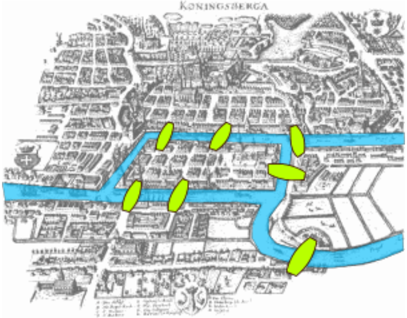
# Graph Theory I

Nashville Math Club

October 6, 2020

# The seven bridges of Konigsberg



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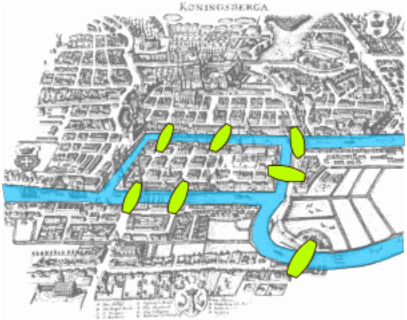


**Question**

*Can we walk through all seven bridges of Konigsberg exactly once?*

# Graphs

**Definition**

A **graph**, $G = (V, E)$, is a collection of points $V$, called **vertices**, and lines $E$, called **edges**, connecting two vertices.

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- Vertices?
- Edges?
- Regions?

## Definitions

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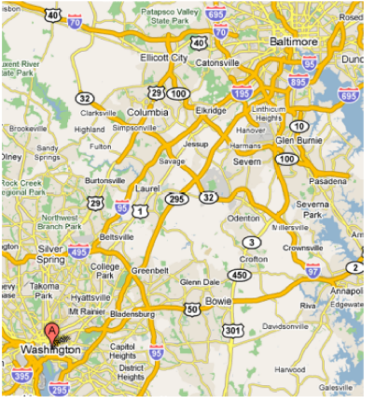
- A **Tree** is a graph with no cycles.
- The **degree** of a vertex is the number of edges that meet at that vertex.
- We say a vertex is **even** (resp. **odd**) if its degree is even (resp. odd).

# Examples of graphs



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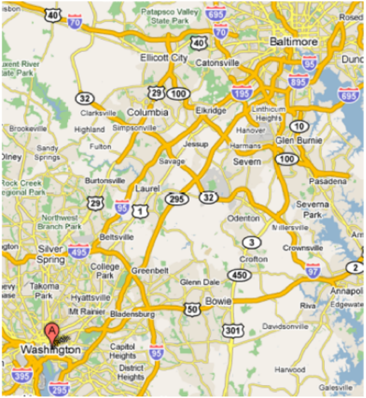


Each place that roads intersect can be thought of as a vertex and each road can be thought of as an edge.

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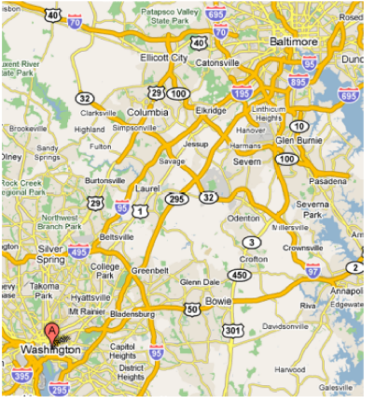


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We can label graphs to add more information.

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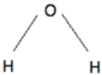

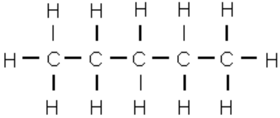

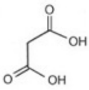

Malonic acid

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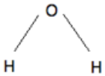


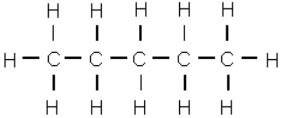


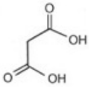


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If we allow multiple edges between two vertices we call the graph a **multigraph**.

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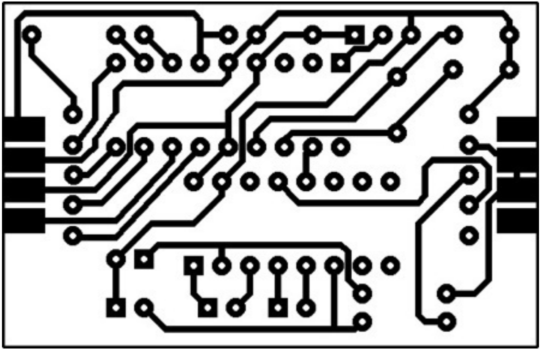

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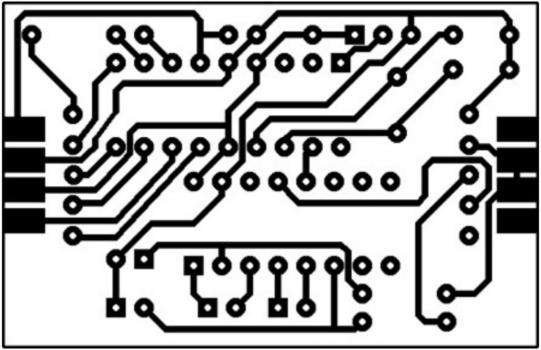

The wires on a circuit board can be viewed as edges between the nodes of the board which can be viewed as vertices.

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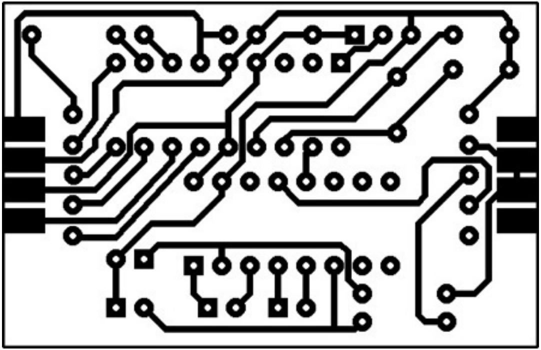
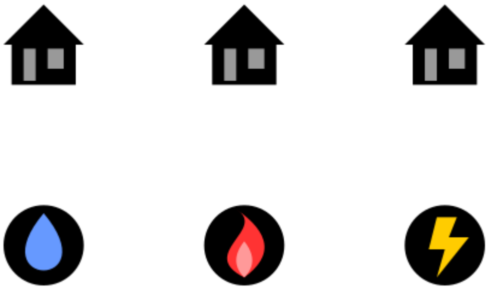


The wires on a circuit board can be viewed as edges between the nodes of the board which can be viewed as vertices. In a circuit board none of the wires can touch. Graphs where the edges only intersect at vertices are called **planar graphs**.

## Three utilities problem



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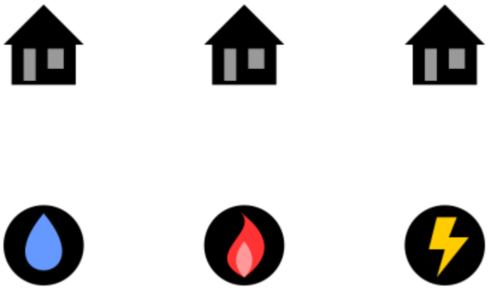

**Question**

*Can you connect water, electricity, and gas to each house without any pipelines crossing over?*

# Examples of graphs

**Question**

*What are some other examples of graphs?*

# Complete graphs

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*Suppose there are $n$ people in a room and we would like all of them to shake hands. How can we represent this as a graph?*

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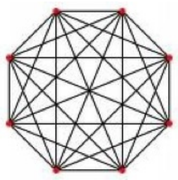

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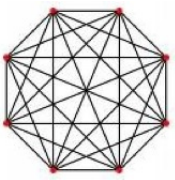

## Definition

A **complete graph** is a graph in which every pair of distinct vertices is connected by a unique edge. The complete graph with $n$ vertices is denoted by $K_n$.

# Complete graphs

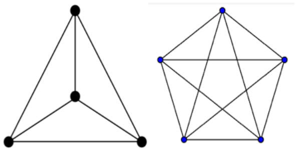

Figure: The graphs $K_4$ and $K_5$

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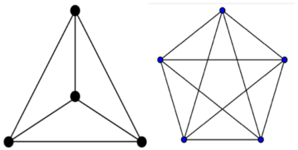

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## Questions

- What is the degree of each vertex in $K_4$?

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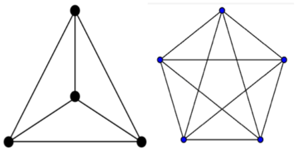

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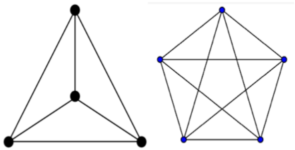

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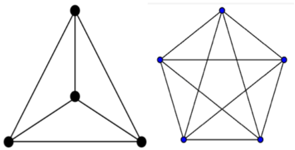

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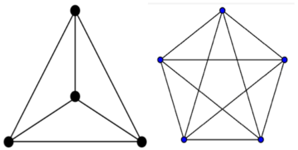

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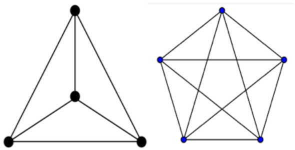

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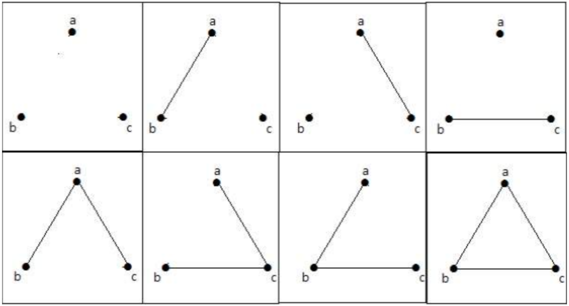
# Even and odd vertices



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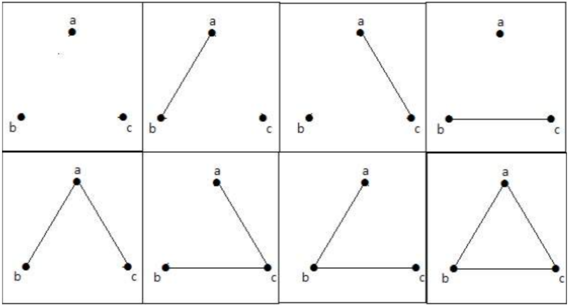


Determine the degree of each vertex in the graphs above.

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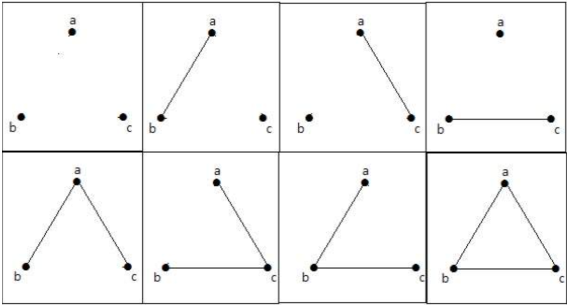


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How many are even?

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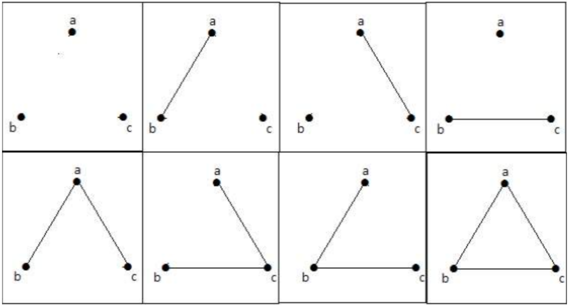


Determine the degree of each vertex in the graphs above. How many are even? How many are odd?

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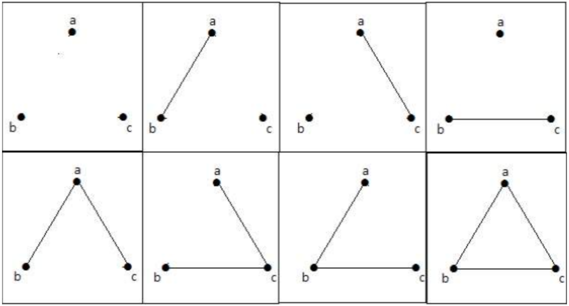


Determine the degree of each vertex in the graphs above.
How many are even? How many are odd?
Take some time to draw your own graphs and answer the same questions.

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**Lemma**

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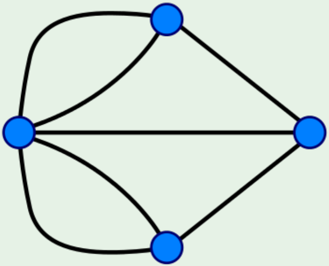

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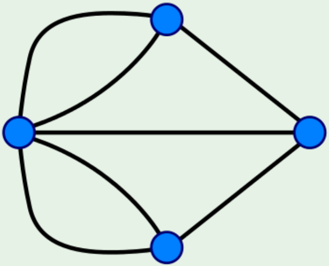

$$5 + 3 + 3 + 3 = 14 = 2 \times 7.$$