

APPROXIMATING THE R-TRANSFORM

- Starting point: Itzykson-Zuber conjecture (1980).

- Two-species particle ensemble:

$$a_1^{(N)} \geq \dots \geq a_N^{(N)} \quad \text{and} \quad b_1^{(N)} \geq \dots \geq b_N^{(N)}, \quad N=1,2,3,\dots$$

- At time N , system consists of N particles of species a , and N particles of species b .



- Associated sequence of entire functions

$$E_N: \mathbb{C} \rightarrow \mathbb{C}, \quad N=1,2,3,\dots,$$

defined by

$$E_N(z) = \int_{U(N)} e^z \text{Tr} \begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)} \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1} dU = \int_{U(N)} e^{z \sum a_i^{(N)} b_j^{(N)} |U_{ij}|^2} dU.$$

- Significance: $E_N(z)$ describes the interaction between unitarily invariant random Hermitian matrices coupled in a chain.
- $\left\{ \begin{array}{l} \text{Coupled random} \\ \text{matrices} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{2D quantum gravity} \\ \text{coupled to matter} \end{array} \right\}$

- Hermitian one-matrix model: A_N

Matrix density: $e^{\text{Tr} V(A)}$

Eigenvalue density: $e^{\sum V(a_i)} \prod_{i < j} (a_i - a_j)^2$

- Hermitian two-matrix model: (A_N, B_N)

Matrix density: $e^{\text{Tr} (V(A) + W(B) + zAB)}$

Eigenvalue density: $e^{\sum V(a_i) + W(b_i)} \prod_{i < j} (a_i - a_j)^2 (b_i - b_j)^2 \int_{U(N)} e^{z \text{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU$

- Suppose the system $a_1^{(N)} \geq \dots \geq a_N^{(N)}$, $b_1^{(N)} \geq \dots \geq b_N^{(N)}$ has the following features:

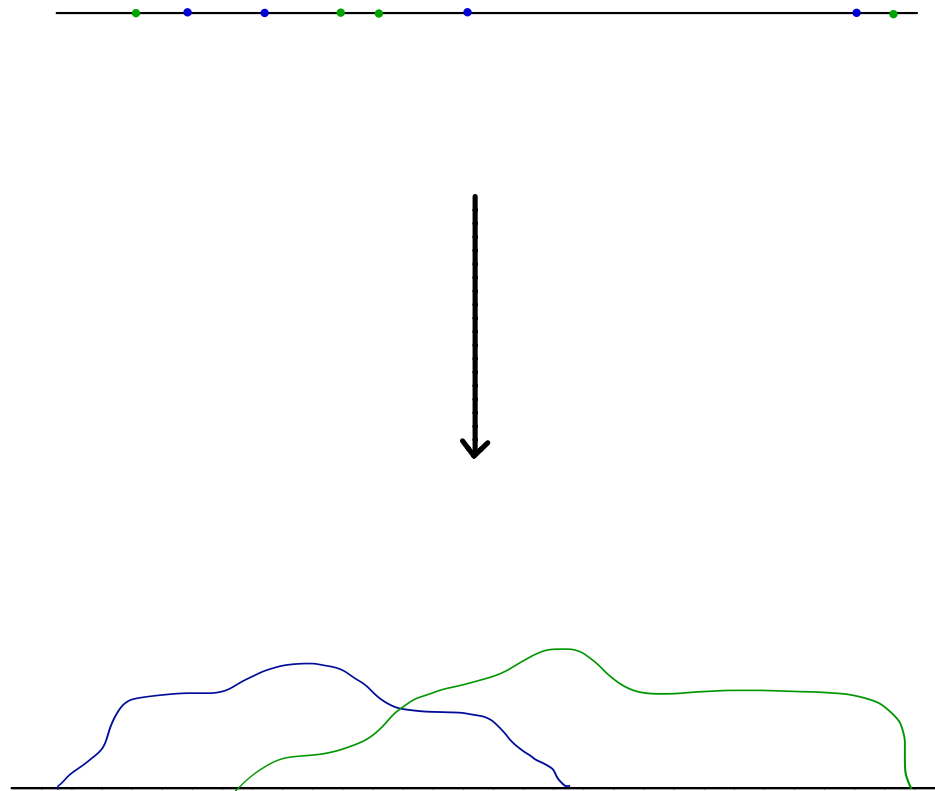
COMPACTNESS: There exists $M > 0$ such that $a_1^{(N)} \geq \dots \geq a_N^{(N)}$, $b_1^{(N)} \geq \dots \geq b_N^{(N)}$ are confined to $[-M\sqrt{N}, M\sqrt{N}]$;

CONVERGENCE: For each $k \in \mathbb{N}$, the limits

$$X_k = \lim_{N \rightarrow \infty} \frac{p_k\left(\frac{a_1^{(N)}}{\sqrt{N}}, \dots, \frac{a_N^{(N)}}{\sqrt{N}}\right)}{N} \quad \text{and} \quad Y_k = \lim_{N \rightarrow \infty} \frac{p_k\left(\frac{b_1^{(N)}}{\sqrt{N}}, \dots, \frac{b_N^{(N)}}{\sqrt{N}}\right)}{N}$$

exist, where $p_k(z_1, \dots, z_N) = z_1^k + \dots + z_N^k$.

- Meaning: after renormalization by $\sqrt{N}$, the empirical distribution of $a_1^{(N)} \geq \dots \geq a_N^{(N)}$ converges to a probability measure μ on $[-M, M]$, and the empirical distribution of $b_1^{(N)} \geq \dots \geq b_N^{(N)}$ converges to a probability measure ν on $[-M, M]$.



Conjecture [Itzykson-Zuber (1980), Collins (2006)]

There exists $\varepsilon > 0$ such that

$$\frac{1}{N^2} \log I_N(z) \rightarrow F(z)$$

as $N \rightarrow \infty$, uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < \varepsilon\}$.

Moreover, for each $d \in \mathbb{N}$, the d^{th} derivative of $F(z)$ at $z=0$ is a polynomial function of $x_1, \dots, x_d$ and $y_1, \dots, y_d$, bihomogeneous of degree d with respect to the grading $\deg(x_k) = \deg(y_k) = k$, with integer coefficients.

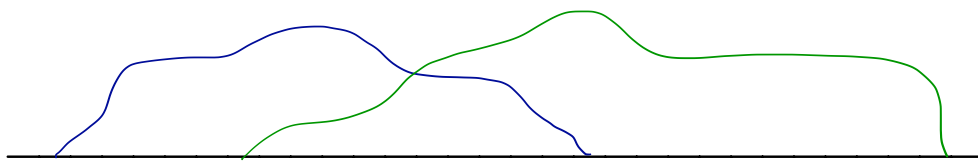
- Itzykson and Zuber were able to explicitly compute eight numbers:

$$\lim_{N \rightarrow \infty} N^{-2} \log I_N^{(d)}(0), \quad d=1, 2, 3, 4, 5, 6, 7, 8.$$

- For example, they showed that the limit of $N^{-2} \log I_N^{(4)}(0)$ exists and equals

$$\begin{aligned} & 6x_4y_4 - 24x_4y_3y_1 - 12x_4y_2^2 + 60x_4y_2y_1^2 - 30x_4y_1^4 \\ & - 24x_3x_1y_4 + 96x_3x_1y_3y_1 + 48x_3x_1y_2^2 - 240x_3x_1y_2y_1^2 + 120x_3x_1y_1^4 \\ & - 12x_2^2y_4 + 48x_2^2y_3y_1 + 18x_2^2y_2^2 - 108x_2^2y_2y_1^2 + 54x_2^2y_1^4 \\ & + 60x_2x_1^2y_4 - 240x_2x_1^2y_3y_1 - 108x_2x_1^2y_2^2 + 576x_2x_1^2y_2y_1^2 - 288x_2x_1^2y_1^4 \\ & - 30x_1^4y_4 + 120x_1^4y_3y_1 + 54x_1^4y_2^2 - 288x_1^4y_2y_1^2 + 144x_1^4y_1^4. \end{aligned}$$

- "Analysis first" approach to the IZ conjecture (Matytsin, Guionnet-Zeitouni):
estimate $\log I_N(z)$ directly.
- Upshot: $F(z) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(z)$ exists pointwise for $z \in \mathbb{R}$, can be described as the unique solution of a complicated variational problem - optimal transport from μ to ν with respect to a complicated functional $\mathcal{E}_z$.



$$F(z) = \inf_{\mu \in \mathcal{M}[-1,1]} \mathcal{E}_z$$

- This approach has nothing to say about complex convergence, or about the combinatorial part of the conjecture.

- "Combinatorics first" approach to IZ conjecture developed by Goulden, Guay-Paquet, and Novak.

- Starting point: the entire function of $2N+1$ complex variables defined by

$$E_N(z | \begin{smallmatrix} a_1 & \dots & a_N \\ b_1 & \dots & b_N \end{smallmatrix}) = \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU.$$

- Harish-Chandra / Itzykson-Zuber integral of rank N .

- The HCIZ integral of rank N has two basic symmetries.

- First is the $S(N) \times S(N)$ symmetry

$$E_N(z \mid \begin{smallmatrix} a_{\sigma(1)} & \dots & a_{\sigma(N)} \\ b_{\sigma(1)} & \dots & b_{\sigma(N)} \end{smallmatrix}) = E_N(z \mid \begin{smallmatrix} a_1 & \dots & a_N \\ b_1 & \dots & b_N \end{smallmatrix}),$$

which follows from left/right invariance of Haar measure.

- Second is the $S(2)$ symmetry

$$E_N(z \mid \begin{smallmatrix} b_1 & \dots & b_N \\ a_1 & \dots & a_N \end{smallmatrix}) = E_N(z \mid \begin{smallmatrix} a_1 & \dots & a_N \\ b_1 & \dots & b_N \end{smallmatrix}),$$

which follows from cyclic invariance of the trace.

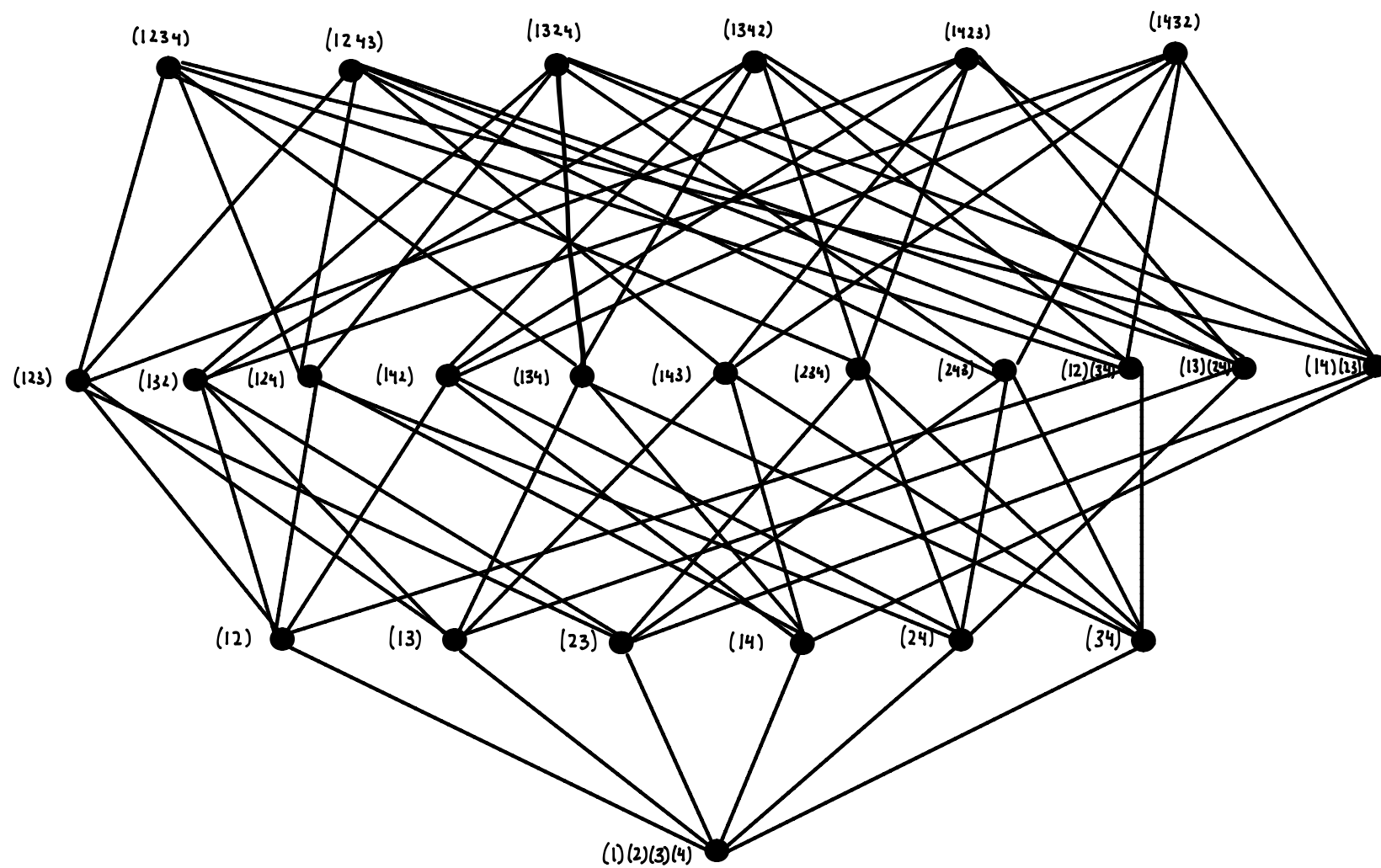
- Symmetries imply that Maclaurin series of E_N can be written

$$E_N(z | \begin{smallmatrix} a_1 \dots a_N \\ b_1 \dots b_N \end{smallmatrix}) = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N) W_N(\alpha, \beta),$$

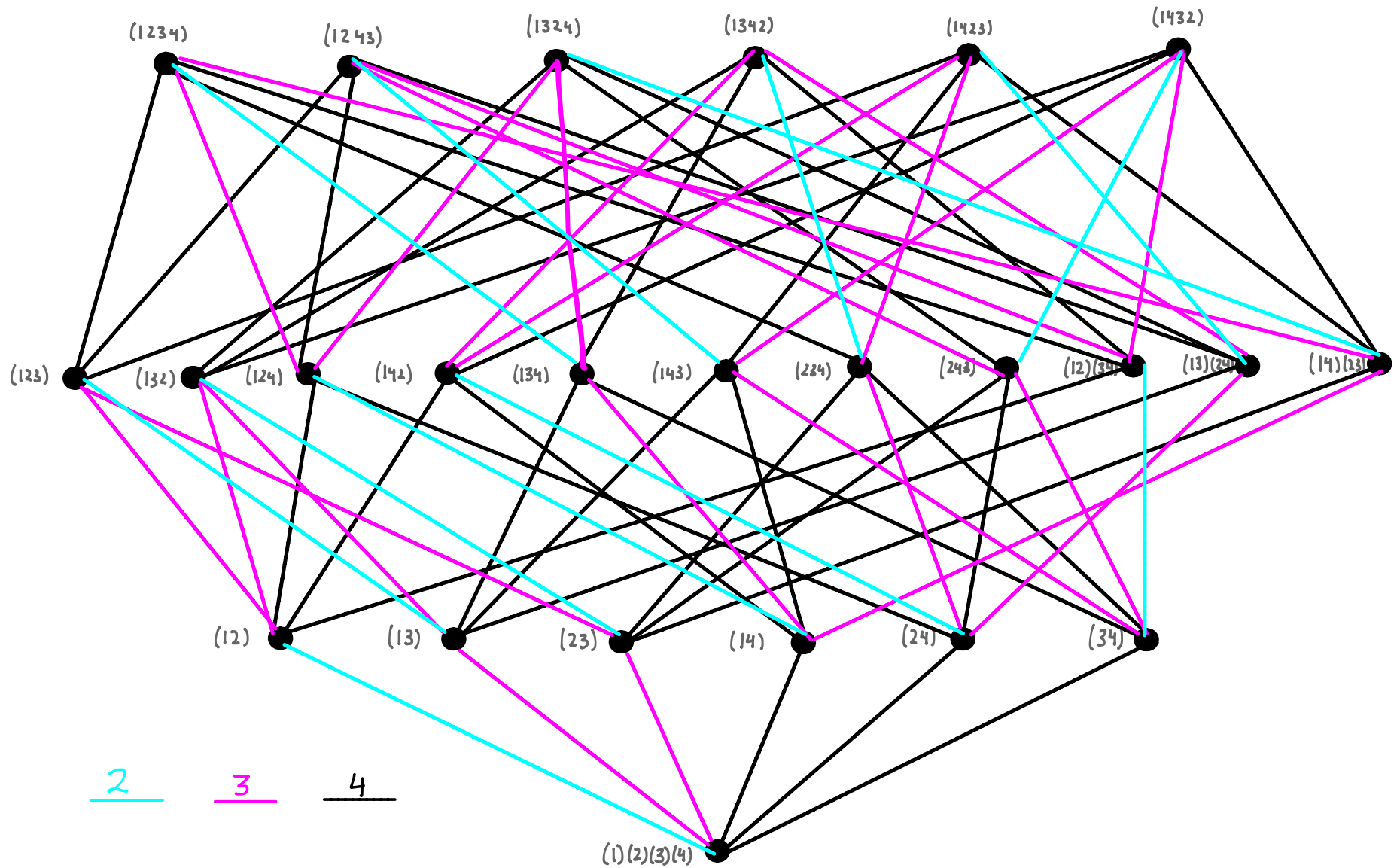
where $W_N(\alpha, \beta) = W_N(\beta, \alpha)$.

- Integrality phenomenon observed by Itzykson - Zuber must be coming from some combinatorial meaning bundled in $W_N(\alpha, \beta)$.

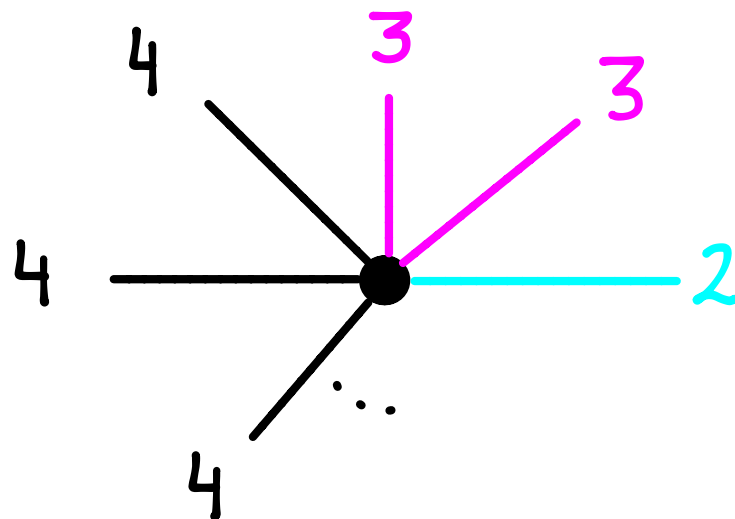
The Group $S(4)$



Biane-Stanley edge-labelling of $S(4)$



A fragment of $S(d)$



$$T = \begin{bmatrix} (12) & (13) & (14) \\ & (23) & (24) \\ & & (34) \end{bmatrix}$$

- Symmetries imply that Maclaurin series of $\mathbb{I}_N$ can be written

$$E_N(z \mid \begin{smallmatrix} a_1, \dots, a_N \\ b_1, \dots, b_N \end{smallmatrix}) = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N) W_N(\alpha, \beta),$$

where $W_N(\alpha, \beta) = W_N(\beta, \alpha)$.

Theorem (Goulden, Guay-Paquet, N., 2014)

For any $1 \leq d \leq N$, we have

$$W_N(\alpha, \beta) = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{W}^r(\alpha, \beta)}{N^r},$$

where $\vec{W}^r(\alpha, \beta)$ is the number of r -step monotone walks from C_{α} to C_{β} on $S(d)$.

- Symmetries imply that Maclaurin series of $\log E_N$ can be written

$$\log E_N(z | \begin{smallmatrix} a_1, \dots, a_N \\ b_1, \dots, b_N \end{smallmatrix}) = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N) H_N(\alpha, \beta),$$

where $H_N(\alpha, \beta) = H_N(\beta, \alpha)$.

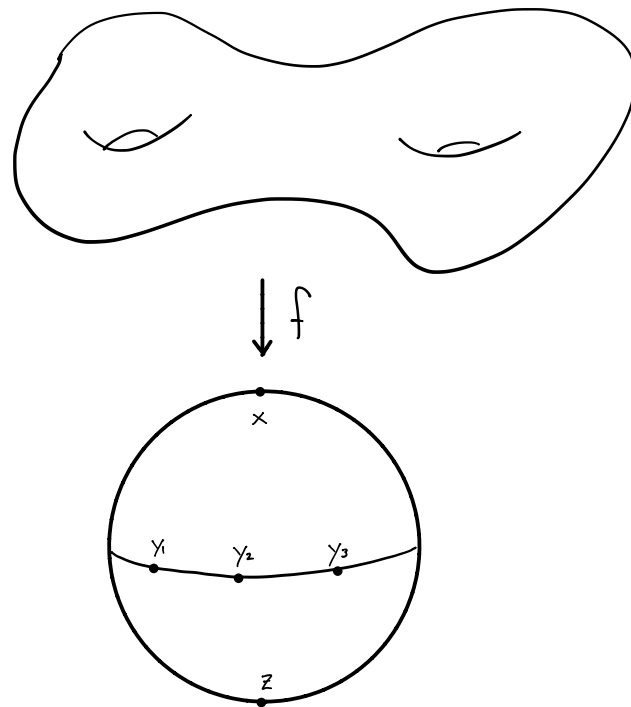
Theorem (Leading Derivatives)

For any $1 \leq d \leq N$, we have

$$H_N(\alpha, \beta) = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{H}^r(\alpha, \beta)}{N^r},$$

where $\vec{H}^r(\alpha, \beta)$ is the number of r -step monotone walks from C_{α} to C_{β} on $S(d)$ whose steps and endpoints generate a transitive subgroup of $S(d)$.

- The total number $H^r(\alpha, \beta)$ of r -step transitive walks from C_α to C_β is a **double Hurwitz number**.
- Enumerative geometry: $H^r(\alpha, \beta)$ counts isomorphism classes of pairs (Σ, f) consisting of a compact Riemann surface Σ and a degree d holomorphic function $f: \Sigma \rightarrow \mathbb{P}^1$ with $r+2$ branch points $x, y_1, \dots, y_r, z \in \mathbb{P}^1$, such that f has branching profile α over x , β over z , and $(2, 1, \dots, 1)$ over each y_i .

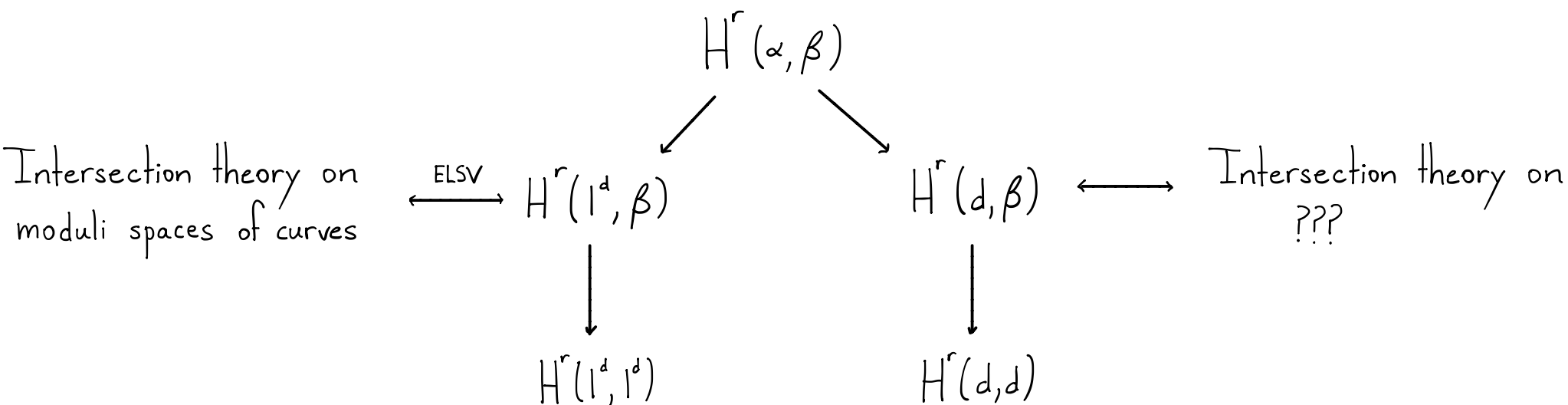


Theorem (Okounkov, 2001)

The generating function

$$H = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} \sum_{r=0}^{\infty} \frac{t^r}{r!} H^r(\alpha, \beta)$$

is a solution of the 2D Toda lattice equations.

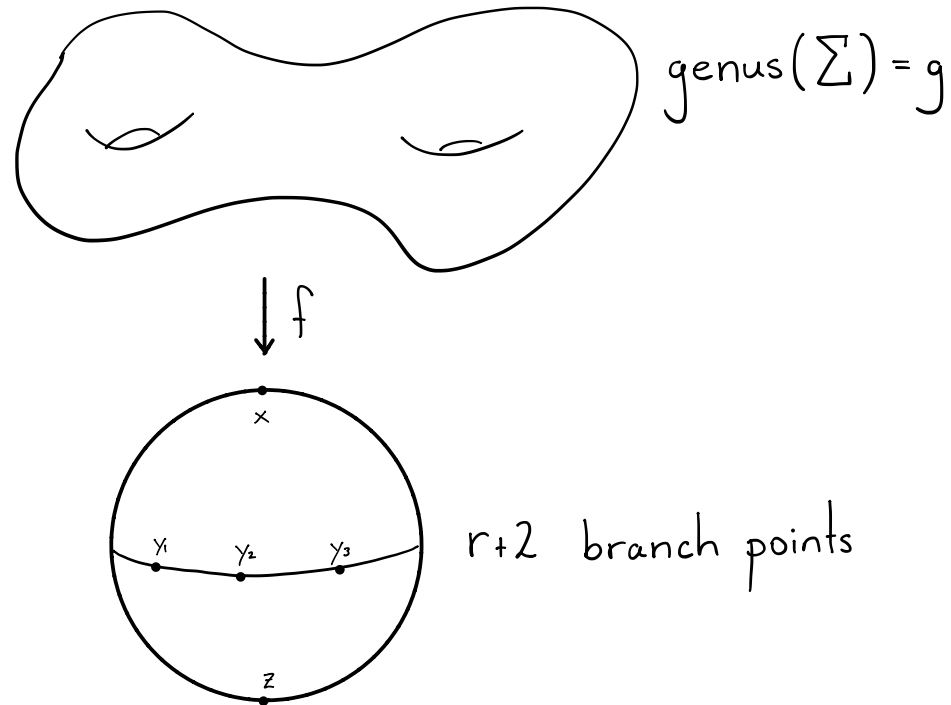


- HCIZ integral: an approximate generating function for **monotone** double Hurwitz numbers.

$$\log E_N \left(z \mid \begin{smallmatrix} a_1 \dots a_N \\ b_1 \dots b_N \end{smallmatrix} \right) = \sum_{d=1}^N \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_\alpha \left(\frac{a_1}{\sqrt{N}}, \dots, \frac{a_N}{\sqrt{N}} \right) p_\beta \left(\frac{b_1}{\sqrt{N}}, \dots, \frac{b_N}{\sqrt{N}} \right) \sum_{r=0}^{\infty} \left(-\frac{1}{N} \right)^r \vec{H}^r(\alpha, \beta) + \dots$$

- In the $N \rightarrow \infty$ limit, the logarithm of the HCIZ integral becomes the monotone version of Okounkov's double Hurwitz generating function.
- Conceptually, explains Itzykson-Zuber conjecture.

- Riemann-Hurwitz: $H^r(\alpha, \beta)$ non-zero if and only if $r+2 = 2g + l(\alpha) + l(\beta)$.



- Genus parameterization:

$$H_g(\alpha, \beta) := H^{2g-2+l(\alpha)+l(\beta)}(\alpha, \beta).$$

- Maclaurin series of $\log E_N(z)$, initial form ($r=r$):

$$\log E_N(z) = \sum_{d=1}^N \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha} \left(\frac{a_1^{(N)}}{\sqrt{N}}, \dots, \frac{a_N^{(N)}}{\sqrt{N}} \right) p_{\beta} \left(\frac{b_1^{(N)}}{\sqrt{N}}, \dots, \frac{b_N^{(N)}}{\sqrt{N}} \right) \sum_{r=0}^{\infty} \left(-\frac{1}{N} \right)^r \vec{H}^r(\alpha, \beta) + \dots$$

- Maclaurin series of $\log E_N(z)$, genus form ($r=2g-2+\ell(\alpha)+\ell(\beta)$):

$$\log E_N(z) = \sum_{d=1}^N \frac{z^d}{d!} N^2 \sum_{\alpha, \beta \vdash d} \frac{p_{\alpha} \left(\frac{a_1^{(N)}}{\sqrt{N}}, \dots, \frac{a_N^{(N)}}{\sqrt{N}} \right)}{N^{\ell(\alpha)}} \frac{p_{\beta} \left(\frac{b_1^{(N)}}{\sqrt{N}}, \dots, \frac{b_N^{(N)}}{\sqrt{N}} \right)}{N^{\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} + \dots$$

- For each fixed $d \in \mathbb{N}$,

$$\lim_{N \rightarrow \infty} \sum_{\alpha, \beta \vdash d} \frac{P_{\alpha} \left(\frac{a_1^{(N)}}{\sqrt{N}}, \dots, \frac{a_N^{(N)}}{\sqrt{N}} \right)}{N^{l(\alpha)}} \frac{P_{\beta} \left(\frac{b_1^{(N)}}{\sqrt{N}}, \dots, \frac{b_N^{(N)}}{\sqrt{N}} \right)}{N^{l(\beta)}} (-1)^{l(\alpha)+l(\beta)} \sum_{g=0}^{\infty} \frac{H_g(\alpha, \beta)}{N^{2g}} = \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{l(\alpha)+l(\beta)} \vec{H}_0(\alpha, \beta).$$

- Means that each fixed derivative of $F_N(z) = \frac{1}{N^2} \log E_N(z)$ converges,

$$\lim_{N \rightarrow \infty} F_N^{(d)}(0) = \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{l(\alpha)+l(\beta)} \vec{H}_0(\alpha, \beta).$$

- Candidate for the Itzykson-Zuber free energy:

$$F(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{l(\alpha)+l(\beta)} \vec{H}_0(\alpha, \beta).$$

- What could go wrong?

1) The candidate $F(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{l(\alpha)+l(\beta)} \vec{H}_0(\alpha, \beta)$ is not absolutely summable; the numbers $\vec{H}_0(\alpha, \beta)$ grow too fast as $d \rightarrow \infty$.

2) The analytic functions $F_N(z) = \frac{1}{N^2} \log E_N(z)$ don't have a common domain of holomorphy; the zeros of $E_N(z)$ pinch the origin as $N \rightarrow \infty$.

- Neither of these occur.

Theorem (Hurwitz, 1891)

$$H_o(1^d, \beta) = |C_\beta| (d + \ell(\beta) - 2)! d^{\ell(\beta)-3} \prod_{i=1}^{\ell(\beta)} \frac{\beta_i^{\beta_i}}{\beta_i!}.$$

Theorem (Goulden, Guay-Paquet, N., 2012)

$$\vec{H}_o(1^d, \beta) = |C_\beta| (2d+1)^{\overline{\ell(\beta)-3}} \prod_{i=1}^{\ell(\beta)} \binom{2\beta_i}{\beta_i}.$$

Theorem (Vakil, 2001)

$$H_1(1^d, \beta) = |C_\beta| (d + \ell(\beta))! \prod_{i=1}^{\ell(\beta)} \frac{\beta_i^{\beta_i}}{\beta_i!} \cdot \frac{1}{24} \left(d^{\ell(\beta)} - d^{\ell(\beta)-1} - \sum_{k=2}^{\ell(\beta)} (k-2)! d^{\ell(\beta)-k} e_k(\beta) \right).$$

Theorem (Goulden, Guay-Paquet, N., 2013)

$$\vec{H}_1(1^d, \beta) = |C_\beta| \prod_{i=1}^{\ell(\beta)} \binom{2\beta_i}{\beta_i} \cdot \frac{1}{24} \left((2d+1)^{\overline{\ell(\beta)}} - 3(2d+1)^{\overline{\ell(\beta)-1}} - \sum_{k=2}^{\ell(\beta)} (k-2)! (2d+1)^{\ell(\beta)-k} e_k(2\beta-1) \right).$$

Theorem (Ekedahl, Lando, Shapiro, Vainshtein, 2001)

For any $g \geq 2$, the generating function

$$H_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\beta \vdash d} \frac{\gamma_{\beta}}{(2g-2+d+l(\beta))!} H_g(1^d, \beta)$$

becomes rational after a complicated change of variables.

Theorem (Goulden, Guay-Paquet, N., 2013)

For any $g \geq 2$, the generating function

$$\vec{H}_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\beta \vdash d} \gamma_{\beta} \vec{H}_g(1^d, \beta)$$

becomes rational after a complicated change of variables.

Theorem (Goulden, Guay-Paquet, N., 2013)

For any $g \geq 2$, the series

$$\vec{H}_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \vec{H}_g(1^d, 1^d)$$

is a rational function of $s(z)$, where

$$s'(z) = {}_2F_1\left(\frac{2}{3}, \frac{4}{3}, \frac{3}{2}; \frac{27}{2}z\right)$$

and

$${}_2F_1(a, b, c; z) = 1 + \frac{z^1}{1!} \frac{ab}{c} + \frac{z^2}{2!} \frac{a(a+1)b(b+1)}{c(c+1)} + \frac{z^3}{3!} \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)} + \dots$$

is the Gauss hypergeometric function.

Corollary

For any $g \geq 0$, the series $\vec{H}_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \vec{H}_g(1^d, 1^d)$ has radius of convergence $\frac{2}{27}$.

Observation (Di Francesco): Fix a prime p . Higman proved

$$\lim_{N \rightarrow \infty} \frac{1}{N^3} \log_p G(N) = \frac{2}{27},$$

where $G(N)$ is number of groups of order p^N .

Theorem (Goulden, Guay-Paquet, N., 2016)

For any $g \geq 0$, the series

$$\vec{H}_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} \vec{H}_g(\alpha, \beta)$$

has radius of convergence at least $\frac{1}{54}$ and at most $\frac{2}{27}$.

- Conclusion: the most obvious necessary condition for the Itzykson-Zuber conjecture to hold is sufficient: if

$$\inf \{ |z| : E_N(z) = 0 \text{ for some } N \in \mathbb{N} \} > 0,$$

then $\exists \varepsilon > 0$ such that

$$\frac{1}{N^2} \log E_N(z) \rightarrow F(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{l(\alpha) + l(\beta)} \vec{H}_0(\alpha, \beta),$$

uniformly on compact subsets of $D_{\varepsilon}(0)$.

- Does the obvious necessary condition always hold?

- Recall the expansions

$$E_N(z \mid \begin{smallmatrix} a_1, \dots, a_N \\ b_1, \dots, b_N \end{smallmatrix}) = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N) W_N(\alpha, \beta)$$

$$\log E_N(z \mid \begin{smallmatrix} a_1, \dots, a_N \\ b_1, \dots, b_N \end{smallmatrix}) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N) H_N(\alpha, \beta).$$

- Leading Derivatives Theorem describes $N \rightarrow \infty$ behavior of $W_N(\alpha, \beta)$ and $H_N(\alpha, \beta)$ for $\alpha, \beta \vdash d$ fixed.

- We need information on $d \rightarrow \infty$ behavior with N fixed.

Theorem

For any $d, N \in \mathbb{N}$, we have

$$W_N(\alpha, \beta) = \frac{|C_\alpha| |C_\beta|}{d!} \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} \chi^\lambda(\alpha) \chi^\lambda(\beta) \prod_{\square \in \lambda} \frac{1}{N + c(\square)},$$

where $\chi^\lambda(\alpha)$ is the trace of a permutation from C_α in the irrep V^λ of $S(d)$.

Theorem

For any $d, N \in \mathbb{N}$, we have

$$W_N(\alpha, \beta) = \mathbb{E} \left[\omega^\lambda(\alpha) \omega^\lambda(\beta) \prod_{\square \in \lambda} \frac{1}{N + c(\square)} \mid \ell(\lambda) \leq N \right],$$

where λ is a Plancherel-random Young diagram, $\mathbb{P}(\lambda) = \frac{(\dim V^\lambda)^2}{d!}$,

and $\omega^\lambda(\alpha)$ is the eigenvalue of C_α acting in V^λ .

- This conditional expectation can be analyzed for N fixed, $d \rightarrow \infty$.

Theorem

There exists $\varepsilon > 0$ such that

$$\frac{1}{N^2} \log \int_{U(N)} e^{z \operatorname{Tr} [a_1^{(N)} \cdots a_N^{(N)}] u [b_1^{(N)} \cdots b_N^{(N)}] u^{-1}} dU \rightarrow \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{\ell(\alpha) + \ell(\beta)} \vec{H}_0(\alpha, \beta)$$

as $N \rightarrow \infty$, uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < \varepsilon\}$.

CONJECTURE

There exists $\varepsilon > 0$ such that

$$\frac{1}{N^2} \log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)} \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU \sim \sum_{g=0}^{\infty} \frac{F_g(z)}{N^{2g}}$$

uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < \varepsilon\}$, where

$$F_g(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} x_{\alpha} y_{\beta} (-1)^{\ell(\alpha) + \ell(\beta)} \vec{H}_g(\alpha, \beta).$$

- Return to formulation of Itzykson-Zuber conjecture, alter it.

- Single species particle ensemble:

$$b_1^{(N)} \geq \dots \geq b_N^{(N)}, \quad N=1,2,3,\dots$$

- At time N , system consists of N particles.



- Associated sequence of entire functions

$$I_N: \mathbb{C}^{N+1} \rightarrow \mathbb{C}, \quad N=1,2,3,\dots,$$

defined by

$$I_N(z; a_1, \dots, a_N) = \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU.$$

- Significance: let $X^{(N)} = [X_{ij}^{(N)}]_{i,j=1}^N$ be uniformly random sample from the compact symplectic manifold

$$\mathcal{O}_N = \left\{ U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1} : U \in U(N) \right\}.$$

Then $I_N(z; a_1, \dots, a_N)$ is the Laplace transform of $X_{11}^{(N)}, \dots, X_{NN}^{(N)}$,

$$I_N(z; a_1, \dots, a_N) = \int_{\mathcal{O}_N} e^{z(a_1 X_{11} + \dots + a_N X_{NN})} dX.$$

- Fix $n \in \mathbb{N}$, set $I_N(z; a_1, \dots, a_n) := I_N(z; a_1, \dots, a_n, 0, \dots, 0)$.

- Want $N \rightarrow \infty$ asymptotics of $I_N(z; a_1, \dots, a_n)$.

- Want to understand dependence on $N \rightarrow \infty$ behaviour of the system

$$b_1^{(N)} \geq \dots \geq b_N^{(N)}, \quad N = 1, 2, 3, \dots$$

- If you know the Leading Derivatives Theorem, it's easy to see scaling limits.

- Maclaurin series of $\log I_N(z; a_1, \dots, a_n)$ is

$$\sum_{d=1}^N \frac{z^d}{d!} N^2 \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \dots, a_n)}{N^{l(\alpha)}} \frac{p_\beta(b_1^{(N)}, \dots, b_n^{(N)})}{N^{l(\beta)}} (-1)^{l(\alpha)+l(\beta)} \frac{1}{N^d} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} + \dots$$

||

$$\sum_{d=1}^N \frac{z^d}{d!} N \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \dots, a_n)}{N^{l(\alpha)-1}} \frac{p_\beta\left(\frac{b_1^{(N)}}{N}, \dots, \frac{b_n^{(N)}}{N}\right)}{N^{l(\beta)}} (-1)^{l(\alpha)+l(\beta)} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} + \dots$$

- Maclaurin series of $\log I_N(z; a_1, \dots, a_n)$ is

$$\sum_{d=1}^N \frac{z^d}{d!} N \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \dots, a_n)}{N^{l(\alpha)-1}} \frac{p_\beta\left(\frac{b_1^{(n)}}{N}, \dots, \frac{b_n^{(n)}}{N}\right)}{N^{l(\beta)}} (-1)^{l(\alpha)+l(\beta)} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} + \dots$$

- If the limit $y_k = \lim_{N \rightarrow \infty} \frac{p_k\left(\frac{b_1^{(n)}}{N}, \dots, \frac{b_n^{(n)}}{N}\right)}{N}$ exists, then

$$\lim_{N \rightarrow \infty} \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \dots, a_n)}{N^{l(\alpha)-1}} \frac{p_\beta\left(\frac{b_1^{(n)}}{N}, \dots, \frac{b_n^{(n)}}{N}\right)}{N^{l(\beta)}} (-1)^{l(\alpha)+l(\beta)} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} = \sum_{\beta \vdash d} p_d(a_1, \dots, a_n) y_\beta (-1)^{1+l(\beta)} \vec{H}_0(d, \beta).$$

Theorem: There exists $\varepsilon > 0$ such that, as $N \rightarrow \infty$,

$$\frac{1}{N} \log I_N(z; a_1, \dots, a_n) \rightarrow F(z; a_1, \dots, a_n),$$

uniformly on compact subsets of $\{(z, a_1, \dots, a_n) \in \mathbb{C}^{n+1} : |za_i| < \varepsilon\}$,

where $F(z; a_1, \dots, a_n) = Q(za_1) + \dots + Q(za_n)$ and

$$Q(z) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\beta \vdash d} \gamma_{\beta} (-1)^{l(\beta)} \vec{H}_0(d, \beta)$$

is a generating function for the genus zero one-part monotone double Hurwitz numbers.

- In this new regime, the Itzykson-Zuber free energy $F_0(z)$ is replaced by a multivariate analytic function $F_0(z; a_1, \dots, a_n)$.
- But, $F_0(z; a_1, \dots, a_n) = \sum_{i=1}^n Q(za_i)$.
- Whereas $F_0(z)$ is a generating function for the monotone double Hurwitz numbers $\vec{H}_0(\alpha, \beta)$, which are complicated, $Q(z)$ is a generating function for the one-part monotone double Hurwitz numbers $\vec{H}_0(d, \beta)$, which are much simpler.

Theorem:
$$\vec{H}_0(d, \beta) = (d-1)! \prod_{i=1}^{\ell(\beta)} \text{Cat}_{\beta_i-1}.$$

Theorem: $\vec{H}_o(d, \beta) = (d-1)! \prod_{i=1}^{l(\beta)} \text{Cat}_{\beta_{i-1}}.$

Definition: The free cumulants of $\gamma_1, \gamma_2, \gamma_3, \dots$ are the numbers $f_1, f_2, f_3, \dots$ defined recursively by

$$\gamma_k = \sum_{\mathcal{P} \in \text{NC}(d)} \prod_{B \in \mathcal{P}} f_{|B|}.$$

The generating function

$$R(z) = f_1 + z f_2 + z^2 f_3 + \dots$$

is the R -transform.

Corollary: $\sum_{\beta \vdash d} \gamma_\beta (-1)^{l(\beta)} \vec{H}_o(d, \beta) = (d-1)! f_d.$

Theorem [Guionnet - Maïda, 2006]: There exists $\varepsilon > 0$ such that, as $N \rightarrow \infty$,

$$\frac{1}{N} \log I_N(z; a_1, \dots, a_n) \rightarrow F(z; a_1, \dots, a_n),$$

uniformly on compact subsets of $\{(z, a_1, \dots, a_n) \in \mathbb{C}^{n+1} : |za_i| < \varepsilon\}$,

where $F(z; a_1, \dots, a_n) = Q(za_1) + \dots + Q(za_n)$ and

$$Q(z) = \sum_{d=1}^{\infty} \frac{z^d}{d} f_d$$

is the antiderivative of the R-transform.

"The R-transform is a limit of orbital Laplace transforms"

CONJECTURE: There exists $\varepsilon > 0$ such that, as $N \rightarrow \infty$,

$$\frac{1}{N} \log I_N(z; a_1, \dots, a_n) \sim \sum_{r=0}^{\infty} \frac{F_r(z; a_1, \dots, a_n)}{N^r}$$

uniformly on compact subsets of $\{(z, a_1, \dots, a_n) \in \mathbb{C}^{n+1} : |za_i| < \varepsilon\}$,

where

$$F_r(z; a_1, \dots, a_n) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\substack{\alpha, \beta, g \\ 2g-1+\ell(\alpha)=r}} p_{\alpha}(a_1, \dots, a_n) \gamma_{\beta} (-1)^{1+\ell(\beta)} \vec{H}_g(\alpha, \beta).$$

- Remark: second order term only receives genus zero contributions.
- Beginings of a new approach to higher-order freeness...

- We have seen that the R -transform arises as a particular scaling limit of the HCIZ integral.
- Combinatorially, free cumulants arise as degenerations of genus zero Hurwitz numbers.
- Let us use the HCIZ integral to peer even deeper into the mysterious mind of Dan-Virgil Voiculescu...

- Schur: up to isomorphism, irreducible rational representations of $GL_N(\mathbb{C})$ are parameterized by configurations $b_1 > \dots > b_N$ of N hard particles on $\mathbb{Z}$.

- For example, $N-1 > N-2 > \dots > 0$ corresponds to the trivial representation $A \mapsto 1$.

The configuration $N > N-2 > N-3 > \dots > 0$ corresponds to the tautological rep. $A \mapsto A$.

- Schur's character formula: if (V, ρ) is the irrep corresponding to $b_1 > \dots > b_N$, then

$$\mathrm{Tr} \rho(A) = \frac{\det [z_i^{b_j}]}{\det [z_i^{N-j}]},$$

where $z_1, \dots, z_N \in \mathbb{C}^\times$ are the eigenvalues of A .

- Given a particle ensemble $b_1^{(N)} > \dots > b_N^{(N)}$, $N \in \mathbb{N}$, have an associated sequence of functions $K_N: (\mathbb{C}^*)^N \rightarrow \mathbb{C}$ defined by

$$K_N(z_1, \dots, z_N) = \frac{\text{Tr } \rho^{(b_1^{(N)}, \dots, b_N^{(N)})} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{bmatrix}}{\text{Tr } \rho^{(b_1^{(N)}, \dots, b_N^{(N)})} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}.$$

- Voiculescu's problem (as reinterpreted by Kerov and Vershik): classify all particle ensembles such that, for each fixed $n \in \mathbb{N}$, the functions

$$K_N(z_1, \dots, z_n) = K_N(z_1, \dots, z_n, 1, \dots, 1), \quad N \in \mathbb{N}$$

converge uniformly on compact subsets of a neighborhood of $(1, \dots, 1) \in \mathbb{C}^n$.

For each such sequence, compute $K(z_1, \dots, z_n) = \lim_{N \rightarrow \infty} K_N(z_1, \dots, z_n)$.

Lecture Notes in Mathematics

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- A new perspective: instead of using Schur's character formula, use the Harish-Chandra character formula.
- If $A \in GL_N(\mathbb{C})$ has eigenvalues $e^{za_1}, \dots, e^{za_N} \in \mathbb{C}^\times$, then normalized character of A in irrep labelled by $b_1, \dots, b_N$ is

$$L_N(z; a_1, \dots, a_N) = \prod_{i < j} \frac{za_i - za_j}{e^{za_i} - e^{za_j}} \cdot \int_{U(N)} e^{z \operatorname{Tr} [a_1 \dots a_N] u [b_1^{(N)} \dots b_N^{(N)}]} u^{-1} du.$$

- Can use this to analyze $N \rightarrow \infty$ asymptotics of $GL_N(\mathbb{C})$ characters.

- For simplicity, look at the case $n=1$, i.e. asymptotics of normalized characters of matrices of the form

$$\begin{bmatrix} e^z & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

- Have

$$L_N(z) = \left(\frac{z}{e^z - 1}\right)^{N-1} \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}} u \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} u^{-1} du.$$

- Thus

$$\log L_N(z) = \log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}} u \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} u^{-1} du - N \log \left(\frac{z}{e^z - 1}\right) + \log \left(\frac{z}{e^z - 1}\right).$$

• Thus

$$\log L_N(z) = \log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU - N \log \left(\frac{z}{e^z - 1} \right) + \log \left(\frac{z}{e^z - 1} \right).$$

$\swarrow$
 N times Q-transform

$\swarrow$
 Q transform
 of $U[0,1]$

• Get convergence precisely when the first two terms cancel.

Theorem: Uniform-on-compact convergence of the sequence of $GL_N(\mathbb{C})$ characters corresponding to $b_1^{(N)} > \dots > b_N^{(N)}$ occurs precisely when the limit distribution of $\frac{b_1^{(N)}}{N} > \dots > \frac{b_N^{(N)}}{N}$ is uniform on $[0,1]$.

- END -