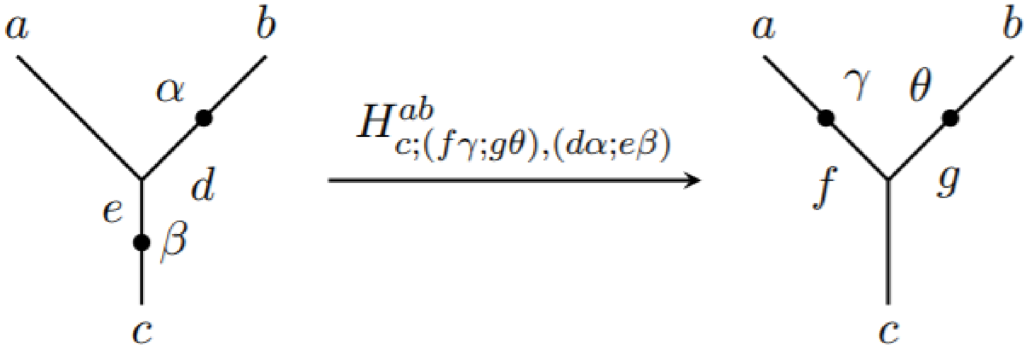
Hopf Monads and Generalized Symmetries of Fusion Categories



Modjtaba Shokrian Zini,

Joint with Zhenghan Wang, Shawn X. Cui

Outline

- From Algebra to Monads and T-Module Category
- Bimonads: a Monoidal T-Module Category
- Hopf Monads (HM): a Fusion T-Module Category
- Diagrammatic Formulation
- Examples of HM
- Condensable Algebras and Condensation
- Generalized Symmetry and Extension Theory

From Algebra to Monads and T-Module Category

Algebra $\mathcal{A}$ in a monoidal category $\mathcal{C}$:

$$\text{object } \mathcal{A} \in \mathcal{C} \text{ with}$$
$$\mu: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \eta: 1 \to \mathcal{A}$$

μ (multiplication) , η (unit) both are $\mathcal{A}$-linear:

associativity: $\mu(\mu \otimes Id_{\mathcal{A}}) = \mu(Id_{\mathcal{A}} \otimes \mu)$

$\mathcal{A}$-linearity of unit: $\mu(\eta \otimes Id_{\mathcal{A}}) = Id_{\mathcal{A}} = \mu(Id_{\mathcal{A}} \otimes \eta)$

- Generalization: Consider (action of $\mathcal{A}$) $\mathcal{A} \otimes -$ instead of the object $\mathcal{A}$. The endomorphism $T(X) = \mathcal{A} \otimes X$: $\mathcal{C} \to \mathcal{C}$ has two associated natural transformations

$\mu: T^2 \to T, \eta: Id_{\mathcal{C}} \to T$ satisfying

$$\mu_X \mu_{T(X)} = \mu_X T(\mu_X),\ \ \mu_X \eta_{T(X)} = Id_{T(X)} = \mu_X T(\eta_X)$$

$T \in End(\mathcal{C})$ is a *monad* and can be defined on any category.
T −module category $\mathcal{C}^T$ has objects

$$\{(M, r) | M \in \mathcal{C}, r \in Hom_{\mathcal{C}}(T(M), M) \text{: } T(M) \to M\}$$

such that r is T −linear: $rT(r) = r\mu_M$

$$\begin{array}{ccc} T^2(M) & \xrightarrow{T(r)} & T(M) \\ \mu_M \downarrow & & \downarrow r \\ T(M) & \xrightarrow{r} & M \end{array}$$

$$r\eta_M = id_M\text{: } M \to T(M) \to M$$

Bimonads: a Monoidal T-Module Category

- Bialgebra $\mathcal{A}$:

$$\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \quad \epsilon: \mathcal{A} \rightarrow 1$$

satisfying co-linearity, but also compatibility with algebra structure, which requires a braiding structure τ on $\mathcal{C}$:

$$(\mu \otimes \mu)\left(Id_{\mathcal{A}} \otimes \tau_{A,A} \otimes Id_{\mathcal{A}}\right)(\Delta \otimes \Delta) = \Delta\mu$$

loosely speaking braiding is needed to exchange y, z in $(x \otimes y).(z \otimes t) = (xz) \otimes (yt)$ before a “component-wise” multiplication.

- Generalize by requiring T to be *comonoidal* :

counit: $T_0: T(1) \rightarrow 1$,

coproduct (comonoidal map), $T_2(X, Y): T(X \otimes Y) \rightarrow T(X) \otimes T(Y)$

satisfying colinearity. T_2 for $T = \mathcal{A} \otimes -$ is $\left(Id_{\mathcal{A}} \otimes \tau_{A,A} \otimes Id_{\mathcal{A}}\right)(\Delta \otimes \Delta)$.

Compatibility with η and μ (have to be comonoidal):

$$T_2(X,Y)\mu_{X\otimes Y} = (\mu_X \otimes \mu_Y)T_2(T(X),T(Y))T(T_2(X,Y));$$
$$T_0\mu_{\mathbb{1}} = T_0T(T_0);$$
$$T_2(X,Y)\eta_{X\otimes Y} = (\eta_X \otimes \eta_Y);$$
$$T_0\eta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}.$$

- ***Bimonad*** T is a monad on a monoidal $\mathcal{C}$ with above structural identities.

Another point of view: Given monad T , when does the monoidal structure of $\mathcal{C}$ lift to a monoidal structure on $\mathcal{C}^T$?

Answer (Moerdijk, 2002): Whenever T is a bimonad

$$(M,r) \otimes (N,s) = \big(M \otimes N, (r \otimes s)T_2(M,N)\big)$$

Hopf Monads (HM): Towards a Fusion T-Module Category

Hopf algebra $\mathcal{A}$ has an antipode $S: \mathcal{A} \to \mathcal{A}, S^2 = Id_{\mathcal{A}}$ such that:

$$\mu(S \otimes Id_{\mathcal{A}})\Delta = \eta\epsilon = \mu(Id_{\mathcal{A}} \otimes S)\Delta$$

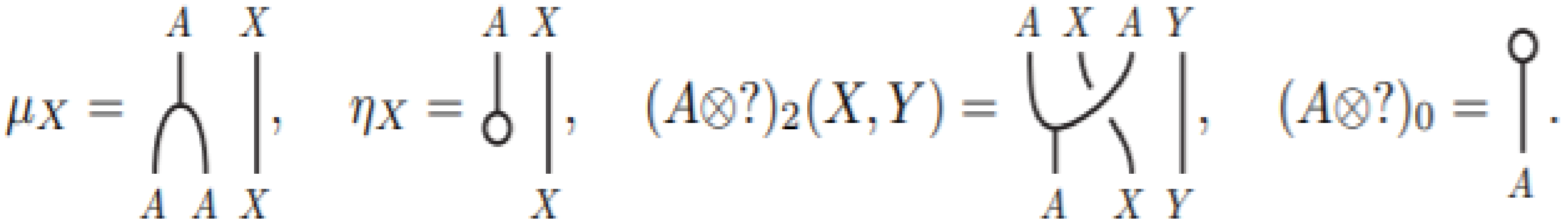
We could start with a (left/right) rigid category $\mathcal{C}$, and define (left/right) antipode satisfying certain equations.

$$s^l = \{s^l_X : T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathrm{Ob}(\mathcal{C})}$$

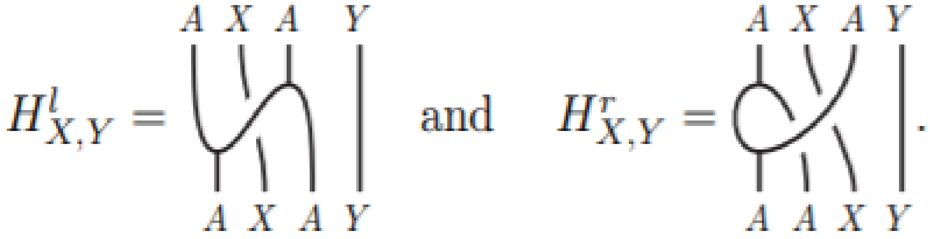
Or we could try to generalize definition of Hopf algebra given by the Fusion operator. Then:

$\mathcal{C}$ only needs to be monoidal!

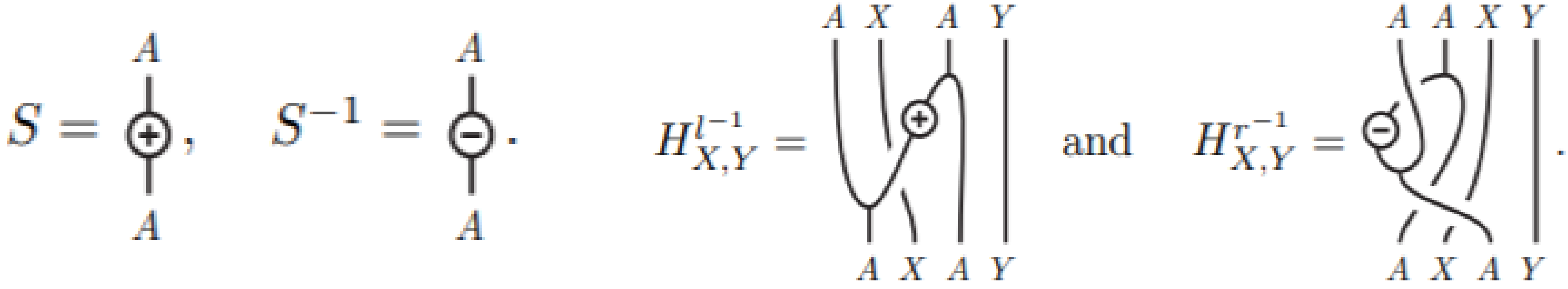
- Diagrams:



- Define fusion operators (left) $H^l_{X,Y}$ and (right) $H^r_{X,Y}$:



They only need to be invertible, and in that antipodes appear:



2.6. **Fusion operators.** Let T be a bimonad on a monoidal category $\mathcal{C}$. The *left fusion operator of* T is the natural transformation $H^l : T(1_\mathcal{C} \otimes T) \to T \otimes T$ defined by:

$$H^l_{X,Y} = (TX \otimes \mu_Y) T_2(X, TY) : T(X \otimes TY) \to TX \otimes TY.$$

The *right fusion operator of* T is the natural transformation $H^r : T(T \otimes 1_\mathcal{C}) \to T \otimes T$ defined by:

$$H^r_{X,Y} = (\mu_X \otimes TY) T_2(TX, Y) : T(TX \otimes Y) \to TX \otimes TY.$$

(left/right) fusion operator invertible → (left/right) ***Hopf monad.***

- *Will always assume a left and right Hopf monad.*
- Another point of view: For a bimonad T, how do we get $\mathcal{C}^T$ to be rigid assuming $\mathcal{C}$ is rigid ?

Answer: if and only if T is a Hopf monad. Then:

$${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r)) \quad \text{and} \quad (M, r)^\vee = (M^\vee, s^r_M T(r^\vee)).$$

More precisely, let T be an endofunctor of a monoidal category $\mathcal{C}$ endowed with a natural transformation $H_{X,Y}: T(X \otimes TY) \to TX \otimes TY$ satisfying the left pentagon equation:

$$(TX \otimes H_{Y,Z})H_{X,Y \otimes TZ} = (H_{X,Y} \otimes TZ)H_{X \otimes TY,Z}T(X \otimes H_{Y,Z}),$$

and with a morphism $T_0: T\mathbb{1} \to \mathbb{1}$ and a natural transformation $\eta_X: X \to TX$ satisfying:

$$H_{X,Y}\eta_{X \otimes TY} = \eta_X \otimes TY, \qquad T_0\eta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}},$$
$$(TX \otimes T_0)H_{X,\mathbb{1}} = T(X \otimes T_0), \qquad (T_0 \otimes TX)H_{\mathbb{1},X}T(\eta_X) = \mathrm{id}_{TX}.$$

Then T admits a unique bimonad structure (T, μ, η, T_2, T_0) having left fusion operator H. The product μ and comonoidal structural morphism T_2 are given by:

$$\mu_X = (T_0 \otimes TX)H_{\mathbb{1},X} \quad \text{and} \quad T_2(X,Y) = H_{X,Y}T(X \otimes \eta_Y).$$

A tuple (T, H, T_0, η) is all you need to define a HM. When H is invertible, T is called a *left HM*.

There is a lot more; e.g.

(**Maschke's Criterion of semisimplicity**) TFAE for a HM T:

- T admits a *cointegral (a notion of bimonads)*
- T is *separable (a notion of monads)*
- T is *semisimple (a notion of monads)*

Also, if T is an additive monad and $\mathcal{C}$ is abelian semisimple, then

$\mathcal{C}^T$ **is abelian semisimple if and only if** T **is semisimple.**

We will always assume $\mathcal{C}$ is fusion, T is linear semisimple HM. So,

$\mathcal{C}^T$ **is a fusion category (and** T **is separable with a cointegral.)**

Other parts of the theory of Hopf algebras can be generalized as well:

- Decomposition Theorem of (left,right) Hopf modules
- Integrals
- Sovereign and involutory HM
- Quasitriangular and ribbon HM

For more see:

- Bruguieres, Alain, and Alexis Virelizier. "Hopf monads." *Advances in Mathematics* 215.2 (2007): 679-733.
- Bruguieres, Alain, Steve Lack, and Alexis Virelizier. "Hopf monads on monoidal categories." *Advances in Mathematics*227.2 (2011): 745-800.

Diagrammatic Formulation

- The theory as explained was the *strictified* version. To do computations, we need a *skeletal* formulation. So associators are no longer trivial but can work with a unique object from isomorphism classes.
- Diagrams make calculations easier and more intuitive!

4.1. **Diagrammatic Notations.** Let $\mathcal{C}$ be a fusion category and $\mathrm{L}(\mathcal{C}) = \{a, b, c, \cdots\}$ be a complete set of representatives, i.e., a set that contains a representative for each isomorphism class of simple objects of $\mathcal{C}$. Let N_{ab}^c be the fusion coefficients,

$$a \otimes b \simeq \oplus_{c\in \mathrm{L}(\mathcal{C})} N_{ab}^c \, c.$$

For each $b \in \mathrm{L}(\mathcal{C})$,

$$T(b) = \oplus_{a\in \mathrm{L}(\mathcal{D})} T_{ab} \, a, \quad T_{ab} \in \mathbb{N}. \tag{5}$$

For each $a, b \in \mathrm{L}(\mathcal{C})$, there exists a natural isomorphism,

$$T_2(a, b) : T(a) \otimes T(b) \quad \xrightarrow{\simeq} \quad T(a \otimes b). \tag{6}$$

The isomorphism $T_2(a, b)$ determines a family of invertible matrices $\{T_c^{ab} : \ a, b \in \mathrm{L}(\mathcal{C}), c \in \mathrm{L}(\mathcal{D})\}$, where the matrix elements of T_c^{ab} are given by

(7)

$$\{T^{ab}_{c;(f\gamma),(d\alpha;e\beta)} : \ f \in \mathrm{L}(\mathcal{C}), d, e \in \mathrm{L}(\mathcal{D}), 1 \le \alpha \le T_{da}, 1 \le \beta \le T_{eb}, 1 \le \gamma \le T_{cf}\}.$$

- Basic diagrams:

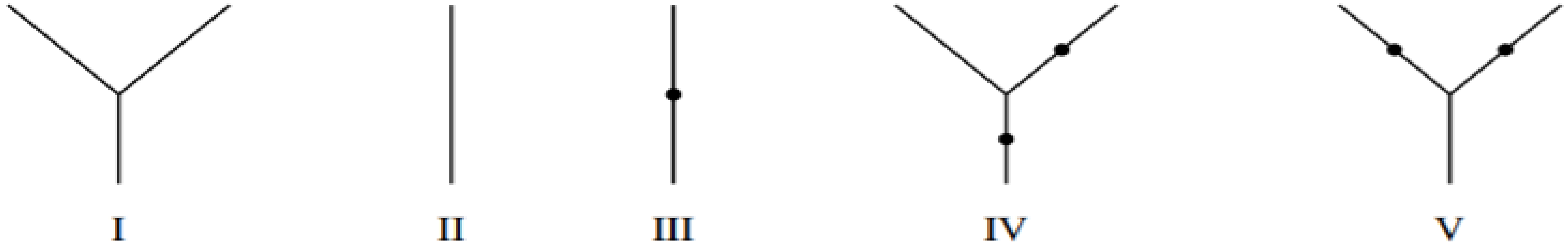


FIGURE 1. (I) $\otimes$; (II) $Id_{\mathcal{C}}$; (III) T; (IV) $T \circ \otimes \circ (Id \times T)$; (V) $\otimes \circ (T \times T)$.

Associator

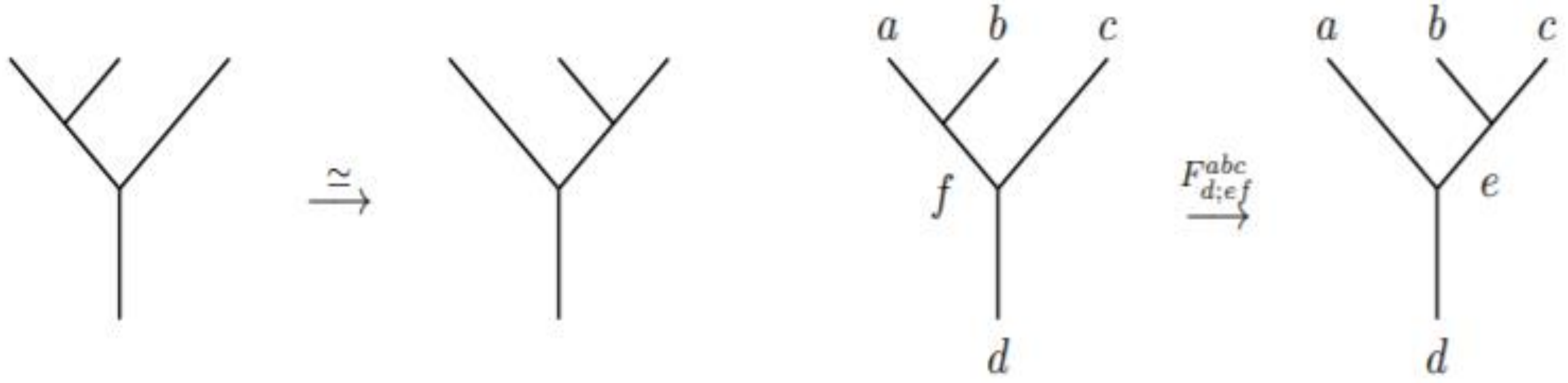


FIGURE 3. (Left) the associator natural isomorphism; (Right) the matrix elements of the associator natural isomorphism in the basis given by labels of internal edges.

$T_2(a,b): T(a \otimes b)$
$\to T(a) \otimes T(b)$

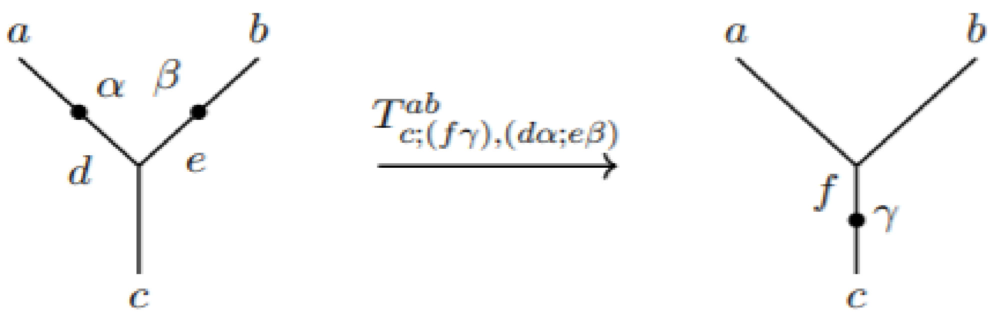


FIGURE 4. The matrix elements of T^{ab}_c.

Co-associativity:

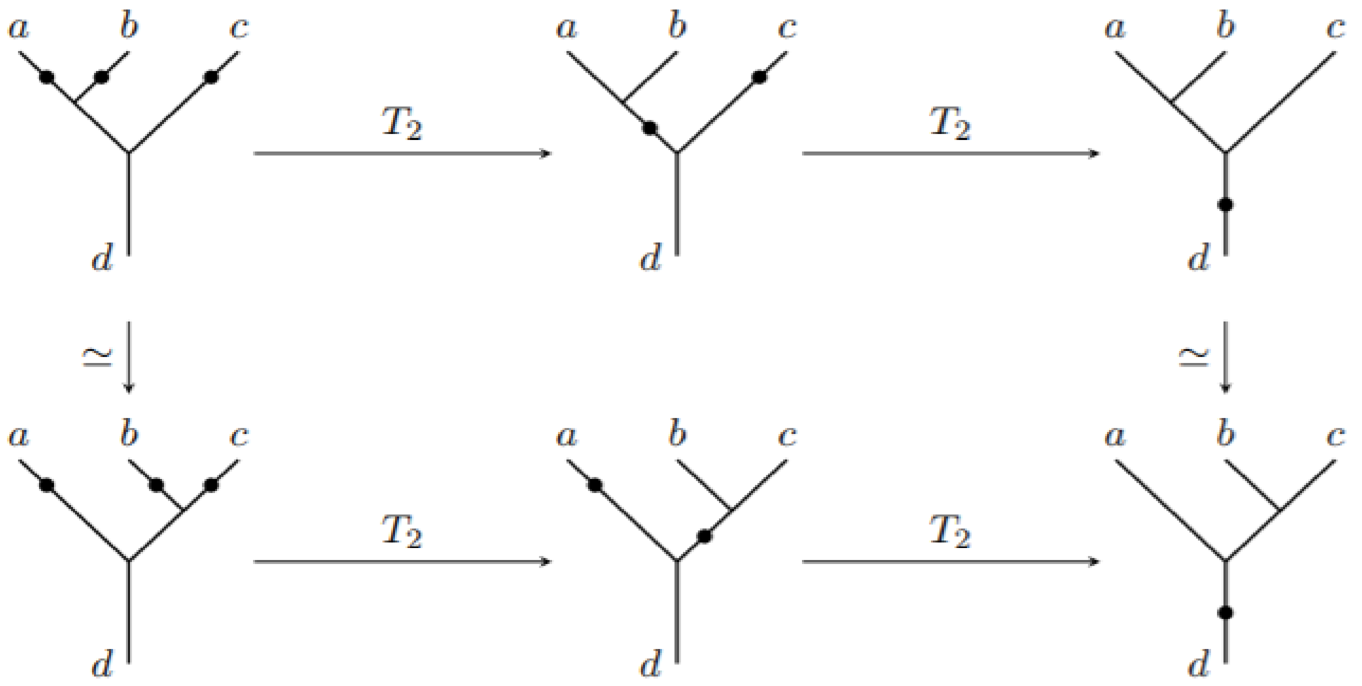


FIGURE 5. The hexagon equation for T_2.

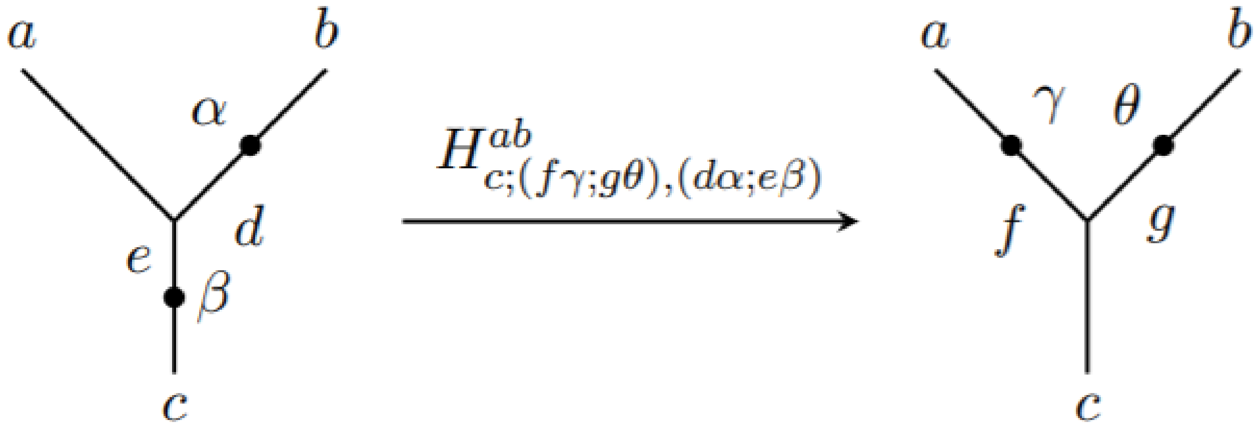
Pentagon equation becomes Heptagon equation as associator is no longer trivial:

- Heptagon Equation[5],

$$(id_{T(X)} \otimes H_{Y,Z})H_{X,Y\otimes Z} =$$

$$a_{T(X),T(Y),T(Z)}(H_{X,Y} \otimes T(Z))H_{X\otimes T(Y),Z}a^{-1}_{X,T(Y),T(Z)}T(id_X \otimes H_{Y,Z}) \quad (2)$$

$$H: T\big(a \otimes T(b)\big) \to T(a) \otimes T(b)$$



$$T(a \otimes T(b \otimes T(c)) \rightarrow T(a) \otimes (T(b) \otimes T(c))$$

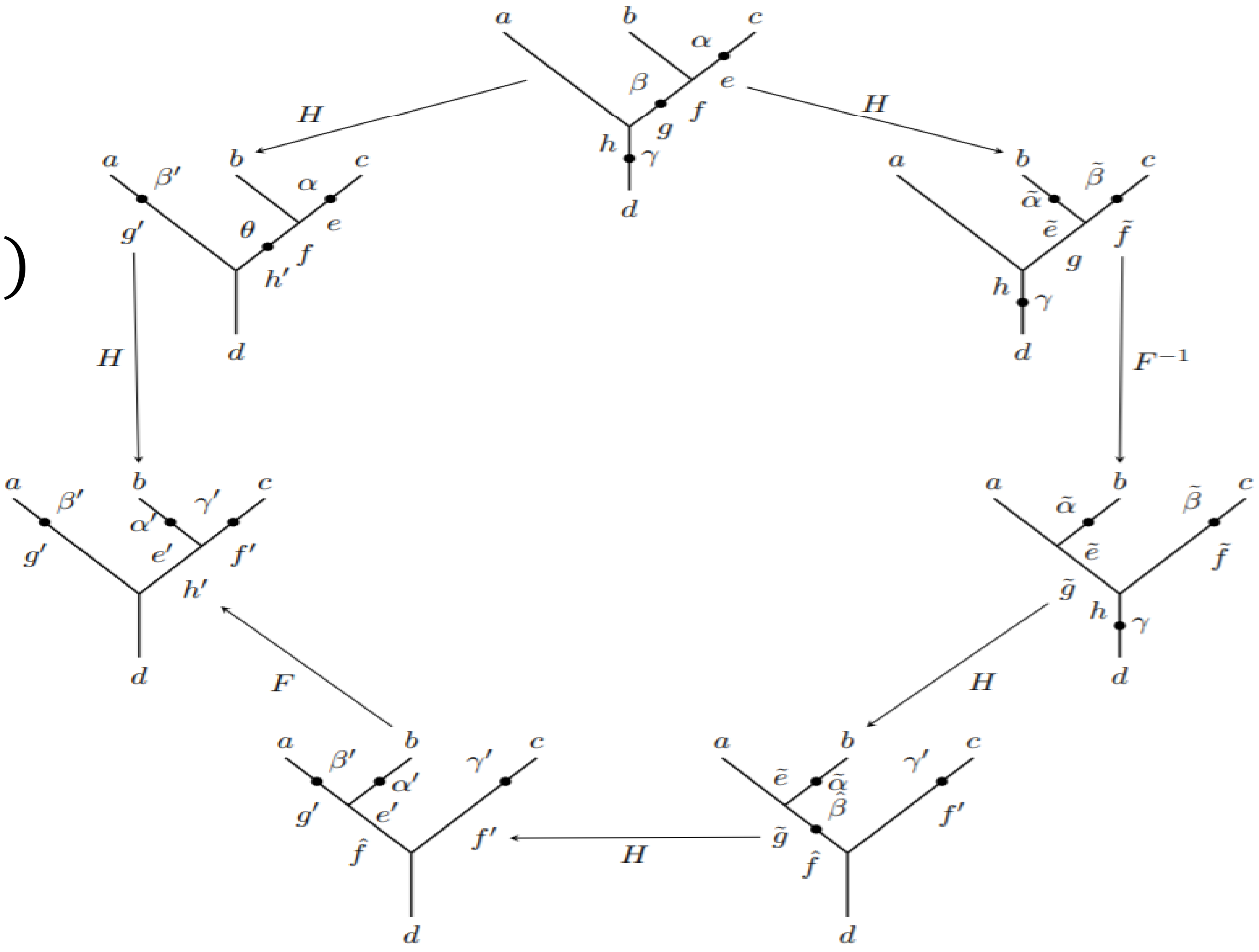


FIGURE 10. Heptagon equation for H

Can also write all equations explicitly. A left HM is a collection of data $\{T_{ab}, H^{ab}_{c;(f\gamma;g\theta),(d\alpha;e\beta)}, \epsilon_\alpha, \eta_{a,\alpha}\}$ satisfying:

$$\sum_\theta H^{af}_{d;(g'\beta';h'\theta),(g\beta;h\gamma)} H^{bc}_{h';(e'\alpha';f'\gamma'),(e\alpha;fw)} =$$

$$\sum_{\tilde{e},\tilde{f},\tilde{g},\hat{f}\tilde{\alpha},\tilde{\beta},\hat{\beta}} H^{bc}_{g;(\tilde{e}\tilde{\alpha};\tilde{f}\tilde{\beta}),(e\alpha;f\beta)} (F^{a\tilde{e}\tilde{f}}_{h})^{-1}_{\tilde{g}g} H^{\tilde{g}c}_{d;(\hat{f}\hat{\beta};f'\gamma'),(\tilde{f}\tilde{\beta};h\gamma)} H^{ab}_{\hat{f};(g'\beta';e'\alpha'),(\tilde{e}\tilde{\alpha};\tilde{g}\hat{\beta})} F^{g'e'f'}_{d;h'\hat{f}}$$

$$\sum_\beta \eta_{c,\beta} H^{ab}_{c;(e\gamma;f\theta),(d\alpha;c\beta)} = \delta_{f,d}\delta_{e,a}\delta_{\theta,\alpha}\eta_{a,\gamma}, \qquad \sum_\alpha \eta_{\mathbf{1},\alpha}\,\epsilon_\alpha = 1,$$

$$\sum_\theta H^{a\mathbf{1}}_{b;(b\gamma;\mathbf{1}\theta),(c\alpha;d\beta)}\,\epsilon_\theta = \delta_{c,\mathbf{1}}\delta_{d,a}\delta_{\gamma,\beta}\epsilon_\alpha, \qquad \sum_{\beta,\gamma} H^{\mathbf{1}a}_{b;(\mathbf{1}\gamma;b\theta),(a\beta;a\alpha)}\,\eta_{a\beta}\,\epsilon_\gamma = \delta_{\theta,\alpha}.$$

$$\sum_{f,g\in L} T_{fa}T_{gb}N^c_{fg} = \sum_{d,e\in L} T_{db}T_{ce}N^e_{ad}.$$

Examples of HM:

- **Hopf algebras**: $\mathcal{A} \otimes -$, or $-\otimes \mathcal{A}$, both are left and right HM due to braiding.
- **Adjunctions**: $\mathcal{C}, \mathcal{D}$ monoidal and $U: \mathcal{D} \to \mathcal{C}$ a *strong monoidal* functor

$$U_2(X, Y): U(X) \otimes U(Y) \to U(X \otimes Y), U_0: 1 \to U(1)$$

both isomorphisms, with a left adjoint $F: \mathcal{C} \to \mathcal{D}$. Then

$T = UF$ is always a bimonad and fusion operators H^l, H^r can be defined. Then

T is a left/right HM if H^l/H^r is invertible.

Also:

$T = UF$ is a (left/right) HM when $\mathcal{C}$ is (left/right) rigid.

Adjunctions "$\cong$" HM

- When $\mathcal{C}, \mathcal{D}$ are fusion, then TFAE:

Existence of left adjoint <-> Existence of right adjoint <-> right exact <-> left exact

- Therefore, for an adjunction to give a HM, all we need is a strong monoidal functor which is left exact; *tensor functors* U.
- If so, F exists and $T = UF$ is HM.
- Further, $\mathcal{D} \cong \mathcal{C}^T$ ((U, F) pair is *monadic*).
- It goes both ways: a HM T, the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ given by

$$U_T\colon\ (M, r) \to M$$

is a tensor functor (Alain Bruguieres,Sonia Natale,2010), with left adjoint

$$F_T\colon M \to (T(M), \mu_M) \Rightarrow T = U_T F_T.$$

Group Symmetries

Definition 4.17. An *action of a group G on a tensor category $\mathcal{C}$ (by tensor autoequivalences)* is a strong monoidal functor

$$\rho : \underline{G} \to \underline{\mathrm{End}}_{\otimes}\mathcal{C}. \tag{4.1}$$

In other words, it consists in the following data:

(1) For each $g \in G$, a tensor endofunctor $\rho^g : \mathcal{C} \to \mathcal{C}$;
(2) For each pair $g, h \in G$, a monoidal isomorphism $\rho_2^{g,h} : \rho^g \rho^h \xrightarrow{\sim} \rho^{gh}$;
(3) A monoidal isomorphism $\rho_0 : \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} \rho^1$;

Examples are many:

- Vec with trivial G symmetry
- $\mathcal{D}(\mathbb{Z}_N)$ with $\mathbb{Z}_2$ symmetry $(a_1, a_2) \to (a_2, a_1)$
- $G < S_n$ symmetry on $\mathcal{C} \boxtimes \mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C}$.
- 3-fermion model $SO(8)_1 = \{1, \psi_1, \psi_2, \psi_3\}$ with S_3 symmetry

- We have

Theorem 4.21. *Let $\mathcal{C}$ be a tensor category over a field $\Bbbk$, and let ρ be an action of a finite group G on $\mathcal{C}$ by tensor autoequivalences. Then:*

(1) The $\Bbbk$-linear exact endofunctor

$$\mathbb{T}^\rho = \bigoplus_{g \in G} \rho^g$$

admits a canonical structure of Hopf monad on $\mathcal{C}$;

(2) There is a canonical isomorphism of categories:

$$\mathcal{C}^G \simeq \mathcal{C}^{\mathbb{T}^\rho}$$

over $\mathcal{C}$, where $\mathcal{C}^G$ denotes the equivariantization of $\mathcal{C}$ under G;

- The structural morphisms are derived similar to Hopf Algebra structure on $\mathbb{C}[G]$.
- This makes sense as $T-$ modules are like *fixed* points of T. Hence equivariantization can be generalized as the process of taking $T-$ modules.

Condensable Algebras & Condensation

In a modular tensor category $\mathcal{B}$, an algebra $\mathcal{A}$ which is :

- Commutative: $\mu\, \tau_{A,A} = \mu$
- Connected: $Hom(1, \mathcal{A}) = \mathbb{C}$.
- Separable: μ admits a splitting $\zeta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, a morphism of $(\mathcal{A}, \mathcal{A})$ −bimodules:

$$(\mu \otimes Id_{\mathcal{A}})(Id_{\mathcal{A}} \otimes \zeta) = (Id_{\mathcal{A}} \otimes \mu)(\zeta \otimes Id_{\mathcal{A}}), \mu\zeta = Id_{\mathcal{A}}$$

Take $\mathcal{A}$ −module $\mathcal{B}_{\mathcal{A}}$. There exist:

Condensation (Induction) functor $D_{\mathcal{A}}: \mathcal{B} \to \mathcal{B}_{\mathcal{A}}$, a tensor functor

Forgetful functor $E_{\mathcal{A}}: \mathcal{B}_{\mathcal{A}} \to \mathcal{B}$, the adjoint

Hence, $T_{\mathcal{A}} = D_{\mathcal{A}} E_{\mathcal{A}}$ is a HM and $\mathcal{B} \cong \mathcal{B}_{\mathcal{A}}^{T_{\mathcal{A}}}$.

Condensation Example

To derive the condensation $\mathcal{B}_{\mathcal{A}}$:

- $Ob(\mathcal{B}_{\mathcal{A}}) = Ob(\mathcal{B})$
- *Frobenius reciprocity*: $\mathrm{Hom}_{\mathcal{B}}(X, \mathcal{A} \otimes Y) = \mathrm{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, Y)$.

5.2.2. $D(S_3)$. Consider the case $\mathcal{B} = D(S_3)$ and the condensable algebra $\mathcal{A} = A + C$. The objects are denoted by $\{A, B, C, D, E, F, G, H\}$ where $\{A, B, C\}$ is the canonical image of $\mathrm{Rep}(S_3)$ in $D(S_3)$. By using the framework in 3.2.4, one derives the condensed category $\mathcal{B}_{\mathcal{A}} = D(\mathbb{Z}_2) \oplus \{X, Y\}$ with the following fusion rules for X, Y[6]:

$$mX = Y, mY = X, \psi Y = X, \psi X = Y, eY = Y, \tag{27}$$
$$X^2 = \mathbf{1} + e + Y, XY = m + \psi + X, Y^2 = \mathbf{1} + e + Y,$$

- One can derive the induction and forgetful functors:

(28)
$$E_{\mathcal{A}} : \mathbf{1} \to A + C, e \to B + C, m \to D, \psi \to E, X \to D + E, Y \to F + G + H,$$
(29)
$$D_{\mathcal{A}} : A \to \mathbf{1}, B \to e, C \to \mathbf{1} + e, D \to m + X, E \to \psi + X, \ F, G, H \to Y.$$

C. Delaney

Then $T_{\mathcal{A}} = D_{\mathcal{A}} E_{\mathcal{A}}$ is $(2+e) \otimes -$ on objects $1, e, Y$ and $(1+Y) \otimes -$ on objects m, ψ, X.

- Observation: If we replace $Y \to 1+e$, and $X \to m+\psi$, fusion rules still hold.
- Also, $\{X, Y\}$ would be all irreducible modules of algebra $1+e$ and as $\mathcal{D}(\mathbb{Z}_2)$ is module category of algebra 1, so $\mathcal{B}_{\mathcal{A}} = (2+e)$ −modules.
- In fact, the algebra $1 \oplus (1+e)$ has a (*unique*) Hopf algebra structure:

The algebra structure is a $\mathbb{Z}_2$ −graded algebra $\mathbb{C}[x,y]/\langle x^2-x, y^2-x, xy-y\rangle$,

where $1, x$ have grade 0 (corresponding to two dimensional v.s. ℂ^2 of **1**) and y has grade one (corresponding to one dimensional v.s. $\mathbb{C}$ of e).

$$\Delta(x) = 1 \otimes x + x \otimes 1 - \frac{3}{2} x \otimes x + \frac{1}{2} y \otimes y,$$

$$\Delta(y) = 1 \otimes y - \frac{3}{2} x \otimes y + y \otimes 1 - \frac{3}{2} y \otimes x.$$

$$\epsilon(x) = \epsilon(y) = 0.$$

$$S(x) = x, \qquad S(y) = -y.$$

And so $\mathcal{H} = \{1 - x\} \oplus \{x, y\}$ is a $\mathbb{Z}_2$-graded hopf algebra. Its modules in $Vec_{\mathbb{Z}_2}$ is $Rep(S_3)$. Its modules in $\mathcal{D}(\mathbb{Z}_2)$ is $\mathcal{D}(\mathbb{Z}_2) \oplus \{X, Y\}$.

Classification of Hopf Algebras is a big problem, even in Vec.

Conjecture 5.1. *In $Vec_{\mathbb{Z}_p}$, the algebra $\mathbf{1} \oplus Vec_{\mathbb{Z}_p}$ admits a categorical Hopf algebra structure whose representation category is the near group category given by $G = \mathbb{Z}_p$ and multiplicity $m = |G| - 1$, if and only if $p = q^m - 1$ for some prime q.*

- Think of the previous example as an extension and then an equivariantization (:= HM gauging) given by taking modules of the Hopf monad $T = (2 + e) \otimes -$ on $\mathcal{D}(\mathbb{Z}_2)$:

$$\mathcal{D}(\mathbb{Z}_2) \to \mathcal{D}(\mathbb{Z}_2) \oplus \{X, Y\} \to \mathcal{D}(S_3)$$

Example of HA symmetry with nontrivial extension:$T_{\mathcal{A}}$ is $(2 + e) \otimes -$ on objects $1, e, Y$ and $(1 + Y) \otimes -$ on objects m, ψ, X.

In general, condensed $\mathcal{B}_{\mathcal{A}} = (deconfined) \oplus (confined)$ and the deconfined part (here is $\mathcal{D}(\mathbb{Z}_2)$) to which $T_{\mathcal{A}}$ should be restricted (by the substitution $Y = 1 + e$) . It should give a (special) case of HM gauging

$$(deconfined) \to \mathcal{B}_{\mathcal{A}} \to \mathcal{B}$$

- Condensable algebras behave like “normal subgroups” for modular categories.

Generalized Symmetry & Extension Theory

- Notice by adjunction, any tensor functor $U: \mathcal{D} \to \mathcal{C}$, gives a HM. Define a category symmetry of fusion $\mathcal{C}$ as a pair $(U, \mathcal{D})$.
- To generalize symmetries, first look examples we have from HM point of view:

Group symmetry: gauging starts with an action $\rho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}\mathcal{C}$

Then a G −graded $\mathcal{C}_G^{\times} = \bigoplus_g \mathcal{C}_g$ extension of $\mathcal{C} (= \mathcal{C}_e)$ is derived.

HA symmetry (condensation $\mathcal{D}(S_3)$): grading is given by how the HA breaks into irreducible algebras: $1 \oplus 1 + e$.

- In general what is the grading for $\mathcal{C}_T^{\times} = \bigoplus_i \mathcal{C}_{m_i}$?

In fact T itself seems to be graded in both cases:

$$T = \bigoplus_g \rho(g), T = \bigoplus_i (m_i \otimes -)$$

Same answer for both:

m_i is the irreducible coalgebra *decomposition* of coalgebra $(T(1), T_2(1,1), T_0)$ as it acts on itself.

Groups: $T^G(1) = \mathbb{C}[G].1$ with $T_2(1,1)\left(1_g\right) = 1_g \otimes 1_g$, so coalgebra decomposition is $T(1) = \bigoplus 1_g$.

HA: $T(1) = \mathcal{A}$ which breaks into its coalgebra decomposition.

- *Observation*: As $(T(X), T_2(1, X))$ is a $T(1)$ comodule, can define $T = \bigoplus_i T_{m_i}$ where $T_{m_i}(X)$ is a m_i-comodule. Further, there is $m_1 = 1$, and X is inside $T_1(X)$ so $\mathcal{C} \subset \mathcal{C}_{m_1}$. Conjectured to be equal.

- Next, what is $\mathcal{C}_{m_i}$?

Group: Fixed points of $\rho(g) = T_{1_g}$, i.e. $T_{1_g}(X) \cong X \cong 1_g \otimes X$.

HA: comodules of m_i.

In general:

$$\mathcal{C}_{m_i} = \{X |\, X \in \mathcal{C}, T_{m_i}(X) \cong m_i \otimes X,\ \ X\ an\ m_i\ comodule\}.$$

Fusion rules and associators

Fusion rules: in case of groups they are derived from a *strong* monoidal functor

$$\phi_G: co - T^G(1) = co - (\mathbb{C}[G].1) = Vec_G \to Bimodc(\mathcal{C}).$$

The image, as ϕ_G is *strong* monoidal and all elements Vec_G are invertible, are the invertible bimodules $Pic(\mathcal{C})$.

$$\phi_G(g) \boxtimes_{\mathcal{C}} \phi_G(h) \cong \phi_G(g \otimes \mathrm{h}) \Rightarrow \quad \otimes: \mathcal{C}_g \boxtimes \mathcal{C}_h \to \mathcal{C}_{gh}$$

In general: monoidal functor ϕ_T from category of $T(1)$ −comodules to $Bimodc(\mathcal{C})$ to get $\otimes$: $\mathcal{C}_{m_i} \boxtimes \mathcal{C}_{m_j} \to \bigoplus_x \mathcal{C}_{m_x}$ assuming $m_i \otimes m_j = \bigoplus_x m_x$.

- More generally, $\mu: T^2 \to T$ restricted to $T_{m_i}(T_{m_j}(X))$ should tell us what the result is for $\mathcal{C}_{m_i} \boxtimes \mathcal{C}_{m_j}$.

Fusion rules for $\mathcal{C}_G^{\times}$ exist iff there is a lifting of $\rho: G \to Aut_{\otimes}(\mathcal{C})$ to a strong monoidal $\rho : \underline{G} \to \underline{\mathrm{Aut}_{\otimes}\mathcal{C}}$.

This is equivalent to a vanishing of an obstruction class in $H_\rho^3(G, Z)$ where Z are invertible elements of $\mathcal{C}$. Then all fusion rules are classified by $H_\rho^2(G, Z)$.

- In general, may need to consider "some cohomology" like

$H_{\phi_T}^3\big(co - T(1)\big)$, something like the Davydov-Yetter (DY) cohomology.

Fact: DY-cohomology $H_F^2(\mathcal{C})$ for a tensor functor $F: \mathcal{C} \to \mathcal{C}'$ (both $\mathcal{C}, \mathcal{C}'$ fusion) parametrizes additively trivial first order deformations of F as a tensor functor modulo equivalence, and $H_F^3(\mathcal{C})$ is the obstruction space for such deformations.

But associators (fusion F-matrices) are not unique and relate to vanishing of a class in $H^4(G, \mathbb{C}^\times)$, classified by $H^3(G, \mathbb{C}^\times)$. In general, may need to consider DY-cohomology like $H^4_{Id}(co - T(1), \mathbb{C}^\times)$.

Fact: $H^3_{Id}(\mathcal{C})$ parametrizes additively trivial first order deformations of C as a tensor category modulo equivalence, and $H^4_{Id}(\mathcal{C})$ is the obstruction space for such deformations

- Example: $H^i_{Id}(Vec_G, \mathbb{C}^\times) = H^i(G, \mathbb{C}^\times)$. So DY-cohomology gives what we want for the case of group symmetry.

Examples of HM extension and a final piece of the puzzle:

- Group symmetry
- Hopf Algebra symmetry:

Given $T = \mathcal{A} \otimes -$, $\mathcal{A}$ a HA in $\mathcal{C}$, then $\mathcal{C}_T^\times = \bigoplus \mathcal{C}_{m_i} = \mathcal{C}^T$. What is the extension of T to $\mathcal{C}_T^\times$? In case of groups, extension is unique. In case of HA, there is always a canonical extension

$$T^\times\big((M,r)\big) = \big(T(M), T(r)\big), \text{ basically } T^\times = \mathcal{A} \otimes -$$

Need not be unique, recall $\mathcal{D}(S_3)$ condensation, where $\mathcal{A} = 2 + e$:

"$T_\mathcal{A} = D_\mathcal{A} E_\mathcal{A}$ is $(2+e) \otimes -$ on objects $1, e, Y$ and $(1+Y) \otimes -$ on objects m, ψ, X."

After deriving $T^\times$, can equivariantize (get $T^\times$-modules) to get $\mathcal{C}_T^{\times, T^\times}$.

6.3.2. *A generalization of the Haagerup category Haag$_p$.* Fib as a fusion category fits into another potential sequence of fusion categories whose fusion rules will be denoted as Haag_p. Haag_p has $2p$ classes of simple objects denotes as $\alpha^i, i = 0, 1, ..., p-1$, and $\rho_i, i = 0, 1, .., p-1$, where $\alpha^0 = \mathbf{1}, \rho_0 = \rho$. The α^i's obey $\mathbb{Z}_p$ fusion rule. The non-group fusion rules are determined by:

$$\alpha^i \otimes \rho = \rho_i = \rho \otimes \alpha^{p-i}, \quad \rho^2 = \mathbf{1} \oplus \sum_{i=0}^{p-1} \rho_i.$$

For $p = 2$, this is the fusion rule of $\mathrm{PSU}(2)_6$, and for $p = 3$, this is the Haagerup fusion rule.

Open Question 6.2. *Is there a HM on $\mathrm{Vec}_{\mathbb{Z}_p}$ with an extension that realizes Haag_p for each prime p?*

6.3.3. *The Doubled Haagerup category $D(Haag_p)$.* The hypothetical modular category $D(\mathrm{Haag}_p)$, defined in [15], for all odd prime p is of rank p^2+3 with anyons $\mathbf{1}, b, a_h, d_l$ with $1 \leq h \leq \frac{p^2-1}{2}, 1 \leq l \leq \frac{p^2+3}{2}$ of quantum dimensions $1, p\delta+1, p\delta+2, p\delta$, and $\delta = \frac{p+\sqrt{p^2+4}}{2}$ satisfying $\delta^2 = 1+p\delta$. It is known to exist for $p \leq 13$ [16].

Proposition 2. *The object $\mathcal{A} = \mathbf{1} + b$ of DHaag has a condensable algebra structure.*

Consider the condensable algebra $\mathcal{A} = \mathbf{1} + b$, then we have[7]

$$D_{\mathcal{A}}(\mathbf{1}) = \mathbf{1}, D_{\mathcal{A}}(b) = \mathbf{1} + X, D_{\mathcal{A}}(a_h) = X + \alpha_{i,j} + \alpha_{i,j}^*, D_{\mathcal{A}}(d_l) = X,$$

where $\alpha_{i,j}$ form $D(\mathbb{Z}_p)$, and $X^2 = D(\mathbb{Z}_p) + p^2 X$. So $\mathcal{C} = D(\mathbb{Z}_p) \oplus \{X\}$ with deconfined $\mathcal{D} = D(\mathbb{Z}_p)$. It follows the HM is

$$T_{\mathcal{A}}(\mathbf{1}) = 2 + X, T_{\mathcal{A}}(\alpha_{ij}) = \alpha_{i,j} + \alpha_{i,j}^* + X, T_{\mathcal{A}}(X) = D(\mathbb{Z}_p) + (p^2+2)X.$$

So $T_{\mathcal{A}}(a) = a + a^* + X \otimes a$ is a HM on $D(\mathbb{Z}_p) \oplus \{X\}$, and the question in general is what T is on $D(\mathbb{Z}_p)$?

Open Question 6.3. *Can $D(Haag_p)$ be realized through gauging a Hopf monad symmetry on $D(\mathbb{Z}_p)$?*

References:

- Cui, S. X., Zini, M. S., & Wang, Z. (2019). On generalized symmetries and structure of modular categories. *Science China Mathematics*, *62*(3), 417-446.
- Etingof, P., Nikshych, D., & Ostrik, V. (2009). Fusion categories and homotopy theory. *arXiv preprint arXiv:0909.3140*.
- Bruguières, A., & Natale, S. (2011). Exact sequences of tensor categories. *International Mathematics Research Notices*, *2011*(24), 5644-5705.
- Barkeshli, M., Bonderson, P., Cheng, M., & Wang, Z. (2014). Symmetry, defects, and gauging of topological phases. *arXiv preprint arXiv:1410.4540*.
- Evans, D. E., & Gannon, T. (2011). The exoticness and realisability of twisted Haagerup–Izumi modular data. *Communications in mathematical physics*, *307*(2), 463.
- Gannon, T., & Morrison, S. (2017). Modular data for the extended Haagerup subfactor. *Communications in Mathematical Physics*, *356*(3), 981-1015.
- Grossman, P., & Snyder, N. (2012). Quantum subgroups of the Haagerup fusion categories. *Communications in Mathematical Physics*, *311*(3), 617-643.