Topological defects in Temperley-Lieb Lattice models

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based on joint work with:
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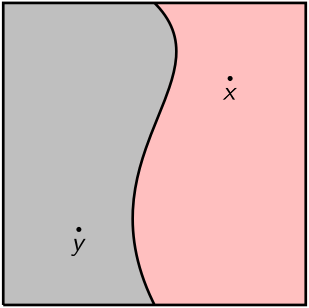
The plan

- Introduction
- Example: the unitary case
- The TL algebras
- The TL lattice models
- More examples

Defects - open case

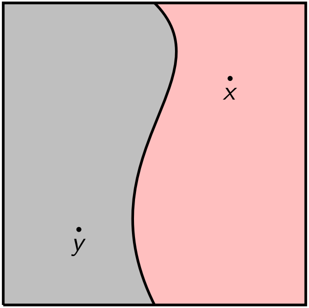


$$\tilde{\phi}(x) \equiv \phi(x)$$

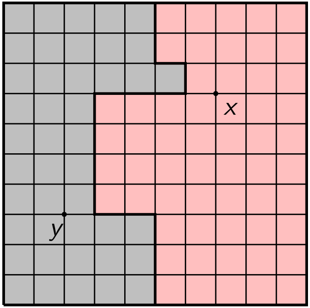
$$\tilde{\phi}(y) \equiv \psi(x)$$

- What's $\tilde{Z}$ or $\tilde{S}$?
- What's $\langle \tilde{\phi}(x) \tilde{\phi}(y) \rangle$?

Defects - open case



$\longrightarrow$



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$$\tilde{\phi}_n(x) \equiv \phi_n(x)$$

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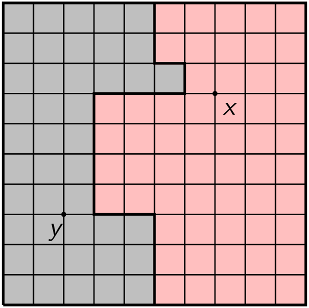
Defects - open case



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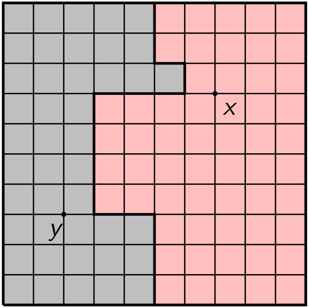
- Not all continuous maps are allowed.
- Width of strips is not free.
- The continuous limit of Z_n is invariant under deformations.

Defects - open case



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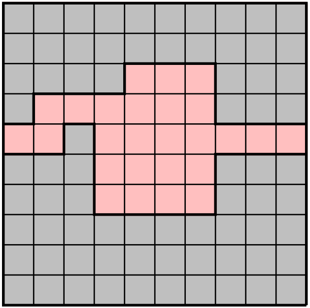
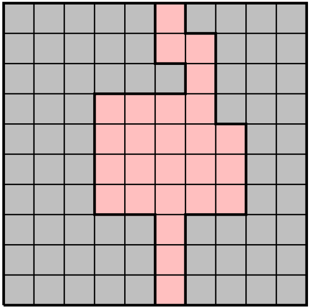
- Not all continuous maps are allowed.
- Width of strips is not free.
- The continuous limit of Z_n is invariant under deformations.
- Equivalent to choosing boundary conditions!

The defects - the closed case

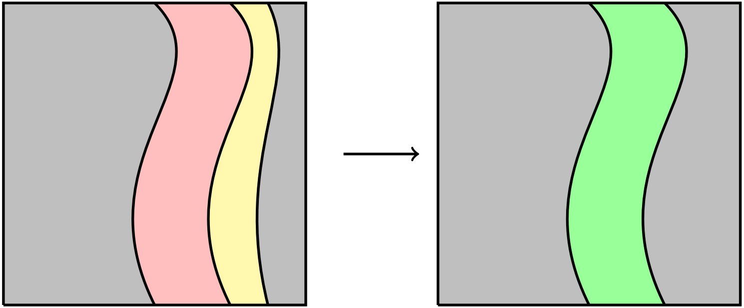
Cross Channel



Direct Channel



Product of defects



$$D_1 \circ D_2(Z) = \sum_k \lambda_k D_k(Z)$$

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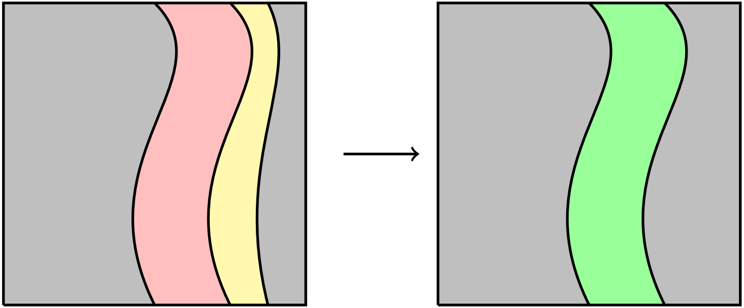
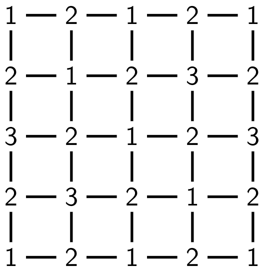


$$D_1 \circ D_2(Z) = \sum_k \lambda_k D_k(Z)$$

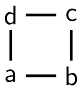
Unitary, rational CFTs: $D_i \leftrightarrows$ Prim. fields, product $\rightarrow$ faithful rep of fusion ring.

Example I: The A_p RSOS models

Includes Ising ($p = 3$), tri-critical Ising ($p = 4$), 3-states Potts ($p = 5$), etc.



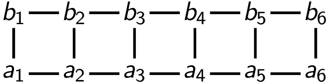
Heights $1, 2, \dots, p$, $\mathfrak{q} \equiv e^{i \frac{\pi}{p+1}}$



$$= \frac{\mathfrak{q} x^{-1} - \mathfrak{q}^{-1} x}{\mathfrak{q} - \mathfrak{q}^{-1}} \delta_{a,c} + \frac{x - x^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}} \delta_{b,d}$$

$$Z_{n,m} = \mathrm{Tr}(T_n)^m$$

$$\langle a_1, \dots, a_6 | T_n | b_1, b_2, \dots, b_6 \rangle =$$



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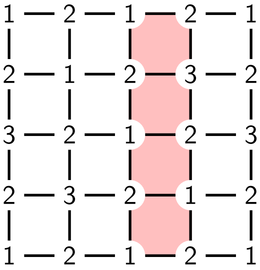
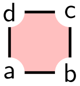
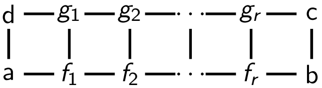


The width $k = 0, 1, 2, \ldots, p-2$



$$= \lim_{x\to 0} \sum_{\substack{f_1,\ldots,f_k \\ g_1,\ldots,g_k}} \lambda_{\substack{f_1,\ldots,f_k \\ g_1,\ldots,g_k}}(a,b,c,d)$$



$$\tilde{Z}_{n,m}(k) = \mathrm{Tr}(\tilde{T}_n(k))^m$$

The functions $\lambda_{\substack{f_1,\ldots,f_k \\ g_1,\ldots,g_k}}(a,b,c,d)$ are implicitly given as products of components of the eigenvectors of some matrix.

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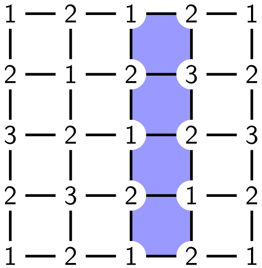

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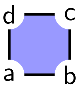
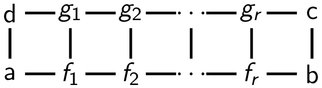


$$= \lim_{x\to\infty} \sum_{\substack{f_1,\dots,f_k \\ g_1,\dots,g_k}} \lambda_{\substack{f_1,\dots,f_k \\ g_1,\dots,g_k}}(a,b,c,d)$$



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Continuum limit: Virasoro minimal model $M(p+1,p)$

$$H \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} (\phi_{r,s} \otimes \bar{\phi}_{r,s}) \tag{1}$$

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Continuum limit: Virasoro minimal model $M(p+1,p)$

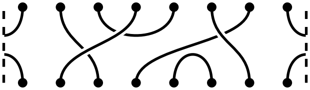
$$H \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} (\phi_{r,s} \otimes \bar{\phi}_{r,s}) \tag{1}$$

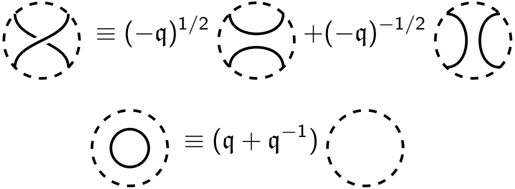
$$\tilde{H}(k) \sim \bigoplus_{s=1}^{p} \bigoplus_{r=1}^{p-1} ((\overbrace{\phi_{r,s} \times_f \phi_{1,k+1}}^{\text{Virasoro fusion!}}) \otimes \bar{\phi}_{r,s}).$$

- Defects give the twisted-boundary RSOS models.
- Computing products require extensive numerics.
- What's the meaning of the λs? How to generalize to other models?

The Temperley-Lieb algebras

Affine n-diagrams:

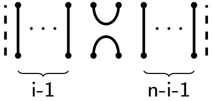




The Temperley-Lieb algebras

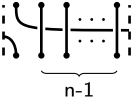
Type A or *regular*

$i = 1, 2, \ldots, n-1$

$e_i =$ 

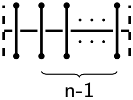
$$e_i e_i = (\mathfrak{q} + \mathfrak{q}^{-1}) e_i,$$
$$e_i e_{i\pm1} e_i = e_i,$$
$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2$$

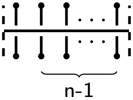
Type B or *affine*

$b = (-\mathfrak{q})^{-3/2}$ 

$$e_1 b e_1 = \overbrace{(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1})}^{-Y} e_1,$$
$$e_i b = b e_i \quad i \neq 1$$

Two *Hoop* operators:

$Y =$ 

$\bar{Y} =$ 

The TL Tower structure

$$\phi^o : \mathsf{aTL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m \to \mathsf{aTL}_{n+m}$$

$$\phi^u : \mathsf{aTL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m \to \mathsf{aTL}_{n+m}$$

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For any $V \in \mathrm{mod}(\mathsf{TL}_m)$, $\mathsf{aTL}_{n+m} \otimes_{\mathsf{TL}_m} V$ is a $(\mathsf{aTL}_{n+m}, \mathsf{aTL}_n)$-bimodule

TL fusions

$$M_n \equiv \text{mod}(\text{aTL}_n),\ V \in \text{mod}(\text{TL}_m)$$

$$- \times_f^{u/o} V$$

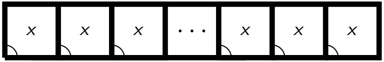
$$M_{n-1} \subset M_n \subset M_{n+1} \subset \cdots \subset M_{n+m-1} \subset M_{n+m} \subset M_{n+m+1} \subset \cdots$$

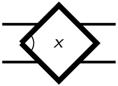
$$- \div_f^{u/o} V$$

$$M \times_f^{u/o} V \equiv (\text{aTL}_{n+m} \otimes_{\text{TL}_m} V) \otimes_{\text{aTL}_n} M,$$ Fusion product

$$\bar{M} \div_f^{u/o} V \equiv \text{Hom}_{\text{aTL}_{n+m}} \left(\text{aTL}_{n+m} \otimes_{\text{TL}_m} V, \bar{M}\right),$$ Fusion quotient

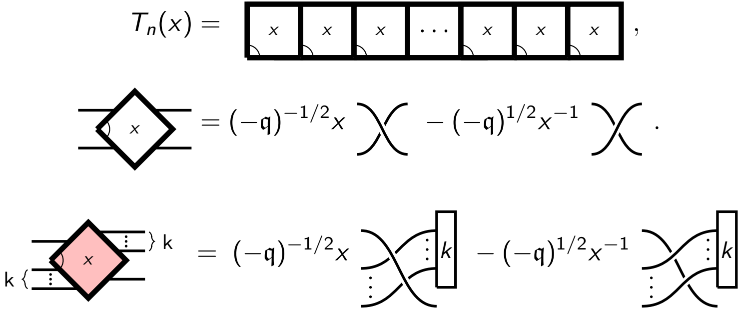
The TL Transfer matrix

$T_n(x) =$  ,



$= (-\mathfrak{q})^{-1/2} x$  $- (-\mathfrak{q})^{1/2} x^{-1}$  .

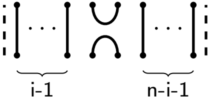
The TL Transfer matrix

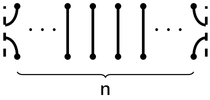


The box  is the Jones-Wenzl projector: unique $\rho_k \in \mathrm{TL}_k$ s.t. $\rho_k^2 = \rho_k$ and $e_i \rho_k = 0$ for all $i = 1, \dots, k-1$. Only exists for $k = 1, \dots, p-1$ ($\mathfrak{q} = e^{i\pi/(p+1)}$).

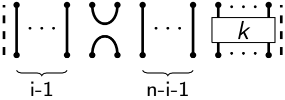
The Hamiltonian

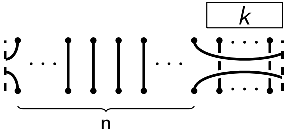
Classical: $H_n = \sum_{i=1}^{n} e_i$

$e_i =$ 

$e_n =$ 

With a defect $\tilde{H}_n^k = \sum_{i=1}^{n-1} \tilde{e}_i$,

$e_i =$ 

$e_n =$ 

The theorem

For $M \in \text{mod}(\text{aTL}_{n+m})$, let $\tilde{M} \equiv \phi^u(1_{\text{aTL}_n} \otimes \rho_k)M$, then

$$\tilde{H}_n^k|_{\tilde{M}} \sim H_n|_{M \div_f^u \text{TL}_k \rho_k} \quad \text{or} \quad H_n|_{M \div_f^o \text{TL}_k \rho_k}$$
$$\tilde{T}_n^k|_{\tilde{M}} \sim T_n|_{M \div_f^u \text{TL}_k \rho_k} \quad \text{or} \quad T_n|_{M \div_f^o \text{TL}_k \rho_k}.$$

In particular,

$$\tilde{Z}_n^k|_M = Z_n|_{M \div_f^u \text{TL}_k \rho_k} \text{or} \quad Z_n|_{M \div_f^o \text{TL}_k \rho_k},$$

i.e. if M describe some model, $M \div_f^{u/o} V$ describes the same model with a V-defect.

Pros and cons

Pros

- Reps can be arranged in families $A[n]$, $B[n]$, etc. which describes the regulariz. of the same sector of the CFT.

$$A[n+m] \div_f^{u/o} B[m] \simeq A[n+k] \div_f^{u/o} B[k],$$

- Products of defects are easy

$$(A \div_f^u B) \div_f^u C \simeq A \div_f^u \underbrace{(B \otimes C)}_{\text{Tensor prod. of the TL cat.}} .$$

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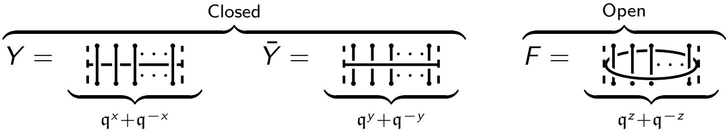
Cons

- Any info. that is basis dependent is lost.
- Computing the iso-class of a fusion quotient is (relatively) easy, finding what it is is difficult.

Example I: Twisted RSOS on A_p



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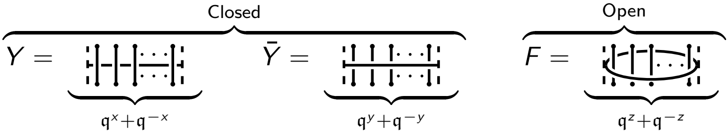


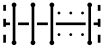
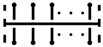
$$(x,y) \div^{o}_{f} (z) \simeq \sum_{\substack{k=|x-z|+1 \\ \text{step}=2}}^{\min(x+z-1,2(p+1)-(x+z)-1)} (k,y)\ ,$$

$$(x,y) \div^{u}_{f} (z) \simeq \sum_{\substack{k=|y-z|+1 \\ \text{step}=2}}^{\min(y+z-1,2(p+1)-(y+z)-1)} (x,k)\ ,$$

$$((1,1) \div^{o}_{f} (x)) \div^{u}_{f} (y) \simeq (x,y)\ .$$

Example I: Twisted RSOS on A_p

Closed

$Y =$  $\underbrace{}_{\mathfrak{q}^x+\mathfrak{q}^{-x}}$ $\bar{Y} =$  $\underbrace{}_{\mathfrak{q}^y+\mathfrak{q}^{-y}}$

Open

$F =$  $\underbrace{}_{\mathfrak{q}^z+\mathfrak{q}^{-z}}$

Non-unitary simple reps:

$$\text{Closed: } (\underbrace{\xi}_{\in\mathbb{C}};\underbrace{\alpha}_{\in\mathbb{Z}}), \qquad \text{Open: } (\underbrace{z}_{\in\mathbb{Z}_{\geq 1}}).$$

$$\underbrace{(\mathfrak{q}^x+\mathfrak{q}^{-x}; x-y)}_{\text{RSOS rep.}} \div^{u}_{f}(z) \simeq 0 \qquad \text{unless } z \leq p.$$

Example II: The XXZ spin-chain

The Hilbert space is $\mathbb{C}_2^{\otimes n}$ and

$$H_n(Q) = \sum_{j=1}^{n} \left(\sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4} \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right),$$

$$\sigma_{n+1}^z \equiv \sigma_1^z, \qquad \sigma_{n+1}^\pm \equiv Q^{\mp 2} \sigma_1^\pm .$$

$$Y = (-1)^n \left(\mathfrak{q}^{S_z} Q^{-1} + \mathfrak{q}^{-S_z} Q \right), \qquad \bar{Y} = \mathfrak{q}^{S_z} Q + \mathfrak{q}^{-S_z} Q^{-1},$$

Example II: The XXZ spin-chain

$$H^u_{n-1}(Q) = \sum_j^{n-1} (a_j^- a_{j+1}^+ + a_{j+1}^- a_j^+ + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}(a_j^z a_{j+1}^z - 1)) + \left((1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-\right)\sigma_n^+,$$

$$\begin{array}{ccc} & (\dots)\otimes|\uparrow\rangle & (\dots)\otimes|\downarrow\rangle \\ \sim & \begin{pmatrix} H_{n-1}(-Q\mathfrak{q}^{-1/2}) & \Delta \\ 0 & H_{n-1}(-Q\mathfrak{q}^{1/2}) \end{pmatrix}, & \end{array}$$

$$\Delta = (1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-$$

where $a_j^k = \sigma_j^k$, $k = z, \pm$, $j = 1, 2, \dots n-1$, and

$$a_n^z \equiv a_1^z, \qquad a_n^\pm \equiv (Q^2\mathfrak{q}^{-\sigma_n^z})^{\mp 1} a_1^\pm. \tag{2}$$

Example II: The XXZ spin-chain

$$Y = (-1)^n(\mathfrak{q}^{S_z}Q^{-1} + \mathfrak{q}^{-S_z}Q),$$

$$\bar{Y} \sim \begin{array}{c} \\ \end{array}\begin{pmatrix} \overset{(\dots)\otimes|\uparrow\rangle}{Q_-\mathfrak{q}^{\tilde{S}_z} + Q_-^{-1}\mathfrak{q}^{-\tilde{S}_z}} & \overset{(\dots)\otimes|\downarrow\rangle}{Q(\mathfrak{q}-\mathfrak{q}^{-1})^2\tilde{S}_-} \\ 0 & Q_+\mathfrak{q}^{\tilde{S}_z} + Q_+^{-1}\mathfrak{q}^{-\tilde{S}_z} \end{pmatrix},$$

where $Q_\pm \equiv -Q\mathfrak{q}^{\pm 1/2}$, and $\tilde{S}_-$, $\mathfrak{q}^{\pm\tilde{S}_z}$ are the standard $U_\mathfrak{q}(\mathfrak{sl}_2)$ generators on $n-1$ spins.

Example III: The XX spin chain

Let f_i, f_i^+, $i = 1, 2, \dots$ be a family of fermion operators, i.e.

$$\{f_i, f_j^+\} = \delta_{i,j}, \qquad \{f_i, f_j\} = \{f_i^+, f_j^+\} = 0.$$

The twisted XX spin chain is defined by the Hamiltonian

$$H_n = \sum_{j=1}^{n} -i(g_j + ig_{j+1})(g_j^+ + ig_{j+1}^+), \tag{2}$$

where $g_j \equiv f_j$ for all $j = 1, \dots, n$ and boundary conditions:

$g_{n+1} \equiv e^{i\theta} g_1,$	$g_{n+1}^+ \equiv e^{-i\theta} g_1^+$	(Classical)
$g_{n+1} \equiv e^{i\theta} g_1 + 2ig_{n+1},$	$g_{n+1}^+ \equiv e^{-i\theta} g_1^+$	(Spin-1/2 u-defect)
$g_{n+1} \equiv e^{i\theta} g_1,$	$g_{n+1}^+ \equiv e^{-i\theta} g_1^+ + 2ig_{n+1}^+$	(Spin-1/2 o-defect)
$g_{n+1} \equiv e^{i\theta} g_1 - 2ig_{n+2},$	$g_{n+1}^+ \equiv e^{-i\theta} g_1^+ + 2ig_{n+1}^+$	(Double Spin-1/2 defect)

Example III: The XX spin chain

$$
\begin{aligned}
g_{n+1} &\equiv e^{i\theta} g_1, & g^+_{n+1} &\equiv e^{-i\theta} g^+_1 & &\text{(Classical)}\\
g_{n+1} &\equiv e^{i\theta} g_1 + 2i g_{n+1}, & g^+_{n+1} &\equiv e^{-i\theta} g^+_1 & &\text{(Spin-1/2 u-defect)}\\
g_{n+1} &\equiv e^{i\theta} g_1, & g^+_{n+1} &\equiv e^{-i\theta} g^+_1 + 2i g^+_{n+1} & &\text{(Spin-1/2 o-defect)}\\
g_{n+1} &\equiv e^{i\theta} g_1 - 2i g_{n+2}, & g^+_{n+1} &\equiv e^{-i\theta} g^+_1 + 2i g^+_{n+1} & &\text{(Double Spin-1/2 defect)}
\end{aligned}
$$

- Eigenvalues are real if and only if $\theta \in \mathbb{R}$,
- Models with defects are diagonalizable if and only if

$$\theta \neq \frac{(2m+1)}{2}\pi n \quad \text{for any } m \in \mathbb{Z}.$$

- They all have the same spectrum (with double or quadruple multiplicities with defects).

So what now?

- Modular transformations?
- Massive gap in the rep. theory of aTL at root of unity.
- Rep. theory when hoops are non-diagonal is (mostly) in-existent.
- Physical meaning of those new boundary conditions?

The end

Thank you!

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