase.