Applications of TQFT to classical and quantum graph polynomials

Slava Krushkal

May 7, 2019

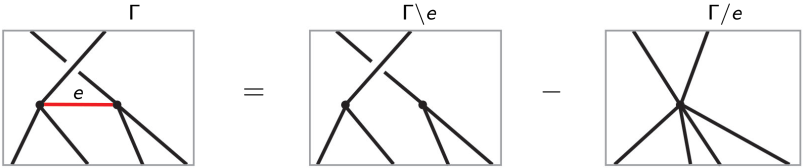
• Background: Identities for the chromatic and flow polynomials of planar graphs from the Temperley-Lieb algebra (with Paul Fendley '08-09)

• Applications to classical and quantum polynomials of graphs (with Ian Agol '17-18)

• Graphs on the torus: TQFT trace, topological Tutte polynomial, and the Pasquier model (with Paul Fendley '19 and work in progress)

The *chromatic polynomial* is defined by the contraction-deletion rule: given any edge e of Γ which is not a loop,

$$\chi_\Gamma(x) = \chi_{\Gamma\setminus e}(x) - \chi_{\Gamma/e}(x)$$



If Γ contains a loop then $\chi_\Gamma \equiv 0$.

If Γ has no edges and V vertices, then $\chi_\Gamma(x) = x^V$.

The chromatic polynomial was defined by Birkhoff in 1912 as a way to approach the 4-color conjecture.

If the parameter is a positive integer n, the value of the chromatic polynomial $\chi_\Gamma(n)$ of Γ at n is the number of colorings of the vertices of Γ with n colors, so that no two adjacent vertices have the same color.

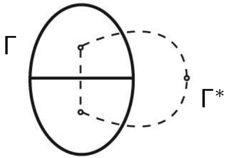
- Contraction-deletion: $\chi_\Gamma(x) = \chi_{\Gamma \setminus e}(x) - \chi_{\Gamma / e}(x)$
- If Γ contains a loop then $\chi_\Gamma \equiv 0$.
- If Γ has no edges and V vertices, then $\chi_\Gamma(x) = x^V$.

Chromatic - flow duality:

For (connected) planar graphs Γ,

$$\mathcal{F}_\Gamma(x) \;=\; \frac{1}{x}\,\chi_{\Gamma^*}(x)$$

where $\mathcal{F}_\Gamma$ is the flow polynomial. Γ^* is the dual graph.



The *flow polynomial* $\mathcal{F}_\Gamma(x)$:

Given any edge e of Γ which is not a bridge,

$$\mathcal{F}_\Gamma(x) = \mathcal{F}_{\Gamma/e}(x) - \mathcal{F}_{\Gamma\setminus e}(x)$$

If Γ contains a bridge then $\mathcal{F}_\Gamma \equiv 0$.

If Γ has a single vertex and n loops, then $\mathcal{F}_\Gamma(x) = x^V$.

For $x \in \mathbb{Z}_+$, $\mathcal{F}_\Gamma(x)$ counts non-zero x-flows on Γ.

For (connected) planar graphs Γ

$$\mathcal{F}_\Gamma(x) = \frac{1}{x}\chi_{\Gamma^*}(x)$$

where Γ^* is the dual graph.

The chromatic and flow polynomials are one variable specializations of the 2-variable *Tutte polynomial* $T_\Gamma(x,y)$: (up to a normalization)

$$\chi_\Gamma(x) = x^{c(\Gamma)}\, T_\Gamma(1-x,0), \quad \mathcal{F}_\Gamma(x) = T_\Gamma(0,1-x).$$

For planar graphs

$$T_\Gamma(x,y) = T_{\Gamma^*}(y,x)$$

W.T. Tutte (1969), Relation I:

The golden identity: for a planar triangulation T,

$$\chi_T(\phi+2) = (\phi+2)\,\phi^{3\,V(T)-10}\,(\chi_T(\phi+1))^2,$$

where $V(T)$ is the number of vertices of the triangulation.

ϕ denotes the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$.

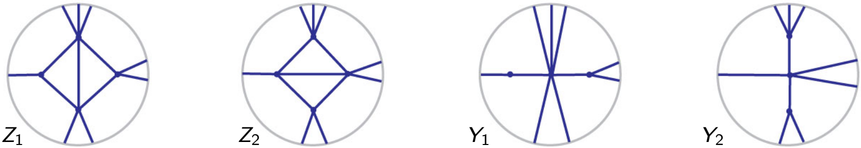
Corollary. For a planar triangulation T, $\chi_T(\phi+2) > 0$.

$(\phi + 2 \approx 3.618...)$

W.T. Tutte (1969), Relation II:

$$\chi_{Z_1}(\phi+1) + \chi_{Z_2}(\phi+1) = \phi^{-3}[\chi_{Y_1}(\phi+1) + \chi_{Y_2}(\phi+1)],$$

where Y_i, Z_i are planar graphs which are locally related as follows:



W.T. Tutte (1969), Relation III:

Let T be a planar triangulation with V vertices. Then

$$|\chi_T(\phi + 1)| \leq \phi^{5-V}$$

In the 1970s Beraha experimentally observed that real zeros of the chromatic polynomial of large planar triangulations seem to accumulate near Beraha numbers $B_n = 2 + 2\cos\left(\frac{2\pi}{n}\right)$, specifically near $B_5 = \phi + 1$.

The Beraha conjecture (that this is the case) is open.

W.T. Tutte (1969), Relation III:

Let T be a planar triangulation with V vertices. Then

$$|\chi_T(\phi+1)| \leq \phi^{5-V}$$

L. Fidkowski, M. Freedman, Ch. Nayak, K. Walker, Z. Wang, *From String Nets to Nonabelions*, CMP (2009):

A sharper bound for large regions of the hexagonal lattice, using shadow evaluation.

The flow polynomial (or dually the chromatic polynomial) of planar graphs is a common specialization of several invariants:

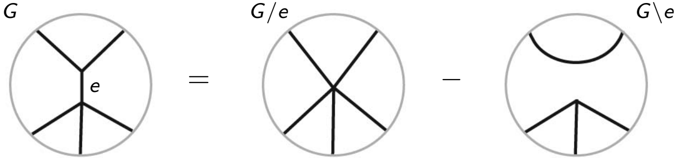
- The flow polynomial of abstract graphs

- The Yamada polynomial of spatial graphs in $\mathbb{R}^3$

- The trace evaluation in $\mathrm{SO}(3)$ TQFTs of graphs on a closed surface Σ_g. (Parameter = root of unity)

- The "topological flow polynomial" of graphs on a surface Σ_g.

The chromatic algebra $\mathcal{C}_n^Q$ consists of $\mathbb{C}$-linear combinations of (isotopy classes of) planar graphs G in the rectangle R with n endpoints at the top and n endpoints at the bottom of the rectangle, modulo local relations:



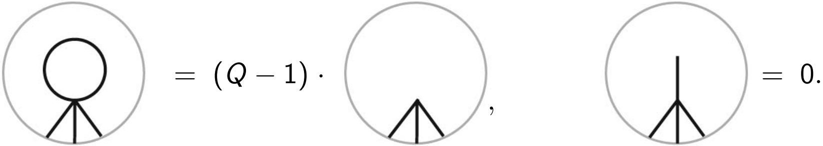


Figure: Examples of graphs in $\mathcal{C}_3$.

The trace, $tr_\chi \colon \mathcal{C}^Q \longrightarrow \mathbb{C}$ is defined on additive generators (graphs) by connecting the endpoints by arcs in the plane and evaluating

$$Q^{-1} \cdot \chi_{G^*}(Q).$$

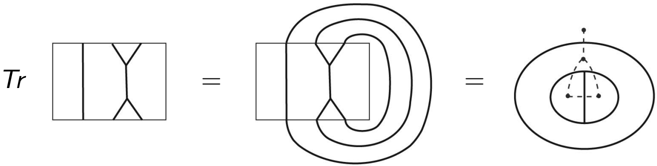
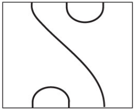


Figure: The trace $= (Q-1)^2(Q-2)$.

The Hermitian product on the chromatic algebra:

$$\langle a, b\rangle = tr(a\,\bar{b})$$

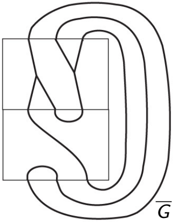
.



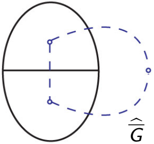
a

b

$\langle a, b\rangle =$



$=$

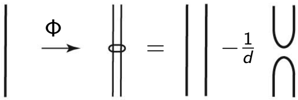


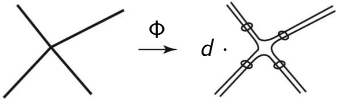
$= (Q-1)(Q-2).$

Consider the algebra homomorphism to the Temperley-Lieb algebra:

$$\Phi\colon \mathcal{C}_n^Q \longrightarrow TL_{2n}^d,$$

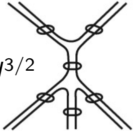
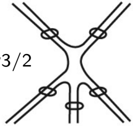
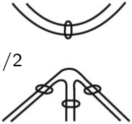
where $Q = d^2$:





The factor corresponding to a k−valent vertex is $d^{(k-2)/2}$.

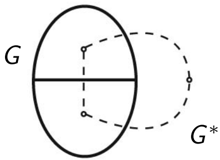
Φ is well-defined:

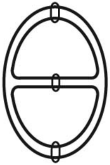
$d^{3/2}$  $=$ $d^{3/2}$  $-$ $d^{1/2}$ 

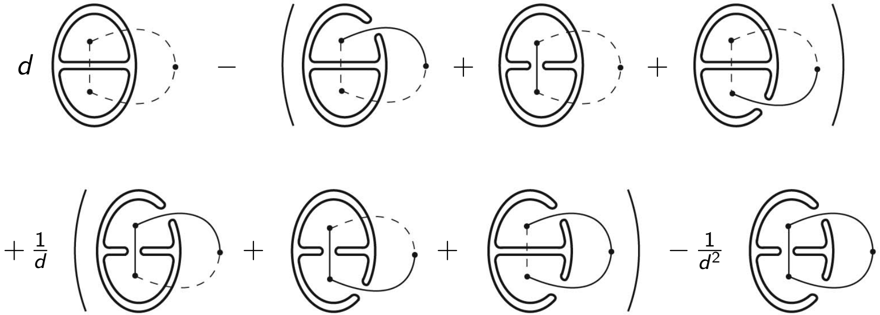
The map Φ is trace-preserving:

$$\begin{array}{ccc} \mathcal{C}_n^Q & \xrightarrow{\Phi} & TL_{2n}^d \\ \downarrow{\scriptstyle tr_\chi} & & \downarrow{\scriptstyle tr_d} \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

For example, for the theta-graph G,



$\Phi(G) = d \cdot$ 



The expansions of $Q^{-1}\chi_Q(G^*)$, $\Phi(G)$ where G is the theta graph.

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$$\begin{array}{ccc} \mathcal{C}_n^Q & \xrightarrow{\Phi} & TL_{2n}^d \\ \Big\downarrow {\scriptstyle tr_\chi} & & \Big\downarrow {\scriptstyle tr_d} \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

Trace radical: $\{a \,|\, \langle a, b\rangle \; =, 0 \text{ for all } b\}$.

It follows that the pullback of the trace radical in TL_{2n}^d to $\mathcal{C}_n^{d^2}$ is in the trace radical of the chromatic algebra.

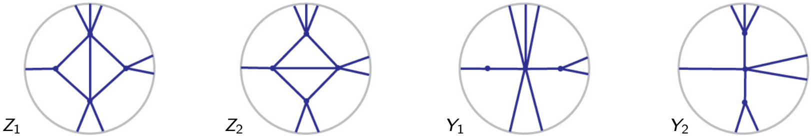
The trace radical in the TL algebra is non-trivial for

$$d = 2\cos\left(\frac{\pi j}{n+1}\right);$$

at this value it is generated by the Jones-Wenzl projector p_n.

Recall Tutte's linear relation at $\phi + 1$:

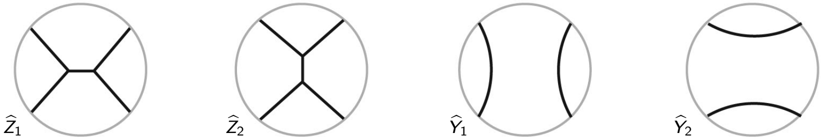
$$\chi_{Z_1}(\phi+1) + \chi_{Z_2}(\phi+1) = \phi^{-3}[\chi_{Y_1}(\phi+1) + \chi_{Y_2}(\phi+1)]$$



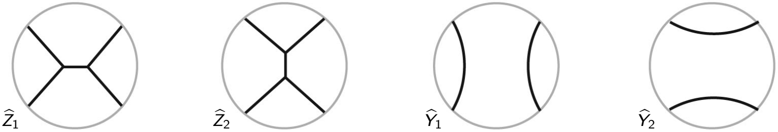
Proof that the relation

$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3}[\,\widehat{Y}_1 + \widehat{Y}_2\,]$$

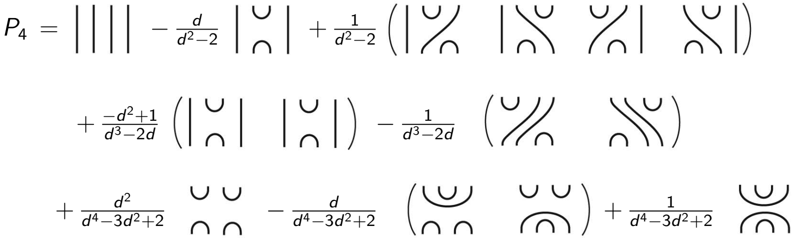
holds in the chromatic algebra $\mathcal{C}_2^{\phi+1}$:



$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3} [\, \widehat{Y}_1 + \widehat{Y}_2 \,]$$



Φ maps the dual of Tutte's relation to the 4-th Jones-Wenzl projector (at $d = \phi$):



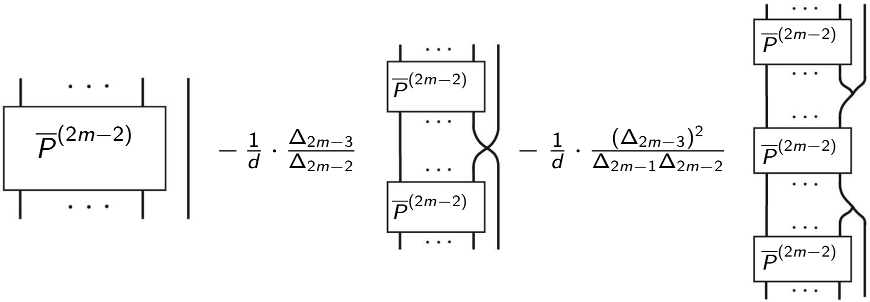


Figure: A generalization of Tutte's relation for the chromatic polynomial at $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$.

Theorem

For a planar triangulation $\widehat{G}$,

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2$$

where $V(\widehat{G})$ is the number of vertices of $\widehat{G}$.

Idea of the proof (Fendley - K., 2008): Construct a map

$$\Psi : \mathcal{C}^{\phi+2} \longrightarrow \mathcal{C}^{\phi+1}/R \otimes \mathcal{C}^{\phi+1}/R$$

and apply the trace:

$$\begin{array}{ccc} \mathcal{C}^{\phi+2} & \longrightarrow & \mathcal{C}^{\phi+1}/R \otimes \mathcal{C}^{\phi+1}/R \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

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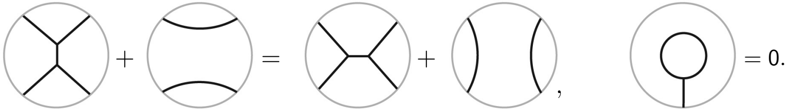

Figure: Relations defining the chromatic algebra.

Key calculation:

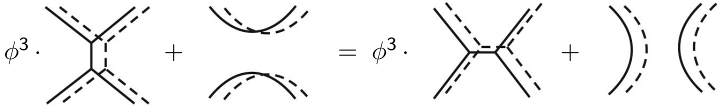

Figure: The image under Ψ of the first relation.

More conceptually, the golden identity

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2$$

is related to level-rank duality:

$SO(3)_4$ and $SO(4)_3$ theories are isomorphic; $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$, and $SO(4)_3$ splits as a product of two copies of $SO(3)_{3/2}$.

The partition function of an $SO(3)$ theory is given by the chromatic polynomial: $\chi(\phi+2)$ for $SO(3)_4$ and $\chi(\phi+1)$ for $SO(3)_{3/2}$.

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A related observation for *knots*:

Scott Morrison, Emily Peters, Noah Snyder, *Knot polynomial identities and quantum group coincidences* (2011):

The 2-colored Jones polynomial of a *knot* at $e^{\pi i/10}$ equals the square of the Jones polynomial at $e^{\pi i/5}$.

Recall Tutte's inequality: for a planar triangulation T with V vertices,

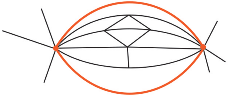
$$|\chi_T(\phi+1)| \leq \phi^{5-V}.$$

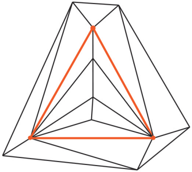
Theorem (Agol-K.) Given a planar triangulation T, let x be either a Beraha number $B_n = 2 + 2\,cos(2\pi/n)$ or a real number ≥ 2. Then

$$|\chi_T(x)| \ \leq \ x(x-1)(x-2)^{(V-2)}.$$

Tutte's inequality is the case $B_5 = \phi + 1$.

Outline of the proof: Use induction to reduce to 4-connected planar triangulations.





By Whitney's theorem any 4-connected planar triangulation has a Hamiltonian cycle. Cut along the cycle to get two outer planar triangulations, and use the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2} \cdot \langle y, y\rangle^{1/2}$$

Question: To what extent do Tutte relations detect planarity?

There are non-planar graphs satisfying the chromatic golden identity:

$$\chi_T(\phi+2) = (\phi+2)\,\phi^{3\,V(T)-10}\,(\chi_T(\phi+1))^2,$$

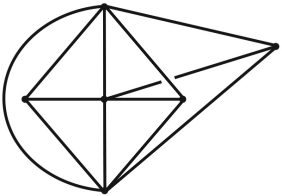


Figure: A non-planar graph G satisfying the chromatic golden identity

The chromatic golden identity: Given a planar triangulation T,

$$\chi_T(\phi+2) = (\phi+2)\,\phi^{3\,V(T)-10}\,(\chi_T(\phi+1))^2.$$

The flow golden identity: Given a planar cubic graph G,

$$F_G(\phi+2) = \phi^E\,(F_G(\phi+1))^2.$$

The chromatic golden identity: Given a planar triangulation T,

$$\chi_T(\phi + 2) = (\phi + 2)\, \phi^{3V(T)-10} (\chi_T(\phi + 1))^2.$$

The flow golden identity: Given a planar cubic graph G,

$$F_G(\phi + 2) = \phi^E (F_G(\phi + 1))^2,$$

Conjecture (Agol-K.) For any trivalent graph G,

$$F_G(\phi + 2) \le \phi^E (F_G(\phi + 1))^2,$$

Moreover, G is planar if and only if equality holds.

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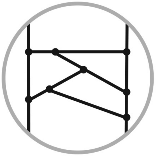
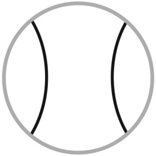
Moreover, G is planar if and only if equality holds.

There is extensive computer evidence. (Thanks to Gordon Royle!)

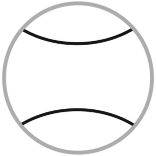
Represent a cubic graph in C_2 as a linear combination of basis elements:



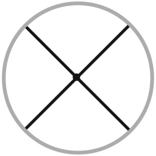
$= \alpha$



$+\beta$



$+\gamma$

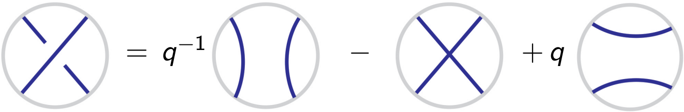


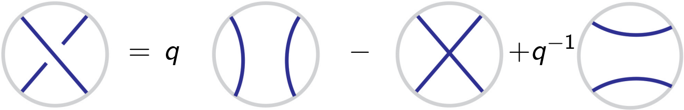
A consequence of the conjecture: at $Q = (3 - \sqrt{5})/2$,

$$(1 + 3\,\phi)\,\alpha\,\beta \;\leq\; \gamma\,(\alpha + \beta + \gamma).$$

The Yamada polynomial $R(q)$ is an invariant of ribbon graphs embedded in $\mathbb{R}^3$. It is defined by the relations:

- The $SO(3)$ Kauffman Skein relations: $\bigcirc = q + 1 + q^{-1}$,





- The contraction-deletion rule

The Yamada polynomial $R(q)$ is an invariant of ribbon graphs embedded in $\mathbb{R}^3$.

• For planar graphs G it coincides with the flow polynomial:

$$R_G(q) \ = \ F_\Gamma(Q), \ \text{where} \ Q \ = \ q + 1 + q^{-1}.$$

• Closely related to the $SO(3)$ Kauffman polynomial of links and graphs.

• For non-planar graphs the Yamada polynomial carries a lot of information about the knotting in 3-space, so in general (for non-planar graphs) it is very different from the flow polynomial.

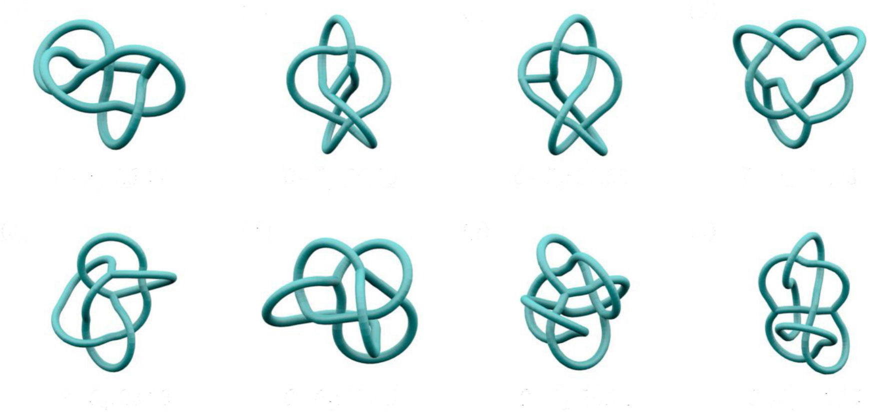


Figure: Examples of knotted theta graphs with few crossings

Let G be a cubic graph with V vertices and E edges; $\phi = \frac{1+\sqrt{5}}{2}$.

Theorem (Agol-K., 2017)
A quadratic identity for the Yamada polynomial of cubic graphs:

$$R_G(e^{\pi i/5}) = (-1)^{V-E}\, \phi^E\, R_G(e^{-2\pi i/5})^2.$$

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This is an extension of the Tutte golden identity for the flow polynomial of planar cubic graphs:

$$F_G(\phi + 2) = \phi^E\, F_G(\phi + 1)^2.$$

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Compare with the **Conjecture** (Agol-K.): *For any cubic graph G,*

$$F_G(\phi+2) \leq \phi^E\,F_G(\phi+1)^2.$$

Moreover, G is planar if and only if this is an equality.

Question (David Treumann, Eric Zaslow):

Let $P(n)$ be the set of polynomials that can occur as the chromatic polynomial of a planar map (triangulation) with n countries. What is known or conjectured about the growth of $|P(n)|$?

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Let $P(n)$ be the set of polynomials that can occur as the chromatic polynomial of a planar map (triangulation) with n countries. What is known or conjectured about the growth of $|P(n)|$?

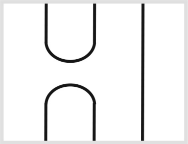
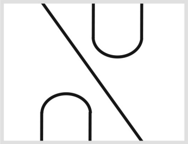
Answer (Agol - K., 2018): Exponential in n.

This may be thought of as an application of the Tits alternative for free semigroups for the chromatic algebra.

An explicit free semi-group in the chromatic algebra $C_3^{\phi+1}$ generated by A, B:

$A \;=$  $B \;=$ 

A, B acts on the 2-dimensional subspace spanned by

$E_1 \;=$  $E_2 \;=$ 

The action of A, B on this 2-dimensional space is represented by the matrices

$$A = \begin{bmatrix} -\phi & -\phi \\ 0 & -\phi^2 \end{bmatrix}, \quad B = \begin{bmatrix} -\phi^2 & 0 \\ -\phi & -\phi \end{bmatrix}$$

A, B generate a free sub-semigroup.

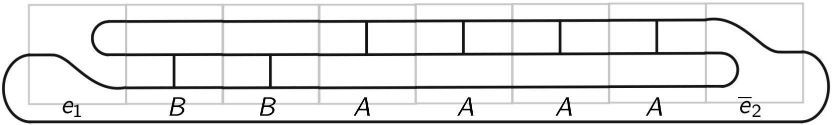


Figure: The flow polynomial of the pictured graph at $(3-\sqrt{5})/2$ equals $\langle A^4 B^2 e_1, e_2 \rangle$.

The flow polynomial (or dually the chromatic polynomial) of planar graphs is a common specialization of several invariants:

• The flow polynomial of abstract graphs

• The Yamada polynomial of spatial graphs in $\mathbb{R}^3$

• The trace evaluation in $\mathrm{SO}(3)$ TQFTs of graphs on a closed surface Σ_g. (Parameter $=$ root of unity)

• The "Topological flow polynomial" of graphs on a surface Σ_g.

Planar graphs:

Loop evaluation,

The flow polynomial,

Partition function of the Potts model

Graphs on the torus:

The trace evaluation in $\mathrm{SO}(3)$ TQFTs,

Topological flow polynomial,

Lattice models

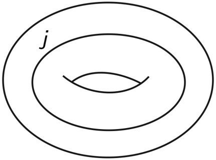
Graphs on the torus:

The trace evaluation in $\mathrm{SO}(3)$ TQFTs.

Given a closed orientable surface Σ, $V_r(\Sigma)$ is the $\mathrm{SU}(2)$, level $r-2$ TQFT vector space (constructed by Blanchet, Habegger, Masbaum, Vogel).

In this talk: $\Sigma =$ torus $\mathbb{T}$.

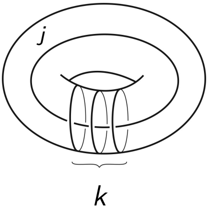
Consider $\mathbb{T}$ as the boundary of a solid torus H. V_r has a basis $\{e_0, \ldots, e_{r-2}\}$, where e_j corresponds to the core curve of H, labeled by the JW projector p_j.



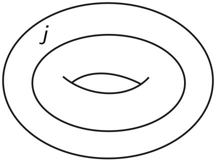
A multi-curve γ in Σ acts as a linear operator on $V_p(\Sigma)$, so associated to γ is an element of $V_r^*(\Sigma) \otimes V_r(\Sigma)$. (May be thought of as an element of the Turaev-Viro theory.)

Given a multi-curve $\gamma \subset \mathbb{T}$, the trace $\mathrm{tr}_r(\gamma)$ is defined as $Z_r(\mathbb{T} \times S^1, \gamma)$, the $\mathrm{SU}(2)$ quantum invariant of the banded link γ in the 3-manifold $\mathbb{T} \times S^1$. (Modular invariant)

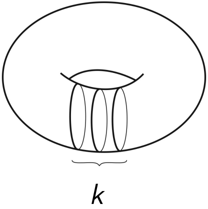
Concretely, $\mathrm{tr}_r(\gamma)$ can be calculated as the trace of the curve operator in $\mathrm{Hom}(V_r, V_r)$ with respect to the usual basis. For example, in V_5:



$$= \left[\frac{\sin(2(j+1)\pi/5)}{\sin((j+1)\pi/5)}\right]^k$$

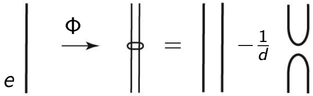


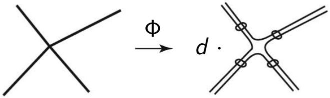
The trace of k non-trivial loops with label 1 on the torus equals

tr_5 

$= \phi^k + (-\phi^{-1})^k.$

Graphs on the torus are labeled with the 2nd JW projectors, and so are considered as linear combinations of multi-curves:

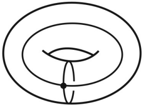
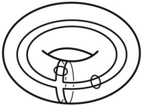


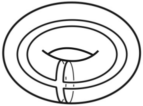
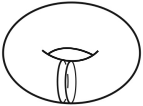
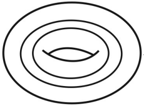
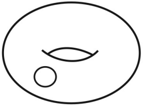


(1)

The factor corresponding to a $k-$valent vertex is $d^{(k-2)/2}$.

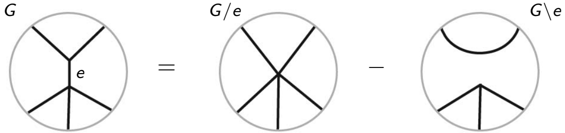
Using this map, graphs G on the torus give rise to elements $\Phi(G) \in \mathrm{Hom}(V_r, V_r)$.

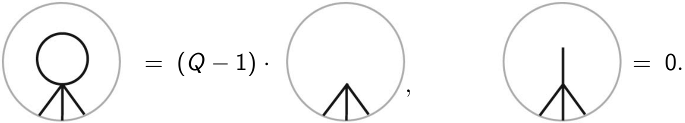
tr_5  $= d\,\mathrm{tr}_5$  $=$

$d\,\mathrm{tr}_5$  $-\,\mathrm{tr}_5$  $-\,\mathrm{tr}_5$  $+\frac{1}{d}\mathrm{tr}_5$ 

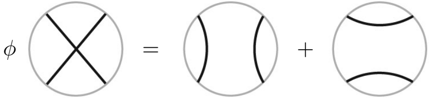
$$= 2\phi^2 - 2(\phi^2 + (\tfrac{1}{\phi})^2) + 2 \;=\; \frac{2}{\phi^2} + 2$$

Considered as elements of $\mathrm{Hom}(V_r, V_r)$, graphs in $\mathbb{T}$ satisfy local relations:





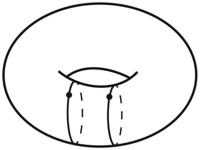
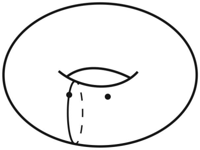
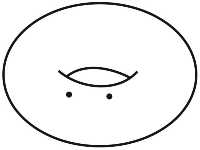
and the local relation corresponding to the JW projector p_{r-1}, for example for $r = 5$:



The "topological flow polynomial" (motivated by the Bollobas-Riordan polynomial):

$$P_G(Y, W, A) := \sum_{H \subset G} (-1)^{E(G)-E(H)} \, Y^{\mathrm{n}(H)} \, W^{\bar{\mathrm{c}}(H)} \, A^{\mathrm{s}(H)},$$

where the summation is taken over all spanning subgraphs of G.

 $+\,2$  $+$ 

$$= Y^2W^2 - 2YW + 1 = (YW - 1)^2.$$

For k non-trivial loops, the polynomial equals $(YW - 1)^k$.

Given a graph $G \subset \mathbb{T}$, consider

$$R_5(G) := P_G(\phi^2, 1, \phi^{-2}) + P_G(\phi^2, \phi^{-4}, \phi^{-2})$$

Theorem. (Fendley - K.) *Given any graph $G \subset \mathbb{T}$, the $\mathrm{SO}(3)$ TQFT trace evaluation $\mathrm{tr}_5(G)$ at $Q = \phi + 1$ (corresponding to $q = e^{2\pi i/5}$) equals $R_5(G)$.*

More generally:

$$R_r(G) := \sum_{j=0}^{r-2} P_G(d^2, W_{j,r}, d^{-2}),$$

$$W_{j,r} := \frac{\sin(2(j+1)\pi/r)}{\sin((j+1)\pi/r)}.$$

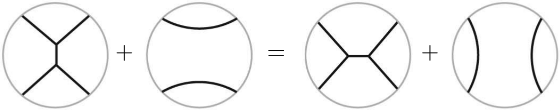
The trace evaluation of G at $q = e^{2\pi i/r}$ equals $R_r(G)$.

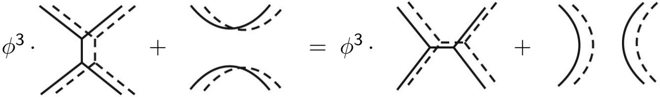
Theorem. (Fendley - K., 2019) *Let $G \subset \mathbb{T}$ be a trivalent graph. Then*

$$R_{10}(G) = \phi^E R_5(G)^2,$$

where E is the number of edges of G.

The proof of is by induction on the number of edges of the cubic graph G, using the theorem of Negami that any two quasi-triangulations of a surface are related by diagonal flips.





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Vincent Pasquier, *Lattice derivation of modular invariant partition functions on the torus*, J. Phys. A: Math. Gen., 20 (1987).

Given a graph G on the torus, the partition function is of the form

$$Z_G = C_G \sum_{S \subseteq G} \left(\frac{y-1}{d} \right)^{|E(S)|-|V|} \mathcal{T}_S$$

where $\mathcal{T}_S$ is a topological weight.

To relate to the usual Tutte polynomial notation, $(x-1)(y-1) = Q = d^2$. The original Pasquier model defined in the self-dual case, $y - 1 = x - 1 = d$

To define the topological weight $\mathcal{T}_S$, consider labelling of the clusters (connected components) of S and the dual $\overline{S}$, with the labels determined by the Dynkin diagram. (Adjacent clusters have heights which are connected by an edge in the Dynkin diagram.) In the $\mathrm{SU}(2)$ case, the heights take integer values $0, \ldots, r-2$.

The topological weight is given by the sum over all height configurations,

$$\mathcal{T}_S = \sum_{\{h\}} w(\mathcal{P}(S), \{h\}) \ .$$

The weight $w(\mathcal{P}(S), \{h\})$ for a height configuration is determined by the components of the eigenvector of the adjacency matrix for the largest eigenvalue.

Conjecture/Work in progress with Paul Fendley:
At roots of unity, the partition function of the (generalized) Pasquier model equals the $\mathrm{SO}(3)$ TQFT trace evaluation, and the sum R_r of evaluations of the topological flow polynomial.

Higher genus?

Question Analogue of the golden identity on surfaces of higher genus?

Conjecture (Agol-K.) For any trivalent graph G,

$$F_G(\phi+2) \leq \phi^E \left(F_G(\phi+1)\right)^2,$$

Moreover, G is planar if and only if equality holds.

Conjecture (Beraha) Real roots of large planar triangulations accumulate near $\phi+1$. (More generally, real roots accumulate near Beraha numbers $B_n = 2 + 2cos(2\pi/n)$.)

Conjecture (Birkhoff-Lewis) The chromatic polynomial of (loopless) planar graphs is positive for $x \geq 4$.
(Known for $x = 4$ and $x \geq 5$.)

This talk is based on:

Fendley - K. "*Tutte chromatic identities from the Temperley-Lieb algebra*" arXiv:0711.0016

Agol - K. "*Tutte relations, TQFT, and planarity of cubic graphs*" arXiv:1512.07339

Agol - K. "*Structure of the flow and Yamada polynomials of cubic graphs*" arXiv:1801.00502

Fendley - K. "*Topological quantum field theory and polynomial identities for graphs on the torus*", arXiv:1902.02760