Problems for Math Circles

February 25, 2019

1 Take-Home Problems

1. Prove that $p(n) < p(n+1)$ for all $n \geq 1$.

2. a). If λ is a partition of n with no distinct parts (in other words, every part that appears in the sum shows up at least twice), what can we say about its conjugate? (Hint: Think about the example where we found that having one as a part meant that the conjugate had a distinct largest part and vice versa. Try to find a similar meaning here.)
b). Write down a product formula that gives the generating function for the number of partitions of n with no distinct parts.

3. An *overpartition* of a number n is a partition of n where the first occurrence of any part may be overlined. For example, the overpartitions of 2 are

$$2, \bar{2}, 1+1, \bar{1}+1,$$

and the overpartitions of 3 are

$$3, \bar{3}, 2+1, 2+\bar{1}, \bar{2}+1, \bar{2}+\bar{1}, 1+1+1, \bar{1}+1+1.$$

a). What are the overpartitions of 4?
b). Let $\bar{p}(n)$ be the number of overpartitions of n. What is the generating function for $\bar{p}(n)$? (Hint: Recall how we found the generating function for $p(n)$.)

4. A partition λ is said to be *self-conjugate* if the conjugate of λ is λ itself. Using Ferrers Diagrams, prove that the number of self-conjugate partitions of a number n is the same as the number of partitions of n into distinct odd parts. (Hint: try "folding" Ferrers diagrams of partitions with distinct odd parts.)

5. Let n be a positive whole number, and for any number x, let $[x]$ be the largest whole number less than or equal to x. For instance, $\pi = 3.14...$, so $[\pi] = 3$, and $[5] = 5$ since 5 is already a whole number.
a). Find a formula for the sum of the numbers $1 + 2 + ... + [\sqrt{n}]$. Conclude that this sum is less than n.

b). Let $S = \{s_1, s_2, \ldots, s_k\}$ be a collection of numbers from the set $\{1, 2, \ldots, [\sqrt{n}]\}$. Show that there is a partition of n which starts with $s_1 + s_2 + ... + s_k$. For instance, if $n = 101$, so that $[\sqrt{101}] = 10$, if S is the collection $S = 1, 3, 4, 9$ of whole numbers less than or equal to 10, then $1 + 3 + 4 + 9 + 84$ is a partition of 101 which "starts" with $1 + 3 + 4 + 9$.
c). Recall from an earlier math club meeting that the number of collections of numbers from $\{1, 2, \ldots, [\sqrt{n}]\}$, that is, the number of subsets, is $2^{[\sqrt{n}]}$.
Conclude that the number of partitions of n grows exponentially with n, and give a function $f(n)$ such that $f(n) < p(n)$.
d). Can you find better approximations to $p(n)$? How many partitions are "missed" by this procedure? Can you find an upper bound for $p(n)$, that is, a function $g(n)$ such that $p(n)$ is always less than $g(n)$?