Quantum fields for Thompson's groups *F* and *T*

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Haagerup CFT? Some things we've tried

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SOME UNITARY REPRESENTATIONS OF THOMPSON'S GROUPS F AND T.

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Abstract. In a "naive" attempt to create algebraic quantum field theories on the circle, we obtain a family of unitary representations of Thompson's groups T and F for any subfactor. The Thompson group elements are the "local scale transformations" of the theory. In a simple case the coefficients of the representations are polynomial invariants of links. We show that all links arise and introduce new "oriented" subgroups $\overrightarrow{F} < F$ and $\overrightarrow{T} < T$ which allow us to produce all *oriented* knots and links.

1. Introduction

This paper is part of an ongoing effort to construct a conformal field theory for every finite index subfactor in such a way that the standard invariant of the subfactor, or at least its quantum double, can be recovered from the CFT. There is no doubt that interesting subfactors arise in CFT nor that in some cases the numerical data of the subfactor appears as numerical data in the CFT. But there are supposedly "exotic" subfactors for which no CFT is known to exist, the first of which was constructed by Haagerup in a tour de force in [14],[1]. But in the last few years ideas of Evans and Gannon (see [8]) have made it seem plausible that CFT's exist for the Haagerup and other exotic subfactors constructed in the Haagerup line (see [20]). This has revived

Subfactors $\longleftrightarrow$ CFTs

S. Doplicher and J. E. Roberts (1989)
M. Bischoff (2015)

Haagerup subfactor

(first irreducible finite-depth subfactor with Jones index>4)

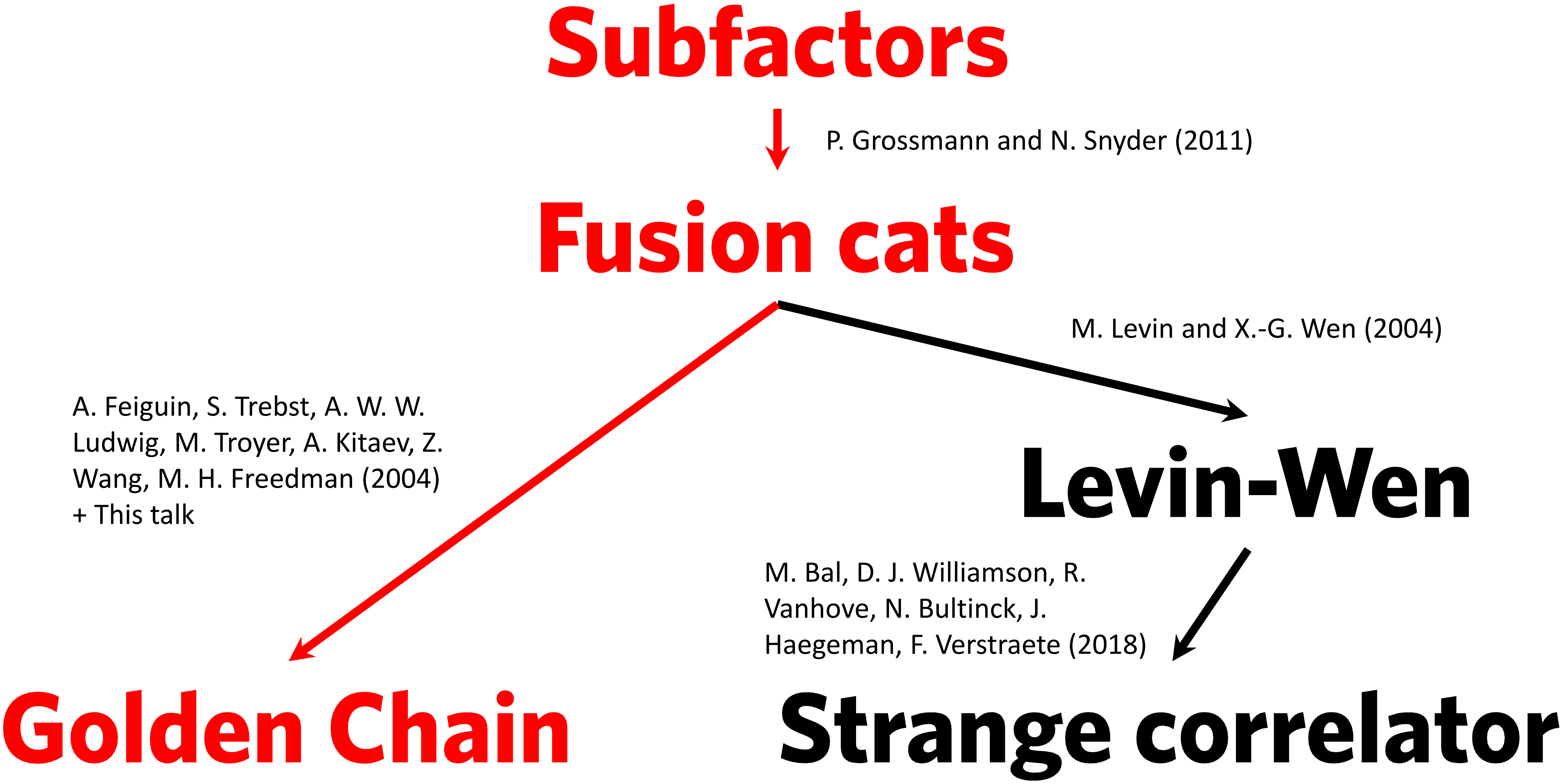
M. Asaeda and U. Haagerup (1999)

Haagerup $\overset{?}{\longleftrightarrow}$ CFT

D. Evans and T. Gannon (2010)

Royal road: build a "Haagerup" microscopic model & find 2nd order phase transition

(2nd order phase transitions “=“ CFTs)


Subfactors
P. Grossmann and N. Snyder (2011)
Fusion cats
M. Levin and X.-G. Wen (2004)
A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, M. H. Freedman (2004) + This talk
Levin-Wen
M. Bal, D. J. Williamson, R. Vanhove, N. Bultinck, J. Haegeman, F. Verstraete (2018)
Golden Chain
Strange correlator

A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, M. H. Freedman(2004)
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P. E. Finch, H. Frahm, M. Lewerenz, A. Milsted, TJO (2014)
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Tensor networks

Jones (2014-2019)

TJO (2019)
+This talk

Golden Chain

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Tensor networks

Jones (2014-2019)

TJO (2019)
+This talk

Continuous limits

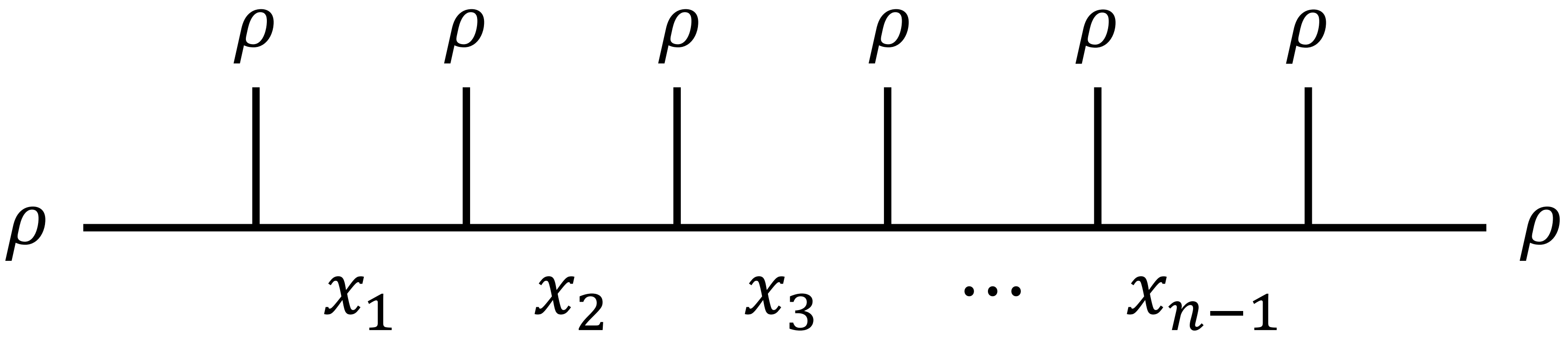
This talk???

CFT

The H3 Fusion Category

	$\mathbf{1}$	α	α^*	ρ	${}_\alpha\rho$	${}_{\alpha^*}\rho$
$\mathbf{1}$	$\mathbf{1}$	α	α^*	ρ	${}_\alpha\rho$	${}_{\alpha^*}\rho$
α	α	α^*	$\mathbf{1}$	${}_\alpha\rho$	${}_{\alpha^*}\rho$	ρ
α^*	α^*	$\mathbf{1}$	α	${}_{\alpha^*}\rho$	ρ	${}_\alpha\rho$
ρ	ρ	${}_{\alpha^*}\rho$	${}_\alpha\rho$	$\mathbf{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$
${}_\alpha\rho$	${}_\alpha\rho$	ρ	${}_{\alpha^*}\rho$	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\mathbf{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$
${}_{\alpha^*}\rho$	${}_{\alpha^*}\rho$	${}_\alpha\rho$	ρ	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\mathbf{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$

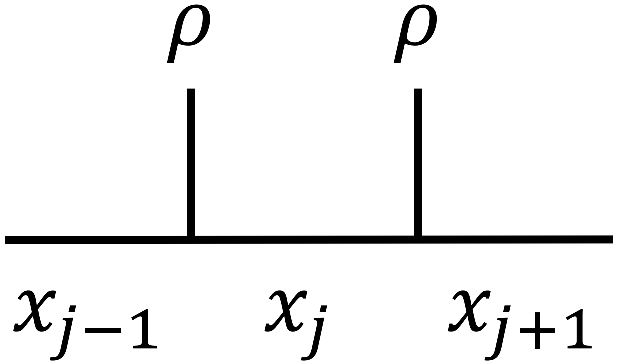
The H3 golden chain: Hilbert space



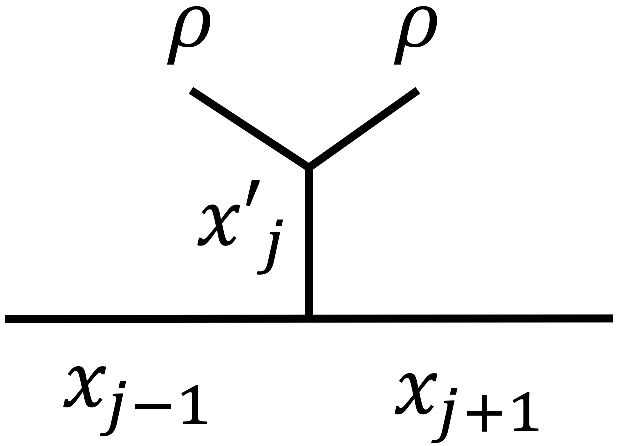
$$\longrightarrow \quad |x_1 x_2 \cdots x_{n-1}\rangle$$

Fusion paths: $x_j \in \{1, \alpha, \alpha^*, \rho, \alpha\rho, \alpha^*\rho\}$

The H3 golden chain: Hamiltonian

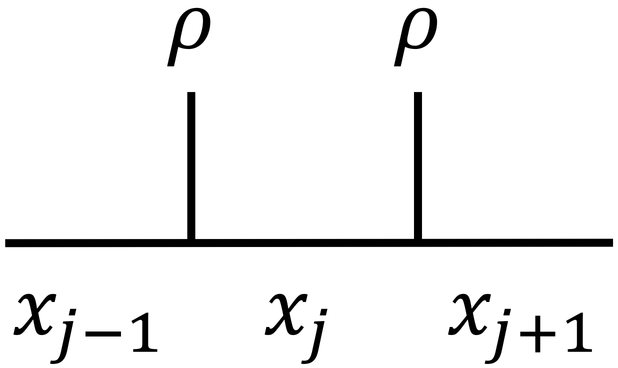


$$= \sum_{x'_j} \left[F^{x_{j-1}\rho\rho}_{x_{j+1}}\right]_{x'_j x_j}$$

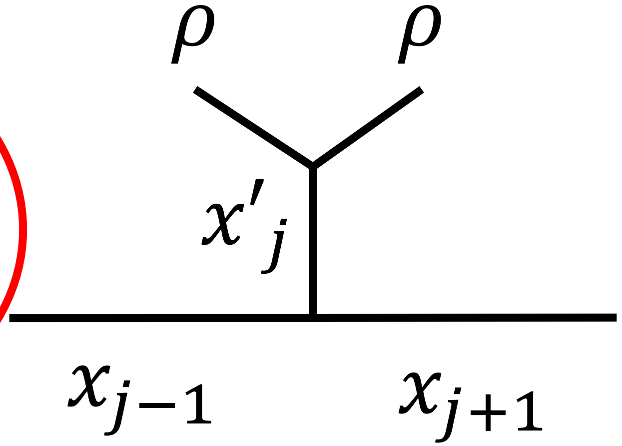


Energy of $x'_j = \mathbf{1}$ is 0, otherwise 1

The H3 golden chain: Hamiltonian



$$= \sum_{x'_j} \left[F^{x_{j-1}\rho\rho}_{x_{j+1}} \right]_{x'_j x_j}$$



?

$$H = \sum_j h_j$$

The Pentagon Equation

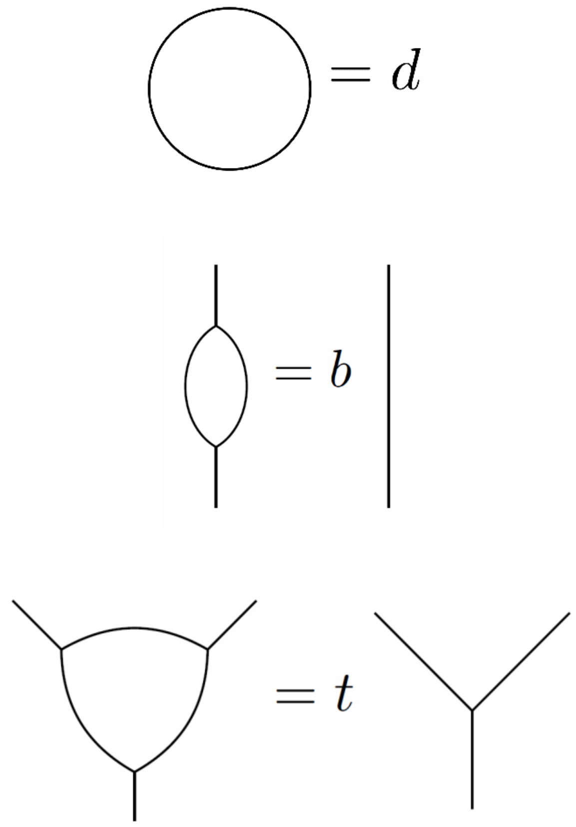
$$\left(F_u^{xyz}\right)_{da}\left(F_u^{azw}\right)_{cb} = \sum_e \left(F_d^{yzw}\right)_{ce}\left(F_u^{xew}\right)_{db}\left(F_b^{xyz}\right)_{ea}$$

objects: $a, b, \ldots \in \{1, \alpha, \alpha^*, \rho, \alpha\rho, \alpha^*\rho\}$

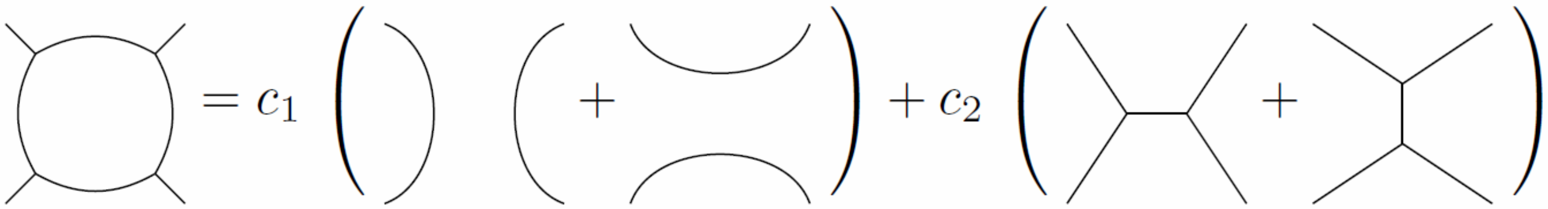
$\Rightarrow$ 41391 equations

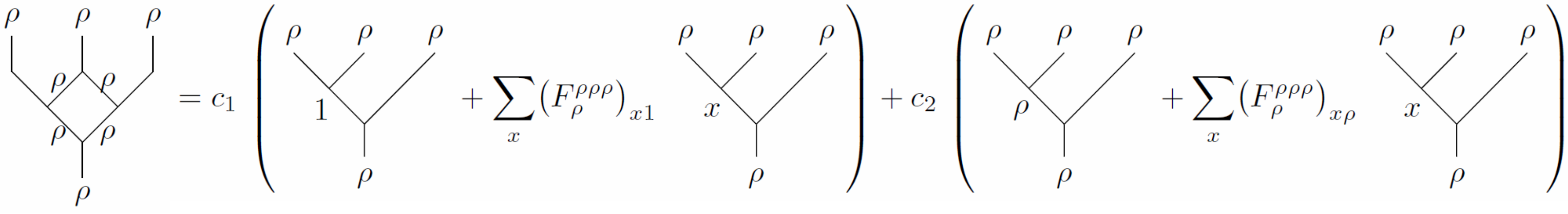
$\Rightarrow$ 1431 variables

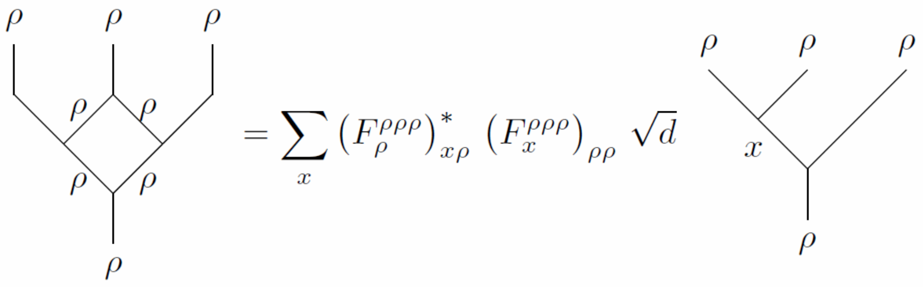
Trivalent categories



Trivalent categories







Trivalent categories

$$\left(F_{\rho}^{\rho\rho\rho}\right)^{*}_{\alpha\rho\rho}\left(F_{\alpha\rho}^{\rho\rho\rho}\right)_{\rho\rho}\sqrt{d} = \mathrm{c}_1\left(F_{\rho}^{\rho\rho\rho}\right)_{\alpha\rho 1} + c_2\left(F_{\rho}^{\rho\rho\rho}\right)_{\alpha\rho\rho}$$

$$\left(F_{\rho}^{\rho\rho\rho}\right)^{*}_{\alpha^*\rho\rho}\left(F_{\alpha^*\rho}^{\rho\rho\rho}\right)_{\rho\rho}\sqrt{d} = \mathrm{c}_1\left(F_{\rho}^{\rho\rho\rho}\right)_{\alpha^*\rho 1} + c_2\left(F_{\rho}^{\rho\rho\rho}\right)_{\alpha^*\rho\rho}$$

Finding solutions

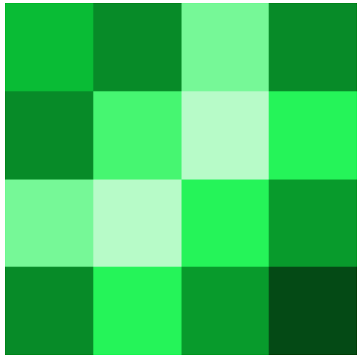
- Equations with one variable
- Gauge freedom
- Small solvable subsystems
- Unitarity

The Solution

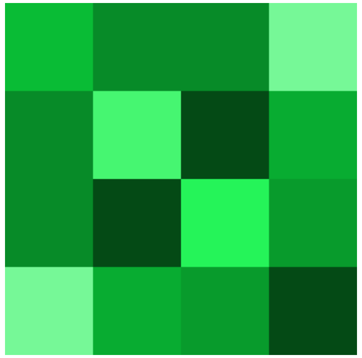
Two parameters: $p_1, p_2 \in \{-1, +1\}$

$$F_{\rho}^{\rho\rho\rho} = \begin{pmatrix} \frac{1}{2}\left(\sqrt{13}-3\right) & \sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)} & -\sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)}\, p_1 & \sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)}\, p_1 \\ \sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)} & \frac{1}{3}\left(2-\sqrt{13}\right) & \frac{1}{12}\left(\sqrt{13}-\sqrt{6\left(\sqrt{13}+1\right)}-5\right)p_1 & -\frac{1}{12}\left(\sqrt{13}+\sqrt{6\left(\sqrt{13}+1\right)}-5\right)p_1 \\ -\sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)}\, p_1 & \frac{1}{12}\left(\sqrt{13}-\sqrt{6\left(\sqrt{13}+1\right)}-5\right)p_1 & \frac{1}{12}\left(-\sqrt{13}-\sqrt{6\left(\sqrt{13}+1\right)}+5\right) & \frac{1}{3}\left(\sqrt{13}-2\right) \\ \sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)}\, p_1 & -\frac{1}{12}\left(\sqrt{13}+\sqrt{6\left(\sqrt{13}+1\right)}-5\right)p_1 & \frac{1}{3}\left(\sqrt{13}-2\right) & \frac{1}{12}\left(-\sqrt{13}+\sqrt{6\left(\sqrt{13}+1\right)}+5\right) \end{pmatrix}$$

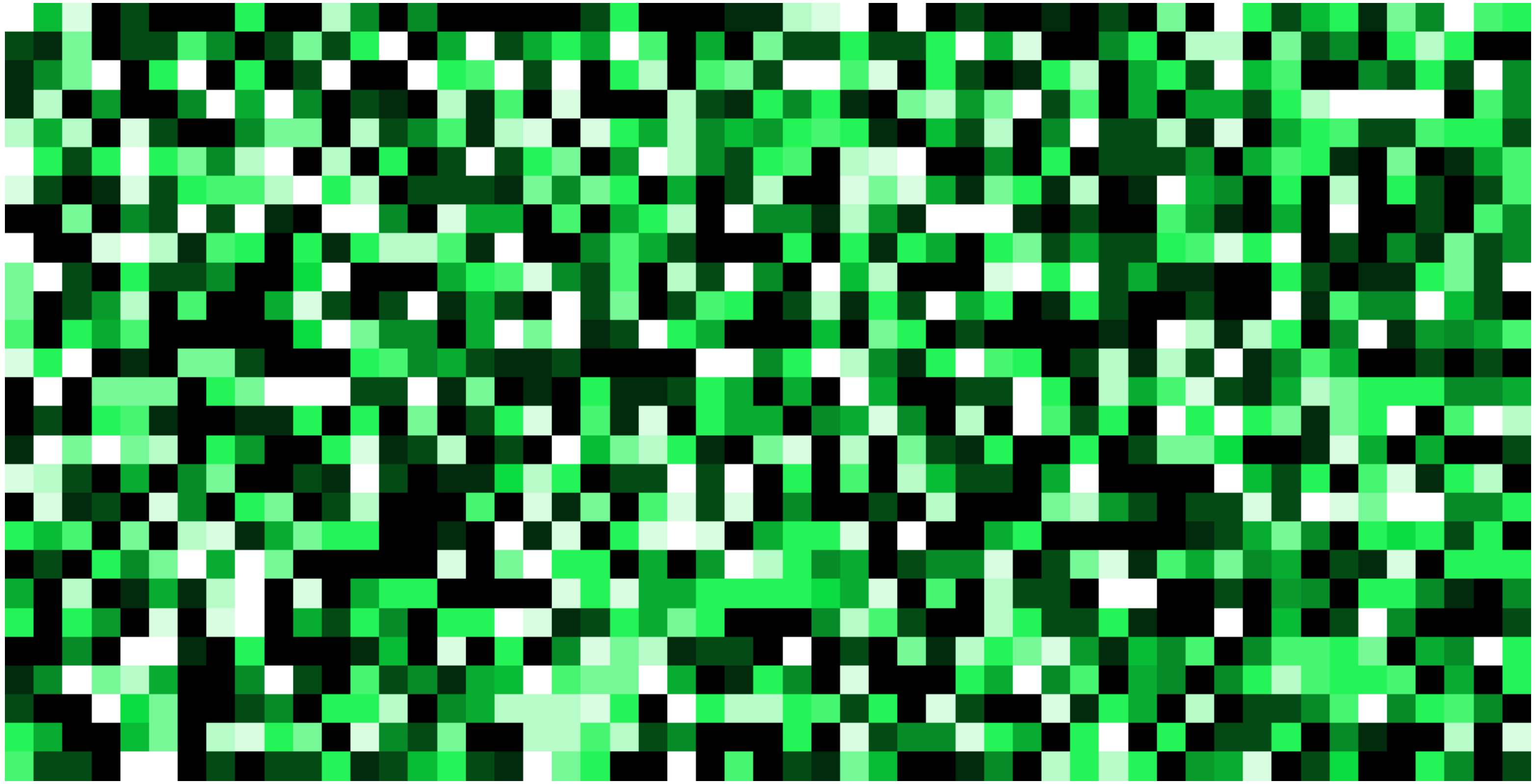
$p_1 = 1$



$p_1 = -1$



The Solution

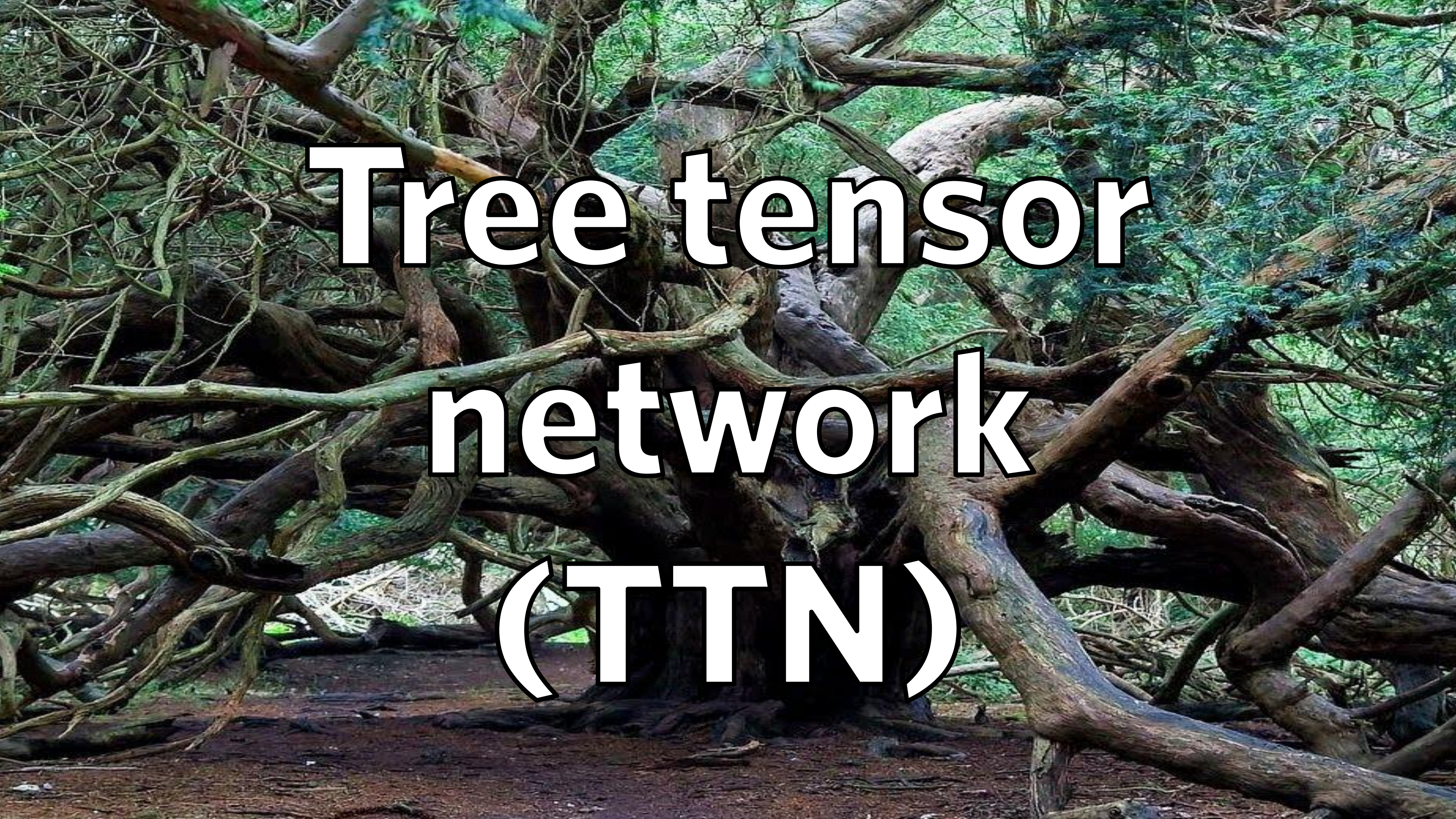


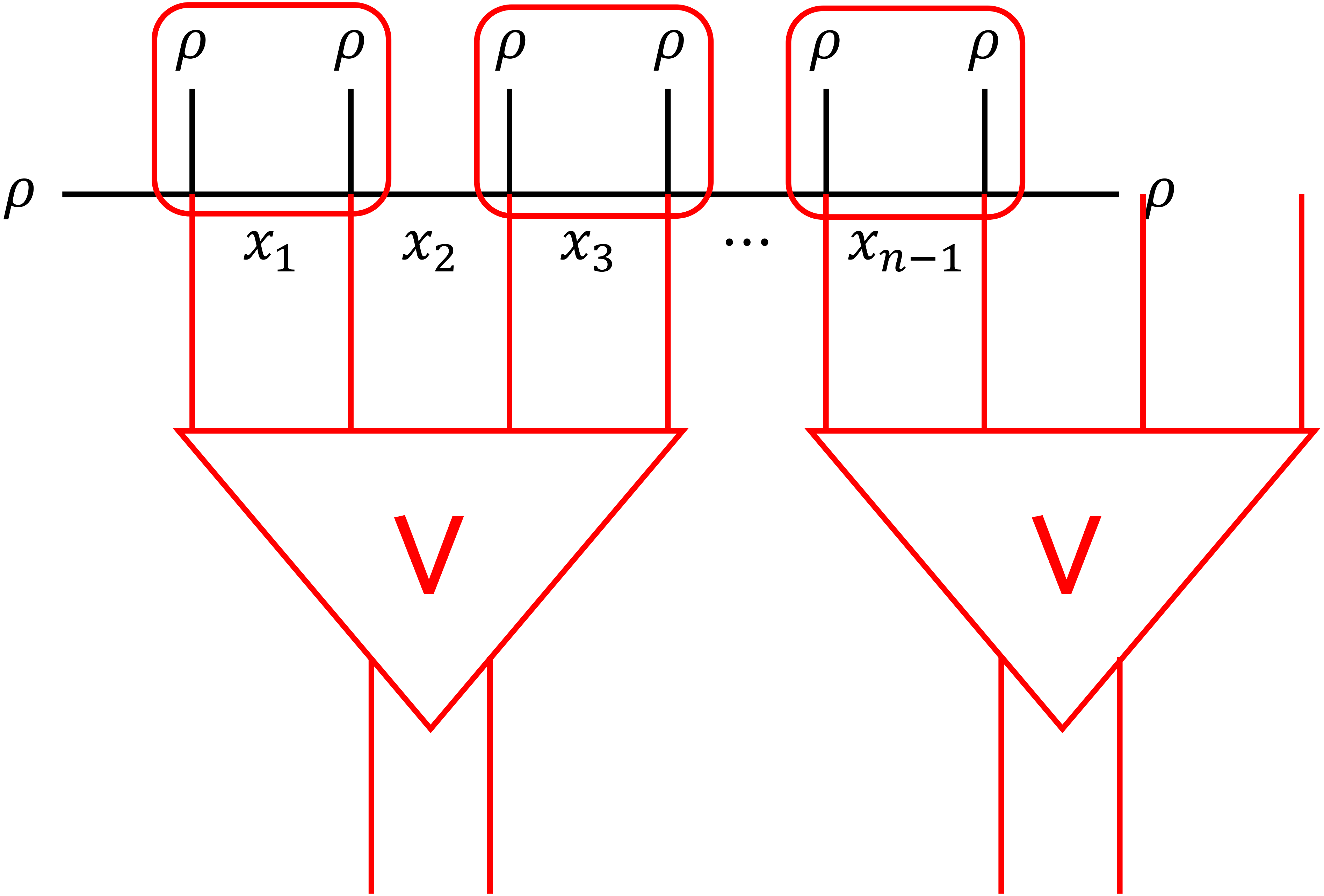
https://github.com/R8monaW/H3Fsymbols

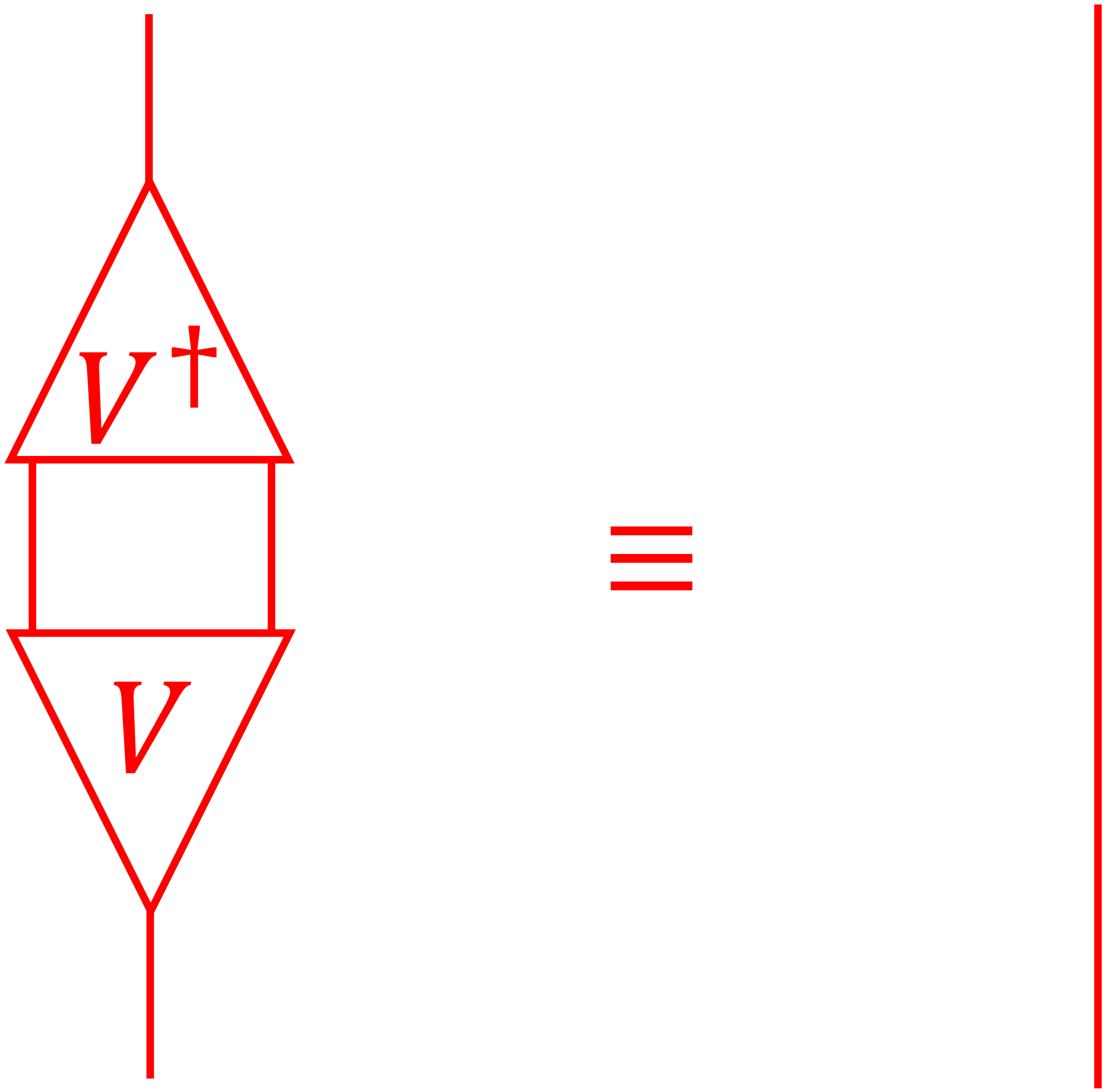
The H3 golden chain: Hamiltonian

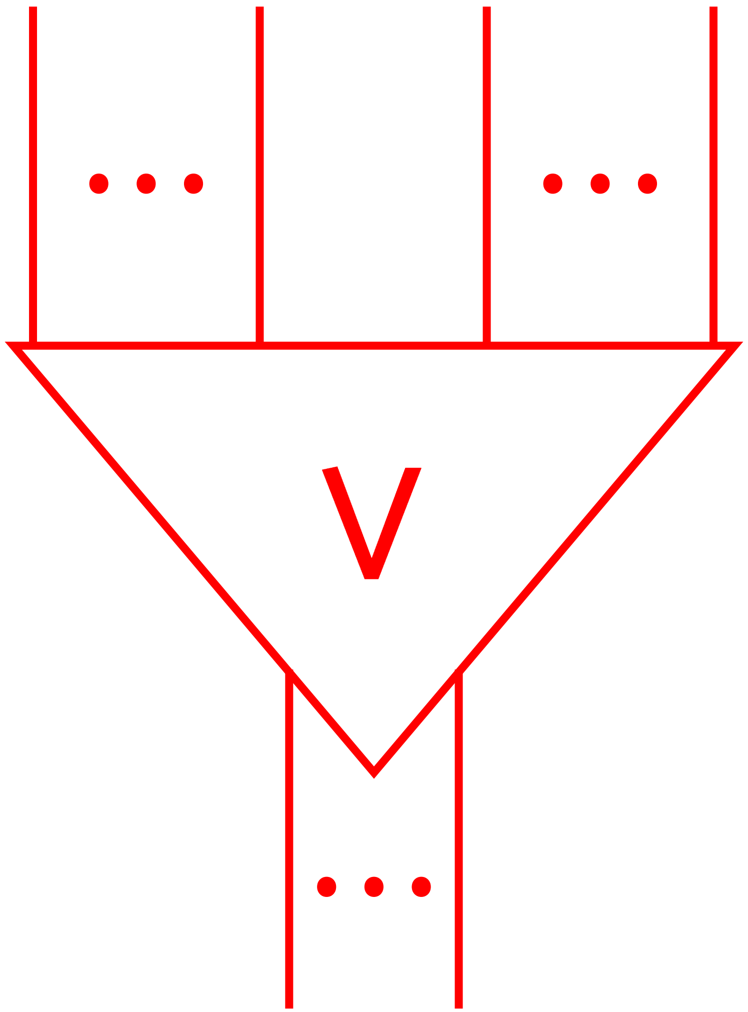
$$H = \sum_j h_j$$

Eigenvalues? Eigenvectors?

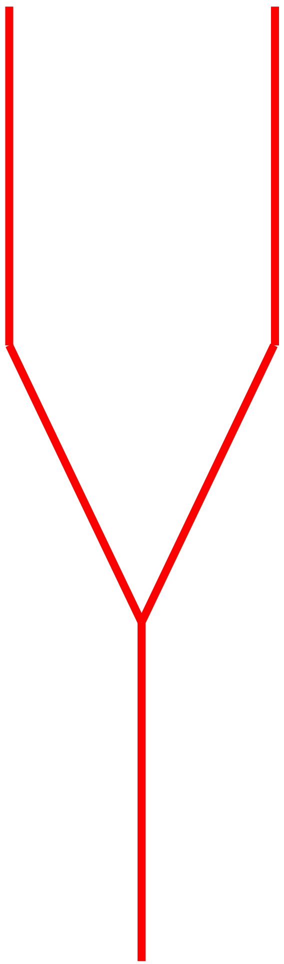

Tree tensor
network
(TTN)


ρ
ρ
ρ
ρ
ρ
ρ
ρ
ρ
x_1
x_2
x_3
⋯
x_{n-1}
V
V


$V^\dagger$
V
$\equiv$



$$\in \mathrm{Mor}(\rho^m \otimes \rho^m, \rho^m)$$



$$\in \mathrm{Mor}(\rho^1 \otimes \rho^1, \rho^1)$$

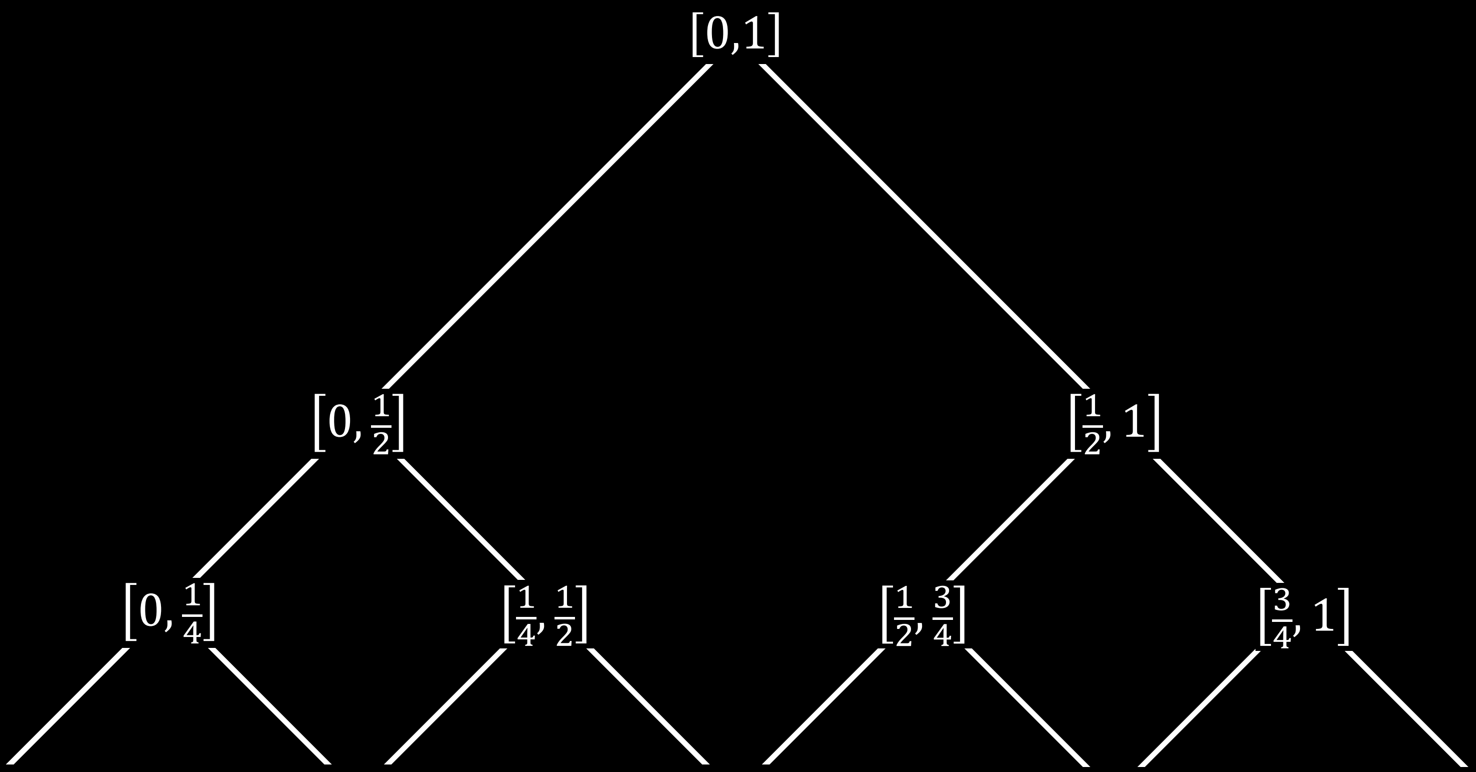
Conformal field theory?

$$\mathrm{conf}(\mathbb{R}^{1,1}) \cong \mathrm{diff}_+(S^1) \times \mathrm{diff}_+(S^1)$$

(Semi)continuous limit

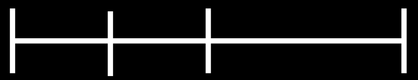
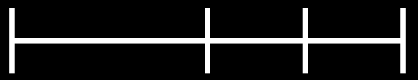
Standard dyadic interval:

interval of form $\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$:



Standard dyadic partitions:

partitions $[0,1]$ into std. dyadic intervals

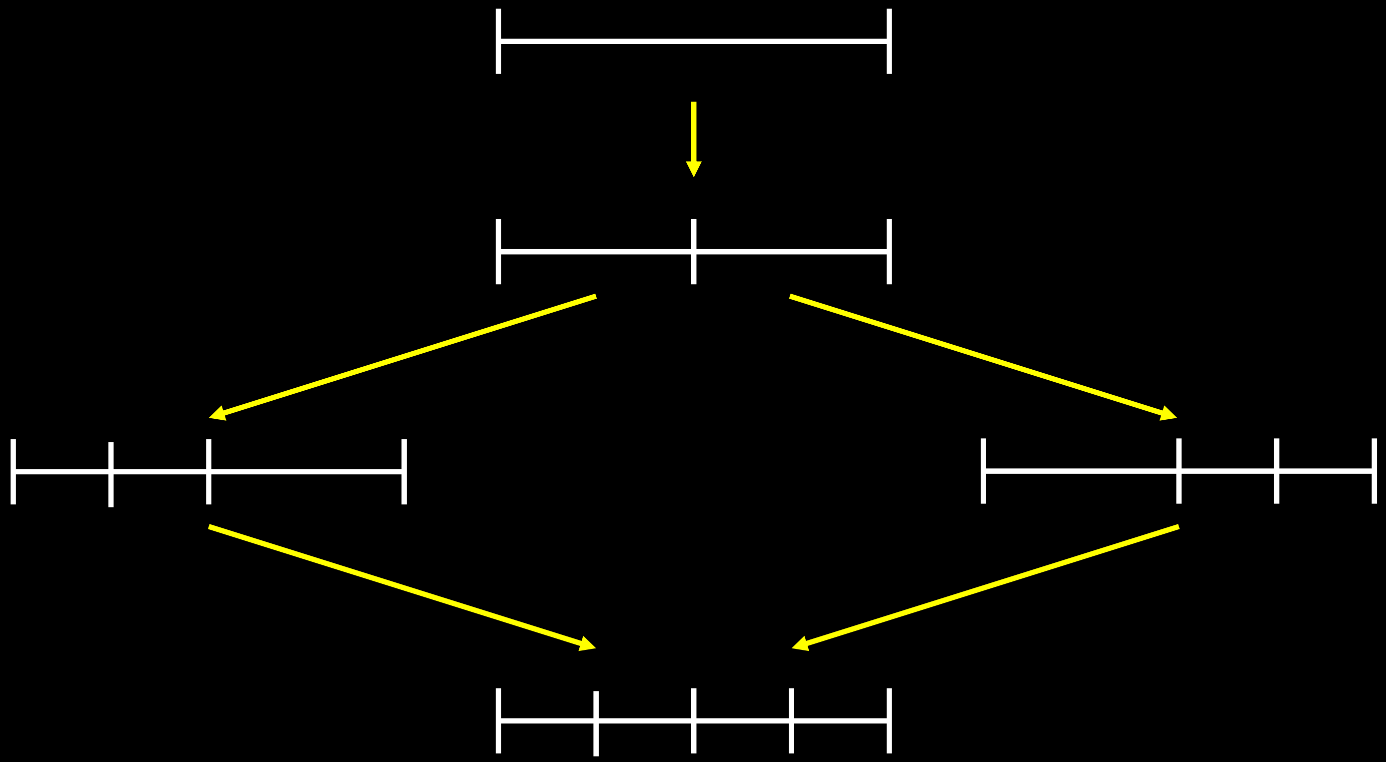
$\mathcal{D} = \Big\{$, …, , , … $\Big\}$

If $P, Q \in \mathcal{D}$ say
"$P \leq Q$" to mean partition
Q is a **refinement** of P

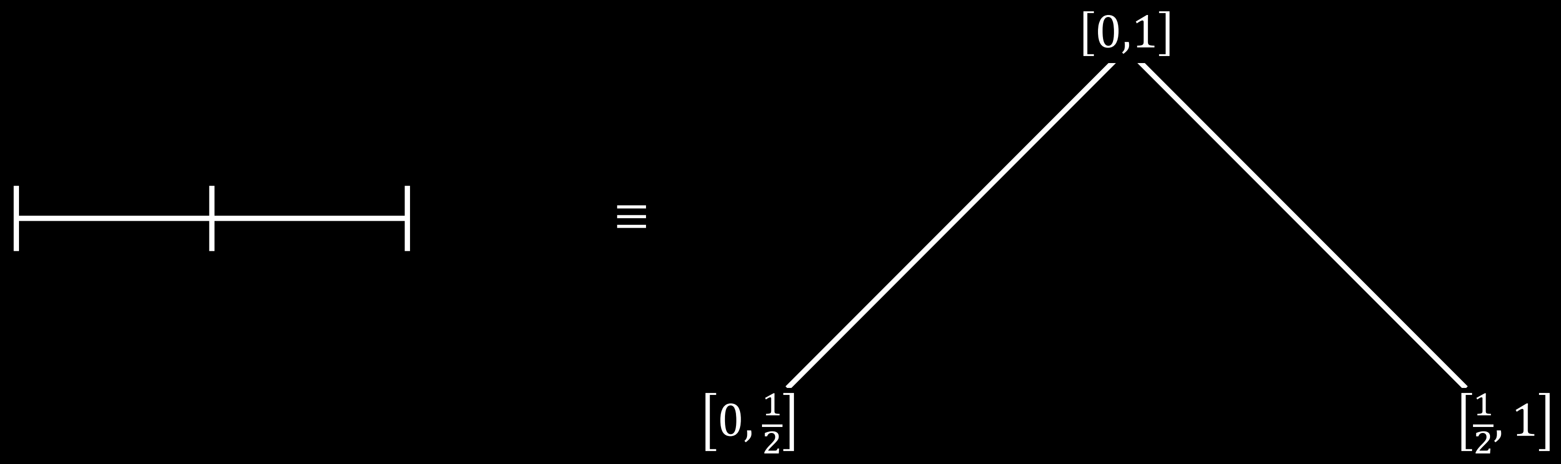
(Q has more cells)

Standard dyadic partition:

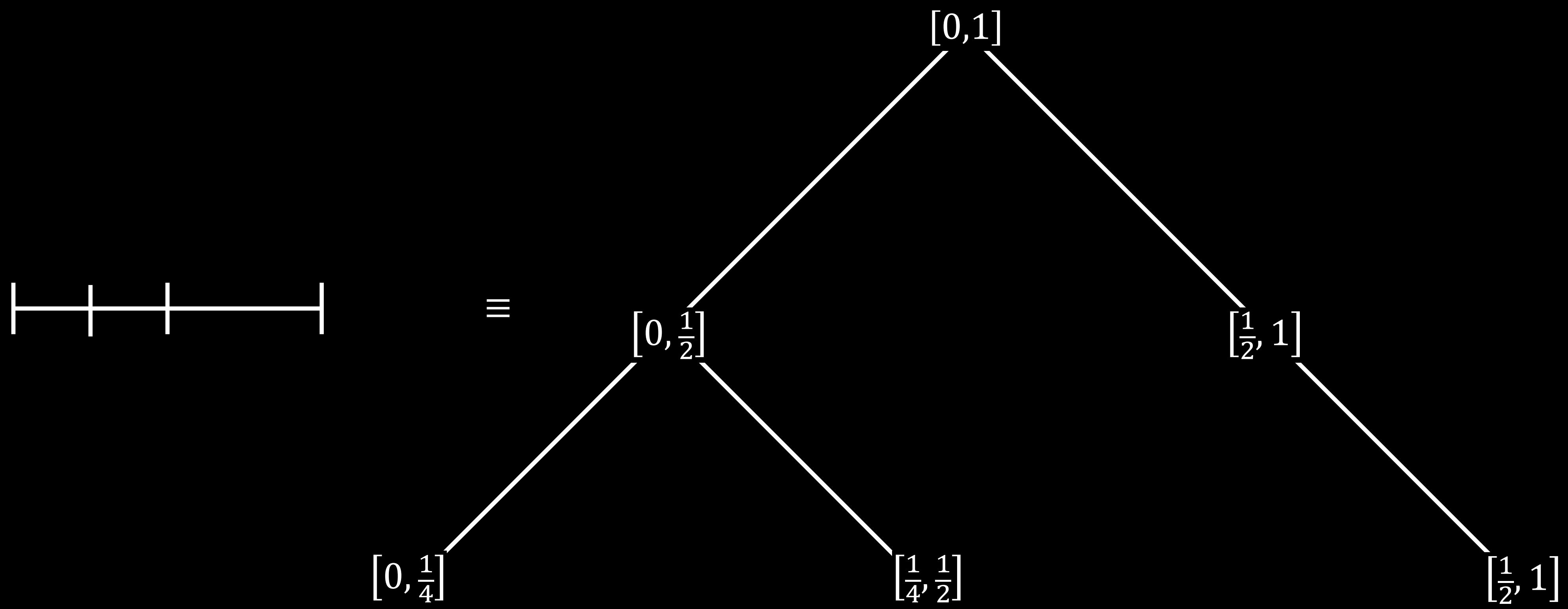
directed set $\mathcal{D}$



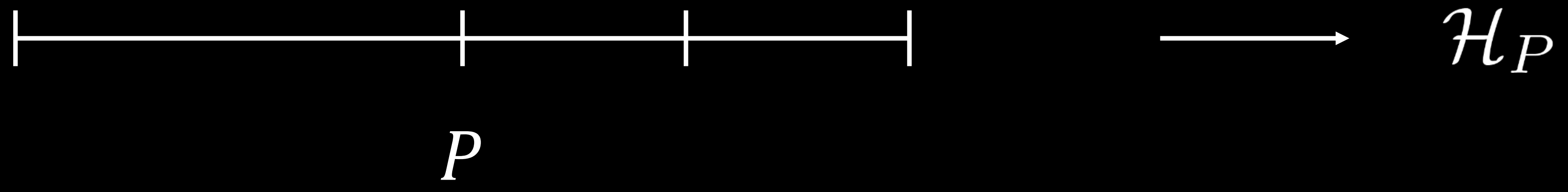
Standard dyadic partitions:
representation via trees



Standard dyadic partitions: representation via trees

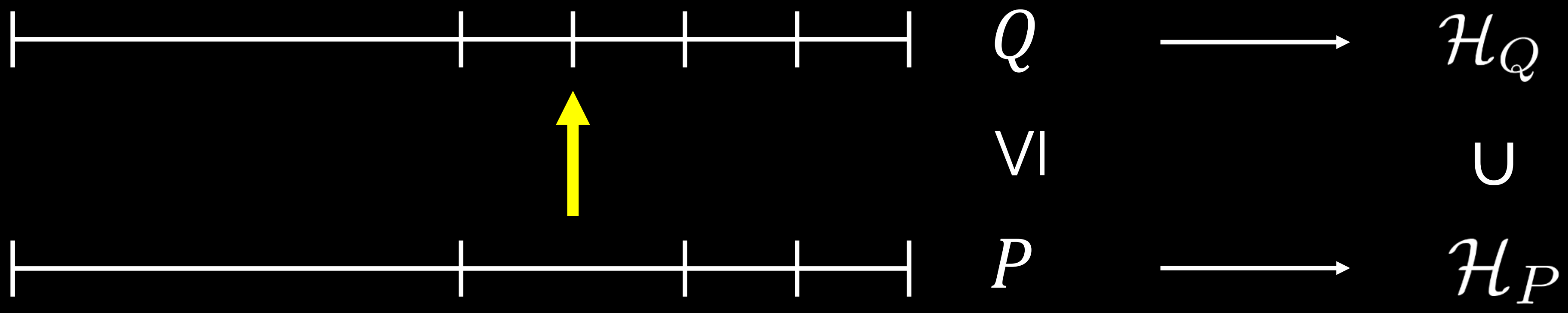


Hilbert space structure

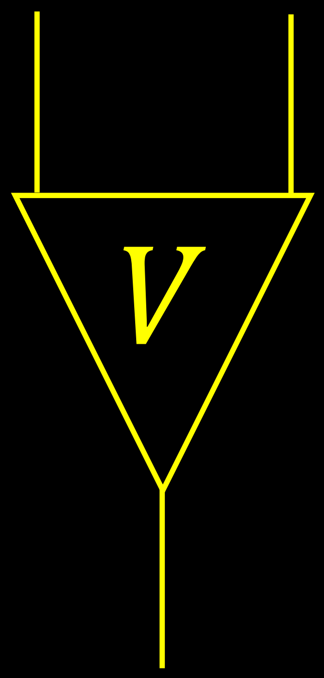


If $P \leq Q$ identify $\mathcal{H}_P \subset \mathcal{H}_Q$
via **isometry:**

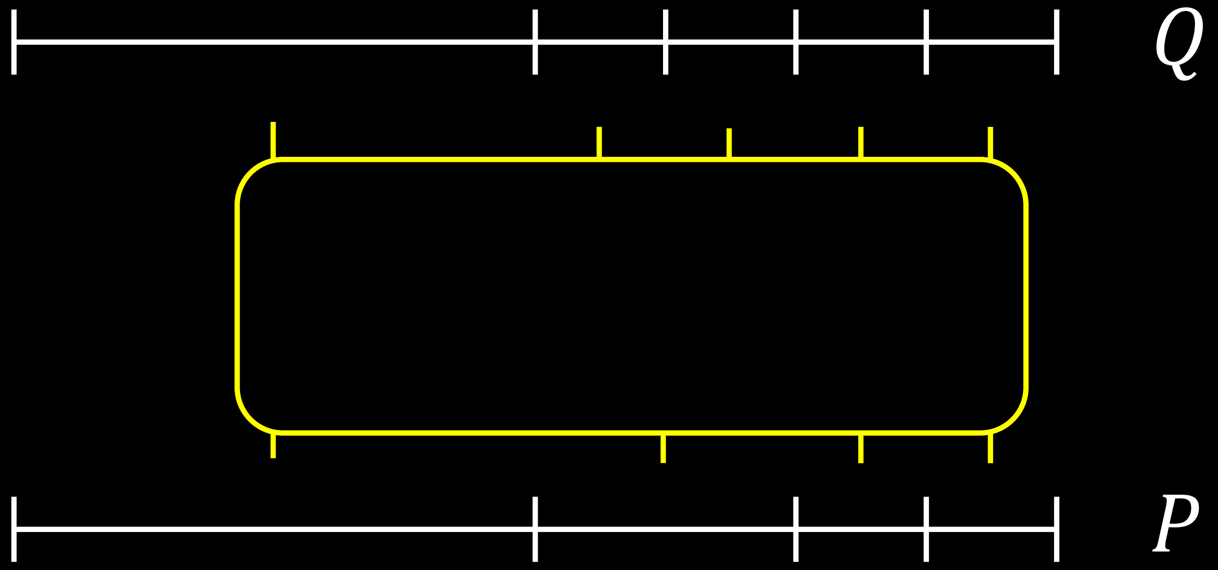
$$T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$$



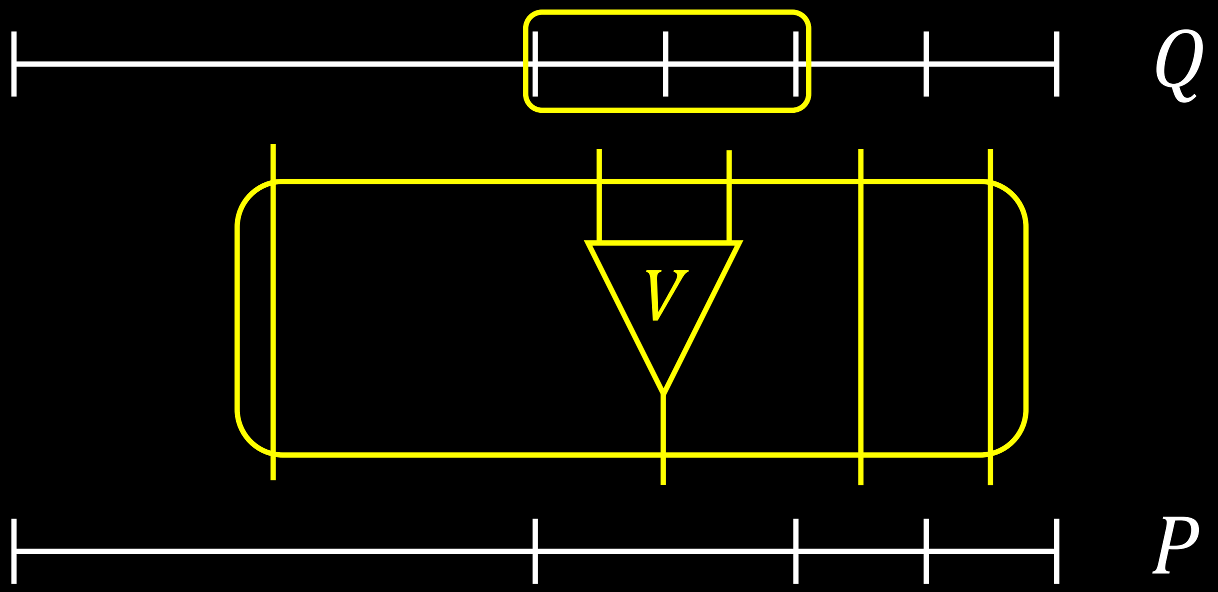
How to build isometries?

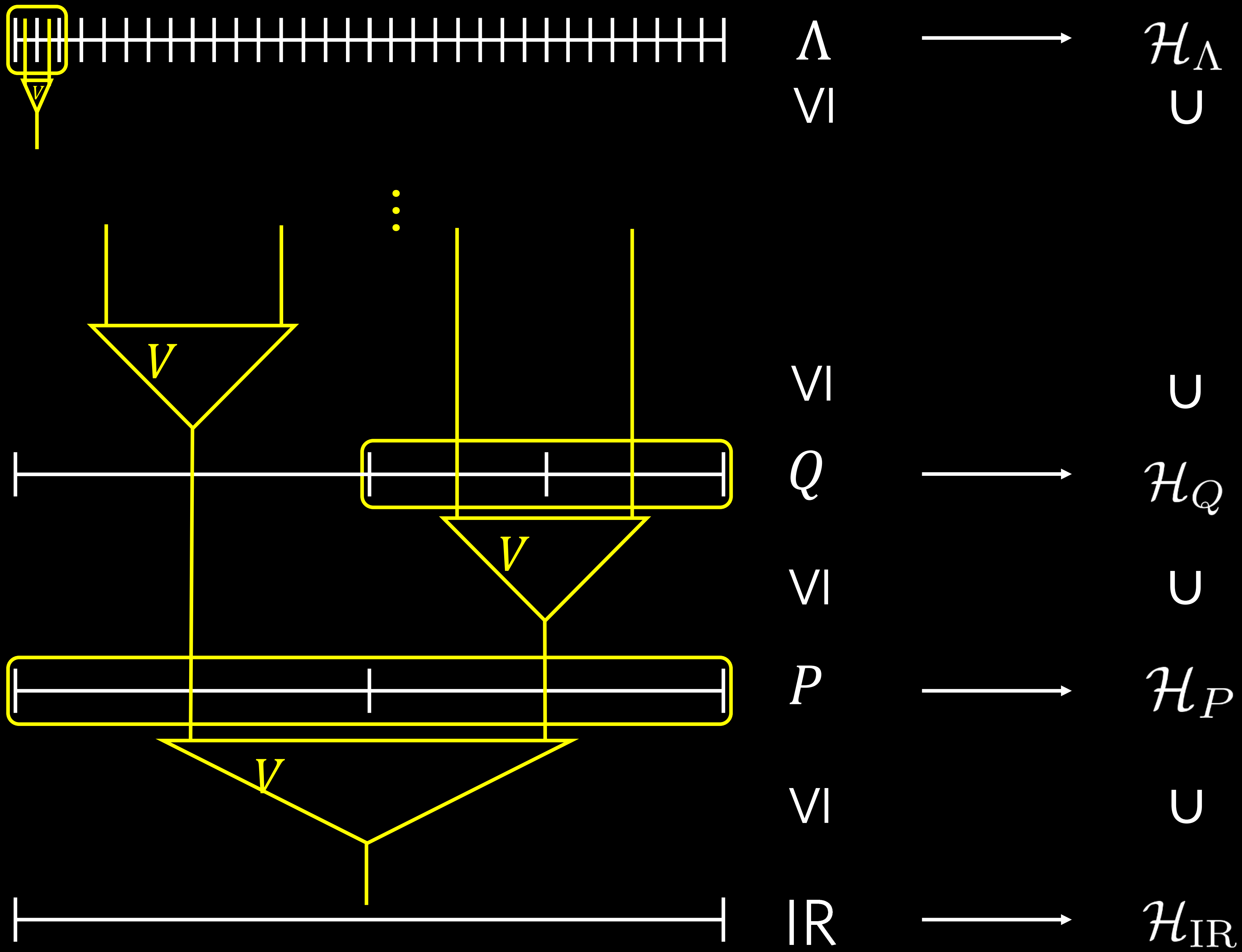


$$T_Q^P =$$



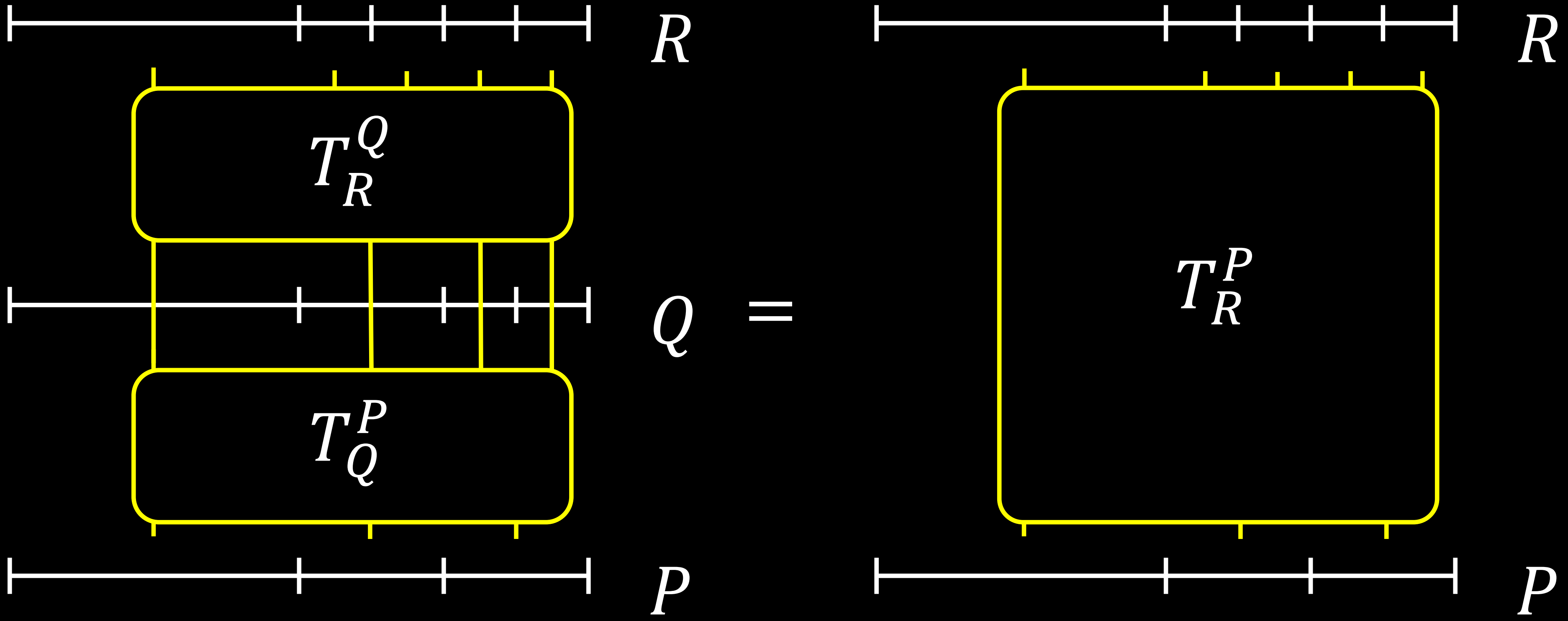
$T_Q^P =$




Λ
$\mathcal{H}_\Lambda$
V
V
Q
$\mathcal{H}_Q$
V
P
$\mathcal{H}_P$
V
IR
$\mathcal{H}_{\mathrm{IR}}$

Demand WLOG

$$T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$$

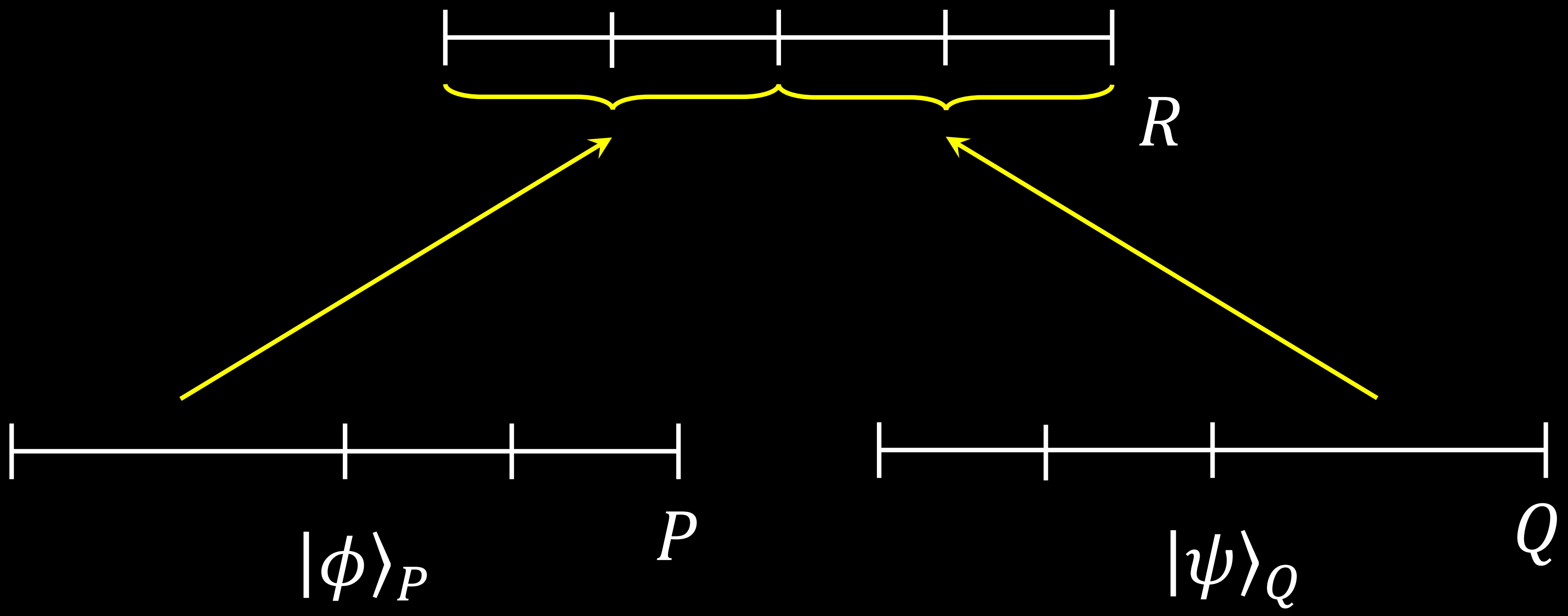


T^Q_R
T^P_Q
R
Q
P
=
T^P_R
R
P

Equivalence: $|\phi\rangle_P \sim |\psi\rangle_Q$

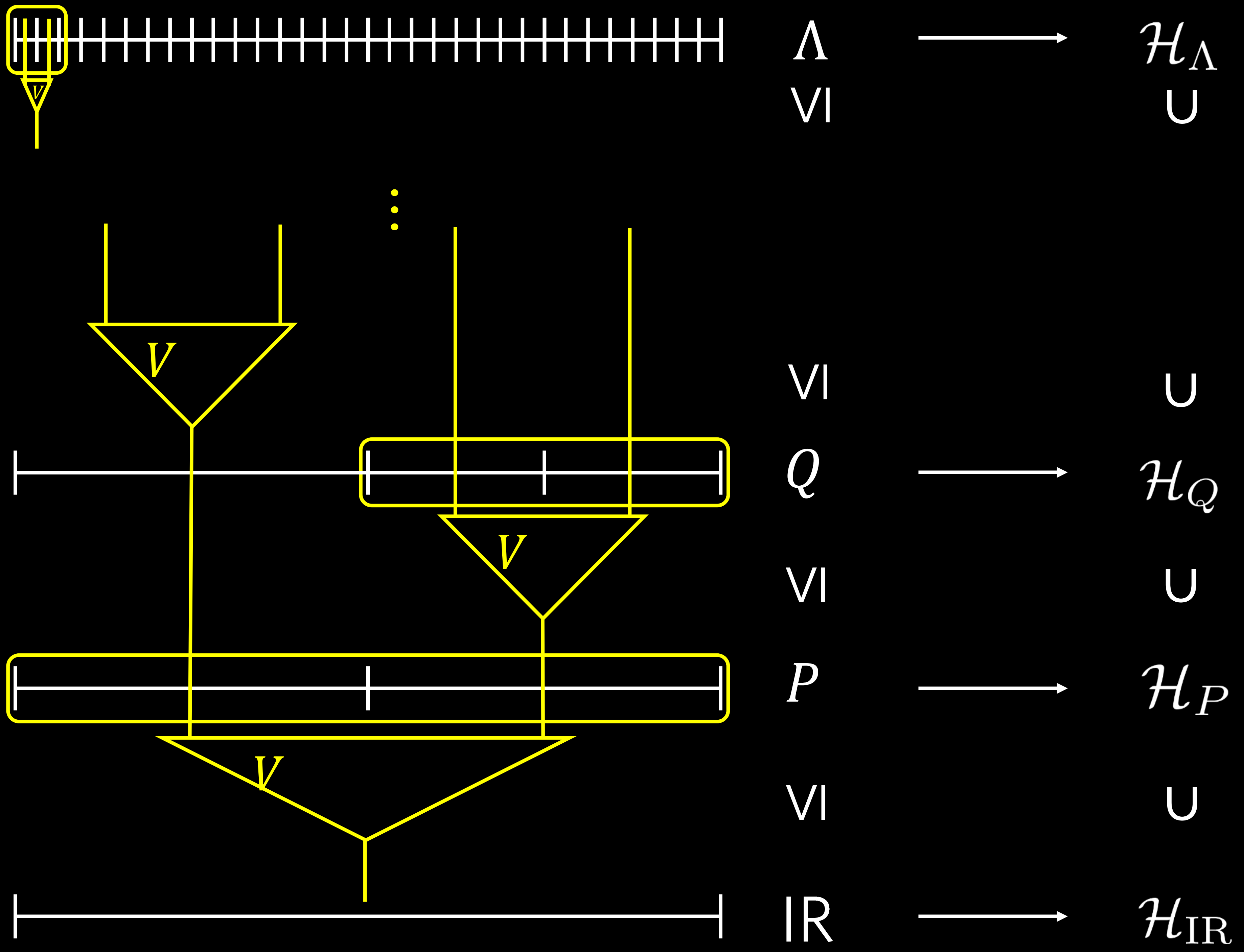
if $\exists R$

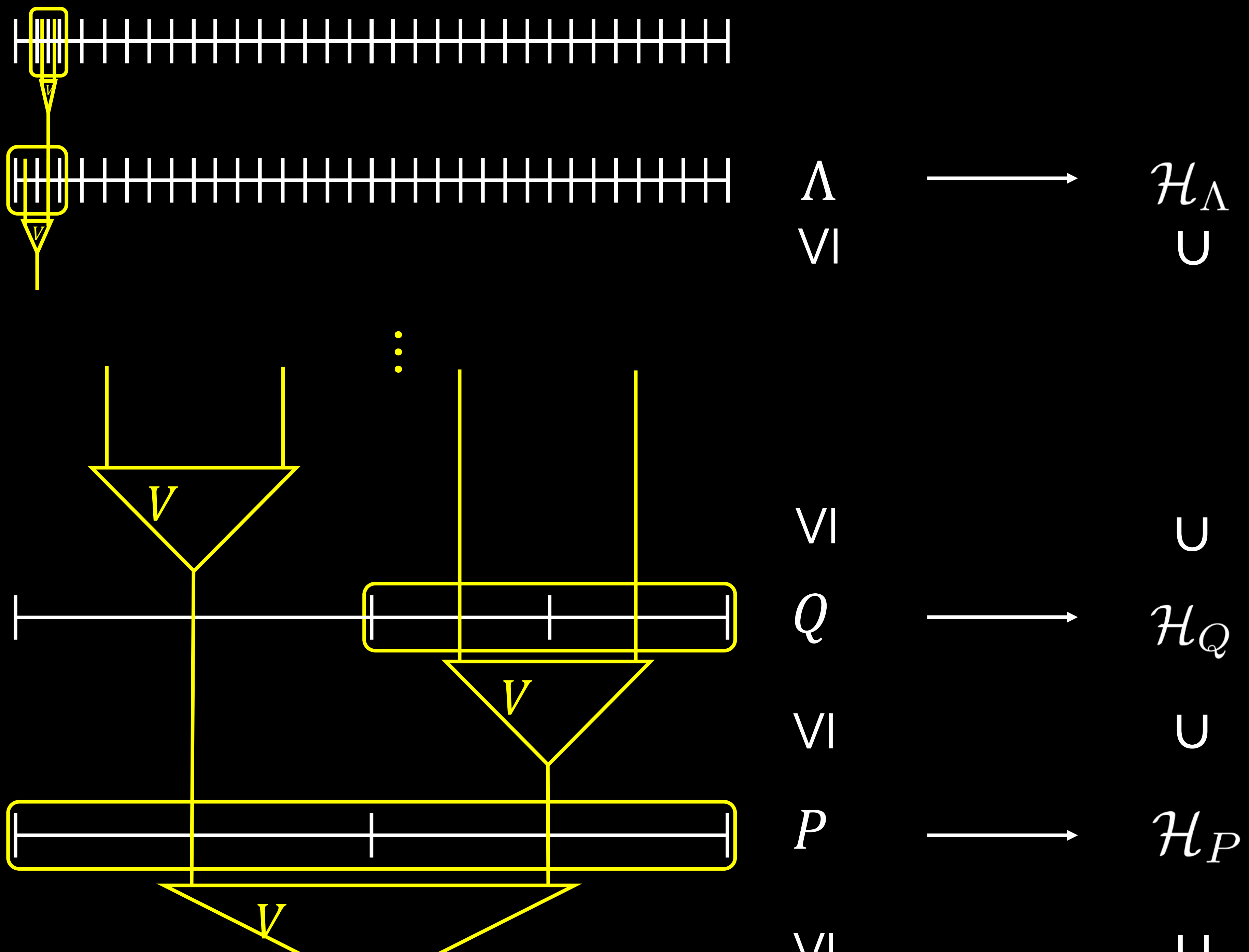
$$T_R^P |\phi\rangle_P = T_R^Q |\psi\rangle_Q$$



Semicontinuous limit: Extrapolate!

T_Q^P embeds into arbitrarily fine (std. dyadic) lattices


Λ
$\mathcal{H}_\Lambda$
V
V
V
Q
$\mathcal{H}_Q$
V
P
$\mathcal{H}_P$
IR
$\mathcal{H}_{\mathrm{IR}}$



V
$\Lambda \longrightarrow \mathcal{H}_\Lambda$
$Q \longrightarrow \mathcal{H}_Q$
$P \longrightarrow \mathcal{H}_P$

Definition: let $(\mathcal{D}, \leq)$ be a directed set. Let a hilbert space $\mathcal{H}_P$ be given for each $P \in \mathcal{D}$ For all $P \leq Q$ let $T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$ be an isometry such that:

(1) T_P^P is the identity

(2) $T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$

Then $(\mathcal{H}_P, T_Q^P)$ is a **directed system**.

Semicontinuous limit:

$$\widehat{\mathcal{H}} \equiv \lim_{\overrightarrow{P \in \mathcal{P}}} \mathcal{H}_P$$

$$= \left\{ \begin{array}{l} \text{the disjoint union of } \mathcal{H}_P \text{ over all } P \in \mathcal{P} \\ \text{modulo the equivalence relation } |\phi\rangle_P \sim |\psi\rangle_Q \\ \text{if there is } R \geq P \text{ and } R \geq Q \text{ such that} \\ \qquad T_R^P |\phi\rangle_P = T_R^Q |\psi\rangle_Q \end{array} \right\}$$

V. F. R. Jones, arXiv:1412.7740 (2014)

Residents of $\widehat{\mathcal{H}}$:

$$[|\psi\rangle_P] \equiv \{|\phi\rangle_Q = T_Q^P |\psi\rangle_P\}$$

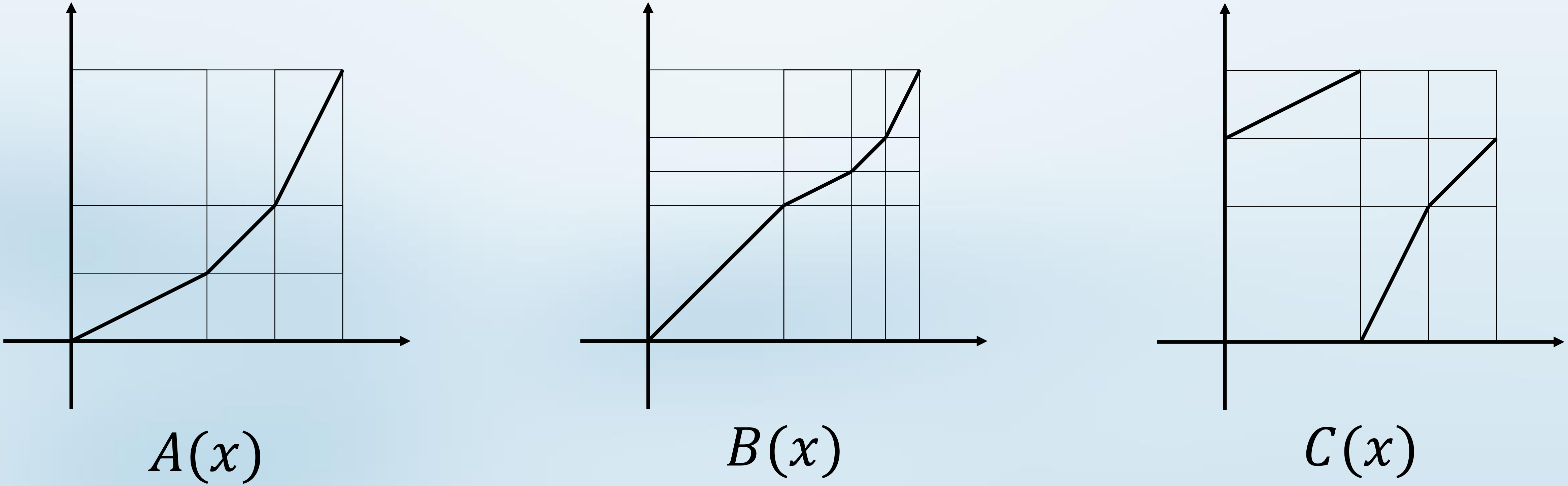
Each hilbert space $\mathcal{H}_P$ is a natural subspace of $\widehat{\mathcal{H}}$:

$$\mathcal{H}_P \hookrightarrow \widehat{\mathcal{H}}$$

via

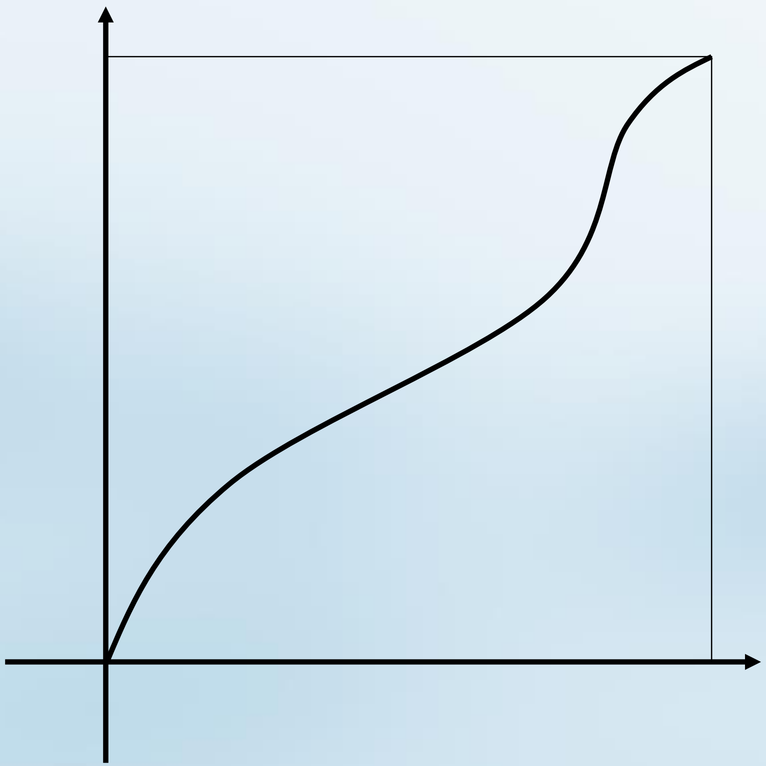
$$|\psi\rangle_P \longmapsto [|\psi\rangle_P]$$

Thompson's group T: generated by $A(x)$, $B(x)$, and $C(x)$ under composition

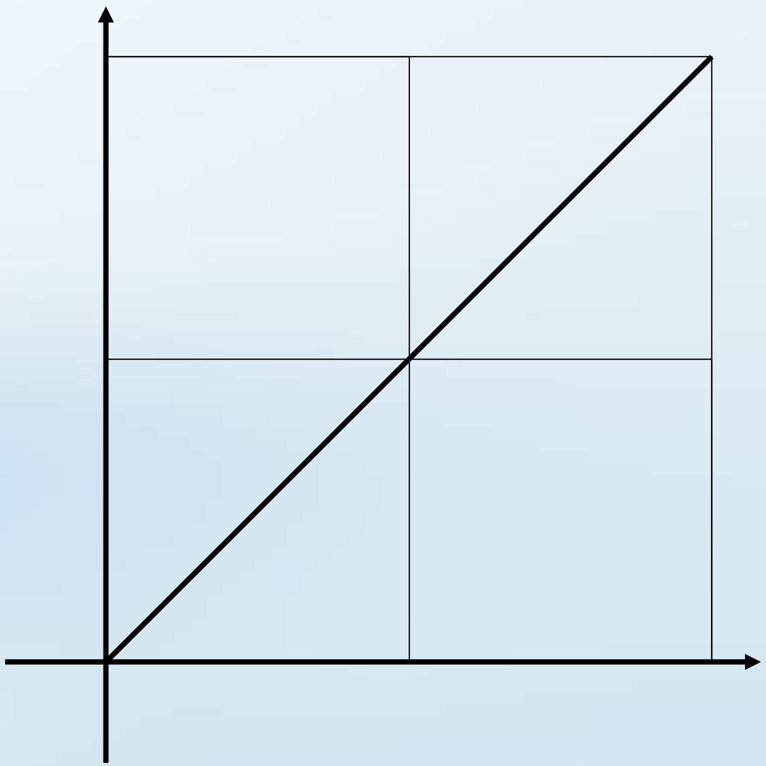


J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

Proposition ("well known"): let $f \in \text{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



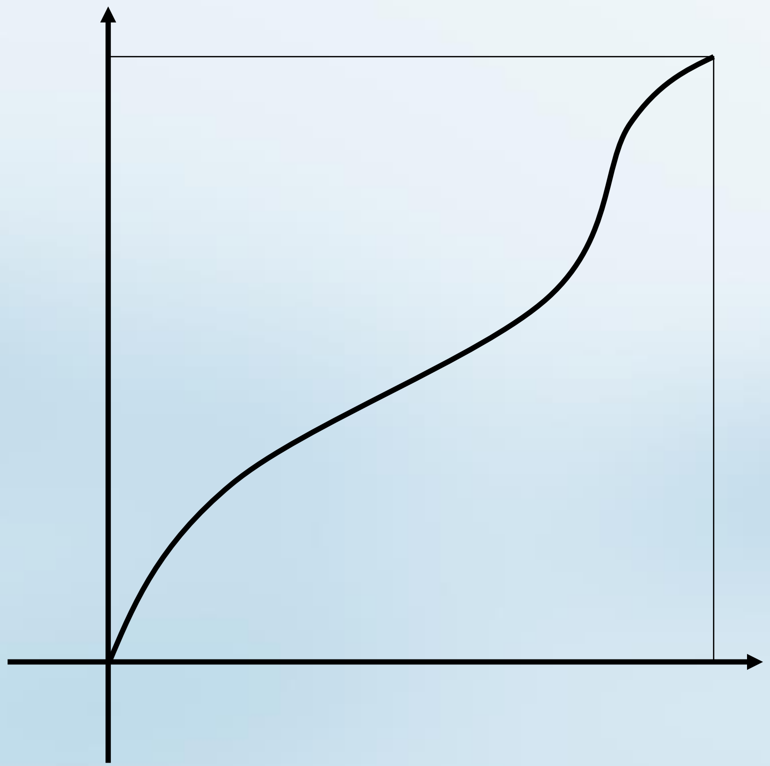
$f(x)$



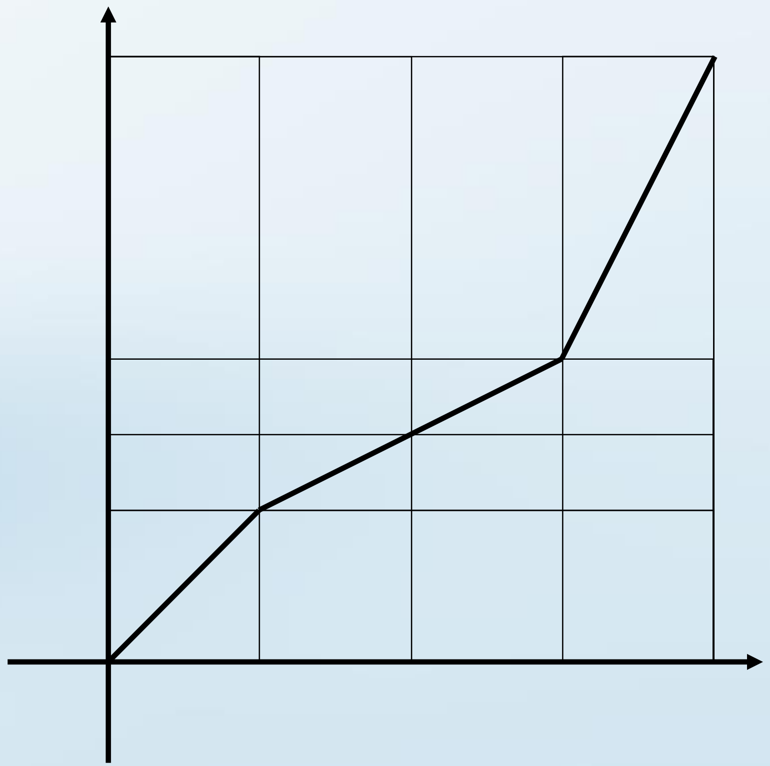
$A_1(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



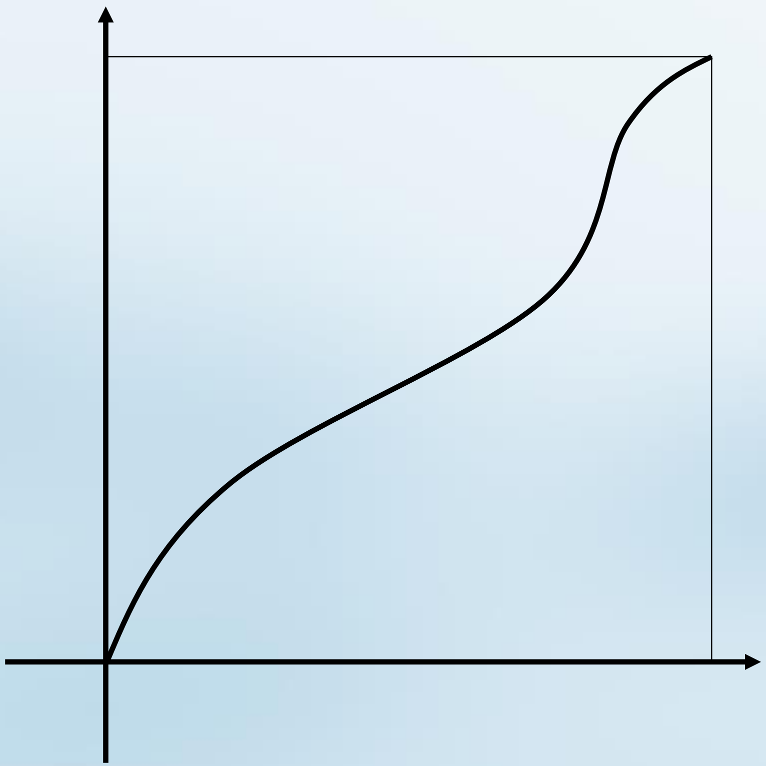
$f(x)$



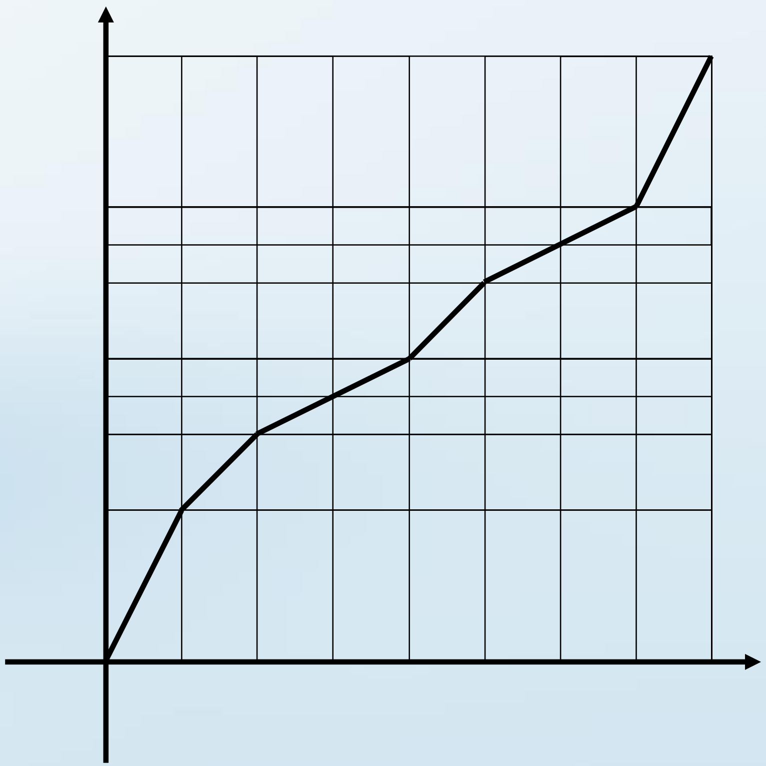
$A_2(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



$f(x)$

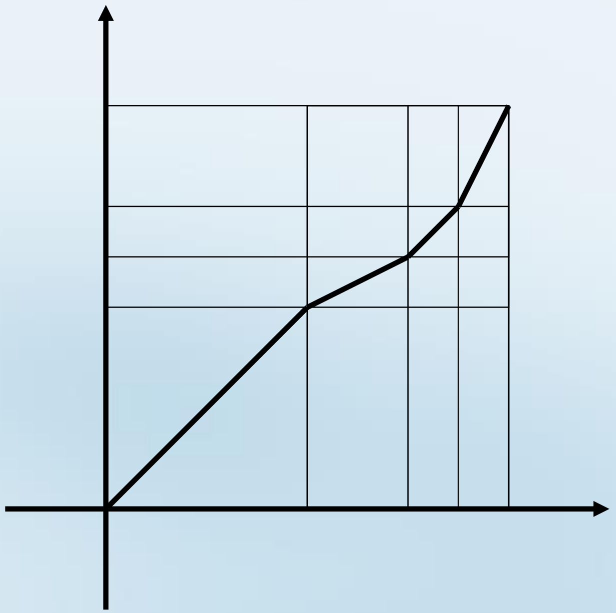


$A_3(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

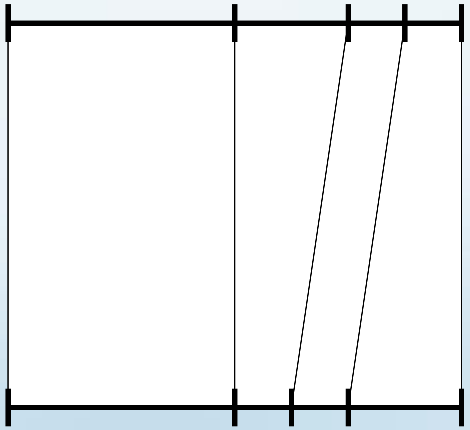
Elements of F and T

Pairs of std. dyadic partitions/trees

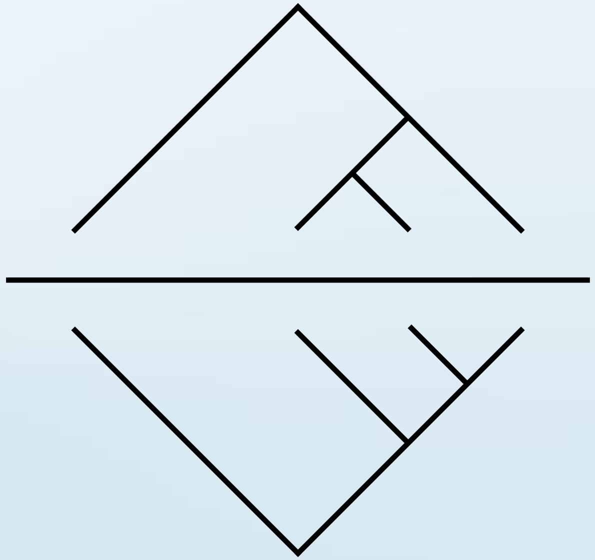


$B(x)$

$\equiv$

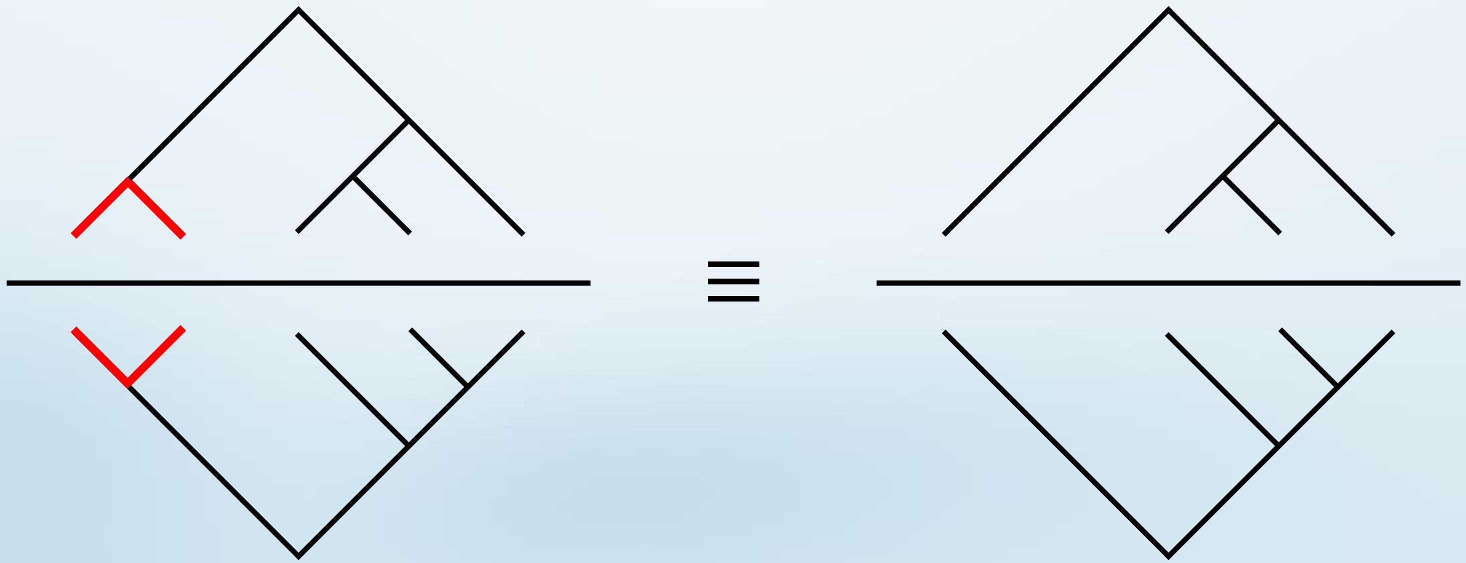


$\equiv$



J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

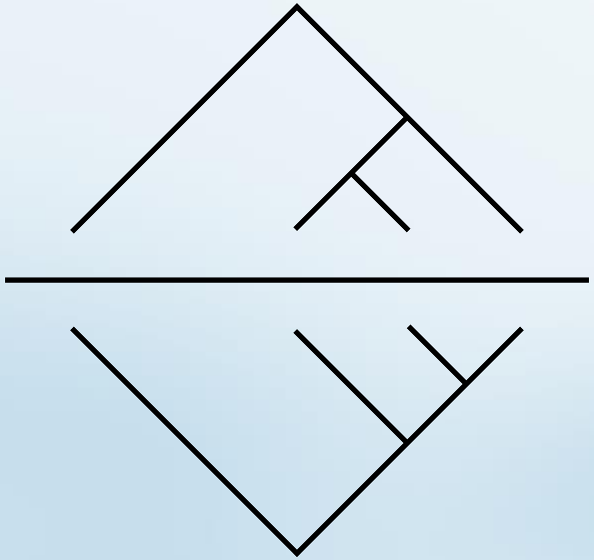
Pairs of std. dyadic partitions/trees



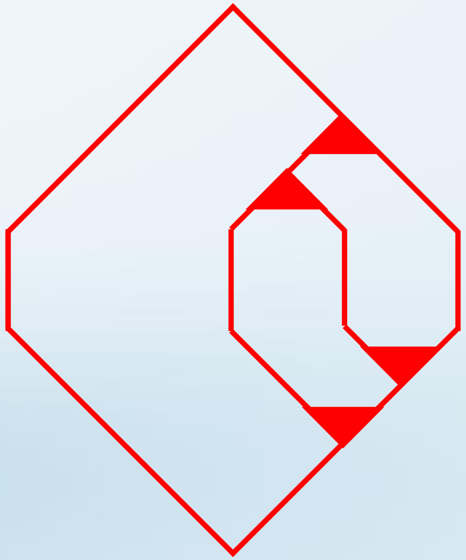
J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

Representing F and T on $\widehat{\mathcal{H}}$

$f =$



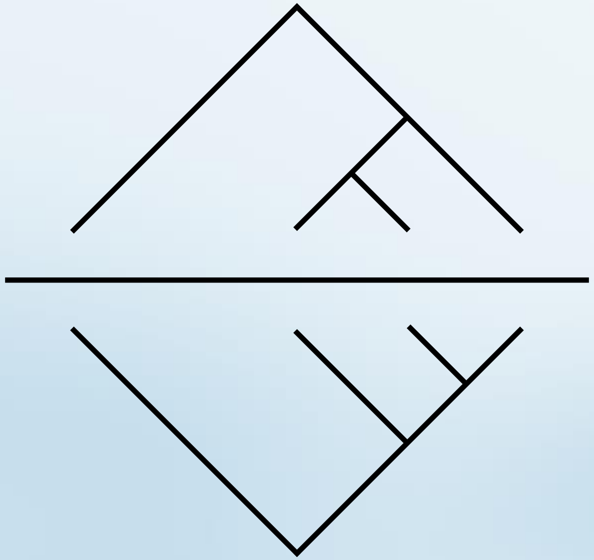
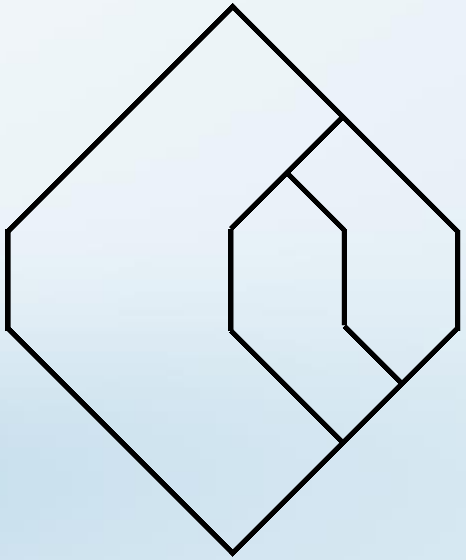
$\to$



$\equiv \langle \Omega | U(f) | \Omega \rangle$

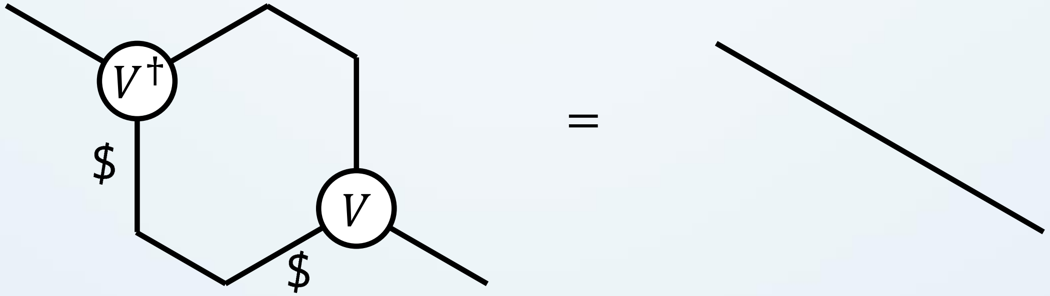
V. Jones (2014)

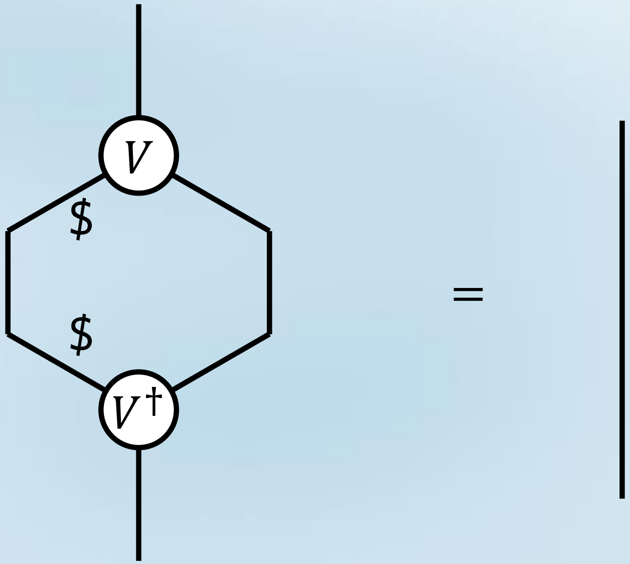
Representing F and T on $\widehat{\mathcal{H}}$

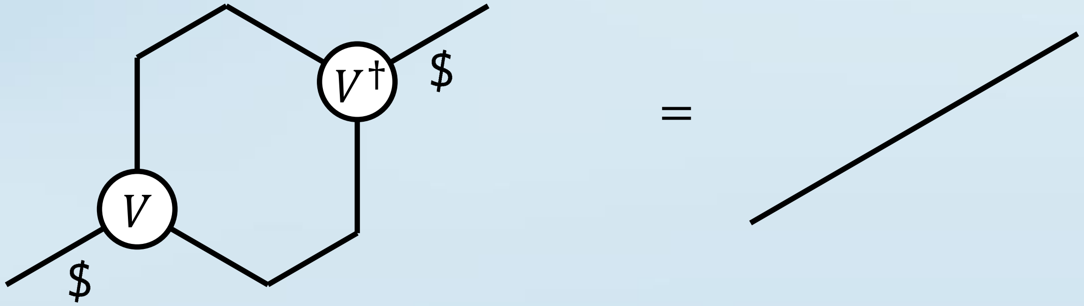
$f =$  $\rightarrow$  $\equiv \langle \Omega | U(f) | \Omega \rangle$

V. Jones (2014)

Perfect tensors and $PSL(2, \mathbb{Z})$ invariance







Perfect tensors and $\mathrm{PSL}(\mathbf{2},\mathbb{Z})$ invariance

$$\langle\Omega|U(a)|\Omega\rangle = \langle\Omega|U(b)|\Omega\rangle = 1$$

$$a^2 = (ab)^3 = 1$$

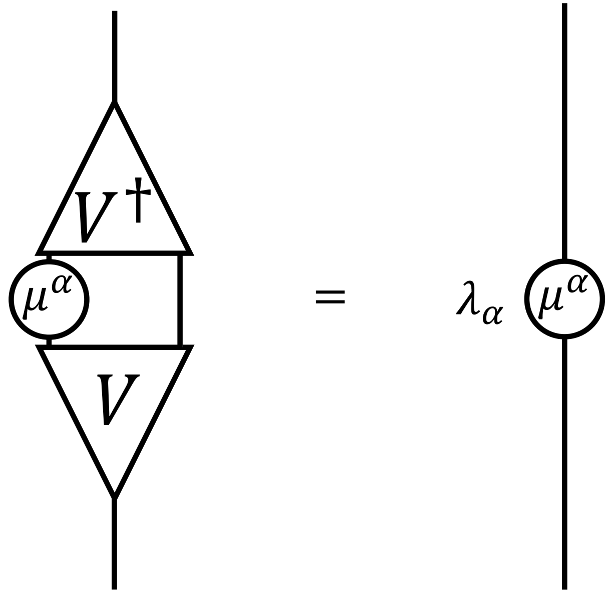
$$a = AC, b = C^{-1}A^{-1}C$$

Observables:

“Thompson field theory”

Definition: an *ascending operator* $\mu_\alpha \in \mathcal{B}(\mathcal{H})$ is an eigenvector of the ascending channel:

$$V^\dagger(\mu^\alpha \otimes \mathbb{I})V = \lambda_\alpha \mu^\alpha$$



Definition: the *discretised field operator of type* α at $z \in S^1$ with respect to a partition $P \equiv (I_1, I_2, \dots, I_n)$ is

$$\phi_P(z) \equiv \sum_{I \in P} \mathbf{I}[z \in I] (\lambda_\alpha)^{\log_2(|I|)} \mu_I^\alpha$$

TJO and D. Stiegemann, arXiv:1903.00318
See also A. Brothier and A. Stottmeister, arXiv:1901.0490

Definition (product of field operators): let $(x_1, x_2, \ldots, x_n)$ be a tuple of positions and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a tuple of types, and P a partition.

$$M_P^{\boldsymbol{\alpha}}(x_1, x_2, \ldots, x_n) \equiv \phi_P^{\alpha_1}(x_1)\phi_P^{\alpha_2}(x_2)\cdots\phi_P^{\alpha_n}(x_n)$$

TJO and D. Stiegemann, arXiv:1903.00318
See also A. Brothier and A. Stottmeister, arXiv:1901.0490

Theorem: the limit

$$C^{\boldsymbol{\alpha}}(x_1, x_2, \ldots, x_n) \equiv \lim_{P} \langle \Omega_{P'} | M_P^{\boldsymbol{\alpha}}(\boldsymbol{x}) | \Omega_{P'} \rangle$$

exists and may be calculated using $O(\log(n))$ operations.

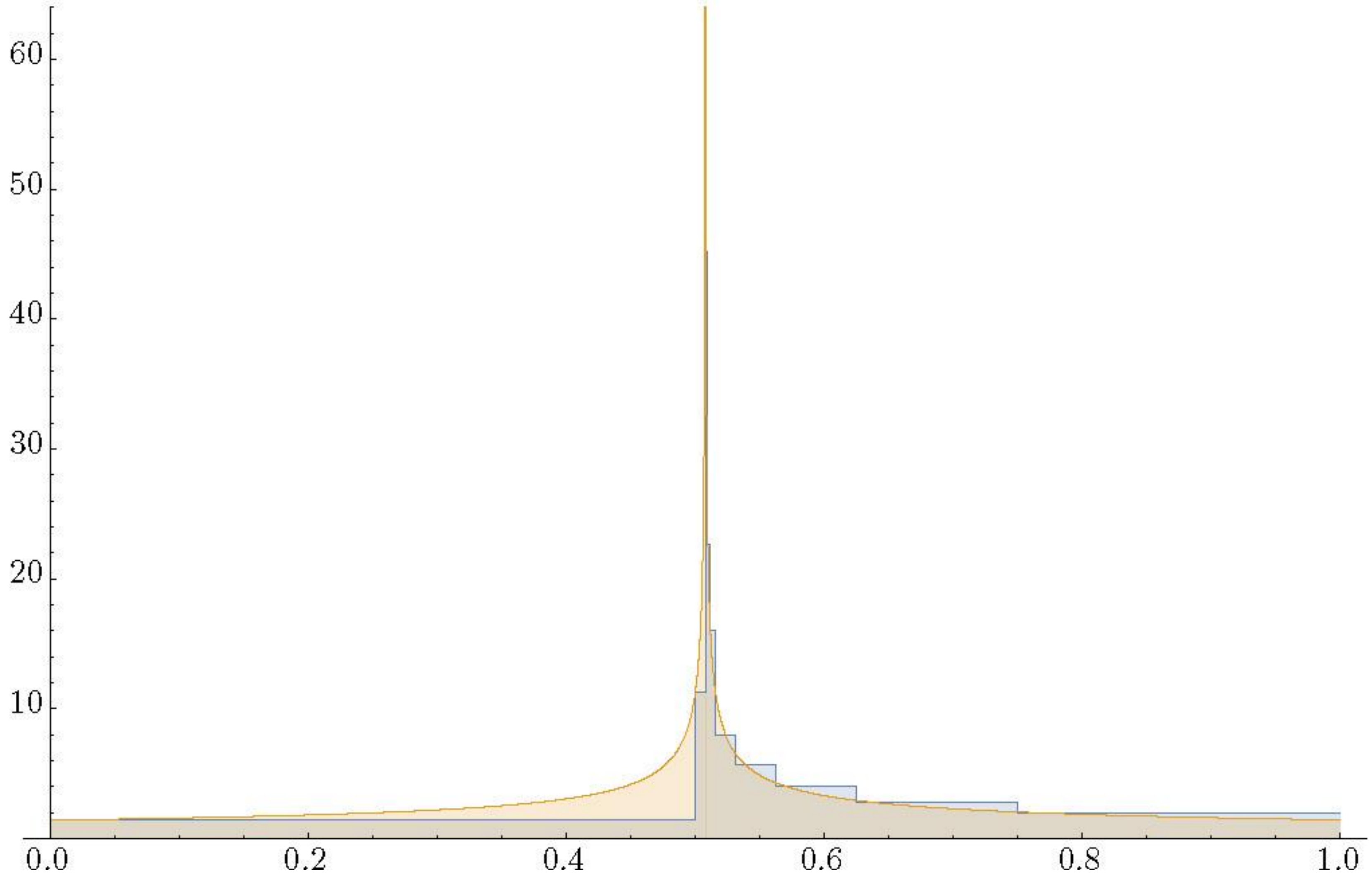
TJO and D. Stiegemann, arXiv:1903.00318

Conjecture (reconstruction):

$$C^{\alpha}(x_1, x_2, \ldots, x_n) \equiv \langle\Omega|\hat{\phi}^{\alpha_1}(x_1) \cdots \hat{\phi}^{\alpha_n}(x_n)|\Omega\rangle$$

TJO and D. Stiegemann, arXiv:1903.00318

$$C\left(\tfrac{1}{2}, x\right) \equiv \lim_{P} \langle \phi_P^{\alpha_1}\left(\tfrac{1}{2}\right) \phi_P^{\alpha_2}(x) \rangle:$$



Lemma: let x and y be two dyadic fractions

$$C^{\alpha\beta}(x,y) = \mathrm{c}(\alpha,\beta,\gamma)\, D(x,y)^{\log\lambda_\alpha + \log\lambda_\beta - \log\lambda_\gamma}$$

where $D(x,y)$ is the *coarse graining distance.*

TJO and D. Stiegemann, arXiv:1903.00318

Short distance expansion:

$$\hat{\phi}^{\alpha}(x)\hat{\phi}^{\beta}(y) \sim f_{\gamma}^{\alpha\beta} D(x,y)^{h_\gamma - h_\alpha - h_\beta} \hat{\phi}^{\gamma}(y)$$

"OPE" coefficients

TJO and D. Stiegemann, arXiv:1903.00318

6j symbols for H3

Golden Chain for H3

Semicontinuous limit

Thompson's groups F and T

Thompson Field Theory