Non-trivial quantum cellular automata (locality-preserving unitaries) in 3 dimensions

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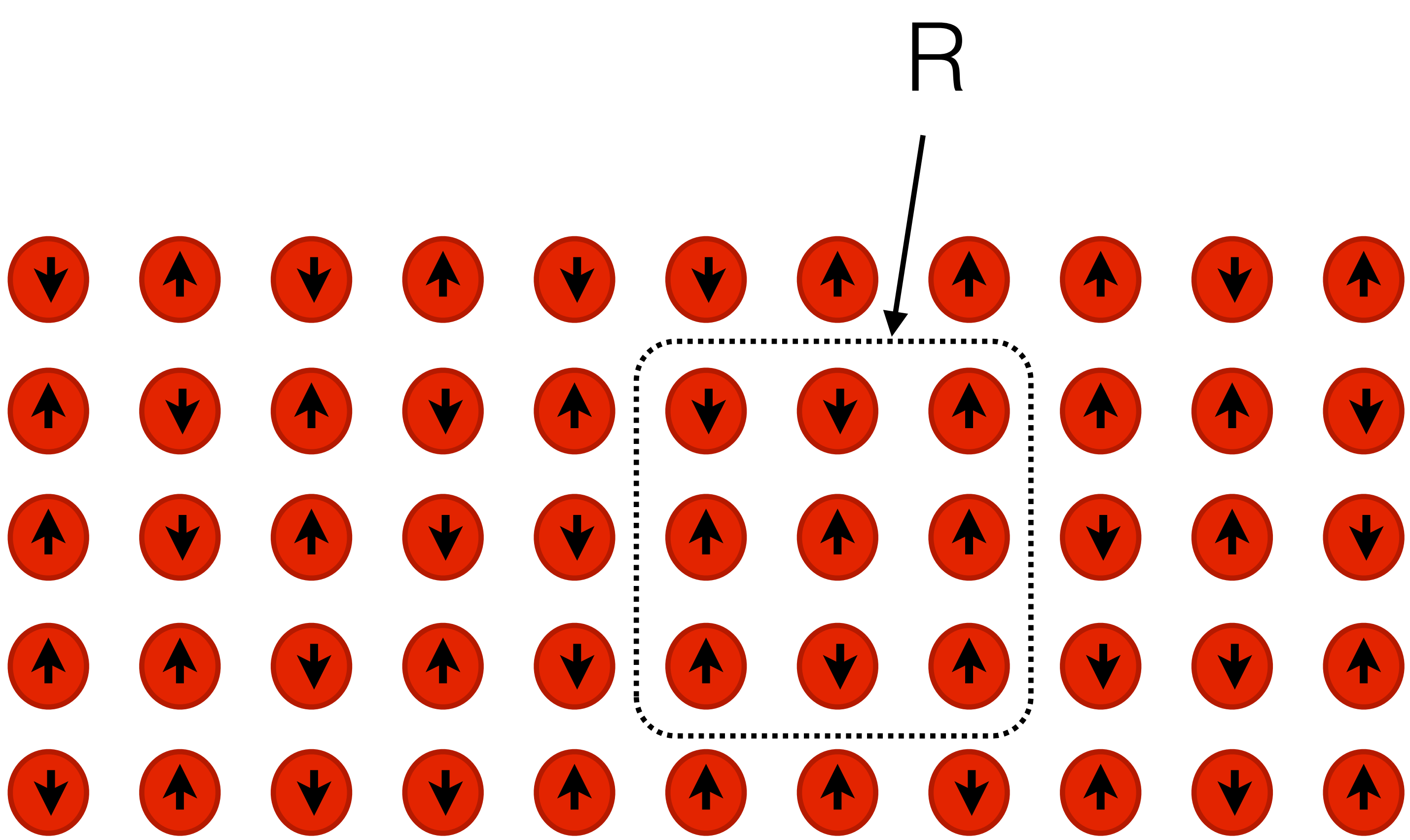
work with Jeongwan Haah and Matthew Hastings

Setting

Hilbert space: $\mathcal{H} = \otimes_{\text{sites i}} \mathcal{H}_i$

local operator: $\mathcal{O} = \mathcal{O}_R \otimes 1_{\bar{R}}$



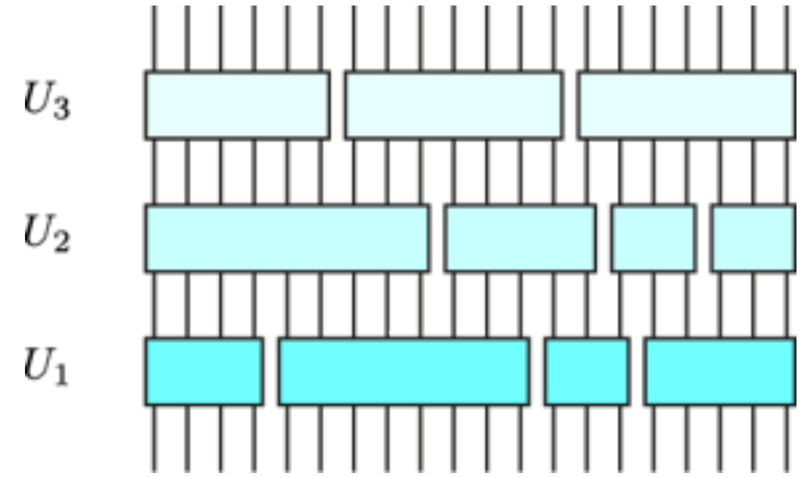
Setting

- a unitary operator U is **locality-preserving** if for every local operator A on site j,

$$U^\dagger A U$$

is supported on a finite number of sites near j.

- a unitary U is **locally-generated** if it is a constant depth circuit of local unitaries:

U = 

- **Compute {locality-preserving} / {locally-generated}**

Plan

- Walker-Wang model (based on UMTC) (Walker & Wang; von Keyserlingk, Burnell, Simon)

  boring in the world of all gapped Hamiltonians, but interesting in the world of commuting projector Hamiltonians.

- Disentangling Walker-Wang models

  can ground state be disentangled with finite depth circuit of **local** unitaries?

  ground state and Hamiltonian can be disentangled with **locality-preserving** unitary U

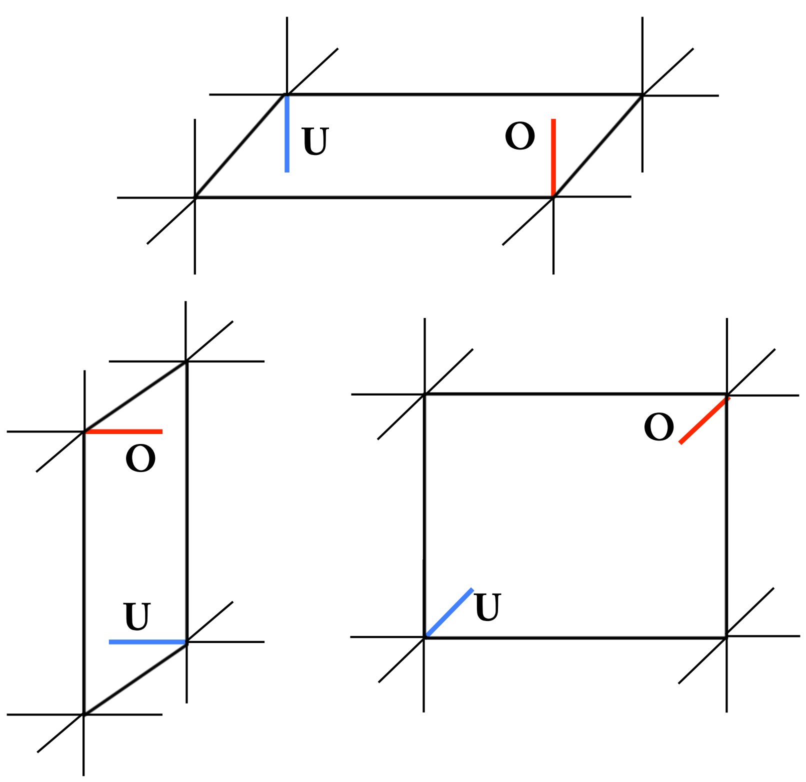
existence of non-trivial 3d locality preserving unitaries

3-fermion Walker-Wang model: Hamiltonian

- gapped (3+1)-D lattice Hamiltonian
- two spin-1/2 degrees of freedom per link ℓ of a cubic lattice

$$H = -\sum_V A_V - \sum_P B_P$$

$$A_V = \prod_{\ell \sim V} \sigma^x_\ell + \prod_{\ell \sim V} \tau^x_\ell \qquad B_P = \sigma^x_O \sigma^x_U \tau^x_U \prod_{\ell \in \partial P} \sigma^z_\ell + \sigma^x_O \tau^x_O \tau^x_U \prod_{\ell \in \partial P} \tau^z_\ell$$



(Burnell, Chen, Fidkowski, Viswhanath)

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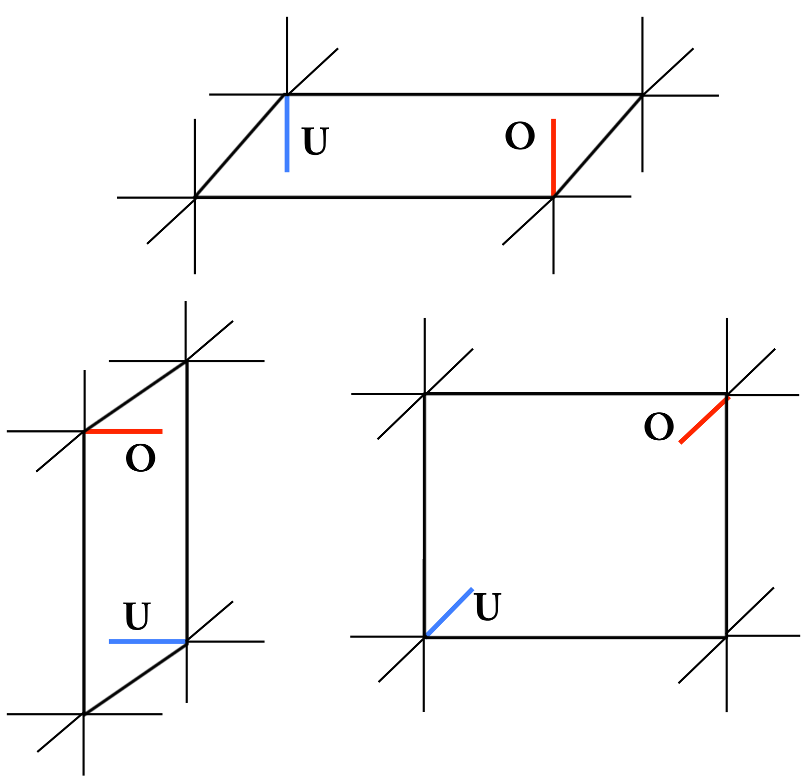


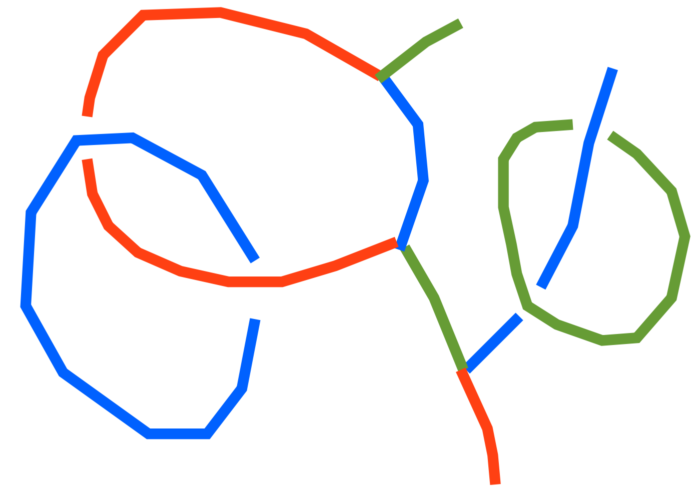
=> no bulk deconfined excitations

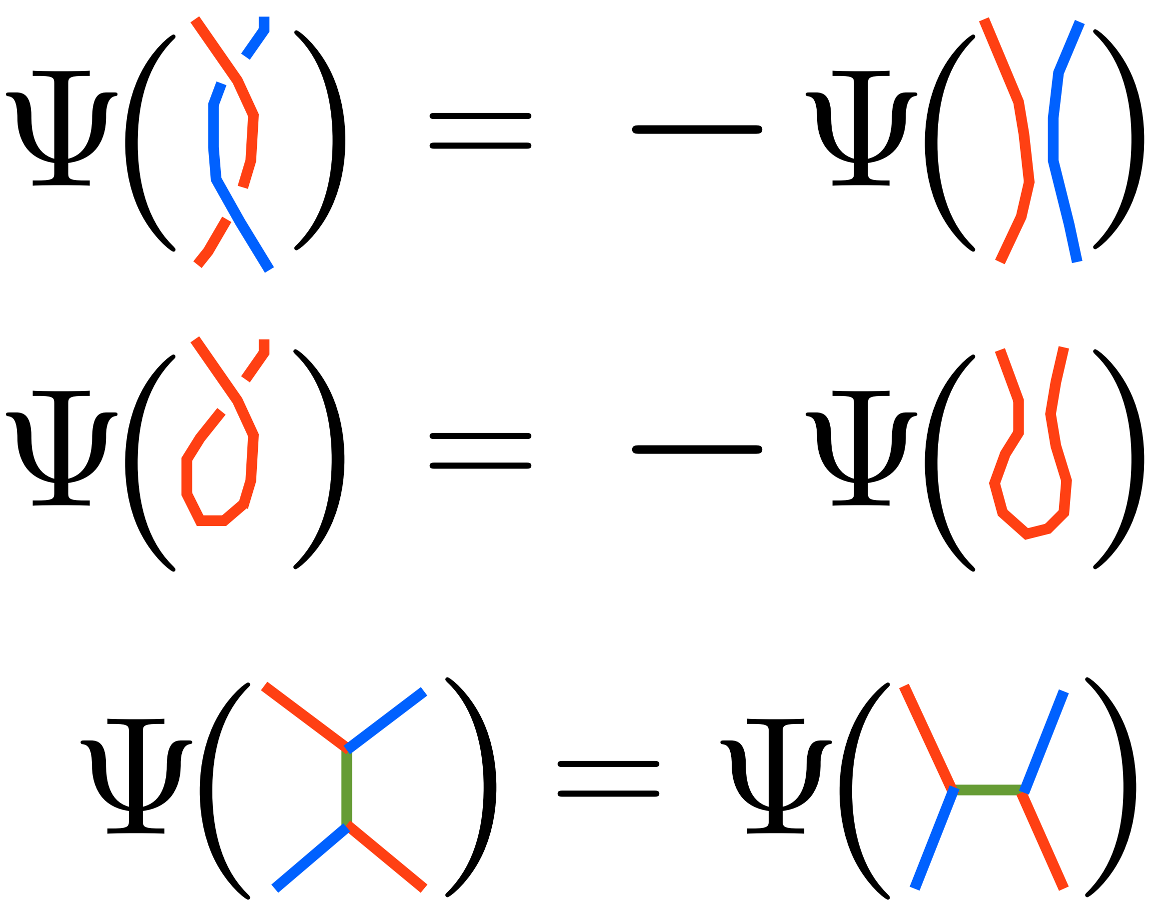
- Pauli stabilizer Hamiltonian

(Burnell, Chen, Fidkowski, Viswhanath)

Walker-Wang model: intuition

- superposition over string net configurations:



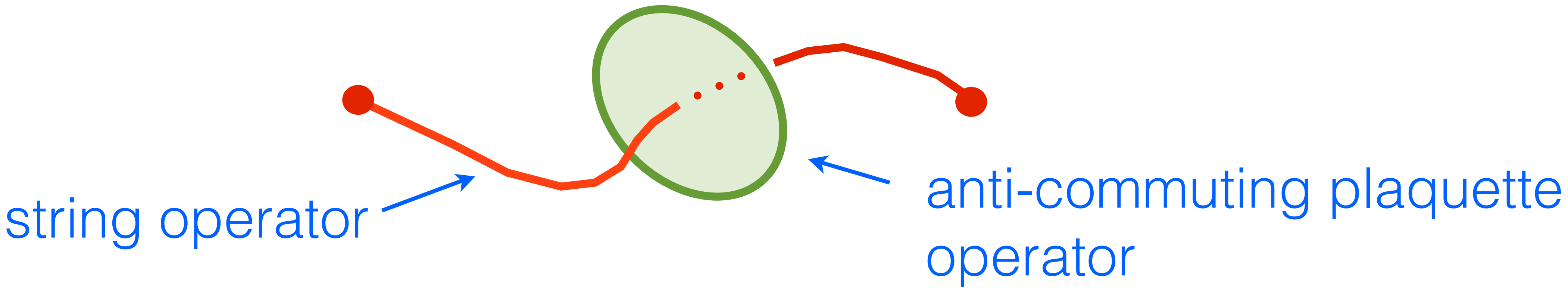


	σ^x	τ^x
	+1	+1
	+1	- 1
	- 1	+1
	- 1	- 1

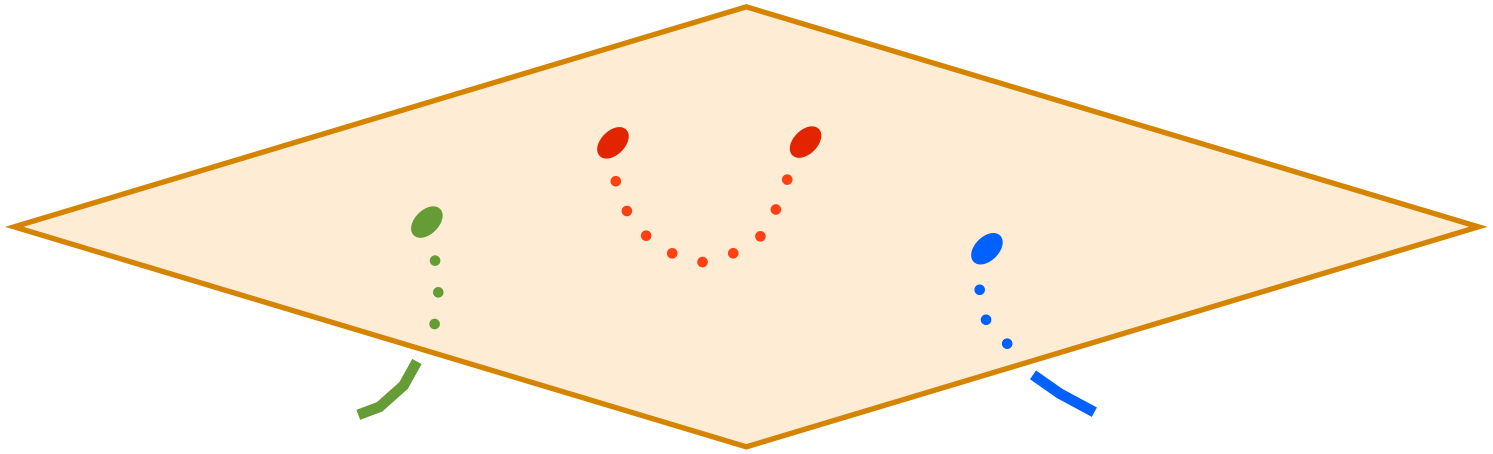
(Walker & Wang; von Keyserlingk, Burnell, Simon)

Walker-Wang model: intuition

- no anyons in 3d bulk:



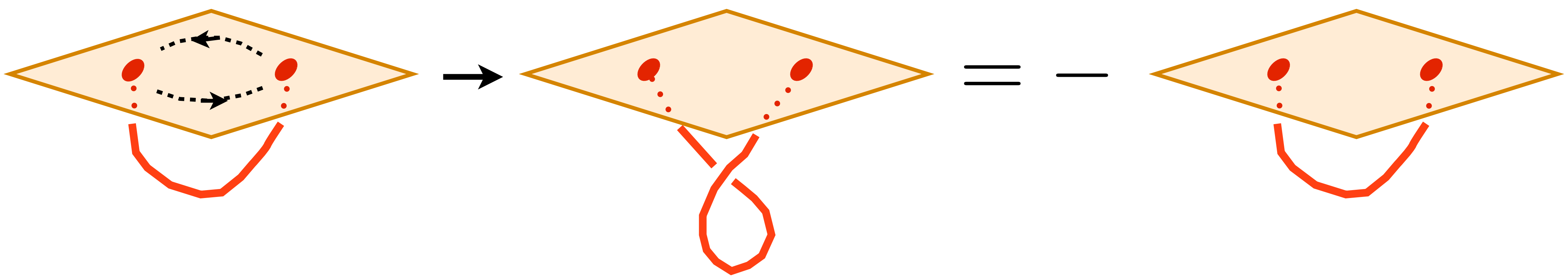
- surface topological order:



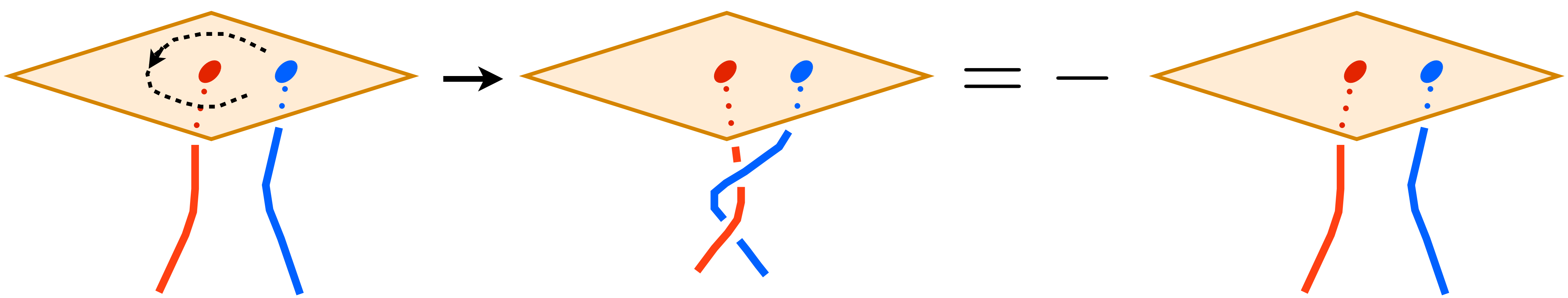
- all statements can be made rigorous in commuting projector context, and proven using Pauli nature of Hamiltonian

Walker-Wang model surface topological order

- quasiparticles are fermions:



- and mutual semions:



“the 3-fermion theory”

- can be generalized to whole class of models (premodular categories)

3-fermion topological order in 2d

$U(1)$ Chern-Simons theory with $K = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$

$$e^{2\pi i c_-/8} = \frac{\sum_a d_a^2 \theta_a}{\sqrt{\sum_a d_a^2}} = \frac{1-1-1-1}{2} = -1$$

=> c_- = 4 mod 8

- nonzero chiral central charge => edge energy current at finite temperature => no 2d commuting projector realization

Analogy

2d theory with 't Hooft anomaly

- cannot be realized by 2d lattice Hamiltonian commuting with onsite symmetry G

- can be realized at surface of 3d lattice Hamiltonian commuting with onsite symmetry G

- 3d Hamiltonian: trivial without symmetry but non-trivial with symmetry (3d 'Symmetry Protected Topological' phase)

3-fermion theory

- cannot be realized by 2d commuting Hamiltonian lattice model

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???

Analogy

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3-fermion theory

- cannot be realized by 2d commuting Hamiltonian lattice model

- can be realized at the surface of Walker-Wang commuting projector model

- 3d Hamiltonian: trivial as a gapped Hamiltonian, but non-trivial as a commuting projector gapped Hamiltonian

Separators and locally flippable separators

- assume Hilbert space is built on qubits on sites j.

- A **separator** is a collection of operators $\mathcal{Z}_j$ such that:

    - $[\mathcal{Z}_j, \mathcal{Z}_k] = 0$

    - $\mathcal{Z}_j$ is supported on sites near j

    - for any set of $z_j = \pm 1$, there is a unique (up to phase) state which is an eigenvalue z_j eigenvector of $\mathcal{Z}_j$ for all j

- A **locally flippable** separator has the additional property that for each j there exists $\mathcal{X}_j$ supported on sites near j such that

$\mathcal{X}_j \mathcal{Z}_j = -\mathcal{Z}_j \mathcal{X}_j$ and $[\mathcal{X}_j, \mathcal{Z}_k] = 0 \ \ (j \neq k)$

Separators: examples

- toric code vertex and plaquette terms: separator (on sphere, with one vertex and one plaquette removed) but not locally flippable

- 1d Ising model: $\sigma_1^z\sigma_2^z,\ \sigma_2^z\sigma_3^z,\ \ldots,\ \sigma_{N-1}^z\sigma_N^z,\ \sigma_N^z$

  also not locally flippable

- 3-fermion Walker-Wang model has another set of stabilizers that define a locally flippable separator

(polynomial formalism for Pauli stabilizer models)

Locally Flippable separators and locality-preserving unitaries

- For any locally flippable separator, the $\mathcal{X}_j$ can be chosen to have the additional property that

$$[\mathcal{X}_j, \mathcal{X}_k] = 0$$

- Together with the commutation relations

$$[\mathcal{Z}_j, \mathcal{Z}_k] = 0, \quad \mathcal{X}_j \mathcal{Z}_j = -\mathcal{Z}_j \mathcal{X}_j, \quad [\mathcal{X}_j, \mathcal{Z}_k] = 0 \ \ (j \neq k)$$

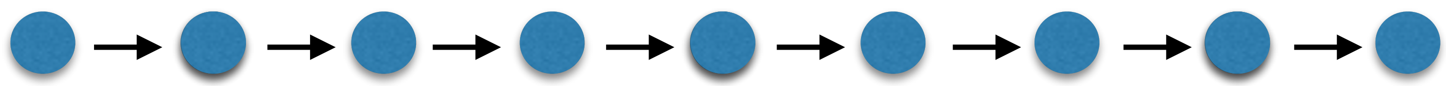
this implies the existence of **locality-preserving** unitary U:

$$U^\dagger \mathcal{Z}_j U = Z_j$$

$$U^\dagger \mathcal{X}_j U = X_j$$

Classification of locality preserving unitaries

- a translation is locality preserving but not constant depth:



$$V^\dagger X_j V = X_{j+1}$$

- that's all in 1d (Gross, Nesme, Voigt, Werner 2012). Open problem in higher dimensions.

- relation to Floquet many-body-localized phases

- the unitary U that disentangles the 3-fermion Walker-Wang model defines a non-trivial locality preserving operator in 3d

(non-trivial = not 'blendable' to identity)

Disentangling the 3-fermion Walker-Wang model

$$H = \sum_j H_j$$

commuting local terms

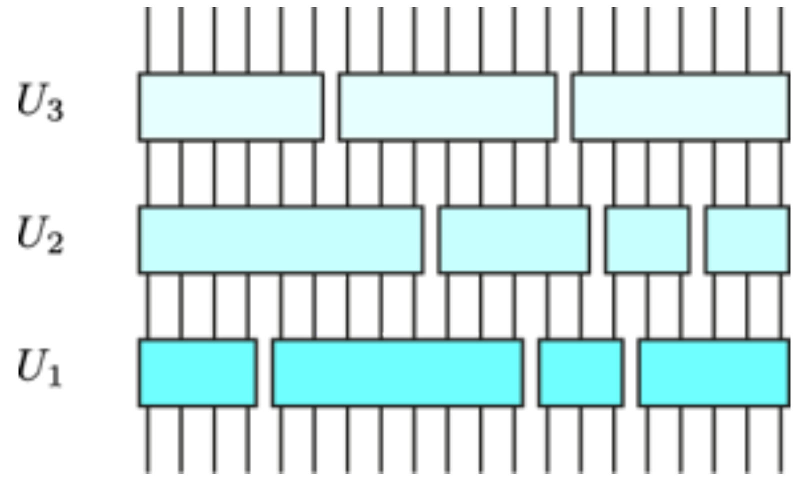
- impossible to find **constant-depth circuit** U such that

$$U^\dagger H_j U = \sigma_j^z$$

Pauli z operators on independent spins

- <u>Proof</u>: by contradiction - truncate U:

$$U =$$



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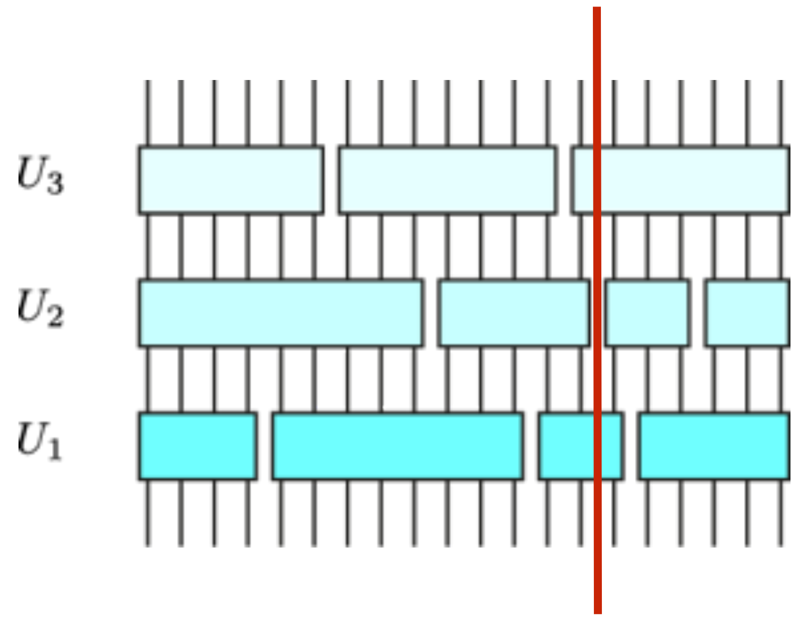

Disentangling the 3-fermion Walker-Wang model

$$H = \sum_j H_j$$ commuting local terms

- impossible to find **constant-depth circuit** U such that

$$U^\dagger H_j U = \sigma_j^z$$ Pauli z operators on independent spins

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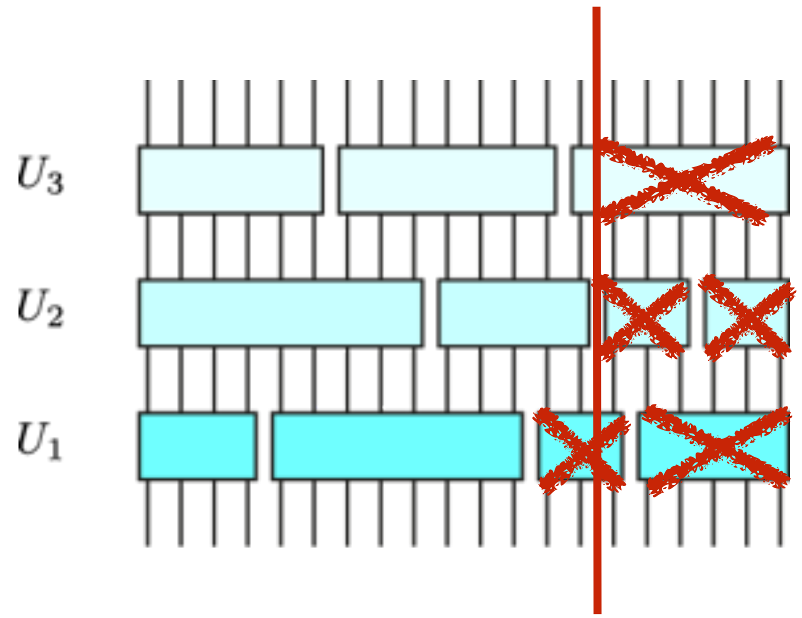
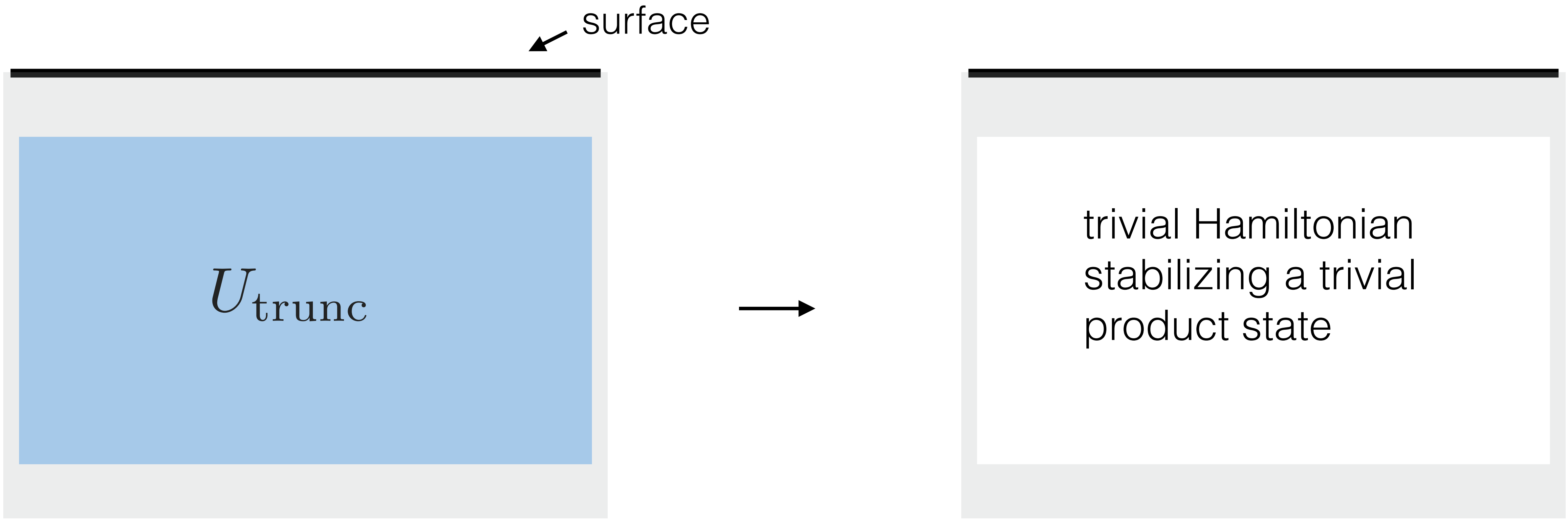
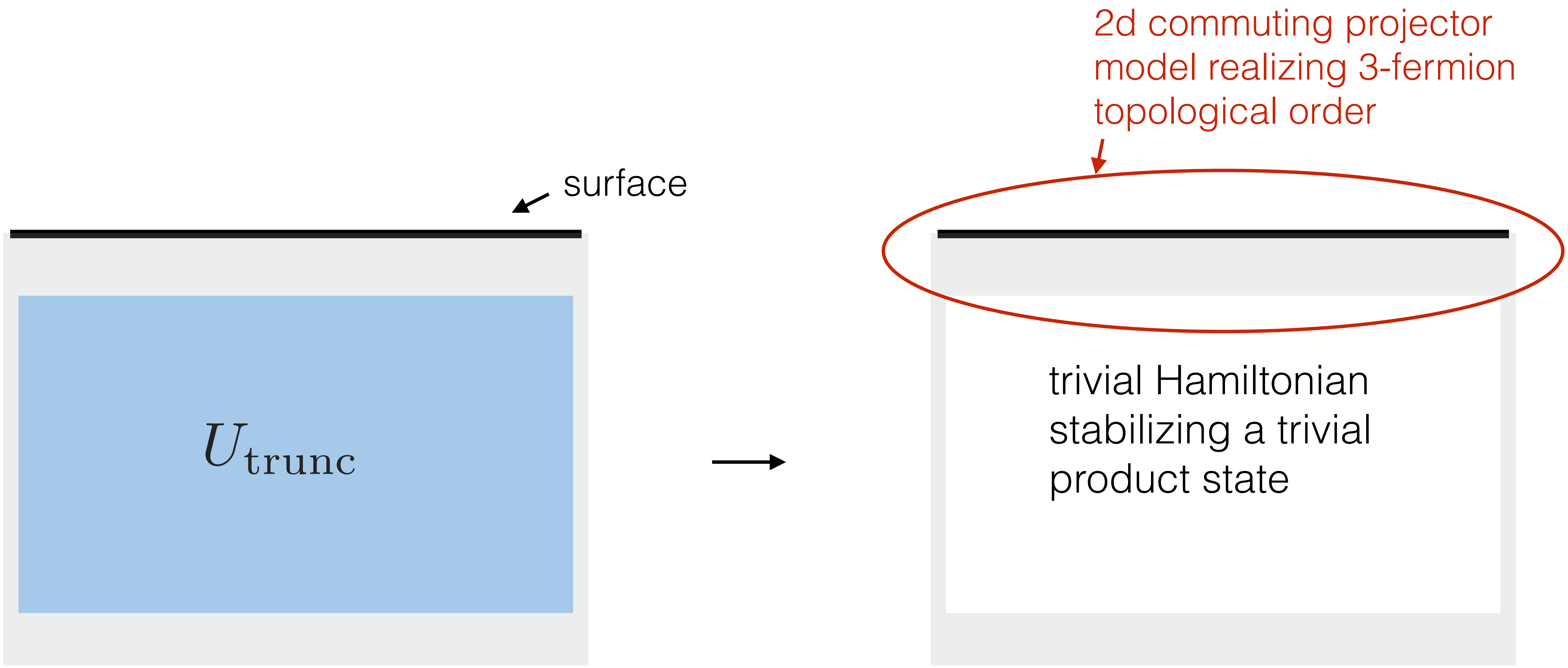


(also true for any locality-preserving unitary blendable to the identity)

Disentangling the 3-fermion Walker-Wang model



Disentangling the 3-fermion Walker-Wang model



contradiction, because then we would have 3-fermion topological order must have thermal Hall response

Conclusions:

- 3-fermion Walker-Wang model can be stabilized by flippable separator

- the corresponding locality preserving unitary operator is not blendable to identity.

- *Open questions:*

  - quantized index for U? I.e. what does U pump?

  - can ground state be disentangled with a finite depth circuit (short range or with tails)?

  - can we find U'^2 = 1? Duality interpretation?

  - rigorous proof that 3-fermion impossible with commuting projectors in 2d?

  - U(1) analogue?