Jeopardy!

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Algebra \$100

How many pairs of non-zero real numbers (a, b) exist such that the sum $a + b$, the product ab, and the quotient $\frac{a}{b}$ are all equal.

Algebra \$100 (Solution)

We have that $ab = \frac{a}{b}$, which implies that

$$ab^2 = a \implies b^2 = 1 \implies b = \pm 1 \tag{1}$$

since a is non-zero. Additionally,

$$a + b = ab \implies a + 1 = a \text{ or } a - 1 = -a \tag{2}$$

Since $a + 1 = a$ is impossible, then $a - 1 = -a \implies a = \frac{1}{2}$. The only possible pair is $(\frac{1}{2}, -1)$ so there is 1 pair in total. (1)

Algebra \$200

Andy rides his bike from his house to the store at 6 miles per hour. Upon arriving at the store, he realizes it's closed and immediately heads back, riding his bike back home at a speed of 2 miles per hour. What is Andy's average speed in miles per hour over the entire trip?

Algebra \$200 (Solution)

Let d be the distance from the house to the store. Let t_1 be the amount of time it took to go from the house to the store and let t_2 be the amount of time it took to go from the store to the house. Using $d = vt$, we get the equations

$$d = 6t_1; \quad d = 2t_2 \tag{3}$$

which together, implies that $t_2 = 3t_1$.
Since average speed is total distance divided by total time, we get

$$\text{average speed} = \frac{2d}{t_1 + t_2} = \frac{2(6t_1)}{t_1 + 3t_1} = \frac{12t_1}{4t_1} = 3 \text{ mph} \tag{4}$$

(1)

Algebra \$300

Evaluate the following:

$$\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}} \tag{5}$$

Algebra \$300 (Solution)

$$\begin{aligned} x &= \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\dots}}}} \\ x &= \sqrt{3+x} \\ x^2 &= 3+x \end{aligned} \tag{6}$$

Then, we apply the quadratic formula and remove the negative answer to get

$$x = \frac{1+\sqrt{13}}{2} \tag{7}$$

(1)

Algebra \$400

Let $n \geq 2$ be a natural number (i.e. $n = 2, 3, 4, \ldots$). For how many n is $\log_n(n+1)$ a rational number?

Algebra $400 Solution

Suppose, by contradiction, that $\log_n(n+1) = \frac{p}{q}$ for some integers p, q. with $q \neq 0$. Since $n+1 > n$, we have $\log_n(n+1) > 1$ and thus, $p/q > 1$, and $p, q > 0$. Then, we have

$$\begin{aligned} n^{p/q} &= n+1 \\ n^p &= (n+1)^q \end{aligned} \tag{8}$$

However, n and $n+1$ have opposite parity (i.e. one is even and one is odd), and for any positive integers p and q, n^p has the same parity as n and $(n+1)^q$ has the same parity as $(n+1)$, so $n^p \neq (n+1)^q$ and we have a contradiction. Thus, $\log_n(n+1)$ is not rational and the answer is 0.
(1)

Number Theory \$100

What is the last digit of 2019^{2019} ?

Number Theory \$100 (Solution)

Since we only care about the ones digit, it suffices to see what happens to 9 when we take subsequent powers.
We observe that 9^1 ends in a 9, 9^2 ends in a 1, 9^3 ends in a 9, 9^4 ends in a 1, and so on. Since the power 2019 is odd, 2019^{2019} ends in a 9.
(1)

Number Theory \$200

How many zeroes in a row occur at the end of the number 100!.

Number Theory \$200 (Solution)

Each factor of $10 = 2 \cdot 5$ adds a zero, so we need to count the number of 2's and 5's in 100!. There are considerably more 2's, so we need to determine how many 5's appear as factors in the numbers from 1 to 100. There are 20 numbers that have at least one factor of 5 (5, 10, 15, ..., 100) and 4 numbers that have an additional factor of 5 (25, 50, 75, 100), so there are 24 5's in 100! and thus, 24 zeroes.

(1)

Number Theory \$300

I have 2 numbers a and b such that $a + b = c$. The sum of the digits of a is 27 and the sum of the digits of b is 32. When I added a and b by hand, I had to carry 3 times. What is the sum of the digits of c?

Number Theory \$300 (Solution)

First, notice that if we didn't have to carry at all, then the sum of the digits of c would just be the sum of the digits of a plus the sum of the digits of b $(27 + 32)$. Every time we carry (in a base 10 number system), we reduce the sum of the digits by 9 (added a 1 to the tens place instead of a 10 to the ones place, so we reduced the digit sum by 10-1). Since there are 3 carries, the sum of the digits of c will be

$$27 + 32 - (3 \cdot 9) = 32. \tag{9}$$

(1)

Number Theory $400

If

$$
\begin{aligned}
X &= \binom{2019}{0} + \binom{2019}{1} + \binom{2019}{2} + \ldots + \binom{2019}{2019} \\
Y &= \binom{2019}{0} - \binom{2019}{1} + \binom{2019}{2} - \ldots - \binom{2019}{2019}
\end{aligned}
\tag{10}
$$

Find $\frac{X+Y}{2}$.
(Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$).

Number Theory $400 (Solution)

Consider the binomial expansion

$$\begin{aligned}(a+b)^{2019} &= \binom{2019}{0}a^{2019}b^0 + \binom{2019}{1}a^{2018}b^1 + \dots \\ (1+1)^{2019} &= \binom{2019}{0} + \binom{2019}{1} + \binom{2019}{2} + \dots \\ (1+(-1))^{2019} &= \binom{2019}{0} - \binom{2019}{1} + \binom{2019}{2} - \dots\end{aligned} \tag{11}$$

So we get that

$$\frac{X+Y}{2} = \frac{(1+1)^{2019} + (1+(-1))^{2019}}{2} = 2^{2018} \tag{12}$$

(1)

Geometry \$100

Suppose you are given a square with side length a and an equilateral triangle with side length b such that both figures have the same area. What is the ratio of a to b?

Geometry \$100 (Solution)

An equilateral triangle with base b will have a height of $\frac{\sqrt{3}}{2}b$, so we get

$$\begin{aligned} a^2 &= \frac{1}{2}b \cdot \frac{\sqrt{3}}{2}b \\ a^2 &= \frac{\sqrt{3}}{4}b^2 \\ \frac{a}{b} &= \frac{\sqrt[4]{3}}{2} \end{aligned} \tag{13}$$

(1)

Geometry \$200

Four circles of radius one are centered at the points $(1,1),(1,-1),(-1,-1)$, and $(1,-1)$. A fifth circle is drawn centered at the origin such that it is tangent to the other four circles. What is the radius of the fifth circle?

Geometry \$200 (Solution)

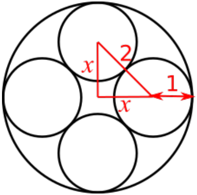

Figure: (Rotated 45°) We observe that the red triangle is an isoceles right triangle, so $x = \sqrt{2}$ and the radius of the larger circle is $1 + \sqrt{2}$.

(1)

Geometry \$300

Four distinct points are arranged on a plane so that the segments connecting them have lengths $a, a, a, a, 2a$, and b. What is the ratio of b to a?

Geometry \$300 (Solution)

If we arrange the points, we get an equilateral triangle of sidelength a adjacent to an obtuse isoceles triangle with angle 120° and lengths a, a and b. (Draw Picture) Using $30-60-90$ right triangles, we get $b = a\sqrt{3}$, so the ratio of b to a is $\sqrt{3}$. (1)

Geometry $400

There are 3 trees, each of height 30 meters, at a distance of 30, 60, and 90 meters, respectively from a mouse sitting on the ground. If the mouse is looking at the top of each of the trees, what is the sum of the three angles made as the mouse sees each of the trees?

Geometry \$400 (Solution)

If we let the angles that the mouse makes with each of the trees to be A, B, and C respectively, then we have the following values:

$$\begin{aligned} \sin A &= \frac{1}{\sqrt{2}}, \quad \cos A = \frac{1}{\sqrt{2}} \\ \sin B &= \frac{1}{\sqrt{5}}, \quad \cos B = \frac{2}{\sqrt{5}} \\ \sin C &= \frac{1}{\sqrt{10}}, \quad \cos C = \frac{3}{\sqrt{10}} \end{aligned} \tag{14}$$

Then, we have

$$\begin{aligned} \sin(A+B+C) &= \sin(A+B)\cos C + \cos(A+B)\sin C \\ &= [\sin A \cos B + \cos A \sin B]\cos C + [\cos A \cos B - \sin A \sin B]\sin C \\ &= \left[\frac{2}{\sqrt{10}} + \frac{1}{\sqrt{10}}\right]\frac{3}{\sqrt{10}} + \left[\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{10}}\right]\frac{1}{\sqrt{10}} = \frac{9}{10} + \frac{1}{10} = 1 \end{aligned} \tag{15}$$

Thus, $A+B+C = \frac{\pi}{2}$ or 90°. (1)

Logic \$100

Allen is taller than Brad and shorter than Clara. Diana is shorter than Clara and taller than Allen. Brad is shorter than Sarah. If the first three statements are true, which of the following must be true? (consider all that apply)

1 Diana is taller than Brad.

2 Brad is the same height as Diana

3 Sarah is taller than Allen

Logic \$100 (Solution)

Ordering the heights, we get

$$C > A > B, \quad C > D > A, \quad S > B \tag{16}$$

So we have

$$C > D > A > B, \text{ and } S > B \tag{17}$$

Option 1 is true as $D > B$, which means 2 is false. Additionally, we don't know whether $S > A$, only that $S > B$. So 1 is the only option that must be true. (1)

Logic $200

Suppose David has the set of integers from 1 to 100 (inclusive) and he hands you 51 numbers at random. Which of the following must be true of the 51 numbers he hands you?

- A At least one number is even
- B At least two numbers are odd
- C At least two numbers have a difference of 1
- D A and B only
- E A and C only

Logic $200 (Solution)

There are 50 even numbers and 50 odd numbers between 1 and 100 (inclusive). If we pick 51 numbers at random, we must pick at least one even number and at least one odd number. Thus A is true and B is false. If we try to pick numbers avoiding a difference of 1, the best we can do is choose either 50 even numbers or 50 odd numbers, in which case the 51st number chosen must be adjacent to one of the 50 we picked (pigeonhole principle). So the answer is (E).

(1)

Logic $300

Given that one and only one answer is correct, which of the following is true?

- A All of the below
- B None of the below
- C One of the above
- D All of the above
- E None of the above
- F None of the above

Logic \$300 (Solution)

Consider the choices. Contradictions between answers, such as between C and D, eliminate A as a possibility. B is false because if it were correct, then C would also be true. It follows that C is false because both A and B are false. Similarly, it follows that D is false. Thus, E must be true. That result makes F false.

(1)

Logic $400

The Mathematician and the Three Sons

Two math professors met in a street. They started talking about their children. "I have three sons," the first one said. "How old are they?" asked the second math professor. "Well, it's easy to figure out," he replied. "The product of their ages (in years) is equal to 36 and the sum of their ages is equal to the number of windows in that house across the street." The second math professor thought for a while and said, "What you've told me is not sufficient to solve your problem." "Oh, yes, also my youngest son has red hair," the first replied. "Ah, now I know the answer to your problem" answered the second math professor happily.
What were the ages of the three sons (in years)?

Logic $ 400 (Solution)

If we write all the products for 36 and their sums, we get

Product=36	Sum
1, 1, 36	38
1, 2, 18	21
1, 3, 12	16
1, 4, 9	14
1, 6, 6	**13**
2, 2, 9	**13**
2, 3, 6	11
3, 3, 4	10

If the number of windows in the house across the street was not 13, then the second mathematician would have known the answer. However, since he didn't, the sum must be 13. Additionally, the last comment tells us that there is a youngest son (in years), which rules out $2, 2, 9$ as a possibility. (1)

Misc. $100

$163,000$ basketball teams are playing in a single elimination bracket. (For each game, two teams play each other where one team is eliminated and the other team moves on to the next round). How many games are needed to determine a winner?

Misc. $100 (Solution)

Each game eliminates only one team, so we need to have $163,000 - 1 = 162,999$ games to determine a winner. (1)

Misc. $200

Suppose you have a regular hexagon, and for every pair of non-adjacent vertices, you draw a line segment connecting them. How many times do the line segments intersect in the interior of the hexagon?

Misc. $200 (Solution)

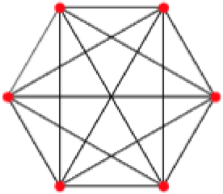

Figure: One way is to draw out the figure by hand and count the intersections. Alternatively, notice that the interior will be a hexagon with each of its sides bisected (12 intersections) and we add one more for the middle intersection which is 13 interior intersections in total.

(1)

Misc. \$300

How many (unique) ways are there to sit 6 people at a round table? Consider two seating arrangements to be the same if one is a rotation of the other.

Misc. \$300 solution

First fix a seat where the first person is to sit. This can be done in only one way. Then there will be 5 possibilities for the person sitting to his or her left, then 4 choices to fill in the seat to the left of the second person, and so on. There are

$$(6-1)! = 5! = 120 \text{ possible seating arrangements} \tag{18}$$

(1)

Misc. $400

One hundred people line up to board an airplane. Each has a boarding pass with an assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes their assigned seat if it is unoccupied, and one of the unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in their assigned seat?

Misc \$400 (Solution)

Look at the situation when the k'th passenger enters. None of the previous passengers showed any preference for the k'th seat vs. the seat of the first passenger. (The passengers before the k'th one are just as likely to sit in the 1st seat as they are to sit in the k'th seat if their seat is taken). This in particular is true when $k = 100$. But the 100th passenger can only occupy his seat or the first passenger's seat. So the fate of the 100th passenger is sealed the moment someone takes the 1st seat or the 100th seat (both of which are equally likely each time a passenger boards). Therefore, the probability is $1/2$.

(1)